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Research Article

Analysis and Control of Fractional Order Generalized Lorenz Chaotic System by Using Finite Time Synchronization

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In this paper, we present a corresponding fractional order three-dimensional autonomous chaotic system based on a new class of integer order chaotic systems. We found that the fractional order chaotic system belongs to the generalized Lorenz system family by analyzing its linear term and topological structure. We also found that the equilibrium point generated by the fractional order system belongs to the unstable saddle point through the prediction correction method and the fractional order stability theory. The complexity of fractional order chaotic system is given by spectral entropy algorithm and C_0 algorithm. We concluded that the fractional order chaotic system has a higher complexity. The fractional order system can generate rich dynamic behavior phenomenon with the values of the parameters and the order changed. We applied the finite time stability theory to design the finite time synchronous controller between drive system and corresponding system. The numerical simulations demonstrate that the controller provides fast and efficient method in the synchronization process.

1. Introduction

Chaos, as a standout in the field of nonlinearity, has shown great vitality over the past five decades since Lorenz introduced a continuous three-dimensional autonomous system from the meteorological problems and studied it as the first chaotic model [1]. It is found that the chaotic system is described by nonlinear differential equations, which is very sensitive to initial conditions [2]. Then a series of systems similar to Lorenz systems have been proposed and analyzed such as the Rossler system [3, 4], Chen system [5], Lü system [6], and Liu system [7–9]. The changes and control of chaotic systems have also been further considered and explored by researchers [10, 11].

With the continuous exploration in the field of chaos, it is found that the chaotic system includes two types: autonomous chaotic system and nonautonomous chaotic system. The control application of autonomous and nonautonomous chaotic systems has always been a research focus in the control field. Scholars have done more research on chaotic systems through explaining the integer order mathematical models [12–14].

In 1983, Mandelbort reported a large number of fractal phenomena in many scientific fields [15], fractional calculus has appeared widely in various fields [16–23], and it has quickly attracted people's attention. The fractional calculus in the nonlinear field has developed rapidly. The main reason is that the fractional order calculus models have many advantages compared with the integer order calculus models [16–20].

(i) Many physical systems reveal fractional dynamic behavior due to their special feature; it is more common to study the fractional calculus models to explain the fractional order system. Integer order calculus is a special case in fractional order calculus, which is approximation to the actual system in mathematical model.

(ii) The fractional order calculus can more accurately describe the dynamic response of the actual system in the mathematical model, while the integral order calculus depends on the partial features of the function.

Chaotic synchronization has become a research focus in the field of control. Many effective methods have been proposed such as coupling synchronization, adaptive synchronization, projection synchronization, and sliding mode control synchronization [24–27]. With further study of fractional order chaotic systems, people recognize that the application to synchronization of fractional order chaotic systems is more wide than that of integer orders in the fields of secure communication and information science [28]. In contrast, in integral order chaotic system, there is less information about the synchronous control in fractional order chaotic system until the fractional order system realized the synchronization through numerical simulation [29], which greatly inspired people's interest in the synchronization aspect of fractional order chaotic systems. Zhao proposed a new synchronization method called the finite time synchronization theory [30]. Compared to the synchronization under general circumstance, this synchronization method is effective.

In this paper, a new fractional order chaotic system is introduced. This system is transformed from a integer order chaotic system. This system has equilibrium points belonging to unstable saddle point. There are parameter and order to control this system, and one remarkable feature of the new chaotic system is that it can generate rich dynamic behavior with parameter and order changed. The fractional order chaotic system can be synchronously controlled in a finite time by certain conditions. It has important research value for the application technology in practical engineering fields, especially in the field of secure communication, which can realize more complex dynamic behavior and improve the overall security of the communication system.

The paper is organized as follows. Section 2 constructs the corresponding fractional order system by analyzing its topological structure, makes a systematic judgment on the fractional order chaotic system, and analyzes the equilibrium point. In Section 3, the discretization equation of the system and the complexity of fractional order chaotic system are concluded. Section 4 analyzes the dynamics behavior of the whole fractional order chaotic system through the variation of parameters and orders in the system. In Section 5, the controller is designed by finite time synchronization theory. Finally, Section 6 draws the conclusions.

2. Theoretical Analysis of Fractional Order Chaotic System

2.1. Description of the New Fractional Order Three-Dimensional System. Various definitions are proposed in the fractional order differential with the further study of the whole fractional order system. However, during the practical application, the Riemann-Liouville (R-L) definition [31, 32] and the Caputo definition are widely used [31]. The R-L definition is usually adopted in the field of pure mathematics; it needs to specify initial conditions and these conditions are required to have the same properties. While the Caputo definition is often adopted in the engineering field. It allows nonhomogeneous initial conditions. Thus, we use the Caputo definition to describe the dynamic behavior of fractional order chaotic system. We introduce the Caputo fractional order differential by related properties and theories.

Property 1:

$$ {}_{a}^{C}D_{t}^{\alpha} x^{u} = \frac{\Gamma(u+1)}{\Gamma(u+1-\alpha)} x^{u-\alpha} \, {}_{a}^{C}D_{t}^{\alpha} x \tag{1} $$

General fractional order differential equations are expressed as follows.

$$ {}_{a}^{C}D_{t}^{\alpha} x(t) = Ax(t) \tag{2} $$

The result to this equation is written as follows.

$$ x(t) = x(0) E_{\alpha}\left(At^{\alpha}\right) \tag{3} $$

The Mittag-Leffler function:

$$ E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(\alpha k+1)} \tag{4} $$

Lemma 1 (see [33]). *For the order $\alpha < 1$ in fractional order system, there is a symmetric matrix P and a state variable $x(x = (x_1, x_2, \ldots x_n)^T)$; the equation satisfies $J = x^T P(d^{\alpha}x/dt^{\alpha}) \leq 0$; and the fractional order system is stable.*

The new chaotic system [34] is expressed as follows:

$$ \begin{aligned} \dot{x} &= y - x \\ \dot{y} &= ay - xz \\ \dot{z} &= xy - b \end{aligned} \tag{5} $$

where $x, y, z \in R$ are the state variables in the system, with a, b representing the system parameters. System (5) has six terms including two multipliers (xy, xz) on the right hand side. System (5) is symmetrical on the z-axis as it has a transformation $(x, y, z) \longrightarrow (-x, -y, z)$, and in system (5), the parameters can be taken as $a = 0.5$, $b = 0.5$. We substitute the integer differential operator with the fractional differential operator in the chaotic system. By the description of the fractional order differential definition, the new fractional order chaotic system in the mathematical model is described as follows:

$$ \begin{aligned} \frac{d^{\alpha}x}{dt^{\alpha}} &= y - x \\ \frac{d^{\beta}y}{dt^{\alpha}} &= 0.5y - xz \\ \frac{d^{\gamma}z}{dt^{\alpha}} &= xy - 0.5 \end{aligned} \tag{6} $$

where α, β, γ are the order of the fractional order differential system (6), with x, y, z representing state variables, and α, β, γ can be different in the fractional order differential system. Here we adopted $0 < \alpha = \beta = \gamma = q \leq 1$ and used q to represent the order of the fractional order differential system. The fractional order chaotic system model is expressed as follows.

$$ \begin{aligned} \frac{d^{q}x}{dt^{q}} &= y - x \\ \frac{d^{q}y}{dt^{q}} &= 0.5y - xz \\ \frac{d^{q}z}{dt^{q}} &= xy - 0.5 \end{aligned} \tag{7} $$

2.2. Determination of the Fractional Order Differential System. The Lorenz system can be generally expressed as [35]

$$\dot{X} = \begin{pmatrix} A & 0 \\ 0 & a_{33} \end{pmatrix} X + x \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} X \tag{8}$$

where $X = \begin{pmatrix} x & y & z \end{pmatrix}^T \in R$, $a_{33} \in R^3$, for the A described as $A = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)$ in Lorenz system. The Lorenz system linear matrix term satisfies $a_{11}a_{22} > 0$, and the Chen system linear matrix term satisfies $a_{11}a_{22} < 0$. There is a transition between Lorenz and Chen system, namely, Lü system, whose linear matrix term satisfies $a_{11}a_{22} = 0$ in matrix A. They constructed a whole Lorenz system family.

By the description of the whole Lorenz system family, the fractional order differential system (7) is expressed as follows.

$$\dot{\mathrm{X}} = \begin{pmatrix} A & 0 \\ 0 & a_{33} \end{pmatrix} X + x \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} X + \begin{pmatrix} 0 \\ 0 \\ -b \end{pmatrix} \tag{9}$$

The fractional order system linear matrix term satisfies $a_{11}a_{22} = 0$. We found that the new order fractional order chaotic system is similar to the Lü system by presenting the analysis of topological structure. And, thus, we considered it as belonging to the entire Lorenz system family.

2.3. Chaotic Analysis of Fractional Order Systems. The fractional order chaotic system is described, and we state the following lemma.

Lemma 2. *A linear fractional order system [36] is considered:*

$$\begin{aligned} \frac{d^\alpha x}{dt^\alpha} &= A\mathrm{x}, \\ x(0) &= x_0 \end{aligned} \tag{10}$$

where $x \in R^n (n \in N)$ is the system state variable and A is the coefficient matrix.

(1) The system is asymptotically stable if it satisfies $\lambda \in \operatorname{spec}(A), |\arg(\lambda)| > \alpha\pi/2$.

(2) The system is stable if it satisfies $\lambda \in \operatorname{spec}(A)$, $|\arg(\lambda)| \geq \alpha\pi/2$.

The chaos character of nonlinear fractional order equations is expressed as follows:

We get two equilibrium points for the fractional order system, which are given by $E_1(0.7071, 0.7071, 0.5)$, $E_2(-0.7071, -0.7071, 0.5)$. The fractional order system (7) is symmetrical about the z-axis. The eigenvalues at the equilibrium point E_1 are given by $\lambda_1 = -1$, $\lambda_2 = 0.25 \pm 0.9682i$, $\lambda_3 = 0.25 - 0.9682i$. By the Lyapunov stability theory, the equilibrium point E_1 is unstable as λ_2, λ_3 have eigenvalues with positive real parts and λ_1 is positive. The eigenvalues at the equilibrium point E_2 are identical to the eigenvalues of the equilibrium point E_1. Thus, by using the same argument, the equilibrium point E_2 becomes unstable.

Following Lemma 2, when $q = 0.95$时, $q > (2/\pi)\tan^{-1}[|\mathrm{Im}(\lambda)|/\mathrm{Re}(\lambda)] = |\tan^{-1}(0.9682/0.25)| = 0.8391$, the system is chaotic. And thus, when the parameters are a= 0.5, b=0.5, q=0.95, the fractional order dynamic system has a chaotic character.

3. Discretization and Complexity Analysis of Fractional Order Chaotic System

3.1. Discretization Analysis. The fractional order differential equation is different from the integer order differential equation. We access two kinds of approximation methods to resolve the fractional order differential equation. The first is the improved Adams-Bashforth-Moulton decomposition method [37–39], which is built on the prediction correction method. The second is the time domain approximation conversion method [32].

The improved Adams-Bashforth-Moulton decomposition method has the features of high computational accuracy and fast convergence speed during numerical approximation. Here we adopt the improved Adams-Bashforth-Moulton decomposition method to settle the fractional order differential equation.

The method description is as follows.

$$D_t^q y(t) = r(t, y(t)), \quad 0 \leq t \leq T \tag{11}$$

$$y^{(k)}(0) = y_0^{(k)}, \quad k = 0, 1, \ldots\ldots m - 1 \tag{12}$$

This differential equation is equivalent to [39]

$$y(t) = \sum_{k=0}^{[q]-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} r(s, y(s))\, ds \tag{13}$$

where $\mathrm{h} = T/N$, $t_n = nh$ $(n = 0, 1, 2 \ldots\ldots N)$; the above equation is discretized into the following.

$$\begin{aligned} y_h(t_{i+1}) = \sum_{k=0}^{[q]-1} y_0^{(k)} \frac{t^k}{k!} &+ \frac{h^q}{\Gamma(q+2)} r\left(t_{i+1}, y_h^p(t_{i+1})\right) \\ &+ \frac{h^q}{\Gamma(q+2)} \sum_{j=0}^{n} a_{j,i+1} r\left(t_j, y_h(t_j)\right) \end{aligned} \tag{14}$$

The approximation error is

$$\max_{j=0,1,\ldots N} \left| y(t_j) - y_h(t_j) \right| = O(h^p) \tag{15}$$

and then

$$p = \min(2, 1+q). \tag{16}$$

The discretization result in the fractional order system equation is as follows.

$$
\begin{aligned}
x_{i+1} &= x_0 + \frac{h^\alpha}{\Gamma(\alpha+2)}\left[y_{i+1}^p - x_{i+1}^p\right] \\
&\quad + \frac{h^\alpha}{\Gamma(\alpha+2)}\sum_{j=0}^{i} r_{j,i+1}\left[y_{i+1}^p - x_{i+1}^p\right] \\
y_{i+1} &= y_0 + \frac{h^\alpha}{\Gamma(\alpha+2)}\left[a y_{i+1}^p - x_{i+1}^p z_{i+1}^p\right] \\
&\quad + \frac{h^\alpha}{\Gamma(\alpha+2)}\sum_{j=0}^{i} r_{j,i+1}\left[a y_{i+1}^p - x_{i+1}^p z_{i+1}^p\right] \\
z_{i+1} &= z_0 + \frac{h^\alpha}{\Gamma(\alpha+2)}\left[x_{i+1}^p y_{i+1}^p - b\right] \\
&\quad + \frac{h^\alpha}{\Gamma(\alpha+2)}\sum_{j=0}^{i} r_{j,i+1}\left[x_{i+1}^p y_{i+1}^p - b\right]
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
x_{i+1}^p &= x_0 + \frac{h^\alpha}{\Gamma(\alpha+1)}\sum_{j=0}^{i} v_{j,i+1}\left[y_{i+1}^p - x_{i+1}^p\right] \\
y_{i+1}^p &= y_0 + \frac{h^\alpha}{\Gamma(\alpha+1)}\sum_{j=0}^{i} v_{j,i+1}\left[a y_{i+1}^p - x_{i+1}^p z_{i+1}^p\right] \\
z_{i+1}^p &= z_0 + \frac{h^\alpha}{\Gamma(\alpha+1)}\sum_{j=0}^{i} v_{j,i+1}\left[x_{i+1}^p y_{i+1}^p - b\right]
\end{aligned}
\tag{18}
$$

$$
r_{j,i+1} = \begin{cases} i^{\alpha+1} - (1-\alpha)(i+1)^\alpha, & j = 0 \\ (i-j+2)^{\alpha+1} + (i-j)^{\alpha+1} - 2(i-j+1)^{\alpha+1} & \\ 1 & j = i+1 \end{cases}
\tag{19}
$$

$$
v_{j,i+1} = (i-j+1)^\alpha - (i-j)^\alpha \tag{20}
$$

3.2. Analysis of Complexity Algorithm. There is no unified definition since complexity involves many fields. Horgan J has pointed out that there are at least 45 definitions of complexity, such as time complexity, spatial complexity, semantic complexity, and Kolmogorov complexity [40]. The complexity of a chaotic system refers to the degree to which the chaotic sequences approach random sequence; the complexity of chaotic systems belongs to the category of chaotic dynamics. Spectral entropy algorithm and C_0 algorithm [41–44] are used to analyze the complexity in the fractional order chaotic system, which shows that the chaotic system has rich dynamic characteristics and also proves the superiority of spectral entropy and C_0 complex algorithm in analyzing the continuous chaotic sequences.

3.2.1. Spectral Entropy Algorithm Analysis. The spectral entropy algorithm mainly uses the Fourier transform domain energy distribution combined with Shannon entropy to get the corresponding spectral entropy value. We considered that the sequence spectrum structure is more complicated, the spectral entropy measure is larger, and the complexity is greater from the spectral entropy. During the system analysis, we take a and q as variable parameters, respectively. By comparing the complexity of the system between parameters $a \in [0, 0.5]$, $q \in [0, 1]$ for a $\in [0.45, 0.5] q \in [0.9, 1]$, the complexity of the system gradually becomes lighter as the order increases. When the fractional order q is fixed, the color within a certain range is constant; that is, the parameter has less influence on the complexity of the system.

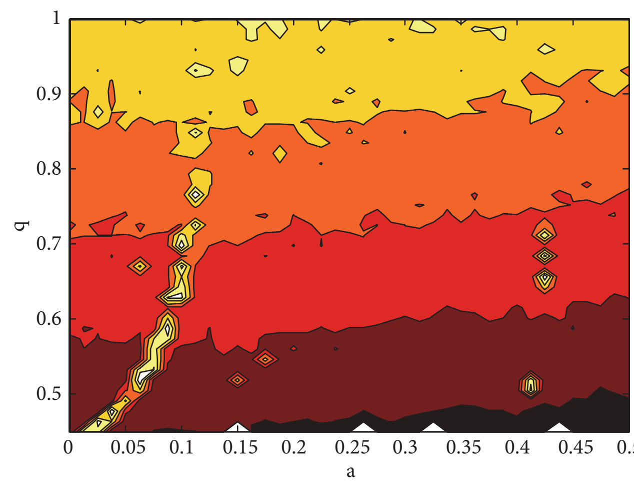


Figure 1: Spectral entropy complexity.

We concluded that there is a higher complexity in the upper right corner from Figure 1.

3.2.2. C_0 Algorithm Analysis. C_0 complexity algorithm is generated form FFT transformation, which refers to deleting the regular part of the sequence and retaining the irregular part. In the process of system analysis, a and q are taken as variable parameters, respectively. By comparing the complexity of the system between the parameters $a \in [0, 0.5]$ and $q \in [0, 1]$, the complexity of the system gradually becomes lighter as the order increases.

We concluded that the C_0 measure value gradually increases with the parameter q decreasing. As the order is reduced, the color is getting darker and darker. The C_0 complexity algorithm graph is shown in Figure 2.

4. Phase Portraits Analysis of Fractional Order Chaotic System

(1) Fix the parameters as $a = 0.5$, $b = 0.5$; the fractional order system is equivalent to the following.

$$
\begin{aligned}
\frac{d^q x}{dt^q} &= y - x \\
\frac{d^q y}{dt^q} &= 0.5y - xz \\
\frac{d^q z}{dt^q} &= xy - 0.5
\end{aligned}
\tag{21}
$$

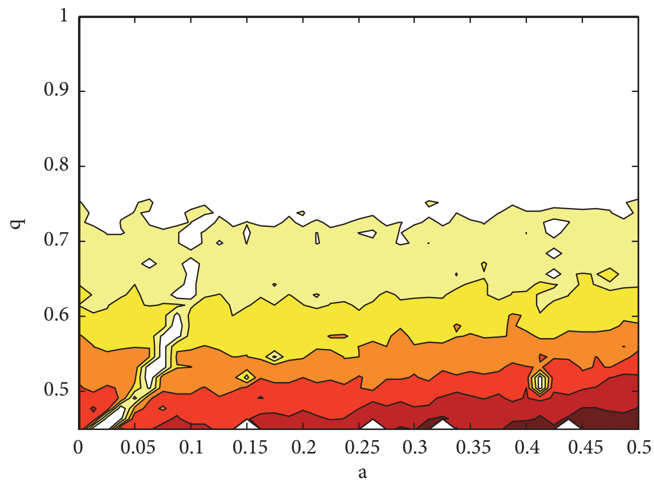


FIGURE 2: C_0 complexity.

The periodic motion in the phase space corresponds to all the closed curves, and the chaotic motion is a never-closed curved trajectory corresponding to the separation in a certain region. During simulating the fractional order chaotic system, we consider whether the system has chaotic characteristics from the phase portraits.

The evolution in the fractional order chaotic system will be presented in phase portraits to demonstrate chaotic traits. We focus on order q in describing the dynamical behavior of the fractional order chaotic system, and hence we set time step size 0.001, keep the absolute and relative error 0.00001, choose the initial conditions $(0.1, 0.2, 0.3)$, set the time within 30s, and adopt Adomian decomposition method in MATLAB. The result will give rise to parameter q. When q is variable in scope $[0.3, 1.0]$, we note that the fractional order chaotic system has different chaotic dynamic feature by observing the phase portraits. When $q \in [0.3, 0.4]$, the fractional order chaotic system in the phase portraits curve converges to a straight line, that is, nonchaotic state, for $q \in [0.5, 1.0]$; the fraction order system has a random separation state, that is, chaotic state. The phase portraits are illustrated in Figure 3. By increasing q, Figure 3 clearly shows the chaotic dynamic characters in fractional order system

(2) Fix the order as $q = 0.8$; the fractional order system is equivalent to the following.

$$\begin{aligned} \frac{d^{0.8}x}{dt^{0.8}} &= y - x \\ \frac{d^{0.8}y}{dt^{0.8}} &= ay - xz \\ \frac{d^{0.8}z}{dt^{0.8}} &= xy - 0.5 \end{aligned} \tag{22}$$

The evolution in the fractional order chaotic system will be presented in phase portraits to demonstrate chaotic characteristics. We focus on parameter a in describing the dynamical behavior of the fractional order chaotic system, and hence we set time step size 0.001, keep the absolute and relative error 0.00001, and choose the initial conditions $(0.1, 0.2, 0.3)$. The result will give rise to parameter a. When a is variable in scope $[0.1, 0.5]$, when $a \in [0.1, 0.2]$, the overall chaos of the system is small and the attractor in the area is visible, for $a \in [0.3, 0.5]$; when a increases to a certain value, such as a=0.48, the whole fractional chaotic system with a larger chaotic region and the chaotic attraction domain covers a large area in phase space. We can see chaotic behavior by phase portraits in Figure 4. We must apply complex mathematical analysis and calculation if we want to further accurately analyze the changes of the whole fractional order chaotic phase portraits and attractor topology.

(3) Parameters and order change: the fractional order system is equivalent to the following.

$$\begin{aligned} \frac{d^{q}x}{dt^{q}} &= y - x \\ \frac{d^{q}y}{dt^{q}} &= ay - xz \\ \frac{d^{q}z}{dt^{q}} &= xy - b \end{aligned} \tag{23}$$

The evolution in the fractional order chaotic system will be presented in phase diagram to demonstrate chaotic characters. We focus on parameter a and order q in describing the dynamical behavior of the fractional order system, and hence we set time step size 0.001, keep the absolute and relative error 0.00001, and choose the initial conditions $(0.1, 0.2, 0.3)$. The result will give rise to varying parameter a in the scope [0-0.5] and the order q in the scope [0-1]. It is noted that, for the order q=1.0, the system is an integer order system, and the state of fractional order system is always chaotic no matter how the parameter changes. The chaotic state differs

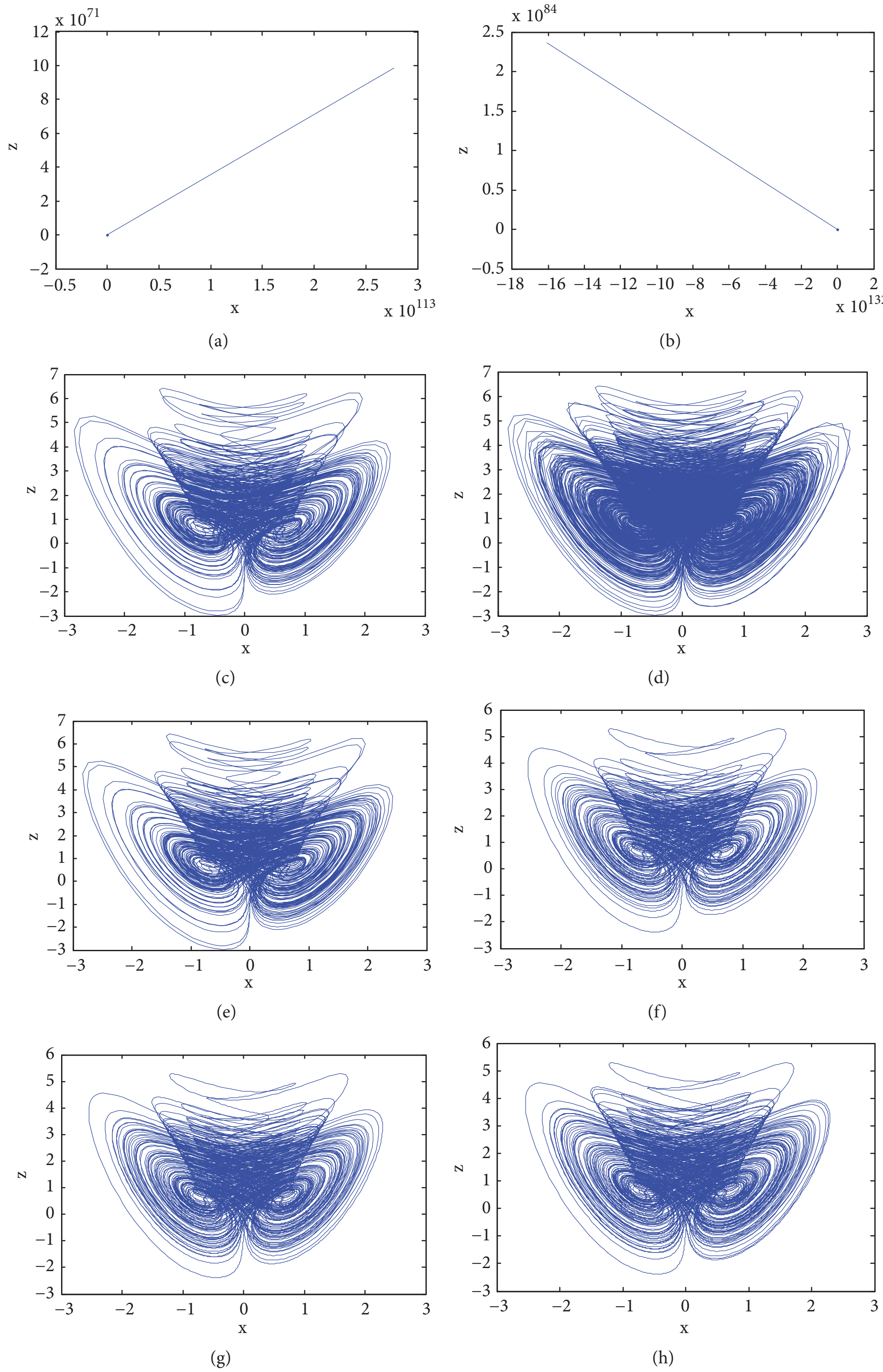


FIGURE 3: Phase portraits of system (20) with parameters $a = 0.5$ and $b = 0.5$ and initial conditions $(0.1, 0.2, 0.3)$. (a) Projection on x-z plane with $q = 0.3$; (b) projection on x-z plane with $q = 0.4$; (c) projection on x-z plane with $q = 0.5$; (d) projection on x-z plane with $q = 0.6$; (e) projection on x-z plane with $q = 0.7$; (f) projection on x-z plane with $q = 0.8$; (g) projection on x-z plane with $q = 0.9$; (h) projection on x-z plane with $q = 1.0$.

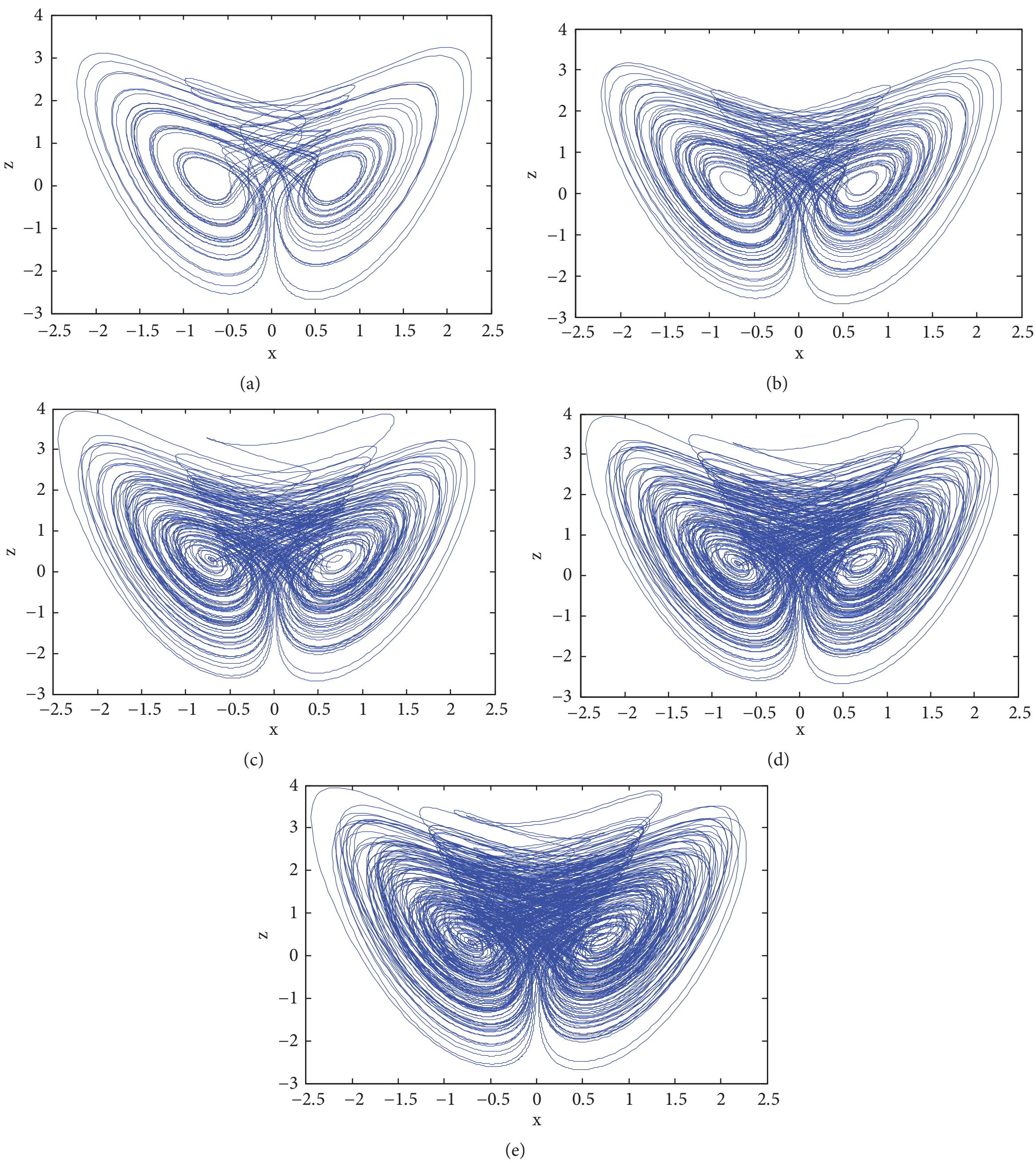


Figure 4: Phase portraits of system (21) with parameter $q = 0.8$ and initial conditions $(0.1, 0.2, 0.3)$. (a) Projection on x-z plane with $a = 0.06$; (b) projection on x-z plane with $a = 0.17$; (c) projection on x-z plane with $a = 0.28$; (d) projection on x-z plane with $a = 0.36$; (e) projection on x-z plane with $a = 0.48$.

in different parameters and orders range. We show the different parameters and orders in detail; the results are shown in Table 1.

5. Finite Time Synchronization Analysis of Fractional Order Chaotic System

5.1. Finite Time Stability Theory [30]

Theorem 3. *General fractional order system satisfies the following.*

$$x\,{}_a^C D_t^{\alpha} x = \frac{\Gamma(2)}{\Gamma(2+\alpha)}\,{}_a^C D_t^{\alpha} x x^{\alpha} \le -k(xx)^{\beta} \quad (24)$$

$$\beta < \frac{\alpha + \alpha^2}{2}, \quad k > 0 \quad (25)$$

The state variable approaches zero in a finite time.

$$t = \left(v(0)^{\alpha - (2\beta/(1+\alpha))} \cdot \frac{\Gamma(1+\alpha-2\beta/(1+\alpha))\,\Gamma(1+\alpha)}{\Gamma(1+2\alpha-2\beta/(1+\alpha))\,k\Gamma(2+\alpha)} \right)^{1/\alpha} \quad (26)$$

Table 1: The chaotic state with different parameters and orders.

parameter	order	State behavior
a = 0.1	$0 < q < 0.4$	non-chaos
	$0.5 < q \le 0.9$	chaos
	q = 1	chaos
a = 0.2	$0 < q < 0.4$	non-chaos
	$0.5 < q \le 0.9$	chaos
	q = 1	chaos
a = 0.3	$0 < q < 0.4$	non-chaos
	$0.5 < q \le 0.9$	chaos
	q = 1	chaos
a = 0.4	$0 < q < 0.4$	non-chaos
	$0.5 < q \le 0.9$	chaos
	q = 1	chaos
a = 0.5	$0 < q < 0.4$	non-chaos
	$0.5 < q \le 0.9$	chaos
	q = 1	chaos

Proof. The fractional differential property is

$$x\,{}_a^C D_t^\alpha = \frac{\Gamma(2)}{\Gamma(2+\alpha)}\,{}_a^C D_t^\alpha (x x^\alpha) \tag{27}$$

$$x\,{}_a^C D_t^\alpha = \frac{\Gamma(2)}{\Gamma(2+\alpha)}\,{}_a^C D_t^\alpha v \le -k v^{2\beta/(1+\alpha)} \tag{28}$$

where $v = \mathrm{x}x^\alpha$, $(x^T x)^\beta = (x x^\alpha)^{2\beta/(1+\alpha)} = v^{2\beta/(1+\alpha)}$, and we have the following formula.

$$x\,{}_a^C D_t^\alpha = \frac{\Gamma(2)}{\Gamma(2+\alpha)}\,{}_a^C D_t^\alpha v \le -k v^{2\beta/(1+\alpha)} \tag{29}$$

$$v^{-2\beta/(1+\alpha)}\,{}_a^C D_t^\alpha v \le -k\frac{\Gamma(2+\alpha)}{\Gamma(2)} \tag{30}$$

$$\begin{aligned}{}_a^C D_t^\alpha v^{\alpha-2\beta/(1+\alpha)} &\le -k\frac{\Gamma(2+\alpha)}{\Gamma(2)} \\ &\cdot \frac{\Gamma(1+2\alpha-2\beta/(1+\alpha))}{\Gamma(1+\alpha-2\beta/(1+\alpha))}\end{aligned} \tag{31}$$

$$\begin{aligned}{}_a^C D_t^\alpha v^{\alpha-2\beta/(1+\alpha)} &= -k\frac{\Gamma(2+\alpha)}{\Gamma(2)} \\ &\cdot \frac{\Gamma(1+2\alpha-2\beta/(1+\alpha))}{\Gamma(1+\alpha-2\beta/(1+\alpha))}\end{aligned} \tag{32}$$

$$\begin{aligned}v(t)^{\alpha-2\beta/(1-\alpha)} - v(0)^{\alpha-2\beta/(1-\alpha)} &= -k\frac{\Gamma(2+\alpha)}{\Gamma(2)} \\ &\cdot \frac{\Gamma(1+2\alpha-2\beta/(1+\alpha))}{\Gamma(1+\alpha-2\beta/(1+\alpha))\,\Gamma(1+\alpha)} t^\alpha\end{aligned} \tag{33}$$

$$\begin{aligned}t = \Bigg(& v(0)^{\alpha-2\beta/(1+\alpha)} \\ & \cdot \frac{\Gamma(1+\alpha-2\beta/(1+\alpha))\,\Gamma(1+\alpha)}{(1+2\alpha-2\beta/(1+\alpha))\,k\Gamma(2+\alpha)}\Bigg)^{1/\alpha}\end{aligned} \tag{34}$$

The state variable achieves stability in a finite time t. The finite time is expressed as follows.

$$\begin{aligned}t = \Bigg(& v(0)^{\alpha-2\beta/(1+\alpha)} \\ & \cdot \frac{\Gamma(1+\alpha-2\beta/(1+\alpha))\,\Gamma(1+\alpha)}{(1+2\alpha-2\beta/(1+\alpha))\,k\Gamma(2+\alpha)}\Bigg)^{1/\alpha}\end{aligned} \tag{35}$$

□

Lemma 4 (see [45]). *When* a > 0, b > 0, 0 < c < 1 *are satisfied, the following inequality is given.*

$$(a+b)^c \le a^c + b^c \tag{36}$$

5.2. Finite Time Synchronization Analysis of Fractional Order Chaotic System. The fractional order chaotic systems have drawn much more attention because of the tremendous potential and practice value in the field of secure communication, and various synchronization methods have been searched for in recent years, for example, the Lyapunov equation method, the driving response method, generalized synchronization method, and linear separation projection. Here we shed light on the finite time stability theory to realize the synchronous control in fractional order chaotic systems.

More practical circumstances are reflected in the fractional order chaotic system. The drive system and the corresponding response system are expressed as follows:

$$\begin{aligned}\frac{d^q x}{dt^q} &= y - x \\ \frac{d^q y}{dt^q} &= 0.5y - xz \\ \frac{d^q z}{dt^q} &= xy - 0.5\end{aligned} \tag{37}$$

and

$$\begin{aligned}\frac{d^q x_1}{dt^q} &= y_1 - x_1 - u_1 \\ \frac{d^q y_1}{dt^q} &= 0.5y_1 - x_1 z_1 - u_2 \\ \frac{d^q z_1}{dt^q} &= x_1 y_1 - 0.5 - u_3\end{aligned} \tag{38}$$

where u_i $(i = 1-3)$ is controller designed by Theorem 3.

Defining the synchronization error as e_i $(i = 1-3)$, we construct the synchronization error system between the drive system and the corresponding response system as follows.

$$\begin{aligned} e_1 &= x_1 - x \\ e_2 &= y_1 - y \\ e_3 &= z_1 - z \end{aligned} \tag{39}$$

We receive the synchronization error system with the combination of (37), (38), and (39). The synchronization error system is described as

$$\begin{aligned} \frac{d^q e_1}{dt^q} &= e_2 - e_1 - u_1 \\ \frac{d^q e_2}{dt^q} &= 0.5e_2 - x_1 e_3 - z e_1 - u_2 \\ \frac{d^q e_3}{dt^q} &= y_1 e_1 + x e_2 - u_3 \end{aligned} \tag{40}$$

where k and β are parameters in the controller; we configure the corresponding controller u_i $(i = 1 - 3)$ as follows.

$$\begin{aligned} u_1 &= (1 - z) e_2 + y_1 e_3 + k e_1^{\beta} \\ u_2 &= x e_3 + k e_2^{\beta} \\ u_3 &= -x_1 e_2 + k e_3^{\beta} \end{aligned} \tag{41}$$

The fractional order error system will be stabilized by adding the controller in a finite time. The time is expressed as

$$\begin{aligned} t = &\Bigg\{ \left[e \left(e^q \right)^T \right]^{q-(1+\beta)/(1+q)} \\ &\cdot \frac{\Gamma\left(1 + q - (1+\beta)/(1+q)\right)\Gamma(1+q)}{\Gamma\left(1 + 2q - (1+\beta)/(1+q)\right) k\Gamma(2+q)} \Bigg\}^{1/q} \end{aligned} \tag{42}$$

where e and e^q are described as $e = [e_1,\ e_2,\ e_3]$, $e^q = [e_1^q,\ e_2^q,\ e_3^q]$.

Proof. We obtain the synchronization error system from designed controller as follows.

$$\begin{aligned} \frac{d^q e_1}{dt^q} &= e_2 - e_1 - (1 - z) e_2 - y_1 e_3 - k e_1^{\beta} \\ \frac{d^q e_2}{dt^q} &= 0.5e_2 - x_1 e_3 - z e_1 - x e_3 - k e_2^{\beta} \\ \frac{d^q e_3}{dt^q} &= e_1 y_1 + x e_2 + x_1 e_2 - k e_3^{\beta} \end{aligned} \tag{43}$$

We deduce it from this equation.

$$\begin{aligned} &\frac{\Gamma(2)}{\Gamma(2+\alpha)} \frac{d^q}{dt^q} [e_1,\ e_2,\ e_3] [e_1^q,\ e_2^q,\ e_3^q]^T \\ &= [e_1,\ e_2,\ e_3] \left[\frac{d^q e_1^q}{dt^q},\ \frac{d^q e_2^q}{dt^q},\ \frac{d^q e_3^q}{dt^q} \right]^T \\ &= e_1 e_2 - e_1^2 - e_1 e_2 (1 - z) - e_1 e_3 y_1 - k e_1^{\beta+1} + 0.5 e_2^2 \\ &\quad - e_2 e_3 x_1 - z e_1 e_2 - x e_2 e_3 - k e_2^{\beta+1} + e_1 e_3 y_1 \\ &\quad + x e_2 e_3 + e_2 e_3 x_1 - k e_3^{\beta+1} \\ &= -e_1^2 + 0.5 e_2^2 - k e_1^{\beta+1} - k e_2^{\beta+1} - k e_3^{\beta+1} \\ &\geq -e_1^2 - k e_1^{\beta+1} - k e_2^{\beta+1} - k e_3^{\beta+1} \\ &= -e_1^2 - k e_1^{\beta+1} - k e_2^{\beta+1} - k e_3^{\beta+1} \\ &\leq -k e_1^{\beta+1} - k e_2^{\beta+1} - k e_3^{\beta+1} \\ &= -k \left(e_1^2\right)^{(\beta+1)/2} - k \left(e_2^2\right)^{(\beta+1)/2} - k \left(e_3^2\right)^{(\beta+1)/2} \end{aligned} \tag{44}$$

We conclude it in Lemma 4.

$$\begin{aligned} &- k \left(e_1^2\right)^{(\beta+1)/2} - k \left(e_2^2\right)^{(\beta+1)/2} - k \left(e_3^2\right)^{(\beta+1)/2} \\ &\quad \leq -k \left(e_1^2 + e_2^2 + e_3^2\right)^{(\beta+1)/2} \end{aligned} \tag{45}$$

$$\begin{aligned} &\frac{\Gamma(2)}{\Gamma(2+\alpha)} \frac{d^q}{dt^q} [e_1,\ e_2,\ e_3] [e_1^q,\ e_2^q,\ e_3^q]^T \\ &\quad \leq -k \left(e_1^2 + e_2^2 + e_3^2\right)^{(\beta+1)/2} \\ &\quad = -k \left([e_1,\ e_2,\ e_3] [e_1,\ e_2,\ e_3]^T \right)^{(\beta+1)/2} \\ &\quad = -k \left(e e^T \right)^{(\beta+1)/2} \end{aligned} \tag{46}$$

□

As the error system remained stable during the finite time from Theorem 3, the drive system and response system attain synchronization in the certain finite time.

$$\begin{aligned} t = &\Bigg\{ \left[e \left(e^q \right)^T \right]^{q-(1+\beta)/(1+q)} \\ &\cdot \frac{\Gamma\left(1 + q - (1+\beta)/(1+q)\right)\Gamma(1+q)}{\Gamma\left(1 + 2q - (1+\beta)/(1+q)\right) k\Gamma(2+q)} \Bigg\}^{1/q} \end{aligned} \tag{47}$$

5.3. Numerical Simulation and Discussion. Numerical results are presented to demonstrate that the designed controller is effective in the fractional order chaotic system. The initial values of the drive system and corresponding system are $[x(0), y(0), z(0)] = (5,\ 0,\ 10)$ and $[x_1(0),\ y_1(0),\ z_1(0)] = (-1,\ -1,\ -1)$, respectively. We set time step size 0.001, keep the absolute and relative error 0.00001, and allow the time within 15s. The parameters of the system are selected as $k = 1.5$, $\beta = 1$, $q = 0.9$. The error systems e_1, e_2, e_3 can converge to zero within 4s after adding the controller, indicating that the error system reaches a steady state in Figure 5. Each state variable of the drive system and the response system can also reach a completely synchronous state within 4s in Figure 6.

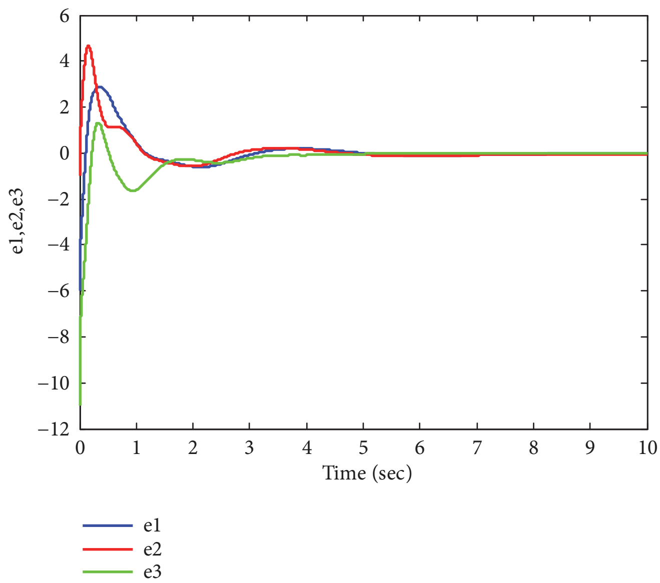
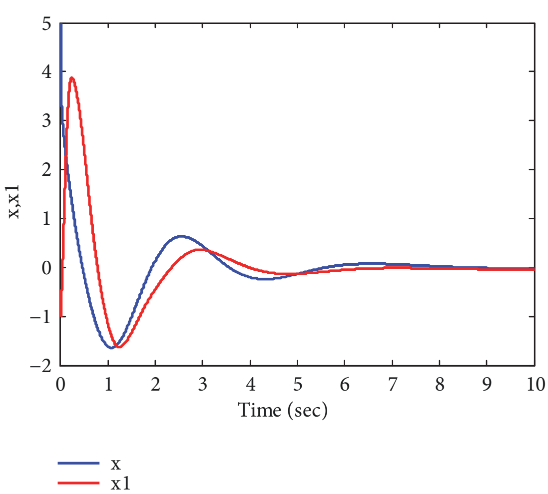
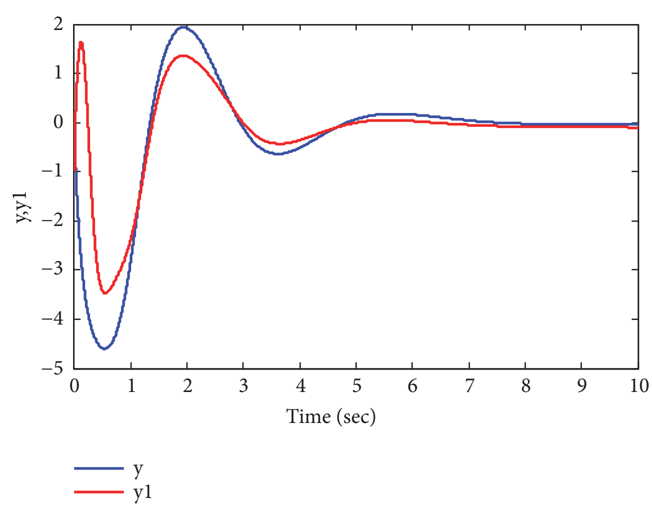


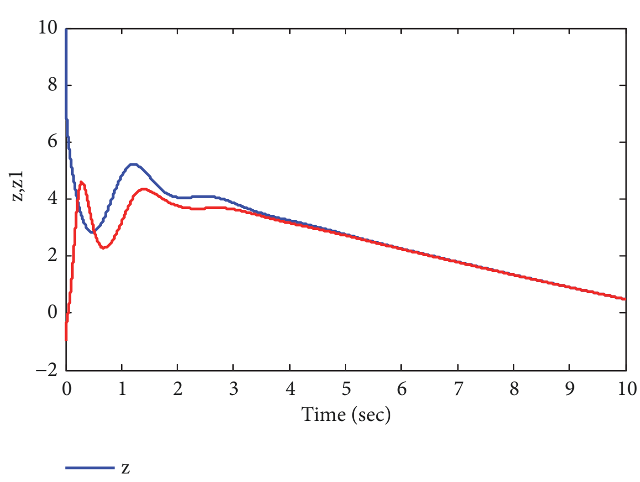
Figure 5: Synchronization errors during $t \in [0, 10]$.



(a)



(b)



(c)

Figure 6: Time responses of the drive system and corresponding system $t \in [0, 10]$: (a) x, x_1, (b) y, y_1, (c) z, z_1.

6. Conclusion

This paper set out to develop a model for a new fractional order chaotic system based on the Lorenz system family; it is different from the well-known integer order system. The search has shown the equilibrium, stability, and a series of basic characteristics. The chaotic relation between parameter and order was accessed by theoretically and numerically observing the phase portraits. When the parameters were fixed and the order of the system changed within a certain scope in the chaotic system, the piecewise chaos phenomenon was exhibited in the fractional system. When the order was fixed and the parameters via the system change in a certain scope, we found that the attraction domain has an increased dynamic characteristic in the chaotic system. The findings of this research provide insights that the fractional order chaotic system has rich dynamic behavior in terms of parameters and order altered.

Spectral entropy and C_0 algorithm were applied to discuss the complexity of fractional systems, which provides a theoretical basis for the application in secure communication and cryptography field. A synchronous controller was designed by the finite time synchronization theory. The synchronous controller proved effective in the numerical simulation. Through the analysis of the fractional order system, the findings from this study make several contributions to the current literature. First, fractional order differential equation was described in the whole chaos system, which extends the range of the whole system and makes a description of the chaotic system more accurate. Second, the controller enhances the unpredictability of the chaotic system in the fields of secure communication and image encryption and improves the reliability and security.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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