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Research Article

Solution to the Problem of a Mass Traveling on a Taut String via Integral Equation

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The problem of a massive taut string, traveled by a heavy point mass, moving with an assigned law, is formulated in a linear context. Displacements are assumed to be transverse, and the dynamic tension is neglected. The equations governing the moving boundary problem are derived via a variational principle, in which the geometric compatibility between the point mass and the string is enforced via a Lagrange multiplier, having the meaning of transverse reactive force. The equations are rearranged in the form of a unique Volterra integral equation in the reactive force, which is solved numerically. A classical Galerkin solution is implemented for comparison. Numerical results throw light on the physics of the phenomenon and confirm the effectiveness of the algorithm.

1. Introduction

The problem of a point mass traveling a taut string is a classical topic of mechanics, which has been extensively studied since 1964 [1] to nowadays [2, 3]. Except few exceptions (see, e.g., [3–5]) the problem has been formulated in a purely linear context, in which the displacements of the string are taken of transverse type only, and the tension is assumed to remain (nearly) equal to its initial value.

In spite of the expected simplicity of the mathematical problem, formulation requires some care, since the moving mass induces a singularity in the slope of the string (circumstance which also occurs in a Timoshenko beam but not in an Euler-Bernoulli beam), which moves with the point mass itself. The singularity calls for breaking the integration interval into two complementary subintervals, which vary in time and in enforcing suitable boundary conditions at singularity; that is, it requires formulating a *moving boundary problem*. If one writes the equations of motion by resorting to balance laws, he has to account for the jump of the velocity field at the singularity, which induces an instantaneous variation of the linear momentum of the string; this contribution enters the local balance equations, providing an extra term with respect to the usual engineering problems with immovable ends. The mechanism, which is correctly described in a few papers (see, e.g., [3, 6]), is instead overlooked in most of papers (see, e.g., [2, 7]), in which use is made of the Dirac Delta function in the partial differential equation governing the motion of the string. Although the procedure is correct, since the Delta Dirac encompasses the abovementioned effect, it certainly shades the phenomenon in an unsatisfactory way. It is worth noting that the use of the Dirac Delta function is prevalent not only when the analyzed travelled structure is a string but also when it is a cable [4, 5], a beam [8–12], or a plate [13, 14].

Concerning the solution strategies, different methods have been used in the literature: in [2], by following a semianalytical approach, the continuous problem has been reduced to a second order ODEs system, which has been numerically integrated; in [15, 16] a space-time finite element method approach is proposed and its efficiency proved; finally, in [17], the moving boundary problem has been transformed into a fixed boundary problem via a suitable coordinate transformation; then, the continuous problem has been discretized via a Galerkin approach by using a set of shape functions having a discontinuity in the slope. An

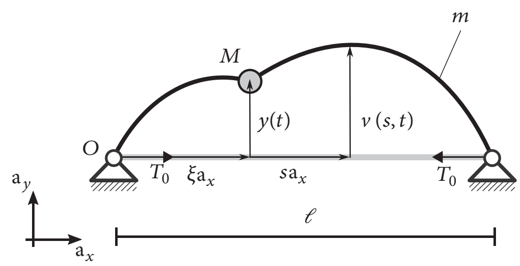


FIGURE 1: Horizontal taut string travelled by a heavy point mass.

integral approach has also been illustrated by Smith [1] and successively used in [18] for the evaluation of the response of a semi-infinite string travelled by a mass particle undergoing a constant horizontal acceleration. In the original paper by Smith, such an integral equation was solved analytically for a massless/massive string, by accounting for successive waves reflections, and therefore consisting in several branches, each valid in a specific time interval. Quite surprisingly, to the best of authors' knowledge, a numerical implementation of the method, valid for any times, seems not to be performed, yet.

In this paper a variational approach is followed, in which the moving boundary problem is consistently approached. A mixed displacement-reactive force is presented, in which the reaction acts as a Lagrangian parameter enforcing permanent contact between the point mass and the string. The equations are rearranged to obtain a single integral equation, which is solved numerically. Results concern different motions of the point mass, monotonic and periodic (back and forth).

2. Equations of Motion

Let us consider a point mass M, of weight $P = Mg$, traveling, with assigned horizontal motion $\xi = \overline{\xi}(t)$, a horizontal string of length ℓ, mass per unit length m, and taut by a tension T_0, fixed at the ends (Figure 1). The problem is formulated in a linear context. Accordingly, it is assumed that the string undergoes small vertical displacements $v(s,t) = O(\epsilon)$, with $s \in [0,\ell]$ a material abscissa and t the time; hence, horizontal displacements $u(s,t) = O(\epsilon^2)$ are ignored as higher-order effects. As a consequence, the spatial abscissa $x = \xi(t) + u(\xi(t),t)$, currently occupied by the point mass, and the associated material abscissa $\xi(t)$ coincide at any time; that is, $x \simeq \xi$. Moreover, by referring to the Green-Lagrange one-dimensional (incremental) strain, it is

$$\varepsilon = u' + \frac{1}{2}\left(u'^2 + v'^2\right) \simeq \frac{1}{2}v'^2, \tag{1}$$

where a dash denotes space differentiation. If a linear elastic behavior is taken for the string, the dynamic tension is $\widetilde{T} = EA\varepsilon = O(EA\epsilon^2)$, with EA the axial stiffness; therefore, if $EA/T_0 \leqslant O(\epsilon^{-1})$, it is $\widetilde{T}/T_0 \leqslant O(\epsilon)$; that is, the current tension $T = T_0 + \widetilde{T}$ coincides, at first order, with the initial tension T_0, in any points at any time, $T \simeq T_0$.

Due to the forces exchanged between the point mass and the string, this latter manifests a discontinuity in the slope at the abscissa $\xi(t)$ instantaneously occupied by the mass. Since the singularity moves with a known law, the problem assumes the form of a *moving boundary problem*. It is formulated here by using a variational principle, in which movable ends are accounted for.

2.1. Variational Principle. The equations of motion are derived by the variational Hamilton principle, in which the mass-string geometric compatibility is appended as a constraint. The modified Hamilton functional reads

$$\begin{aligned}\widetilde{\mathscr{H}}\left[v, y, R_y\right] := \int_{t_1}^{t_2} \Bigg[& \frac{1}{2}\int_0^\ell m\dot{v}^2 \mathrm{d}s - \frac{1}{2}\int_0^\ell T_0 v'^2 \mathrm{d}s \\ & + \frac{1}{2}M\dot{y}^2 - Py - R_y\left(y - v(\xi,t)\right)\Bigg] \mathrm{d}t.\end{aligned} \tag{2}$$

Here, the first term is the kinetic energy of the string, the dot denoting time differentiation; the second term is the quadratic part of potential energy of the string; the third term is the kinetic energy of the point mass, in which $y(t)$ is its ordinate; the fourth term is the potential energy of the force P; the last term is the compatibility condition $y = v(\xi,t)$, expressing permanent contact between point mass and string, affected by the Lagrange multiplier R_y, which assumes the meaning of vertical reactive force exchanged by the two bodies.

The principle requires $\delta\widetilde{\mathscr{H}} = 0$ for any admissible arguments and arbitrary times t_1, t_2. The string is fixed at the external boundary; namely,

$$v(0,t) = v(\ell,t) = 0, \quad \forall t, \tag{3}$$

so that the admissibility at these boundaries requires $\delta v(0,t) = \delta v(\ell,t) = 0$, $\forall t$. Admissibility at singularity (internal boundary), located at $s = \xi$, requires that

$$\delta v_- = \delta v_+, \tag{4}$$

where $\delta v_\pm = \lim_{\epsilon \longrightarrow 0}\delta v(\xi \pm \epsilon, t)$, $\epsilon > 0$.

The first variation of Equation (2) reads

$$\begin{aligned}\delta\widetilde{\mathscr{H}} = \int_{t_1}^{t_2} \Bigg[& \int_0^\ell m\dot{v}\delta\dot{v}\,\mathrm{d}s - \int_0^\ell T_0 v'\delta v' \mathrm{d}s + M\dot{y}\delta\dot{y} \\ & - P\delta y - R_y\left(\delta y - \delta v_\pm\right) - \delta R_y\left(y - v_\pm\right)\Bigg] \mathrm{d}t = 0.\end{aligned} \tag{5}$$

Integration by parts of $\dot{y}\delta\dot{y}$ is straightforward; in contrast, integration by parts of the terms which involve integrals on space requires attention, since it calls for properly accounting for the presence of a singularity at $s = \xi$. By breaking the integration interval as $[0,\ell] = [0,\xi] \cup [\xi,\ell]$,

$$\begin{aligned}&\int_{t_1}^{t_2}\int_0^\ell m\dot{v}\delta\dot{v}\mathrm{d}s\,\mathrm{d}t \\ &\quad = -\int_{t_1}^{t_2}\int_0^\ell m\ddot{v}\delta v\,\mathrm{d}s\,\mathrm{d}t + \int_{t_1}^{t_2}\int_0^\ell \left(m\dot{v}\delta v\right)^{\bullet}\mathrm{d}s\,\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&= -\int_{t_1}^{t_2}\int_0^{\ell} m\ddot{v}\delta v\,\mathrm{d}s\,\mathrm{d}t \\
&\quad + \int_{t_1}^{t_2}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^{\ell} m\dot{v}\delta v\,\mathrm{d}s\right) + \dot{\xi}\,[\![m\dot{v}\delta v]\!]\right]\mathrm{d}t \\
&= -\int_{t_1}^{t_2}\int_0^{\ell} m\ddot{v}\delta v\,\mathrm{d}s\,\mathrm{d}t + \int_{t_1}^{t_2}\dot{\xi}\,[\![m\dot{v}\delta v]\!]\,\mathrm{d}t, \\
&\int_0^{\ell} T_0 v'\delta v'\,\mathrm{d}s \\
&= -\int_0^{\ell} T_0 v''\delta v\,\mathrm{d}s + \left[T_0 v'\delta v\right]_0^{\ell} - [\![T_0 v'\delta v]\!] \\
&= -\int_0^{\ell} T_0 v''\delta v\,\mathrm{d}s - [\![T_0 v'\delta v]\!],
\end{aligned}
\tag{6}
$$

where the arbitrariness of t_1, t_2 and the admissibility at the external boundary conditions have been accounted for; the Leibniz integral rule has been used; and the double square bracket $[\![f]\!] := f_+ - f_-$ denotes the *jump* of the argument at the singularity.

By collecting all previous results, the variational principle (5) reads

$$
\begin{aligned}
\delta\widetilde{\mathscr{H}} &= \int_{t_1}^{t_2}\int_0^{\ell}\left(T_0 v'' - m\ddot{v}\right)\delta v\,\mathrm{d}s\,\mathrm{d}t + \int_{t_1}^{t_2}\left(\dot{\xi}\,[\![m\dot{v}\delta v]\!]\right. \\
&\quad \left. + [\![T_0 v'\delta v]\!]\right)\mathrm{d}t + \int_{t_1}^{t_2}\left[-M\ddot{y}\delta y - P\delta y\right. \\
&\quad \left. - R_y\left(\delta y - \delta v_{\pm}\right) - \delta R_y\left(y - v_{\pm}\right)\right]\mathrm{d}t = 0, \\
&\quad \forall\left(\delta v, \delta y, \delta R_y\right),
\end{aligned}
\tag{7}
$$

or, by using admissibility at singularity, Equation (4):

$$
\begin{aligned}
\delta\widetilde{\mathscr{H}} &= \int_{t_1}^{t_2}\int_0^{\ell}\left(T_0 v'' - m\ddot{v}\right)\delta v\,\mathrm{d}s\,\mathrm{d}t + \int_{t_1}^{t_2}\left(\dot{\xi}\,[\![m\dot{v}]\!]\right. \\
&\quad \left. + [\![T_0 v']\!]\right)\delta v_{\pm}\mathrm{d}t + \int_{t_1}^{t_2}\left[-M\ddot{y}\delta y - P\delta y\right. \\
&\quad \left. - R_y\left(\delta y - \delta v_{\pm}\right) - \delta R_y\left(y - v_{\pm}\right)\right]\mathrm{d}t = 0, \\
&\quad \forall\left(\delta v, \delta y, \delta R_y\right).
\end{aligned}
\tag{8}
$$

From this latter, the equations of motion are finally derived:

$$T_0 v'' = m\ddot{v}, \tag{9}$$

$$-P - R_y = M\ddot{y}, \tag{10}$$

$$m\dot{\xi}\,[\![\dot{v}]\!] + T_0\,[\![v']\!] + R_y = 0, \tag{11}$$

$$y - v(\xi, t) = 0, \tag{12}$$

in which the abovementioned inertial term in $[\![\dot{v}]\!]$ appears at the singularity.

These equations must be sided by the boundary conditions (3), the initial conditions

$$
\begin{aligned}
v(s, 0) &= v_0(s), \\
\dot{v}(s, 0) &= \dot{v}_0(s),
\end{aligned}
\tag{13}
$$

and the continuity conditions $[\![v]\!] = 0$. From this latter, Hadamard's condition follows:

$$[\![\dot{v}]\!] + \dot{\xi}\,[\![v']\!] = 0, \tag{14}$$

which allows Equation (11) to be rewritten as

$$\left(T_0 - m\dot{\xi}^2\right)[\![v']\!] + R_y = 0. \tag{15}$$

The field equation (9) and the mechanical boundary condition (15) can be incorporated, as customary, in a unique equation, in which the Dirac delta appears:

$$T_0 v'' + R_y\delta(s - \xi) = m\ddot{v}, \tag{16}$$

so that the problem is governed by Equations (16), (10), and (12).

2.2. Nondimensional Equations. By introducing the following nondimensional quantities:

$$
\begin{aligned}
\widetilde{s} &:= \frac{s}{\ell}, \\
\widetilde{t} &:= \frac{1}{\ell}\sqrt{\frac{T_0}{m}}t, \\
\widetilde{v} &:= \frac{v}{\ell}, \\
\widetilde{\xi} &:= \frac{\xi}{\ell}, \\
\mu &:= \frac{M}{m\ell}, \\
\widetilde{P} &:= \frac{P}{T_0}, \\
\widetilde{R}_y &:= \frac{R_y}{T_0},
\end{aligned}
\tag{17}
$$

the whole problem is recast in nondimensional form:

$$v'' + R_y\delta(s - \xi) = \ddot{v}, \tag{18}$$

$$-P - R_y = \mu\ddot{y}, \tag{19}$$

$$y = v(\xi, t), \tag{20}$$

with relevant boundary and initial conditions:

$$
\begin{aligned}
v(0, t) &= 0, \\
v(1, t) &= 0,
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
v(s, 0) &= v_0(s), \\
\dot{v}(s, 0) &= \dot{v}_0(s).
\end{aligned}
\tag{22}
$$

Here, tilde has been omitted, and dash and dot denote differentiation with respect to the nondimensional independent variables. In the following the string will be taken to be at the rest at the initial time; that is, $v_0(s) = 0$, $\dot{v}_0(s) = 0$.

Equations (18)-(20) can, of course, be combined to eliminate R_y and y, in order to obtain an equation, well known in literature (see, e.g., [2], for the case of assigned uniform horizontal motion of the point mass), in which the string displacements only appear; that is,

$$v'' = \left[P + \mu\left(\ddot{\xi}v' + \dot{\xi}^2 v''_{\pm} + 2\dot{\xi}\dot{v}'_{\pm} + \ddot{v}_{\pm}\right)\right]\delta(s-\xi) + \ddot{v}. \quad (23)$$

However, a limited use will be made of this equation, since the solution discussed ahead refers, instead, to Equations (18)-(20).

3. Solution

The semianalytical strategy of solution proposed by Smith [1] is followed. It consists of (i) solving Equations (18) and (19) by the way of the convolution integral, in order to express $v(s,t)$ and $y(t)$ in terms of the unknown $R_y(t)$ and (ii) using the remaining Equation (20) to obtain an integral equation in the unknown $R_y(t)$.

3.1. The Integral Equation. The solution to Equation (18) is sought in the form of an infinite series:

$$v(s,t) = \sum_{k=1}^{\infty} \phi_k(s)\, q_k(t), \quad (24)$$

where $\phi_k(s) \coloneqq \sqrt{2}\sin(\omega_k s)$, $\omega_k \coloneqq k\pi$, $k = 1, 2, 3, \cdots$, are the eigenfunctions of the linear taut string, suitably normalized, and $q_k(t)$, $k = 1, 2, 3, \cdots$, are unknown Lagrangian coordinates. By substituting Equation (24) in Equation (18) and by projecting on the eigenfunction basis, the following equations are obtained:

$$\ddot{q}_k + \omega_k^2 q_k = R_y \phi_k(\xi), \quad k = 1, 2, \ldots, \quad (25)$$

which admit the solution

$$q_k(t) = \frac{\sqrt{2}}{\omega_k}\int_0^t R_y(\tau)\sin(\omega_k \xi(\tau))\sin(\omega_k(t-\tau))\,\mathrm{d}\tau. \quad (26)$$

Then, the solution (24) becomes

$$v(s,t) = \int_0^t R_y(\tau)\, K(s,t,\tau)\,\mathrm{d}\tau, \quad (27)$$

where the kernel is defined as follows:

$$K(s,t,\tau) \coloneqq \sum_{k=1}^{\infty}\frac{2}{\omega_k}\sin(\omega_k s)\sin(\omega_k \xi(\tau))\sin(\omega_k(t-\tau)), \quad (28)$$

or, by performing trigonometric manipulations

$$\begin{aligned} K(s,t,\tau) = \frac{1}{4}\big[&k(s-t+\tau+\xi(\tau)) \\ &- k(s-t+\tau-\xi(\tau)) + k(s+t-\tau-\xi(\tau)) \\ &- k(s+t-\tau+\xi(\tau))\big] \end{aligned} \quad (29)$$

where the following definition holds:

$$k(\eta) \coloneqq \sum_{k=1}^{\infty}\frac{2}{\omega_k}\sin(\omega_k \eta). \quad (30)$$

Remarkably, the limit of this series converges to

$$k(\eta) = \begin{cases} (2n+1) - \eta & 2n < \eta < 2(n+1) \\ 0 & \eta = n \end{cases}, \quad n = 0, \pm 1, \pm 2, \cdots. \quad (31)$$

On the other hand, the solution to Equation (19) is

$$y(t) = -\frac{Pt^2}{2\mu} - \frac{1}{\mu}\int_0^t R_y(\tau)(t-\tau)\,\mathrm{d}\tau. \quad (32)$$

By substituting Equations (27) and (32) into Equations (20), the following equation is found:

$$\int_0^t R_y(\tau)\,\mathscr{A}(t,\tau)\,\mathrm{d}\tau = f(t), \quad (33)$$

where

$$\begin{aligned} \mathscr{A}(t,\tau) &\coloneqq K(\xi(t), t, \tau) + \frac{t-\tau}{\mu}, \\ f(t) &\coloneqq -\frac{Pt^2}{2\mu}. \end{aligned} \quad (34)$$

Equation (33) is a Volterra integral equation of the first kind in the unknown $R_y(t)$. Once it has been solved, the response of the string is evaluated via Equation (27).

3.2. The Numerical Algorithm. To solve the final equation (33), a numerical procedure is applied, in which the trapezoidal rule is adopted to evaluate the integral. The time interval $[0, t_f]$ is divided into $N_s > 2$ equispaced subintervals of amplitude $\Delta t = t_f/N_s$. The following notation is used:

$$\begin{aligned} t_i &= i\Delta t, \\ \tau_j &= j\Delta t, \\ &\qquad i, j = 0, 1, 2, \ldots, N_s, \\ R_{yi} &= R_y(t_i), \\ f_i &= f(t_i), \\ \mathscr{A}_{ij} &= \mathscr{A}(t_i, \tau_j), \\ \overline{K}_{ij} &= K(\overline{s}, t_i, \tau_j), \end{aligned} \quad (35)$$

The integral equation (33) is approximated as

$$\left(\frac{1}{2}\mathscr{A}_{10}R_{y0}+\frac{1}{2}\mathscr{A}_{11}R_{y1}\right)\Delta t=f_1,$$
$$\left(\frac{1}{2}\mathscr{A}_{i0}R_{y0}+\sum_{j=1}^{i-1}\mathscr{A}_{ij}R_{yj}+\frac{1}{2}\mathscr{A}_{ii}R_{yi}\right)\Delta t=f_i \qquad (36)$$
$$i=2,\ldots,N_s,$$

Equations (36) can be solved in cascade by the sequence

$$R_{y0}=\frac{2f_1}{\mathscr{A}_{10}\Delta t},$$
$$R_{y1}=\frac{f_2}{\mathscr{A}_{21}\Delta t}-\frac{\mathscr{A}_{20}}{2\mathscr{A}_{21}}R_{y0},$$
$$R_{y(i-1)}=\frac{f_i}{\mathscr{A}_{i(i-1)}\Delta t} \qquad (37)$$
$$-\frac{1}{\mathscr{A}_{i(i-1)}}\left(\frac{1}{2}\mathscr{A}_{i0}R_{y0}+\sum_{j=1}^{i-2}\mathscr{A}_{ij}R_{yj}\right),$$
$$i=3,\ldots,N_s,$$

in which $\mathscr{A}_{ii}=0$ has been accounted for (i.e., $\mathscr{A}(t,t)=0$).

Once R_y has been determined, the displacement of the string at any given position $s=\bar{s}$ and at any given time $t=t_i$ is evaluated by the discrete counterpart of Equation (27); that is,

$$v(\bar{s},t_1)=\left(\frac{1}{2}\overline{K}_{10}R_{y0}+\frac{1}{2}\overline{K}_{11}R_{y1}\right)\Delta t,$$
$$v(\bar{s},t_i)=\left(\frac{1}{2}\overline{K}_{i0}R_{y0}+\sum_{j=1}^{i-1}\overline{K}_{ij}R_{yj}+\frac{1}{2}\overline{K}_{ii}R_{yi}\right)\Delta t, \qquad (38)$$
$$i=2,\ldots,N_s,$$

4. Numerical Results

Some case studies have been analyzed, in which the point mass moves: (a) of uniform motion $\xi=U_0t$ or (b) of uniformly accelerated/decelerated motion $\xi=U_0t+(1/2)\dot{U}_0t^2$ or (c) of periodic (back and forth) motion $\xi=\xi_0+(1/2)a(1-\cos((2\pi/t_t)t))$. Due to the nondimensionalization adopted, U_0 is the ratio between the dimensional velocity and the celerity of the transverse waves in the string, which has to be chosen always smaller than 1 (i.e., only subsonic motions have been analyzed here). In the periodic motion, ξ_0 is the initial position of the mass, a is the amplitude of the motion, and t_t is the (nondimensional) period. Results obtained by the integral approach are compared with those provided by the Galerkin approach, described in the Appendix, for the motions (a) and (b); the motion (c) is instead analyzed only with the Galerkin approach.

4.1. Uniform Motion. A sample system is considered, in which the mass ratio has been fixed to $\mu=0.1$ and the load-to-tension ratio to $P=0.01$. Two different uniform velocities, (i) $U_0=0.3$ and (ii) $U_0=0.9$, have been enforced to the point mass. The relevant results are plotted in Figures 2(a), 2(c), and 2(e) for the case (i) and Figures 2(b), 2(d), and 2(f) for the case (ii). Figures display, in the order: the trajectory $y(x)$ of the mass and the transverse displacement at midspan *vs* the position of the point mass; some deflections of the string, sampled at different times. The plots show that the point mass initially moves downward along a nearly straight line and then rises along a curve. The abscissa x_0 at which inversion of the vertical motion occurs is found to be nearly equal at the point at which the transverse wave (moving with celerity 1) meets the point mass (moving with velocity U_0), after reflection at the right end; that is, $x_0=2U_0/(1+U_0)$. Accordingly, $x_0\simeq 0.46$ in the case (i) and $x_0\simeq 0.95$ in the case (ii). Along the ascendant branch the effect of further reflections on the approaching point mass is noticed.

Graphs of the deflections of the string show a step-wise almost-linear pattern before the transverse wave has reached the right end; that is, when the nondimensional time $0\le t\le 1$; after the wave has reached the right end, the deflection of the string starts to change its shape due to reflections of waves between the mass and the two supports. In general, it is observed that the maximum deflection of the string below the mass increases with the travelling velocity of the point mass.

In Figure 2 the results of the integral approach (black curves) are compared with those of the Galerkin approach (gray curves). It is seen that an excellent agreement exists between the two methods. When the number of eigenfunctions N_e used in the Galerkin approach is increased, the relevant results tend to those of the integral method, in which, as observed, an infinite number of eigenfunctions has been considered.

4.2. Uniformly Accelerated/Decelerated Motion. The same string as before is taken, but with velocity of the point mass no more constant, but linear in time. Two cases have been considered: (i) initial velocity $U_0=0.7$ and different accelerations $\dot{U}_0$ and (ii) acceleration $\dot{U}_0=-0.065$ and different initial velocities. The relevant point-mass trajectories, plotted in Figures 3(a) and 3(b) for the two cases, respectively, show a character similar to that of the uniform velocity. Again, the inversion of the vertical motion is found to be very close to the point x_0 at which the point mass meets the reflected wave, which is solution to

$$x_0=U_0(2-x_0)+\frac{1}{2}\dot{U}_0(2-x_0)^2. \qquad (39)$$

Similar considerations hold for the (not shown) deflected shape of the string. Comparison between the two methods of solution adopted is, again, excellent.

4.3. Periodic (Back and Forth) Motion. Periodic back and forth motions are now addressed. The same system as before is taken; moreover, the initial position of the point mass is fixed at $\xi_0=0.01$ and the traveling period at $t_t=4$. Two

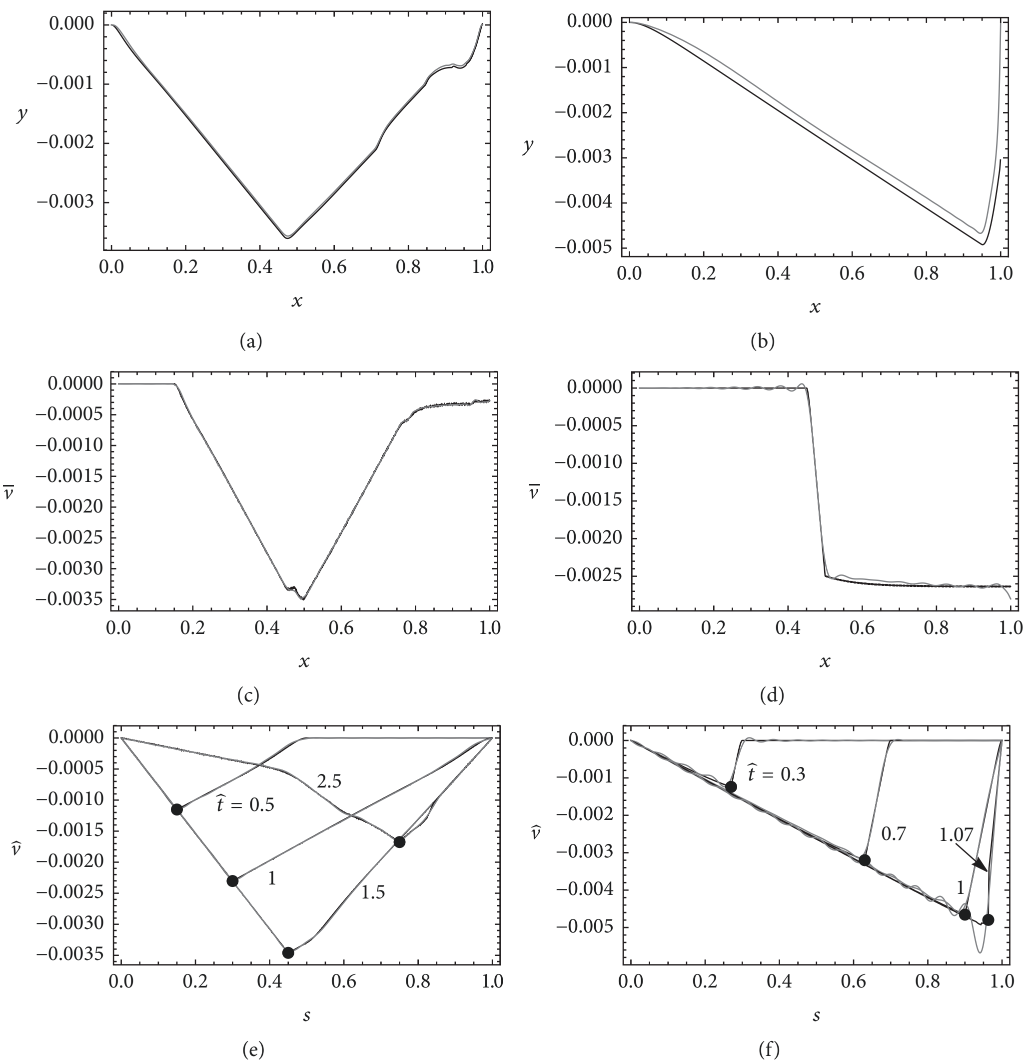


FIGURE 2: Comparison between the integral approach (black lines) and the Galerkin approach (gray lines), when $P = 0.01$ and $\mu = 0.1$; $N_e = 30$: (a), (c), (e) $U_0 = 0.3$ and $N_s = 6000$; (b), (d), (f) $U_0 = 0.9$ and $N_s = 2000$. The symbols $\overline{v}$ and $\widehat{v}$ denote $v(1/2, t)$ and $v(s, \hat{t})$, respectively.

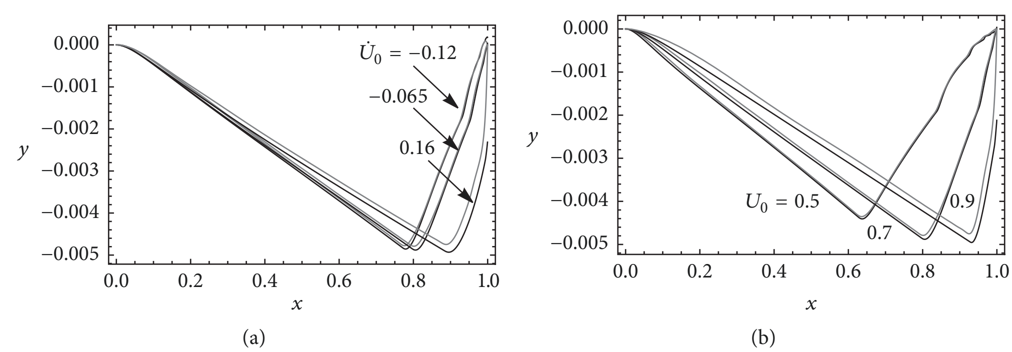


FIGURE 3: Comparison between the integral approach (black lines) and the Galerkin approach (gray lines), when $P = 0.01$ and $\mu = 0.1$; $N_s = 3000$ and $N_e = 30$: (a) $U_0 = 0.7$ and $\dot{U}_0$ variable; (b) U_0 variable and $\dot{U}_0 = -0.065$.

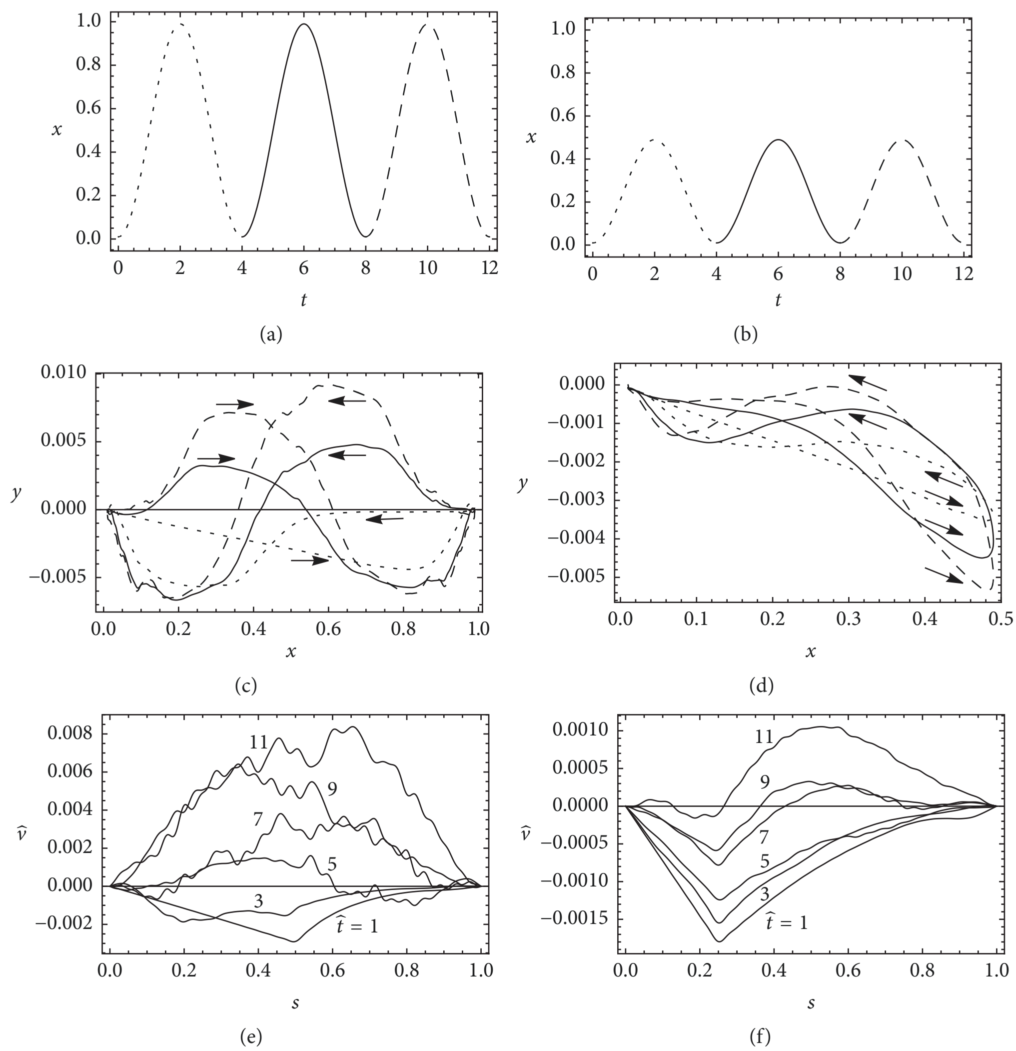


Figure 4: Back and forth motion, when $P = 0.01$, $\mu = 0.1$, $\xi_0 = 0.01$, and $t_t = 4$; $N_e = 50$: (a), (c), (e) $a = 0.98$; (b), (d), (f) $a = 0.48$. The symbol $\hat{v}$ denotes $v(s, \hat{t})$.

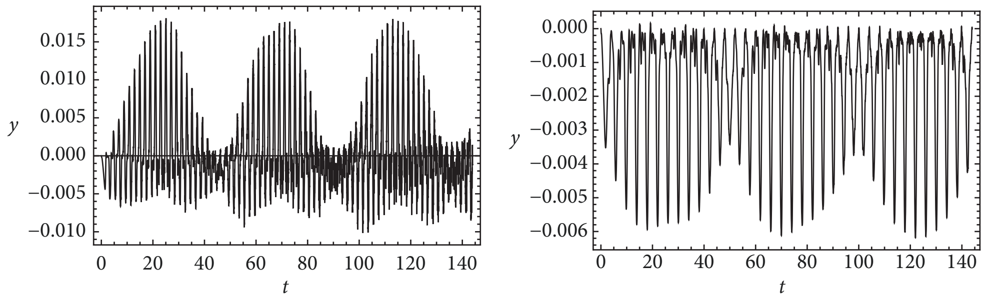
different amplitudes are considered: (i) $a = 0.98$ (nearly complete length excursion) and (ii) $a = 0.48$ (about half-length excursion). Numerical results have been obtained, for computational convenience, by the Galerkin method, in conjunction with the use of an optimized commercial code, although this entails adopting a truncated series. The choice has been suggested by the fact that, in a long time interval, the response of the string becomes more and more wrinkled, requiring time-step adaptation.

Figures 4 and 5 report, for the two cases, the response of the system to three back and forth cycles. Figures 4(c) and 4(d), which depict the point-mass trajectories, show that the forward paths are different from the backward paths, while all the forward (backward) branches are quite similar among them. The deflection patterns of the string, sampled at regular time intervals, are represented in Figures 4(e) and 4(f). The responses are found to be much more complex than those relevant to monotonic motions, caused by the numerous reflections of waves at boundaries and at the point mass. Since this latter moves slower in case (ii), the relevant deflections are more smooth than in case (i).

When the number of cycles is pushed to 36, the trajectories depicted in Figure 5 are found for the two cases. A quasiperiodic behavior emerges, which seems to exhibit a displacement accumulation phenomenon, similar to that discussed in [19], where repeated one-directional passages of forces were studied. It is worth noticing that such a phenomenon is more pronounced in case (i), probably for the different velocity considered. Such behavior should deserve deeper investigations, by a more sophisticated model, able to account for large change of tension and large displacements.

5. Conclusions

The classical problem of a point mass, moving with assigned motion along a taut string, has been studied. The problem has been formulated in a linear context, by neglecting higher



(a) (b)

Figure 5: Asymptotic behavior of the back and forth motion, when $P = 0.01$, $\mu = 0.1$, $\xi_0 = 0.01$, and $t_t = 4$; $N_e = 50$: (a) $a = 0.98$; (b) $a = 0.48$.

order effects, namely, longitudinal displacements and incremental strain and tension. A variational procedure has been followed, in which the compatibility condition of permanent contact between the two bodies has been introduced in the functional via a Lagrange multiplier, having the meaning of transverse reactive internal force. The procedure supplies the correct boundary conditions for the moving boundary problem, in terms of the jumps of the derivative of the displacement at the moving singular point.

With the help of a series representation and the convolution integral, the equations have been rearranged in the form of a single Volterra integral equation in the reactive force unknown. This has been solved in a numerical way, by using the trapezoidal rule, which leads to a sequence of linear equations to be solved in cascade. As an alternative, a Galerkin procedure has been implemented, once the equations have been combined to contain the transverse displacement only. This leads to a set of ordinary differential equations in the Lagrangian coordinates.

Numerical results lead to the following conclusions:

(1) The integral and the Galerkin method give very close results; while, however, the integral method accounts for an infinite number of eigenfunctions, the Galerkin method takes a finite number, so that the approximation in the first approach concerns the numerical implementation, only (not optimized, here).

(2) When the point mass moves of uniform motion, the first part of its trajectory is almost linear: the point mass first goes down and then gets up, the inversion of motion occurring when the reflected backward wave hits it for the first time. Accordingly, the deflection of the string is initially almost stepwise linear and then modifies in a curve, due to successive wave reflections.

(3) When the point mass moves of uniformly accelerated/decelerated motion, a similar qualitative behavior is detected.

(4) When the point mass executes periodic back and forth motions, the repeated reflections at the boundaries and at the point mass lead to a much more complex behavior. Repetitive cycles of the motion of the mass may produce quasiperiodic motions in the system with possible accumulation phenomena.

In conclusion, the integral method followed here, and numerically implemented, has been found to be an effective tool of analysis for the problem at hand. Further studies must be performed in order to include the dynamic tension in the equations of motion and to evaluate the limit of the linear model here analyzed.

Appendix

Galerkin Discretization of the Equations of Motion

A discrete model is obtained by discretizing Equation (23) via the Galerkin method. The transverse displacement is assumed as in Equation (24), but truncating the series at the first N_e terms. By substituting Equation (24) into Equation (20) and weighting the residual by ϕ_i, the following equations are obtained:

$$
\begin{aligned}
q_j \sum_{j=1}^{N_e} \int_0^1 \phi_i \phi_j'' \mathrm{d}s &= \ddot{q}_j \sum_{j=1}^{N_e} \int_0^1 \phi_i \phi_j \mathrm{d}s + \Bigg\{ \phi_i \Bigg[P \\
&\quad + \mu \sum_{j=1}^{N_e} \left(\ddot{\xi} \phi_j' q_j + \dot{\xi}^2 \phi_j'' q_j + 2\dot{\xi} \phi_j' \dot{q}_j + \phi_j \ddot{q}_j \right) \Bigg] \Bigg\}_{s=\xi}, \\
&\qquad i = 1, \dots, N_e.
\end{aligned} \tag{A.1}
$$

Using the normalized eigenfunction and its orthonormalization conditions, Equation (A.1) reads

$$
\begin{aligned}
&\ddot{q}_i + \omega_i^2 q_i + a_i P \\
&\quad + \mu \left(\ddot{\xi} \sum_{j=1}^{N_e} b_{ij} q_j + \dot{\xi}^2 \sum_{j=1}^{N_e} c_{ij} q_j + 2\dot{\xi} \sum_{j=1}^{N_e} b_{ij} \dot{q}_j + \sum_{j=1}^{N_e} a_{ij} \ddot{q}_j \right) \\
&\quad = 0, \quad i = 1, \dots, N_e,
\end{aligned} \tag{A.2}
$$

where

$$
\begin{aligned}
a_i &:= \left[\phi_i\right]_{s=\xi}, \\
a_{ij} &:= \left[\phi_i\phi_j\right]_{s=\xi}, \\
b_{ij} &:= \left[\phi_i\phi_j'\right]_{s=\xi}, \\
c_{ij} &:= \left[\phi_i\phi_j''\right]_{s=\xi}.
\end{aligned} \tag{A.3}
$$

The discretized model is governed by Eq (A.2).

It is worth noticing that the same equations can be obtained by following a Ritz approach. Indeed, the substitution of Equation (24) into Equation (2) and the imposition of the stationary of the functional lead to the same equations.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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