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Research Article

The Improved $\exp(-\Phi(\xi))$-Expansion Method and New Exact Solutions of Nonlinear Evolution Equations in Mathematical Physics

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The $\exp(-\Phi(\xi))$-expansion method is improved by presenting a new auxiliary ordinary differential equation for $\Phi(\xi)$. By using this method, new exact traveling wave solutions of two important nonlinear evolution equations, i.e., the ill-posed Boussinesq equation and the unstable nonlinear Schrödinger equation, are constructed. The obtained solutions contain Jacobi elliptic function solutions which can be degenerated to the hyperbolic function solutions and the trigonometric function solutions. The present method is very concise and effective and can be applied to other types of nonlinear evolution equations.

1. Introduction

Nonlinear evolution equations have widely applied in various areas of science and engineering, such as nonlinear optics, fluid dynamics, biophysics, and plasma physics. Since the exact solutions of nonlinear evolution equations play a determining role in comprehension the nonlinear phenomena, many effective methods have been proposed, such as the direct algebraic method [1], the extended direct algebraic method [2–5], the function transformation method [1, 6–8], the ansatz function method [9], the extended simple equation method [10, 11], the variational method [12–14], and the extended auxiliary equation method [15].

Recently, Khan and Akbar introduced the $\exp(-\Phi(\xi))$-expansion method to obtain new traveling wave solutions for the Modified Benjamin-Bona-Mahony Equation in [16]. Afterwards, many nonlinear partial differential equations were solved by means of this method, such as the Vakhnenko-Parkes equation [17], the Zakharov Kuznetsov-Benjamin-Bona-Mahony (ZK-BBM) equation and the ill-posed Boussinesq equation [18], the space time-fractional (3+1)-dimensional nonlinear Jimbo-Miwa equation and nonlinear Hirota-Satsuma coupled KdV system [19], the time-fractional modified BBM equation and the time-fractional Cohn-Hilliard equation [20], the Pochhammer- Chree equation [21], and the Gerdjikov-Ivanov equation [22].

The traveling wave solutions of the nonlinear evolution equations solved by the $\exp(-\Phi(\xi))$-expansion method have the form

$$u(\xi) = \sum_{i=0}^{n} l_i \exp(-\Phi(\xi))^i, \tag{1}$$

where l_i $(i = 1, \ldots, n)$ are undetermined constants, $l_n \neq 0$, and $\Phi(\xi)$ satisfies the auxiliary equation:

$$\Phi'(\xi) = \exp(-\Phi(\xi)) + \lambda + \mu \cdot \exp(\Phi(\xi)). \tag{2}$$

In this paper, we improve the $\exp(-\Phi(\xi))$-expansion method by presenting a new auxiliary equation:

$$\begin{aligned}&\left(\Phi'(\xi)\right)^2\\&\quad = k^2\left(r \cdot \exp(-2\Phi(\xi)) + a + b \cdot \exp(2\Phi(\xi))\right).\end{aligned} \tag{3}$$

From (2), we get

$$\begin{aligned}\left(\Phi'(\xi)\right)^2 &= \exp(-2\Phi(\xi)) + 2\lambda \exp(-\Phi(\xi)) + \lambda^2 + 2\mu\\&\quad + 2\lambda\mu \exp(\Phi(\xi)) + \mu^2 \exp(2\Phi(\xi)).\end{aligned} \tag{4}$$

Obviously, if $\lambda = 0$, $k = 1$, $r = 1$, $a = 2\mu$, and $b = \mu^2$, then (4) becomes (3). That is, in special case of $\lambda = 0$, the improved $\exp(-\Phi(\xi))$-expansion method is a generalization of the $\exp(-\Phi(\xi))$-expansion method.

In [16], the hyperbolic function solutions, the trigonometric solutions, and the rational function solutions of (2) were given, while, in this paper, the Jacobi elliptic function solutions of (3) are shown in Section 2. We know that when $m \longrightarrow 1$ ($m \longrightarrow 0$) (where m is the modulus of the Jacobi elliptic function), the Jacobi elliptic functions degenerate to hyperbolic functions (trigonometric functions), so the improved $\exp(-\Phi(\xi))$-expansion method extend the tanh function method [23], the solitary wave ansatz method [24], the sine-cosine method [25], etc.

In addition, besides the Jacobi elliptic function solutions, (3) also has many combined Jacobi elliptic function solutions which we omit for simplicity in Section 2. In fact, substituting these solutions into (2), we get the power series of the reciprocal of Jacobi elliptic functions or combined Jacobi elliptic functions. Hence, the improved $\exp(-\Phi(\xi))$-expansion method is more general than the Jacobi elliptic function method [26], the F-function method [27], and the indirect F-function method [28].

We implement the improved $\exp(-\Phi(\xi))$-expansion method to construct new traveling wave solutions for two nonlinear partial differential equations. The first one is the ill-posed Boussinesq equation:

$$u_{tt} = u_{xx} + \left(u^2\right)_{xx} + u_{xxxx}. \quad (5)$$

This equation was first introduced by the French Scientist J. Boussinesq, which can describe the propagation of long wave in shallow water with a small amplitude in one-dimension nonlinear lattices and in nonlinear strings [29–31]. Equation (5) differs from the well-posed Boussinesq equation only in the sign of the last dispersive term. Recently, the numerical and analytical solutions of (5) were extensively reported in the literature. For instance, in [31], the approximate solutions were obtained by applying filtering and regularization techniques. In [32], the exact homoclinic orbits solutions with periodic boundary condition and even constraint were studied. In [33], Jafari et al. gave the solitary wave solutions by using the sine-cosine and extended tanh function method. In [34], Seadawy A derived the dark and bright solitary wave solutions by means of the direct algebraic method. In [35, 36], the authors used the theory of Lie group to obtain the symmetry reductions and the group invariant solutions. In [24, 37], the authors obtained the new traveling wave solutions and the nonlinear self-adjoint substitution to construct new conservation laws.

The second nonlinear partial differential equation is the unstable nonlinear Schrödinger equation:

$$iu_t + u_{xx} + 2\lambda |u|^2 u - 2\gamma u = 0. \quad (6)$$

This equation is a type of nonlinear Schrödinger equation with space and time exchanged. Its behavior occurs in case of lossless symmetric two-stream plasma instability [38] and the two-layer baroclinic instability [39]. Many traveling wave solutions have been derived by different methods, such as the semi-inverse variational principle [40], the sine-cosine method [25], the new Jacobi elliptic function rational expansion method and exponential rational function method [26], the (G'/G)-expansion method [41], the $\tan(\Phi(\xi)/2)$-expansion method [42], and the extended simple equation method [43]. To the best of our knowledge, no previous work has been done by the improved $\exp(-\Phi(\xi))$-expansion method.

The organization of this paper is as follows: firstly, we give the methodology of the improved $\exp(-\Phi(\xi))$ expansion method. Then we apply this method to the ill-posed Boussinesq equation and the unstable nonlinear Schrödinger equation for finding new exact traveling wave solutions. Also we discuss the obtained solutions with the already existing results in the previous literature. At last, a conclusion is given.

2. Description of the Improved $\exp(-\Phi(\xi))$-Expansion Method

In this section, we describe the improved $\exp(-\Phi(\xi))$-expansion method. Consider a nonlinear partial differential equation with independent variables x and t

$$P\left(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \ldots\right) = 0, \quad (7)$$

where $u = u(x,t)$ is an unknown function and P is a polynomial in $u = u(x,t)$ and its partial derivatives.

Firstly, applying the traveling wave transformation $u(x,t) = u(\xi)$, $\xi = x - vt$ to (7), we reduce (7) to an ordinary differential equation (ODE):

$$Q\left(u, u', u'', \ldots\right) = 0, \quad (8)$$

where prime denotes the derivative with respect to ξ. If possible, integrate (8) term by term one or more times. For simplicity, we set the integration constant(s) to zero.

Secondly, suppose that the solution of (8) can be expressed as

$$u(\xi) = \sum_{i=0}^{n} l_i \exp(-\Phi(\xi))^i, \quad (9)$$

where $\Phi(\xi)$ satisfies the following auxiliary ODE:

$$\left(\Phi'(\xi)\right)^2 = k^2\left(r \cdot \exp(-2\Phi(\xi)) + a + b \cdot \exp(2\Phi(\xi))\right). \quad (10)$$

l_i $(i = 1, 2, \ldots, n)$, v, k, r, a and b are constants to be determined later, $l_n \neq 0$, and n is a positive integer which can be determined by homogenous balance in (8).

We know that the auxiliary equation (10) has many different Jacobi elliptic function solutions. For simplicity, we only list several typical solutions:

(1) If $r = m^2$, $a = -1 - m^2$, $b = 1$, then $\Phi_1 = \ln \mathrm{ns}(k\xi, m)$.

(2) If $r = -m^2$, $a = -1 + 2m^2$, $b = 1 - m^2$, then $\Phi_2 = \ln \mathrm{nc}(k\xi, m)$.

(3) If $r = 1$, $a = -(m^2 + 1)$, $b = m^2$, then $\Phi_3 = \ln \mathrm{sn}(k\xi, m)$.

(4) If $r = 1 - m^2$, $a = 2m^2 - 1$, $b = -m^2$, then $\Phi_4 = \ln \mathrm{cn}(k\xi, m)$.

(5) If $r = 1 - m^2$, $a = 2 - m^2$, $b = 1$, then $\Phi_5 = \ln \mathrm{cs}(k\xi, m)$.

(6) If $r = 1$, $a = 2 - m^2$, $b = 1 - m^2$, then $\Phi_6 = \ln \mathrm{sc}(k\xi, m)$,

where m is the modulus of the Jacobi elliptic function.

Thirdly, substituting (9) along with (10) into (8) and equating coefficients of the same power in $(\exp(-\Phi(\xi)))$ to zero yields a set of algebraic equations for l_i $(i = 1, 2, \ldots, n)$, k and v. Solving these equations, we can construct a variety of exact solutions for (7).

3. Application of the Improved $\exp(-\Phi(\xi))$-Expansion Method

In this section, two examples are presented to illustrate how the method works.

3.1. The Ill-Posed Boussinesq Equation. Using the traveling wave transformation we can reduce (5) to the following ODE:

$$\left(v^2 - 1\right) u'' - 2\left(u'\right)^2 - 2uu'' - u'''' = 0, \quad (11)$$

where prime denotes the derivation with respect to ξ. Integrating (11) twice with respect to ξ, and taking the integration constants as zero, we get

$$\left(v^2 - 1\right) u - u^2 - u'' = 0. \quad (12)$$

By homogeneous balance, we get $n = 2$. So the solution of (12) can be expressed as

$$u(\xi) = l_0 + l_1 \exp(-\Phi(\xi)) + l_2 \left(\exp(-\Phi(\xi))\right)^2. \quad (13)$$

Substituting (13) along with (10) into (12) and setting each coefficient of $\exp(-i\Phi(\xi))$, $(i = 0, 1, 2, 3, 4)$ to zero, we obtain a set of algebraic equations as follows:

$$\begin{aligned}
\exp(-4\Phi(\xi)) &: -6l_2k^2r - l_2^2 = 0, \\
\exp(-3\Phi(\xi)) &: -2l_1l_2 - 2l_1k^2r = 0, \\
\exp(-2\Phi(\xi)) &: v^2l_2 - l_2 - 4l_2k^2a - 2l_0l_2 - l_1^2 = 0, \\
\exp(-\Phi(\xi)) &: v^2l_1 - l_1 - l_1k^2a - 2l_0l_1 = 0, \\
\exp^0 &: v^2l_0 - l_0^2 - l_0 - 2l_2k^2b = 0.
\end{aligned} \quad (14)$$

Solving the above system, we get

$$\begin{aligned}
k &= k, \\
v^2 &= 1 \pm 4k^2\sqrt{a^2 - 3rb}, \\
l_0 &= 2k^2\left(-a \pm \sqrt{a^2 - 3rb}\right), \\
l_1 &= 0, \\
l_2 &= -6k^2r.
\end{aligned} \quad (15)$$

Consequently, we have the following different cases for the exact traveling solutions of the ill-posed Boussinesq equation:

(1) If $r = m^2$, $a = -1 - m^2$, $b = 1$, then

$$\begin{aligned}
u_1(x,t) &= 2k^2\left(1 + m^2 \pm \sqrt{1 - m^2 + m^4}\right) \\
&\quad - 6k^2m^2\mathrm{sn}^2\left(k\left(x \mp \sqrt{1 \pm 4k^2\sqrt{1 - m^2 + m^4}}\,t\right),\right. \\
&\quad \left. m\right);
\end{aligned} \quad (16)$$

(2) If $r = -m^2$, $a = -1 + 2m^2$, $b = 1 - m^2$, then

$$\begin{aligned}
u_2(x,t) &= 2k^2\left(1 - 2m^2 \pm \sqrt{1 - m^2 + m^4}\right) \\
&\quad + 6k^2m^2\mathrm{cn}^2\left(k\left(x \mp \sqrt{1 \pm 4k^2\sqrt{1 - m^2 + m^4}}\,t\right),\right. \\
&\quad \left. m\right);
\end{aligned} \quad (17)$$

(3) If $r = 1$, $a = -(m^2 + 1)$, $b = m^2$, then

$$\begin{aligned}
u_3(x,t) &= 2k^2\left(m^2 + 1 \pm \sqrt{1 - m^2 + m^4}\right) \\
&\quad - 6k^2\mathrm{ns}^2\left(k\left(x \mp \sqrt{1 \pm 4k^2\sqrt{1 - m^2 + m^4}}\,t\right), m\right);
\end{aligned} \quad (18)$$

(4) If $r = 1 - m^2$, $a = 2m^2 - 1$, $b = -m^2$, then

$$\begin{aligned}
u_4(x,t) &= 2k^2\left(1 - 2m^2 \pm \sqrt{1 - m^2 + m^4}\right) \\
&\quad + 6k^2\left(m^2 - 1\right) \\
&\quad \cdot \mathrm{nc}^2\left(k\left(x \mp \sqrt{1 \pm 4k^2\sqrt{1 - m^2 + m^4}}\,t\right), m\right);
\end{aligned} \quad (19)$$

(5) If $r = 1 - m^2$, $a = 2 - m^2$, $b = 1$, then

$$\begin{aligned}
u_5(x,t) &= 2k^2\left(m^2 - 2 \pm \sqrt{1 - m^2 + m^4}\right) \\
&\quad - 6k^2\left(1 - m^2\right) \\
&\quad \cdot \mathrm{sc}^2\left(k\left(x \mp \sqrt{1 \pm 4k^2\sqrt{1 - m^2 + m^4}}\,t\right), m\right);
\end{aligned} \quad (20)$$

(6) If $r = 1$, $a = 2 - m^2$, $b = 1 - m^2$, then

$$u_6(x,t) = 2k^2\left(m^2 - 2 \pm \sqrt{1 - m^2 + m^4}\right) - 6k^2 \mathrm{cs}^2\left(k\left(x \mp \sqrt{1 \pm 4k^2\sqrt{1 - m^2 + m^4}}\,t\right), m\right); \quad (21)$$

Remark 1. When the modulus $m \longrightarrow 1$, we have $\mathrm{sn}(\xi, m) \longrightarrow \tanh(\xi)$, $\mathrm{cn}(\xi, m) \longrightarrow \mathrm{sech}(\xi)$, $\mathrm{ns} \longrightarrow \coth(\xi)$, $\mathrm{cs} \longrightarrow \mathrm{csch}(\xi)$. Taking $k = \lambda/2$ and $m \longrightarrow 1$ in $u_1(x,t)$ and $u_4(x,t)$, where λ is an arbitrary constant, we get the following hyperbolic function solutions, respectively:

$$u_{1_1}(x,t) = \frac{3}{2}\lambda^2 - \frac{3}{2}\lambda^2 \tanh^2\left(\frac{\lambda}{2}\left(x \pm \sqrt{1 + \lambda^2}t\right)\right);$$
$$u_{1_2}(x,t) = \frac{1}{2}\lambda^2 - \frac{3}{2}\lambda^2 \tanh^2\left(\frac{\lambda}{2}\left(x \pm \sqrt{1 - \lambda^2}t\right)\right). \quad (22)$$

$$u_{3_1}(x,t) = \frac{3}{2}\lambda^2 - \frac{3}{2}\lambda^2 \coth^2\left(\frac{\lambda}{2}\left(x \pm \sqrt{1 + \lambda^2}t\right)\right);$$
$$u_{3_2}(x,t) = \frac{1}{2}\lambda^2 - \frac{3}{2}\lambda^2 \coth^2\left(\frac{\lambda}{2}\left(x \pm \sqrt{1 - \lambda^2}t\right)\right). \quad (23)$$

Taking $k = B$ and $m \longrightarrow 1$ in $u_2(x,t)$ and $u_8(x,t)$, where B is an arbitrary constant, we get

$$u_{2_1}(x,t) = 6B^2 \mathrm{sech}^2\left(B\left(x \mp \sqrt{1 + 4k^2}t\right)\right);$$
$$u_{2_2}(x,t) = -4B^2 + 6B^2 \mathrm{sech}^2\left(k\left(x \mp \sqrt{1 - 4k^2}t\right)\right). \quad (24)$$

$$u_{6_1}(x,t) = -6B^2 \mathrm{csch}^2\left(k\left(x \mp \sqrt{1 + 4k^2}t\right)\right);$$
$$u_{6_2}(x,t) = -4B^2 - 6B^2 \mathrm{csch}^2\left(k\left(x \mp \sqrt{1 - 4k^2}t\right)\right). \quad (25)$$

The above solutions coincide with solutions of Tchier et al. [24]:

(i) our solutions $u_{1_1}(x,t)$ and $u_{1_2}(x,t)$ match with solutions (36) and (34) in [24], respectively;

(ii) obtained solution $u_{2_1}(x,t)$ is identical with solution (8) in [24];

(iii) the solutions $u_{3_1}(x,t)$ and $u_{3_2}(x,t)$ turn out solutions (37) and (35) in [24], respectively;

(iv) obtained solution $u_{6_1}(x,t)$ comes into solution (15) in [24].

When the modulus $m \longrightarrow 0$, we get $\mathrm{ns}(\xi, m) \longrightarrow \csc(\xi)$, $\mathrm{nc}(\xi, m) \longrightarrow \sec(\xi)$, $\mathrm{sc}(\xi, m) \longrightarrow \tan(\xi)$, and $\mathrm{cs}(\xi, m) \longrightarrow \cot(\xi)$; from $u_3(x,t)$, $u_4(x,t)$, $u_5(x,t)$, and $u_6(x,t)$ we obtain the following trigonometric function solutions, respectively:

$$u_{3_3}(x,t) = 4k^2 - 6k^2 \csc^2\left(k\left(x \mp \sqrt{1 + 4k^2}t\right)\right). \quad (26)$$

$$u_{4_1}(x,t) = 4k^2 - 6k^2 \sec^2\left(k\left(x \mp \sqrt{1 + 4k^2}t\right)\right). \quad (27)$$

$$u_{5_1}(x,t) = -2k^2 - 6k^2 \tan^2\left(k\left(x \mp \sqrt{1 + 4k^2}t\right)\right);$$
$$u_{5_2}(x,t) = -6k^2 - 6k^2 \tan^2\left(k\left(x \mp \sqrt{1 - 4k^2}t\right)\right). \quad (28)$$

$$u_{6_3}(x,t) = -2k^2 - 6k^2 \cot^2\left(k\left(x \mp \sqrt{1 + 4k^2}t\right)\right);$$
$$u_{6_4}(x,t) = -6k^2 - 6k^2 \cot^2\left(k\left(x \mp \sqrt{1 - 4k^2}t\right)\right). \quad (29)$$

To the best of our knowledge, the solutions $u_{3_3}(x,t), u_{4_1}(x,t), u_{5_1}(x,t), u_{5_2}(x,t), u_{6_3}(x,t)$, and $u_{6_4}(x,t)$ have not been reported in the previous literature.

3.2. The Unstable Nonlinear Schrödinger Equation. Using the following traveling wave transformations

$$u(x,t) = U(\xi) e^{i\theta}, \quad \xi = x - vt \ \textit{and}\ \theta = \beta x + r_1 t \quad (30)$$

to (6), we obtain the ODE

$$U'' - \left(\beta^2 + r_1 + 2\gamma\right)U + 2\lambda U^3 = 0, \quad (31)$$

where $v = 2\beta$ and β is a constant which will be determined later, while γ, r_1 and λ are arbitrary constants. We get the balancing number as $n = 1$. Thus the solution can be written as

$$U(\xi) = l_0 + l_1 \exp(-\Phi(\xi)). \quad (32)$$

Substituting (32) along with (10) into (31) and collecting the coefficients of each power $\exp(-i\Phi(\xi))$ $(i = 0, 1, 2, 3)$ and setting each of the coefficients to zero, we get a set of algebraic equations of l_0, l_1, β, and k as follows:

$$\begin{aligned} \exp(-\Phi(\xi))^3 &: 2l_1k^2r + 2\lambda l_1^3 = 0, \\ \exp(-\Phi(\xi))^2 &: 6\lambda l_0 l_1^2 = 0, \\ \exp(-\Phi(\xi)) &: l_1ak^2 - \beta^2 l_1 - r_1 l_1 - 2\gamma l_1 + 6\lambda l_0^2 l_1 = 0, \\ \exp^0 &: r_1 l_0 - \beta^2 l_0 + 2\lambda l_1^3 - 2\gamma l_0 = 0. \end{aligned} \quad (33)$$

Solving the above system with the aid of Maple, we get

$$\begin{aligned} k &= \pm\sqrt{\frac{\beta^2 + r_1 + 2\gamma}{a}}, \\ \beta &= \beta, \\ l_0 &= 0, \\ l_1 &= \pm\sqrt{\frac{-r\left(\beta^2 + r_1 + 2\gamma\right)}{\lambda a}}. \end{aligned} \quad (34)$$

Consequently, we have the following different cases for the exact solutions of Schrödinger equation:

(1) If $r = m^2$, $a = -1 - m^2$, $b = 1$, then

$$u_1(x,t) = \pm m\sqrt{\frac{\beta^2 + r_1 + 2\gamma}{\lambda(1+m^2)}}\operatorname{sn}\left(\pm\sqrt{\frac{\beta^2 + r_1 + 2\gamma}{-1-m^2}}(x - 2\beta t), m\right)e^{i(\beta x + r_1 t)}; \tag{35}$$

(2) If $r = -m^2$, $a = -1 + 2m^2$, $b = 1 - m^2$, then

$$u_2(x,t) = \pm m\sqrt{\frac{\beta^2 + r_1 + 2\gamma}{\lambda(-1+2m^2)}}\operatorname{cn}\left(\pm\sqrt{\frac{\beta^2 + r_1 + 2\gamma}{-1+2m^2}}(x - 2\beta t), m\right)e^{i(\beta x + r_1 t)}; \tag{36}$$

(3) If $r = 1$, $a = -(m^2 + 1)$, $b = m^2$, then

$$u_3(x,t) = \pm\sqrt{\frac{\beta^2 + r_1 + 2\gamma}{\lambda(1+m^2)}}\operatorname{ns}\left(\pm\sqrt{\frac{\beta^2 + r_1 + 2\gamma}{-1-m^2}}(x - 2\beta t), m\right)e^{i(\beta x + r_1 t)}; \tag{37}$$

(4) If $r = 1 - m^2$, $a = 2m^2 - 1$, $b = -m^2$, then

$$u_4(x,t) = \pm\sqrt{\frac{(m^2-1)(\beta^2 + r_1 + 2\gamma)}{\lambda(2m^2-1)}}\operatorname{nc}\left(\pm\sqrt{\frac{\beta^2 + r_1 + 2\gamma}{2m^2-1}}(x - 2\beta t), m\right)e^{i(\beta x + r_1 t)}; \tag{38}$$

(5) If $r = 1 - m^2$, $a = 2 - m^2$, $b = 1$, then

$$u_5(x,t) = \pm\sqrt{\frac{(m^2-1)(\beta^2 + r_1 + 2\gamma)}{\lambda(2-m^2)}}\operatorname{sc}\left(\pm\sqrt{\frac{\beta^2 + r_1 + 2\gamma}{2-m^2}}(x - 2\beta t), m\right)e^{i(\beta x + r_1 t)}; \tag{39}$$

(6) If $r = 1$, $a = 2 - m^2$, $b = 1 - m^2$, then

$$u_6(x,t) = \pm\sqrt{\frac{\beta^2 + r_1 + 2\gamma}{\lambda(m^2-2)}}\operatorname{cs}\left(\pm\sqrt{\frac{\beta^2 + r_1 + 2\gamma}{2-m^2}}(x - 2\beta t), m\right)e^{i(\beta x + r_1 t)}; \tag{40}$$

Remark 2. When the modulus $m \longrightarrow 1$, our solutions $u_1(x,t), u_2(x,t), u_3(x,t)$ and $u_6(x,t)$ turn into the following hyperbolic function solutions, respectively:

$$u_{1_1}(x,t) = \pm\sqrt{\frac{\beta^2 + r_1 + 2\gamma}{2\lambda}} \cdot \tanh\left(\sqrt{\frac{\beta^2 + r_1 + 2\gamma}{-2}}(x - 2\beta t)\right)e^{i(\beta x + r_1 t)}; \tag{41}$$

$$u_{2_1}(x,t) = \pm\sqrt{\frac{\beta^2 + r_1 + 2\gamma}{\lambda}} \cdot \operatorname{sech}\left(\sqrt{\beta^2 + r_1 + 2\gamma}(x - 2\beta t)\right)e^{i(\beta x + r_1 t)}; \tag{42}$$

$$u_{3_1}(x,t) = \pm\sqrt{\frac{\beta^2 + r_1 + 2\gamma}{2\lambda}} \cdot \coth\left(\sqrt{\frac{\beta^2 + r_1 + 2\gamma}{-2}}(x - 2\beta t)\right)e^{i(\beta x + r_1 t)}; \tag{43}$$

$$u_{6_1}(x,t) = \pm\sqrt{\frac{\beta^2 + r_1 + 2\gamma}{-\lambda}} \cdot \operatorname{csch}\left(\sqrt{\beta^2 + r_1 + 2\gamma}(x - 2\beta t)\right)e^{i(\beta x + r_1 t)};. \tag{44}$$

The above solutions coincide with solutions in [25, 40, 41]:

(i) our solution $u_{1_1}(x,t)$ matches with solutions $u_{31}(x,t)$ and $u_{35}(x,t)$ in [41];

(ii) obtained solution $u_{2_1}(x,t)$ identical with solutions (45) and (46) in [40];

(iii) our solution $u_{3_1}(x,t)$ matches with solutions $u_{32}(x,t)$ and $u_{36}(x,t)$ in [41];

(iv) the solution $u_{6_1}(x,t)$ turns out solution (52) in [25].

When the modulus $m \longrightarrow 0$, our solutions $u_3(x,t), u_4(x,t), u_5(x,t)$ and $u_6(x,t)$ come into the following trigonometric function solutions, respectively:

$$u_{3_2}(x,t) = \pm\sqrt{\frac{\beta^2 + r_1 + 2\gamma}{\lambda}} \cdot \csc\left(\sqrt{-(\beta^2 + r_1 + 2\gamma)}(x - 2\beta t)\right)e^{i(\beta x + r_1 t)}; \tag{45}$$

$$u_{4_1}(x,t) = \pm\sqrt{\frac{\beta^2 + r_1 + 2\gamma}{\lambda}} \cdot \sec\left(\sqrt{-(\beta^2 + r_1 + 2\gamma)}(x - 2\beta t)\right)e^{i(\beta x + r_1 t)}; \tag{46}$$

$$u_{5_1}(x,t) = \pm\sqrt{\frac{\beta^2+r_1+2\gamma}{-2\lambda}} \cdot \tan\left(\sqrt{\frac{\beta^2+r_1+2\gamma}{2}}(x-2\beta t)\right)e^{i(\beta x+r_1 t)}; \tag{47}$$

$$u_{6_2}(x,t) = \pm\sqrt{\frac{\beta^2+r_1+2\gamma}{-2\lambda}} \cdot \cot\left(\sqrt{\frac{\beta^2+r_1+2\gamma}{2}}(x-2\beta t)\right)e^{i(\beta x+r_1 t)};. \tag{48}$$

The above solutions coincide with solutions in [25, 41]:

(i) the solutions $u_{3_2}(x,t)$ and $u_{4_1}(x,t)$ match with solutions (50) and (51), respectively, in [25];

(ii) the solutions $u_{5_1}(x,t)$ comes into solutions $u_{33}(x,t)$ and $u_{37}(x,t)$ in [41];

(iii) the solution $u_{6_2}(x,t)$ is identical with solutions $u_{34}(x,t)$ and $u_{38}(x,t)$ in [41].

Remark 3. If we take $k = 1$ in the auxiliary equation (10), then the solution of (33) becomes

$$\begin{aligned} k &= 1, \\ \beta &= \pm\sqrt{a-r_1-2\gamma}, \\ l_0 &= 0, \\ l_1 &= \pm\sqrt{\frac{-r}{\lambda}}. \end{aligned} \tag{49}$$

Hence we get the following solutions of (6).

(1) If $r = m^2$, $a = -1-m^2$, $b = 1$, then

$$u_7(x,t) = \pm m\sqrt{\frac{-1}{\lambda}}\operatorname{sn}\left(x \mp 2\sqrt{-1-m^2-r_1-2\gamma}t, m\right) \cdot e^{i(\pm\sqrt{-1-m^2-r_1-2\gamma}x+r_1 t)}; \tag{50}$$

(2) If $r = -m^2$, $a = -1+2m^2$, $b = 1-m^2$, then

$$u_8(x,t) = \pm m\sqrt{\frac{1}{\lambda}}\operatorname{cn}\left(x \mp 2\sqrt{-1+2m^2-r_1-2\gamma}t, m\right) \cdot e^{i(\pm\sqrt{-1+2m^2-r_1-2\gamma}x+r_1 t)}; \tag{51}$$

(3) If $r = -m^2$, $a = 2-m^2$, $b = m^2-1$, then

$$u_9(x,t) = \pm m\sqrt{\frac{1}{\lambda}}\operatorname{dn}\left(x \mp 2\sqrt{2-m^2-r_1-2\gamma}t, m\right) \cdot e^{i(\pm\sqrt{2-m^2-r_1-2\gamma}x+r_1 t)}. \tag{52}$$

Our solutions $u_7(x,t), u_8(x,t)$ and $u_9(x,t)$ match with (10), (15), and (22) in [26], respectively.

4. Conclusion

The $\exp(-\Phi(\xi))$-expansion method has been successfully generalized to establish new exact solutions for the ill-posed Boussinesq equation and the unstable nonlinear Schrödinger equation. In special cases, our new Jacobi elliptic function solutions can be degenerated to hyperbolic function solutions and trigonometric function solutions which have been reported in the previous literature. That shows the improved $\exp(-\Phi(\xi))$-expansion method is effective and powerful. This method can be used to solve many other nonlinear evolution equations in mathematical physics, engineering, and other nonlinear sciences.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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