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Research Article

Transformation Method for Generating Periodic Solutions of Abel's Differential Equation

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This paper deals with Abel's differential equation. We suppose that $r = r(t)$ is a periodic particular solution of Abel's differential equation and, then, by means of the transformation method and the fixed point theory, present an alternative method of generating the other periodic solutions of Abel's differential equation.

1. Introduction

The nonlinear Abel type first-order differential equation

$$\frac{dx}{dt} = a(t)x^3 + b(t)x^2 + c(t)x + d(t) \tag{1}$$

plays an important role in many physical and technical applications [1, 2]. The mathematical properties of Equation (1) have been intensively investigated in the mathematical and physical literature [3–11]. S. S. Misthry, S. D. Maharaj, and P. G. L. Leach [12] introduced a new transformation at the boundary that leads to an Abel's equation and showed explicitly that a variety of exact solutions can be generated from the Abel equation. R. Mohanlal, R. Narain, and S. D. Maharaj [13] systematically studied the differential equations that arise using the Lie symmetry infinitesimal generators and showed that several nonlinear equations, including Bernoulli equations and Abel equations of the second kind, in addition to Riccati equations, are generated by assuming functional relationships on the gravitational potentials. They demonstrated that these equations may be solved exactly. The models found admit a linear equation of state for the radial pressure and the energy density. The energy conditions are satisfied and the matter variables are well behaved. Mak et al. [14] and Mak and Harko [15] have presented a solution generating technique for Abel's type ordinary differential equation; both suppose that $y = y_1(t)$ is a particular solution of Equation (1) and then, by means of the transformation methods, present an alternative method of generating the general solution of the Abel's differential Equation (1) from a particular one.

Stimulated by the works of [14, 15], in this paper, we suppose that $r = r(t)$ is a periodic particular solution of Abel's differential equation and then, by means of the transformation method, Equation (1) is turned into Bernoulli's equation, thereby we get two other periodic solutions of Abel's equation; on the other hand, Equation (1) is turned into Abel's type equation; by using the fixed point theory, we obtain another periodic solution of Abel's equation.

The rest of the paper is arranged as follows: in Section 2, we give four lemmas to be used later; in Section 3, we give four theorems about the existence of a unique nonzero periodic solution on Abel's type differential equation; in Section 4, suppose $r = r(t)$ is a periodic particular solution of Abel's differential equation; we derive the existence of other periodic solutions of Abel's differential equation. We end this paper with a brief discussion.

2. Some Lemmas and Abbreviations

Lemma 1 (see [16]). *Consider the following:*

$$\frac{dx}{dt} = a(t)x + b(t), \tag{2}$$

where $a(t), b(t)$ are $\omega-$periodic continuous functions; if $\int_0^{\omega} a(t)dt \neq 0$, then Equation (2) has a unique $\omega-$periodic *continuous solution* $\eta(t)$, $\operatorname{mod}(\eta) \subset \operatorname{mod}(a(t), b(t))$, *and* $\eta(t)$ *can be written as follows:*

$$\eta(t) = \begin{cases} \int_{-\infty}^{t} e^{\int_s^t a(\tau)d\tau} b(s)\, ds, & \int_0^{\omega} a(t)\, dt < 0 \\ -\int_{t}^{+\infty} e^{\int_s^t a(\tau)d\tau} b(s)\, ds, & \int_0^{\omega} a(t)\, dt > 0 \end{cases} \tag{3}$$

Lemma 2 (see [17]). *Suppose that an* ω*−periodic sequence* $\{f_n(t)\}$ *is convergent uniformly on any compact set of* R, $f(t)$ *is an* ω*−periodic function, and* $\operatorname{mod}(f_n) \subset \operatorname{mod}(f)(n = 1, 2, \cdots)$*; then* $\{f_n(t)\}$ *is convergent uniformly on* R.

Lemma 3 (see [18]). *Suppose* V *is a metric space and* C *is a convex closed set of* V*; its boundary is* ∂C*; if* $T : V \longrightarrow V$ *is a continuous compact mapping, such that* $T(\partial C) \subset C$*, then* T *has a fixed point on* C.

Consider one-dimensional periodic differential equation

$$\frac{dx}{dt} = f(t.x), \tag{4}$$

here, $f : R \times I \longrightarrow R$ is a continuous function, and $f(t+\omega, x) = f(t, x), \omega > 0, I \subset R$.

Lemma 4 (see [19]). *If* $f(t, x)$ *has three-order continuous partial derivatives on* x*, and* $f'''_{xxx}(t, x) \neq 0$*, then (4) has at most three periodic continuous solutions.*

For the sake of convenience, suppose that $f(t)$ is an ω-periodic continuous function on R; we denote

$$f_M = \sup_{t\in[0,\omega]} f(t), \qquad f_L = \inf_{t\in[0,\omega]} f(t), \tag{5}$$

3. A Unique Nonzero Periodic Solution on Abel's Type Equation

In this section, we consider Abel's type equation and give four results about the existence and uniqueness of the nonzero periodic solution on Abel's type equation.

Theorem 5. *Consider Abel's type equation:*

$$\frac{dx}{dt} = a(t) x^3 + b(t) x^2, \tag{6}$$

where $a(t), b(t)$ *are* ω*−periodic continuous functions; suppose that the following conditions hold:*

$$(H_1)\ a(t) < 0, \qquad (H_2)\ b(t) > 0. \tag{7}$$

Then Equation (6) has a unique positive ω*−periodic continuous solution* $\gamma(t)$*, and*

$$-\left(\frac{b}{a}\right)_M \le \gamma(t) \le -\left(\frac{b}{a}\right)_L. \tag{8}$$

Proof. (1) Firstly, we prove the existence of a positive ω−periodic continuous solution of Equation (6).

Suppose

$$S = \{\varphi(t) \in C(R, R) \mid \varphi(t+\omega) = \varphi(t)\}. \tag{9}$$

Given any $\varphi(t), \psi(t) \in S$, the distance is defined as follows:

$$\rho(\varphi, \psi) = \sup_{t\in[0,\omega]} |\varphi(t) - \psi(t)|. \tag{10}$$

Thus (S, ρ) is a complete metric space; take a convex closed set of S as follows:

$$B = \left\{\varphi(t) \in C(R, R) \mid \varphi(t+\omega) = \varphi(t),\ -\left(\frac{b}{a}\right)_M \le \varphi(t) \le -\left(\frac{b}{a}\right)_L,\ \operatorname{mod}(\varphi) \subset \operatorname{mod}(a, b)\right\}. \tag{11}$$

Given any $\varphi(t) \in B$, consider the following equation:

$$\frac{dx}{dt} = a(t) x^3 + b(t)\varphi(t) x. \tag{12}$$

since $b(t), \varphi(t)$ are ω−periodic continuous functions, it follows that $b(t)\varphi(t)$ is an ω−periodic continuous function; let

$$x^{-2}(t) = u(t). \tag{13}$$

Then Equation (12) can be turned into

$$\frac{du}{dt} = -2b(t)\varphi(t) u - 2a(t). \tag{14}$$

By (11) and (H_2), it follows that

$$-2b_M\left(-\left(\frac{b}{a}\right)_L\right) \le -2b(t)\varphi(t) \le -2b_L\left(-\left(\frac{b}{a}\right)_M\right) < 0. \tag{15}$$

That is

$$2b_M\left(\frac{b}{a}\right)_L \le -2b(t)\varphi(t) \le 2b_L\left(\frac{b}{a}\right)_M < 0. \tag{16}$$

Thus it follows that

$$-\frac{2}{\omega}\int_0^{\omega} b(t)\varphi(t)\, dt < 0. \tag{17}$$

According to Lemma 1, Equation (14) has a unique ω−periodic continuous solution as follows:

$$\eta(t) = -2\int_{-\infty}^{t} e^{-2\int_s^t b(\tau)\varphi(\tau)d\tau} a(s)\, ds. \tag{18}$$

By (11),(16), and (18), we have

$$\begin{aligned}\eta(t) &= -2\int_{-\infty}^{t} e^{-2\int_s^t b(\tau)\varphi(\tau)d\tau} b(s)\frac{a(s)}{b(s)}ds \\ &\geq -2\left(\frac{a}{b}\right)_M \int_{-\infty}^{t} e^{-2\int_s^t b(\tau)\varphi(\tau)d\tau} b(s)\,ds \\ &\geq -2\left(\frac{a}{b}\right)_M \int_{-\infty}^{t} e^{2(b/a)_L\int_s^t b(\tau)d\tau} b(s)\,ds \\ &= \frac{(a/b)_M}{(b/a)_L}\left[e^{2(b/a)_L\int_s^t b(\tau)d\tau}\right]_{-\infty}^{t} \\ &= \frac{(a/b)_M}{(b/a)_L}\left[1-e^{2(b/a)_L\int_{-\infty}^t b(\tau)d\tau}\right] \\ &\geq \frac{(a/b)_M}{(b/a)_L}\left[1-e^{2(b/a)_L\int_{-\infty}^t b_L d\tau}\right] = \left(\frac{a}{b}\right)_M^2.\end{aligned} \tag{19}$$

and

$$\begin{aligned}\eta(t) &= -2\int_{-\infty}^{t} e^{-2\int_s^t b(\tau)\varphi(\tau)d\tau} b(s)\frac{a(s)}{b(s)}ds \\ &\leq -2\left(\frac{a}{b}\right)_L \int_{-\infty}^{t} e^{-2\int_s^t b(\tau)\varphi(\tau)d\tau} b(s)\,ds \\ &\leq -2\left(\frac{a}{b}\right)_L \int_{-\infty}^{t} e^{2(b/a)_M\int_s^t b(\tau)d\tau} b(s)\,ds \\ &= \frac{(a/b)_L}{(b/a)_M}\left[e^{2(b/a)_M\int_s^t b(\tau)d\tau}\right]_{-\infty}^{t} \\ &= \frac{(a/b)_L}{(b/a)_M}\left[1-e^{2(b/a)_M\int_{-\infty}^t b(\tau)d\tau}\right] \\ &\leq \frac{(a/b)_L}{(b/a)_M}\left[1-e^{2(b/a)_M\int_{-\infty}^t b_M d\tau}\right] = \left(\frac{a}{b}\right)_L^2.\end{aligned} \tag{20}$$

Thus

$$\left(\frac{a}{b}\right)_M^2 \leq \eta(t) \leq \left(\frac{a}{b}\right)_L^2. \tag{21}$$

By (13), we get that Equation (6) has a unique positive ω–periodic continuous solution as follows:

$$\zeta(t) = \frac{1}{\sqrt{-2\int_{-\infty}^t e^{-2\int_s^t b(\tau)\varphi(\tau)d\tau} a(s)\,ds}}. \tag{22}$$

and we have

$$-\left(\frac{b}{a}\right)_M \leq \zeta(t) \leq -\left(\frac{b}{a}\right)_L. \tag{23}$$

Thus $\zeta(t) \in B$.

Define a mapping as follows:

$$(T\varphi)(t) = \zeta(t) = \frac{1}{\sqrt{-2\int_{-\infty}^t e^{-2\int_s^t b(\tau)\varphi(\tau)d\tau} a(s)\,ds}}. \tag{24}$$

Thus, if given any $\varphi(t) \in B$, then $(T\varphi)(t) \in B$; hence $T : B \longrightarrow B$.

Now, we prove that the mapping T is a compact operator.

Consider any sequence $\{\varphi_n(t)\} \subset B (n = 1, 2, \cdots)$; then it follows that

$$\begin{gathered}-\left(\frac{b}{a}\right)_M \leq \varphi_n(t) \leq -\left(\frac{b}{a}\right)_L, \\ \operatorname{mod}(\varphi_n) \subset \operatorname{mod}(a,b),\ (n = 1,2,\cdots).\end{gathered} \tag{25}$$

On the other hand, $(T\varphi_n)(t) = x_{\varphi_n}(t)$ satisfies

$$\frac{dx_{\varphi_n}(t)}{dt} = a(t)\,x_{\varphi_n}^3(t) + b(t)\,\varphi_n(t)\,x_{\varphi_n}(t). \tag{26}$$

Thus we have

$$\begin{gathered}\left|\frac{dx_{\varphi_n}(t)}{dt}\right| \leq a_L\left(\left(\frac{b}{a}\right)_L\right)^3 + b_M\left(-\left(\frac{b}{a}\right)_L\right)^2, \\ \operatorname{mod}\left(x_{\varphi_n}(t)\right) \subset \operatorname{mod}(a,b).\end{gathered} \tag{27}$$

Hence $\{dx_{\varphi_n}(t)/dt\}$ is uniformly bounded; therefore, $\{x_{\varphi_n}(t)\}$ is uniformly bounded and equicontinuous on R; by the theorem of Ascoli-arzela, for any sequence $\{x_{\varphi_n}(t)\} \subset B$, there exists a subsequence (also denoted by $\{x_{\varphi_n}(t)\}$) such that $\{x_{\varphi_n}(t)\}$ is convergent uniformly on any compact set of R; also combined with Lemma 2, $\{x_{\varphi_n}(t)\}$ is convergent uniformly on R; that is to say, T is relatively compact on B.

Next, we prove that T is a continuous operator.

Suppose $\{\varphi_k(t)\} \subset B, \varphi(t) \in B$, and

$$\varphi_k(t) \longrightarrow \varphi(t), \quad (k \longrightarrow \infty). \tag{28}$$

By (24), we have that

$$\begin{aligned}&\left|(T\varphi_k)(t) - (T\varphi)(t)\right| \\ &= \left|\frac{1}{\sqrt{-2\int_{-\infty}^t e^{-2\int_s^t b(\tau)\varphi_k(\tau)d\tau} a(s)\,ds}} - \frac{1}{\sqrt{-2\int_{-\infty}^t e^{-2\int_s^t b(\tau)\varphi(\tau)d\tau} a(s)\,ds}}\right| \\ &= \left|-\frac{1}{2\sqrt{(\xi_1)^3}}\left(-2\int_{-\infty}^t\left(e^{-2\int_s^t b(\tau)\varphi_k(\tau)d\tau} - e^{-2\int_s^t b(\tau)\varphi(\tau)d\tau}\right)a(s)\,ds\right)\right|\end{aligned}$$

$$= \left| -\frac{1}{2\sqrt{(\xi_1)}^3} \left(-2\int_{-\infty}^{t} e^{\xi_2} \left(-2\int_{s}^{t} b(\tau) \cdot (\varphi_k(\tau) - \varphi(\tau))\, d\tau \right) a(s)\, ds \right) \right|. \quad (29)$$

Here, ξ_1 is between $-2\int_{-\infty}^{t} e^{-2\int_s^t b(\tau)\varphi_k(\tau)d\tau} a(s)ds$ and $-2\int_{-\infty}^{t} e^{-2\int_s^t b(\tau)\varphi(\tau)d\tau} a(s)ds$; thus ξ_1 is between $(a/b)_M^2$ and $(a/b)_L^2$; ξ_2 is between $-2\int_s^t b(\tau)\varphi_k(\tau)d\tau$ and $-2\int_s^t b(\tau)\varphi(\tau)d\tau$; thus ξ_2 is between $2b_M(b/a)_L(t-s)$ and $2b_L(b/a)_M(t-s)$, so we have

$$\begin{aligned} &\left|(T\varphi)_k(t) - (T\varphi)(t)\right| \\ &\le \left| -\frac{1}{2\sqrt{\left((a/b)_M^2\right)}^3} \left(-2\int_{-\infty}^{t} e^{2b_L(b/a)_M(t-s)} \left(-2\int_{s}^{t} b(\tau) \cdot (\varphi_k(\tau) - \varphi(\tau))\, d\tau \right) a(s)\, ds \right) \right| \\ &\le \frac{-2b_M a_L}{\sqrt{\left((a/b)_M^2\right)}^3} \left(\int_{-\infty}^{t} e^{2b_L(b/a)_M(t-s)} (t - s)\, ds \right) \rho(\varphi_k, \varphi) \\ &= \frac{-2b_M a_L}{\sqrt{\left((a/b)_M^2\right)}^3 \left(2b_L(b/a)_M\right)^2} \rho(\varphi_k, \varphi). \end{aligned} \quad (30)$$

By (28), it follows that

$$(T\varphi_k)(t) \longrightarrow (T\varphi)(t), \quad (k \longrightarrow \infty). \quad (31)$$

Therefore, T is continuous; by (24), it is easy to see that $T(\partial B) \subset B$; according to Lemma 3, T has at least a fixed point on B; the fixed point is the ω−periodic continuous solution $\gamma(t)$ of Equation (6), and

$$-\left(\frac{b}{a}\right)_M \le \gamma(t) \le -\left(\frac{b}{a}\right)_L. \quad (32)$$

(2) We prove that Equation (6) has exactly a unique nonzero periodic solution $\gamma(t)$.

Let

$$f(t, x) = a(t)x^3 + b(t)x^2. \quad (33)$$

Then

$$f'''_{xxx}(t, x) = 6a(t) < 0. \quad (34)$$

By (34), according to Lemma 4, Equation (6) has at most three periodic continuous solutions; we know that Equation (6) has three periodic continuous solutions: $\gamma(t)$ and double periodic solutions $\gamma_1(t) = \gamma_2(t) = 0$; thus it follows that Equation (6) has exactly a unique positive ω−periodic continuous solution $\gamma(t)$, and

$$-\left(\frac{b}{a}\right)_M \le \gamma(t) \le -\left(\frac{b}{a}\right)_L. \quad (35)$$

This is the end of the proof of Theorem 5. □

Theorem 6. *Consider Equation (6); $a(t), b(t)$ are ω−periodic continuous functions; suppose that the following conditions hold:*

$$\begin{aligned} &(H_1)\ a(t) < 0, \\ &(H_2)\ b(t) < 0. \end{aligned} \quad (36)$$

Then Equation (6) has a unique negative ω−periodic continuous solution $\gamma(t)$, and

$$-\left(\frac{b}{a}\right)_M \le \gamma(t) \le -\left(\frac{b}{a}\right)_L. \quad (37)$$

Proof. (1) Firstly, we prove the existence of a negative ω−periodic continuous solution of Equation (6).

Suppose

$$S = \{\varphi(t) \in C(R, R) \mid \varphi(t + \omega) = \varphi(t)\}. \quad (38)$$

Given any $\varphi(t), \psi(t) \in S$, the distance is defined as follows:

$$\rho(\varphi, \psi) = \sup_{t\in[0,\omega]} \left|\varphi(t) - \psi(t)\right|. \quad (39)$$

Thus (S, ρ) is a complete metric space; take a convex closed set of S as follows:

$$\begin{aligned} B = \Bigg\{&\varphi(t) \in C(R, R) \mid \varphi(t + \omega) = \varphi(t),\ -\left(\frac{b}{a}\right)_M \\ &\le \varphi(t) \le -\left(\frac{b}{a}\right)_L,\ \operatorname{mod}(\varphi) \subset \operatorname{mod}(a, b)\Bigg\}. \end{aligned} \quad (40)$$

Given any $\varphi(t) \in B$, consider the following equation:

$$\frac{dx}{dt} = a(t)x^3 + b(t)\varphi(t)x. \quad (41)$$

Since $b(t), \varphi(t)$ are ω−periodic continuous functions; it follows that $b(t)\varphi(t)$ is an ω−periodic continuous function; let

$$x^{-2}(t) = u(t). \quad (42)$$

Then (41) can be turned into

$$\frac{du}{dt} = -2b(t)\varphi(t)u - 2a(t). \quad (43)$$

By (40) and (H_2), it follows that

$$\begin{aligned} -2b_L\left(-\left(\frac{b}{a}\right)_M\right) &\le -2b(t)\varphi(t) \le -2b_M\left(-\left(\frac{b}{a}\right)_L\right) \\ &< 0. \end{aligned} \quad (44)$$

That is

$$2b_L\left(\frac{b}{a}\right)_M \le -2b(t)\varphi(t) \le 2b_M\left(\frac{b}{a}\right)_L < 0. \tag{45}$$

Thus it follows that

$$-\frac{2}{\omega}\int_0^{\omega} b(t)\varphi(t)\,dt < 0. \tag{46}$$

According to Lemma 1, Equation (43) has a unique $\omega-$periodic continuous solution as follows:

$$\eta(t) = -2\int_{-\infty}^{t} e^{-2\int_s^t b(\tau)\varphi(\tau)d\tau} a(s)\,ds. \tag{47}$$

By (40),(45), and (47), we have

$$\begin{aligned}\eta(t) &= -2\int_{-\infty}^{t} e^{-2\int_s^t b(\tau)\varphi(\tau)d\tau} b(s)\frac{a(s)}{b(s)}ds\\ &\ge -2\left(\frac{a}{b}\right)_L\int_{-\infty}^{t} e^{-2\int_s^t b(\tau)\varphi(\tau)d\tau} b(s)\,ds\\ &\ge -2\left(\frac{a}{b}\right)_L\int_{-\infty}^{t} e^{2(b/a)_M\int_s^t b(\tau)d\tau} b(s)\,ds\\ &= \frac{(a/b)_L}{(b/a)_M}\left[e^{2(b/a)_M\int_s^t b(\tau)d\tau}\right]_{-\infty}^{t}\\ &= \frac{(a/b)_L}{(b/a)_M}\left[1-e^{2(b/a)_M\int_{-\infty}^t b(\tau)d\tau}\right]\\ &\ge \frac{(a/b)_L}{(b/a)_M}\left[1-e^{2(b/a)_M\int_{-\infty}^t b_M d\tau}\right] = \left(\frac{a}{b}\right)_L^2.\end{aligned} \tag{48}$$

and

$$\begin{aligned}\eta(t) &= -2\int_{-\infty}^{t} e^{-2\int_s^t b(\tau)\varphi(\tau)d\tau} b(s)\frac{a(s)}{b(s)}ds\\ &\le -2\left(\frac{a}{b}\right)_M\int_{-\infty}^{t} e^{-2\int_s^t b(\tau)\varphi(\tau)d\tau} b(s)\,ds\\ &\le -2\left(\frac{a}{b}\right)_M\int_{-\infty}^{t} e^{2(b/a)_L\int_s^t b(\tau)d\tau} b(s)\,ds\\ &= \frac{(a/b)_M}{(b/a)_L}\left[e^{2(b/a)_L\int_s^t b(\tau)d\tau}\right]_{-\infty}^{t}\\ &= \frac{(a/b)_M}{(b/a)_L}\left[1-e^{2(b/a)_L\int_{-\infty}^t b(\tau)d\tau}\right]\\ &\le \frac{(a/b)_M}{(b/a)_L}\left[1-e^{2(b/a)_L\int_{-\infty}^t b_L d\tau}\right] = \left(\frac{a}{b}\right)_M^2.\end{aligned} \tag{49}$$

Thus

$$\left(\frac{a}{b}\right)_L^2 \le \eta(t) \le \left(\frac{a}{b}\right)_M^2. \tag{50}$$

By (42), we get that Equation (6) has a unique negative $\omega-$periodic continuous solution as follows:

$$\zeta(t) = -\frac{1}{\sqrt{-2\int_{-\infty}^{t} e^{-2\int_s^t b(\tau)\varphi(\tau)d\tau} a(s)\,ds}}. \tag{51}$$

and we have

$$-\left(\frac{b}{a}\right)_M \le \zeta(t) \le -\left(\frac{b}{a}\right)_L. \tag{52}$$

Thus $\zeta(t) \in B$.

Define a mapping as follows:

$$(T\varphi)(t) = \zeta(t) = -\frac{1}{\sqrt{-2\int_{-\infty}^{t} e^{-2\int_s^t b(\tau)\varphi(\tau)d\tau} a(s)\,ds}}. \tag{53}$$

Thus if given any $\varphi(t) \in B$, then $(T\varphi)(t) \in B$; hence $T : B \longrightarrow B$.

Now, we prove that the mapping T is a compact operator.

Consider any sequence $\{\varphi_n(t)\} \subset B(n = 1, 2, \cdots)$; then it follows that

$$\begin{gathered}-\left(\frac{b}{a}\right)_M \le \varphi_n(t) \le -\left(\frac{b}{a}\right)_L,\\ \operatorname{mod}(\varphi_n) \subset \operatorname{mod}(a,b),\ \ (n = 1,2,\cdots).\end{gathered} \tag{54}$$

On the other hand, $(T\varphi_n)(t) = x_{\varphi_n}(t)$ satisfies

$$\frac{dx_{\varphi_n}(t)}{dt} = a(t)x_{\varphi_n}^3(t) + b(t)\varphi_n(t)x_{\varphi_n}(t). \tag{55}$$

Thus we have

$$\begin{gathered}\left|\frac{dx_{\varphi_n}(t)}{dt}\right| \le -a_L\left(\left(\frac{b}{a}\right)_M\right)^3 - b_L\left(-\left(\frac{b}{a}\right)_M\right)^2,\\ \operatorname{mod}\left(x_{\varphi_n}(t)\right) \subset \operatorname{mod}(a,b).\end{gathered} \tag{56}$$

Hence $\{dx_{\varphi_n}(t)/dt\}$ is uniformly bounded; therefore, $\{x_{\varphi_n}(t)\}$ is uniformly bounded and equicontinuous on R; by the theorem of Ascoli-arzela, for any sequence $\{x_{\varphi_n}(t)\} \subset B$, there exists a subsequence (also denoted by $\{x_{\varphi_n}(t)\}$) such that $\{x_{\varphi_n}(t)\}$ is convergent uniformly on any compact set of R; also combined with Lemma 2, $\{x_{\varphi_n}(t)\}$ is convergent uniformly on R; that is to say, T is relatively compact on B.

Next, we prove that T is a continuous operator.
Suppose $\{\varphi_k(t)\} \subset B, \varphi(t) \in B$, and

$$\varphi_k(t) \longrightarrow \varphi(t), \quad (k \longrightarrow \infty). \tag{57}$$

$$\begin{aligned}
&\left|(T\varphi_k)(t) - (T\varphi)(t)\right| \\
&= \left| -\frac{1}{\sqrt{-2\int_{-\infty}^{t} e^{-2\int_s^t b(\tau)\varphi_k(\tau)d\tau} a(s)\,ds}} + \frac{1}{\sqrt{-2\int_{-\infty}^{t} e^{-2\int_s^t b(\tau)\varphi(\tau)d\tau} a(s)\,ds}} \right| \\
&= \left| -\frac{1}{2\sqrt{(\xi_1)^3}} \left(-2\int_{-\infty}^{t} \left(e^{-2\int_s^t b(\tau)\varphi_k(\tau)d\tau} - e^{-2\int_s^t b(\tau)\varphi(\tau)d\tau}\right) a(s)\,ds\right) \right| \\
&= \left| -\frac{1}{2\sqrt{(\xi_1)^3}} \left(-2\int_{-\infty}^{t} e^{\xi_2} \left(-2\int_s^t b(\tau) \cdot (\varphi_k(\tau) - \varphi(\tau))\,d\tau\right) a(s)\,ds\right) \right|,
\end{aligned} \tag{58}$$

Here, ξ_1 is between $-2\int_{-\infty}^{t} e^{-2\int_s^t b(\tau)\varphi_k(\tau)d\tau} a(s)ds$ and $-2\int_{-\infty}^{t} e^{-2\int_s^t b(\tau)\varphi(\tau)d\tau} a(s)ds$; thus ξ_1 is between $(a/b)_L^2$ and $(a/b)_M^2$; ξ_2 is between $-2\int_s^t b(\tau)\varphi_k(\tau)d\tau$ and $-2\int_s^t b(\tau)\varphi(\tau)d\tau$; thus ξ_2 is between $2b_L(b/a)_M(t-s)$ and $2b_M(b/a)_L(t-s)$, so we have

$$\begin{aligned}
&\left|(T\varphi_k)(t) - (T\varphi)(t)\right| \\
&\le \left| -\frac{1}{2\sqrt{\left((a/b)_L^2\right)^3}} \left(-2\int_{-\infty}^{t} e^{2b_M(b/a)_L(t-s)} \left(-2\int_s^t b(\tau) \cdot (\varphi_k(\tau) - \varphi(\tau))\,d\tau\right) a(s)\,ds\right) \right| \\
&\le \frac{2b_L a_L}{\sqrt{\left((a/b)_L^2\right)^3}} \left(\int_{-\infty}^{t} e^{2b_M(b/a)_L(t-s)} (t - s)\,ds\right) \rho(\varphi_k, \varphi) \\
&= \frac{2b_L a_L}{\sqrt{\left((a/b)_L^2\right)^3} \left(2b_M (b/a)_L\right)^2} \rho(\varphi_k, \varphi).
\end{aligned} \tag{59}$$

By (57), it follows that

$$(T\varphi_k)(t) \longrightarrow (T\varphi)(t), \quad (k \longrightarrow \infty). \tag{60}$$

Therefore, T is continuous, by (53), it is easy to see that $T(\partial B) \subset B$; according to Lemma 3, T has at least a fixed point on B; the fixed point is the negative ω–periodic continuous solution $\gamma(t)$ of Equation (6), and

$$-\left(\frac{b}{a}\right)_M \le \gamma(t) \le -\left(\frac{b}{a}\right)_L. \tag{61}$$

(2) We prove that Equation (6) has exacatly a unique nonzero periodic solution $\gamma(t)$.

Let

$$f(t, x) = a(t)x^3 + b(t)x^2. \tag{62}$$

Then

$$f'''_{xxx}(t, x) = 6a(t) < 0. \tag{63}$$

By (63), according to Lemma 4, Equation (6) has at most three periodic continuous solutions; we know that Equation (6) has three periodic continuous solutions: $\gamma(t)$ and double periodic solutions $\gamma_1(t) = \gamma_2(t) = 0$; thus it follows that Equation (6) has exactly a unique negative ω–periodic continuous solution $\gamma(t)$, and

$$-\left(\frac{b}{a}\right)_M \le \gamma(t) \le -\left(\frac{b}{a}\right)_L. \tag{64}$$

This is the end of the proof of Theorem 6. □

Similar to the proofs of Theorems 5 and 6, we can get the following.

Theorem 7. *Consider Equation (6); $a(t), b(t)$ are ω–periodic continuous functions; suppose that the following conditions hold:*

$$\begin{aligned} &(H_1)\ a(t) > 0, \\ &(H_2)\ b(t) > 0. \end{aligned} \tag{65}$$

Then Equation (6) has a unique negative ω–periodic continuous solution $\gamma(t)$, and

$$-\left(\frac{b}{a}\right)_M \le \gamma(t) \le -\left(\frac{b}{a}\right)_L. \tag{66}$$

Theorem 8. *Consider Equation (6); $a(t), b(t)$ are ω–periodic continuous functions; suppose that the following conditions hold:*

$$\begin{aligned} &(H_1)\ a(t) > 0, \\ &(H_2)\ b(t) < 0. \end{aligned} \tag{67}$$

Then Equation (6) has a unique positive ω–periodic continuous solution $\gamma(t)$, and

$$-\left(\frac{b}{a}\right)_M \le \gamma(t) \le -\left(\frac{b}{a}\right)_L. \quad (68)$$

4. Existence of Periodic Solutions on Abel's Equation

In this section, we discuss the existence of periodic solutions on Abel differential equation.

We suppose that $r = r(t)$ is an $\omega-$ periodic continuous particular solution of Equation (1) and, then, by means of the transformation method, present an alternative method of generating the other $\omega-$ periodic continuous solutions (solution) of Equation (1) from a particular one.

Theorem 9. *Consider Equation (1); $a(t), b(t), c(t), d(t)$ are all $\omega-$periodic continuous functions and $r_1 = r_1(t)$ is an $\omega-$periodic continuous particular solution of Equation (1); suppose that the following conditions hold:*

$$\begin{gathered}(H_1)\ r_1(t) = -\frac{b(t)}{3a(t)},\\ (H_2)\ a(t) < 0,\\ (H_3)\ c(t) - \frac{b^2(t)}{3a(t)} > 0.\end{gathered} \quad (69)$$

Then Equation (1) has other two $\omega-$periodic continuous solutions as follows:

$$\gamma_2(t) = \frac{1}{\sqrt{-2\int_{-\infty}^{t} e^{-2\int_s^t [c(\tau)-b^2(\tau)/3a(\tau)]d\tau} a(s)\,ds}} - \frac{b(t)}{3a(t)}, \quad (70)$$

and

$$\gamma_3(t) = \frac{1}{-\sqrt{-2\int_{-\infty}^{t} e^{-2\int_s^t [c(\tau)-b^2(\tau)/3a(\tau)]d\tau} a(s)\,ds}} - \frac{b(t)}{3a(t)}. \quad (71)$$

Proof. Let

$$y(t) = x(t) - r_1(t), \quad (72)$$

where $x(t)$ is the unique solution with the initial value $x(t_0) = x_0$ of Equation (1); differentiating both sides of (72) along the solution of Equation (1), we get

$$\begin{aligned}\frac{dy}{dt} &= \frac{dx}{dt} - \frac{dr_1}{dt} = a(t)\left[x^3(t) - r_1^3(t)\right]\\ &+ b(t)\left[x^2(t) - r_1^2(t)\right] + c(t)\left[x(t) - r_1(t)\right]\\ &= a(t)\left(x(t) - r_1(t)\right)^3\\ &+ \left[b(t) + 3a(t)r_1(t)\right]\left(x(t) - r_1(t)\right)^2\\ &+ \left[2b(t)r_1(t) + c(t) + 3a(t)r_1^2(t)\right]\left(x(t) - r_1(t)\right)\\ &= a(t)y^3 + \left[b(t) + 3a(t)r_1(t)\right]y^2\\ &+ \left[2b(t)r_1(t) + c(t) + 3a(t)r_1^2(t)\right]y = a(t)y^3\\ &+ \left[2b(t)r_1(t) + c(t) + 3a(t)r_1^2(t)\right]y = a(t)y^3\\ &+ \left[c(t) - \frac{b^2(t)}{3a(t)}\right]y.\end{aligned} \quad (73)$$

Let

$$y^{-2}(t) = u(t). \quad (74)$$

Equation (73) is turned into

$$\frac{du}{dt} = -2\left[c(t) - \frac{b^2(t)}{3a(t)}\right]u - 2a(t). \quad (75)$$

By (H_3), according to Lemma 1, Equation (75) has a unique $\omega-$periodic continuous solution as follows:

$$\zeta(t) = -2\int_{-\infty}^{t} e^{-2\int_s^t [c(\tau)-b^2(\tau)/3a(\tau)]d\tau} a(s)\,ds. \quad (76)$$

By (H_2) and (76), it follows that

$$\zeta(t) > 0. \quad (77)$$

By (74), it follows that Equation (73) has two $\omega-$periodic continuous solutions as follows:

$$\eta_1(t) = \frac{1}{\sqrt{-2\int_{-\infty}^{t} e^{-2\int_s^t [c(\tau)-b^2(\tau)/3a(\tau)]d\tau} a(s)\,ds}}. \quad (78)$$

and

$$\eta_2(t) = \frac{1}{-\sqrt{-2\int_{-\infty}^{t} e^{-2\int_s^t [c(\tau)-b^2(\tau)/3a(\tau)]d\tau} a(s)\,ds}}. \quad (79)$$

By (72), it follows that Equation (1) has other two $\omega-$periodic continuous solutions as follows:

$$\gamma_2(t) = \frac{1}{\sqrt{-2\int_{-\infty}^{t} e^{-2\int_s^t [c(\tau)-b^2(\tau)/3a(\tau)]d\tau} a(s)\,ds}} - \frac{b(t)}{3a(t)}, \quad (80)$$

and

$$\gamma_3(t) = \frac{1}{-\sqrt{-2\int_{-\infty}^{t} e^{-2\int_s^t [c(\tau)-b^2(\tau)/3a(\tau)]d\tau} a(s)\,ds}} - \frac{b(t)}{3a(t)}. \tag{81}$$

This is the end of the proof of Theorem 9. □

Similar to the proof of Theorem 9, we can get the following.

Theorem 10. *Consider Equation (1); $a(t), b(t), c(t), d(t)$ are all ω−periodic continuous functions and $r_1 = r_1(t)$ is an ω−periodic continuous particular solution of Equation (1); suppose that the following conditions hold:*

$$\begin{aligned} &(H_1)\ r_1(t) = -\frac{b(t)}{3a(t)}, \\ &(H_2)\ a(t) > 0, \\ &(H_3)\ c(t) - \frac{b^2(t)}{3a(t)} < 0. \end{aligned} \tag{82}$$

Then Equation (1) has other two ω−periodic continuous solutions as follows:

$$\gamma_2(t) = \frac{1}{\sqrt{2\int_{t}^{+\infty} e^{2\int_s^t [b^2(\tau)/3a(\tau)-c(\tau)]d\tau} a(s)\,ds}} - \frac{b(t)}{3a(t)}, \tag{83}$$

and

$$\gamma_3(t) = \frac{1}{-\sqrt{2\int_{t}^{+\infty} e^{2\int_s^t [b^2(\tau)/3a(\tau)-c(\tau)]d\tau} a(s)\,ds}} - \frac{b(t)}{3a(t)}. \tag{84}$$

Theorem 11. *Consider Equation (1); $a(t), b(t), c(t), d(t)$ are all ω−periodic continuous functions and $r_1 = r_1(t)$ is an ω−periodic continuous particular solution of Equation (1); suppose that the following conditions hold:*

$$\begin{aligned} &(H_1)\ r_1(t) = \frac{-b(t) + \sqrt{b^2(t) - 3a(t)c(t)}}{3a(t)}, \\ &(H_2)\ a(t) < 0, \\ &(H_3)\ b^2(t) - 3a(t)c(t) > 0. \end{aligned} \tag{85}$$

Then Equation (1) has another nonzero ω−periodic continuous solution $\gamma_2(t)$.

Proof. Let

$$y(t) = x(t) - r_1(t), \tag{86}$$

where $x(t)$ is the unique solution with the initial value $x(t_0) = x_0$ of Equation (1); differentiating both sides of (86) along the solution of Equation (1), we get

$$\begin{aligned} \frac{dy}{dt} &= \frac{dx}{dt} - \frac{dr_1}{dt} = a(t)\left[x^3(t) - r_1^3(t)\right] \\ &+ b(t)\left[x^2(t) - r_1^2(t)\right] + c(t)\left[x(t) - r_1(t)\right] \\ &= a(t)\left(x(t) - r_1(t)\right)^3 \\ &+ \left[b(t) + 3a(t)r_1(t)\right]\left(x(t) - r_1(t)\right)^2 \\ &+ \left[2b(t)r_1(t) + c(t) + 3a(t)r_1^2(t)\right]\left(x(t) - r_1(t)\right) \\ &= a(t)y^3 + \left[b(t) + 3a(t)r_1(t)\right]y^2 \\ &+ \left[2b(t)r_1(t) + c(t) + 3a(t)r_1^2(t)\right]y = a(t)y^3 \\ &+ \sqrt{b^2(t) - 3a(t)c(t)}\,y^2. \end{aligned} \tag{87}$$

By $(H_2), (H_3)$, Equation (87) satisfies all the conditions of Theorem 5; according to Theorem 5, Equation (87) has a unique nonzero ω−periodic continuous solution $\zeta(t)$; by (86), it follows that Equation (1) has another nonzero ω−periodic continuous solution $\gamma_2(t) = \zeta(t) + \gamma_1(t)$.

This is the end of the proof of Theorem 11. □

Theorem 12. *Consider Equation (1); $a(t), b(t), c(t), d(t)$ are all ω−periodic continuous functions and $r_1 = r_1(t)$ is an ω−periodic continuous particular solution of Equation (1); suppose that the following conditions hold:*

$$\begin{aligned} &(H_1)\ r_1(t) = \frac{-b(t) - \sqrt{b^2(t) - 3a(t)c(t)}}{3a(t)}, \\ &(H_2)\ a(t) < 0, \\ &(H_3)\ b^2(t) - 3a(t)c(t) > 0. \end{aligned} \tag{88}$$

Then Equation (1) has another nonzero ω−periodic continuous solution $\gamma_2(t)$.

Proof. Let

$$y(t) = x(t) - r_1(t), \tag{89}$$

where $x(t)$ is the unique solution with the initial value $x(t_0) = x_0$ of Equation (1); differentiating both sides of (89) along the solution of Equation (1), we get

$$\begin{aligned} \frac{dy}{dt} &= \frac{dx}{dt} - \frac{dr_1}{dt} = a(t)\left[x^3(t) - r_1^3(t)\right] \\ &+ b(t)\left[x^2(t) - r_1^2(t)\right] + c(t)\left[x(t) - r_1(t)\right] \\ &= a(t)\left(x(t) - r_1(t)\right)^3 \\ &+ \left[b(t) + 3a(t)r_1(t)\right]\left(x(t) - r_1(t)\right)^2 \\ &+ \left[2b(t)r_1(t) + c(t) + 3a(t)r_1^2(t)\right]\left(x(t) - r_1(t)\right) \end{aligned}$$

$$= a(t)y^3 + [b(t) + 3a(t)r_1(t)]y^2 + [2b(t)r_1(t) + c(t) + 3a(t)r_1^2(t)]y = a(t)y^3 - \sqrt{b^2(t) - 3a(t)c(t)}y^2. \quad (90)$$

By $(H_2), (H_3)$, Equation (90) satisfies all the conditions of Theorem 6; according to Theorem 6, Equation (90) has a unique nonzero ω−periodic continuous solution $\zeta(t)$; by (89), it follows that Equation (1) has another nonzero ω−periodic continuous solution $\gamma_2(t) = \zeta(t) + \gamma_1(t)$.

This is the end of the proof of Theorem 12. □

Remark 13. In Theorems 11 and 12, if $a(t) > 0$, it is not difficult for us to get similar results; we omit them here.

5. Discussion

In this paper, firstly, we consider Abel's type differential equation:

$$\frac{dx}{dt} = a(t)x^3 + b(t)x^2. \quad (91)$$

Here, $a(t), b(t)$ are both ω−periodic continuous functions on R; by using fixed point theory, we get that if

$$a(t)b(t) < 0, \quad (92)$$

then Equation (91) has a unique positive nonzero ω−periodic continuous solution; if

$$a(t)b(t) > 0, \quad (93)$$

then Equation (91) has a unique negative nonzero ω−periodic continuous solution. A simple method for judging the existence and uniqueness of nonzero periodic solution of Abel's type equation (91) is given.

Then, consider the following Abel's differential equation:

$$\frac{dx}{dt} = a(t)x^3 + b(t)x^2 + c(t)x + d(t), \quad (94)$$

where $a(t), b(t), c(t), d(t)$ are all ω−periodic continuous functions on R, and $a(t), b(t)$ are derivable on R; by using transformation method, we get other periodic solutions of nonlinear Abel differential equations with a particular periodic solution. When the four coefficients of the equation satisfy certain conditions, there does exist a particular periodic solution; for example, in Theorems 9 and 10, if

$$d(t) = \frac{b^3(t)}{27a^2(t)} + \frac{b(t)}{3a(t)}\left(c(t) - \frac{b^2(t)}{3a(t)}\right) - \frac{d}{dt}\left(\frac{b(t)}{3a(t)}\right), \quad (95)$$

then Equation (94) has an ω−periodic continuous solution $\gamma_1(t) = -b(t)/3a(t)$.

In Theorem 11, if

$$d(t) = -\frac{\left(b(t) + \sqrt{b^2(t) - 3a(t)c(t)}\right)^3}{27a^2(t)} - \frac{\sqrt{b^2(t) - 3a(t)c(t)}\left(b(t) + \sqrt{b^2(t) - 3a(t)c(t)}\right)^2}{9a^2(t)} + \frac{d}{dt}\left(\frac{b(t) + \sqrt{b^2(t) - 3a(t)c(t)}}{3a(t)}\right), \quad (96)$$

then Equation (94) has an ω−periodic continuous solution $\gamma_1(t) = (-b(t) - \sqrt{b^2(t) - 3a(t)c(t)})/3a(t)$.

In Theorem 12, if

$$d(t) = \frac{\left(-b(t) + \sqrt{b^2(t) - 3a(t)c(t)}\right)^3}{27a^2(t)} - \sqrt{b^2(t) - 3a(t)c(t)} \cdot \frac{\left(-b(t) + \sqrt{b^2(t) - 3a(t)c(t)}\right)^2}{9a^2(t)} - \frac{d}{dt}\left(\frac{-b(t) + \sqrt{b^2(t) - 3a(t)c(t)}}{3a(t)}\right), \quad (97)$$

then Equation (94) has an ω−periodic continuous solution $\gamma_1(t) = (-b(t) + \sqrt{b^2(t) - 3a(t)c(t)})/3a(t)$.

This is very important for Abel's differential equation, which plays an important role in the fields of science, technology, and physics. When the coefficients are periodic functions, it is very meaningful and interesting to judge the number of periodic solutions of the equation.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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