Hindawi
Advances in Mathematical Physics
Volume 2019, Article ID 7297090, 8 pages
https://doi.org/10.1155/2019/7297090

Research Article

The Effect of Gain and Strong Dissipative Structures on Nonlinear Schrödinger Equations in Optical Fiber

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Received 30 September 2018; Accepted 11 February 2019; Published 3 March 2019

Academic Editor: Antonio Scarfone

We consider the initial value problem for the nonlinear Schrödinger equation satisfying the strong dissipative condition $\Im\lambda < 0$ and $|\Im\lambda| > ((p-1)/2\sqrt{p})|\Re\lambda|$ in one space dimension. Our purpose in this paper is to study how the gain coefficient $\mu(t)$ and strong dissipative nonlinearity $\lambda|v|^{p-1}v$ affect solutions to the nonlinear Schrödinger equation for large initial data. We prove global existence of solutions and present some time decay estimates of solutions for large initial data.

1. Introduction and Main Results

We consider the Cauchy problem of nonlinear Schrödinger equation:

$$i\partial_t v + \frac{1}{2}\partial_x^2 v = \lambda |v|^{p-1} v + i\mu(t) v, \qquad v(0,x) = v_0(x), \tag{1}$$

where $v = v(t,x)$ is a complex valued unknown function, $t \geq 0$, $x \in \mathbb{R}$, $p > 1$, the gain coefficient $\mu(t)$ is a real valued function, and $\lambda \in \mathbb{C}$. Equation (1) is applied in problems of dispersion-managed optical fibers and soliton lasers (see [1]). The coefficients λ and $\mu(t)$ are, respectively, nonlinearity and amplification. In this work, we study the global existence and investigate time decay estimates of solutions to (1) with the gain coefficient $\mu(t)$ and the strong dissipative nonlinearity $\lambda|v|^{p-1}v$ satisfying $\Im\lambda < 0$ and $|\Im\lambda| > ((p-1)/2\sqrt{p})|\Re\lambda|$ for large initial data, where $\Im\lambda$ and $\Re\lambda$ are the imaginary and real part of λ, respectively.

Over the past few decades, the field of fiber optics has made rapid progress. The damped Schrödinger equation

$$i\partial_t v + \frac{1}{2}\partial_x^2 v = \lambda |v|^2 v + i\mu v, \tag{2}$$

where $v = v(t,x)$ is a complex valued unknown function, $t \in \mathbb{R}$, $x \in \mathbb{R}$, λ, and $\mu \in \mathbb{C}$, is one of the simplest nonlinear Schrödinger equations for studying cubic nonlinear effects in optical fibers (see [1]). Equation (2) is applied in several different aspects of optics (see, e.g., [2]). It has been studied extensively in the context of solitons (see [1]). In the case of $\mu(t) \equiv 0$, (1) is reduced to

$$i\partial_t v + \frac{1}{2}\partial_x^2 v = \lambda |v|^{p-1} v, \qquad v(0,x) = v_0(x), \tag{3}$$

where $t \geq 0$, $x \in \mathbb{R}$, $p > 1$, and $\lambda \in \mathbb{C}$. The nonlinearity $\lambda|v|^{p-1}v$ with $\Im\lambda < 0$ and $|\Im\lambda| > ((p-1)/2\sqrt{p})|\Re\lambda|$ is called strong dissipative. In [3, 4], the large initial problem for (3) with the strong dissipative nonlinearities was investigated. The global solution $v(t,x)$ to (3) decays like $t^{-1/2}(\log t)^{-1/2}$ in the sense of L^∞ for $t > 1$, when $p = 3$. Moreover $\|v\|_{L^\infty} \leq Ct^{-1/(p-1)}$ for $t > 1$, if $2.686 \approx (5+\sqrt{33})/4 < p < 3$ in [3] and $2.586 \approx (19+\sqrt{145})/12 < p < 3$ in [4], respectively. To study the time decays of solutions to (3), the estimate of $\|\mathcal{F}U(-t)v\|_{L^\infty}$ is useful. To get a better estimate of $\|\mathcal{F}U(-t)v\|_{L^\infty}$, the contradiction argument was used in [3] and the method of [5] was applied in [4]. In [6], the time decay estimates of global solutions to (3) with $1+\sqrt{2} < p < 3$ in the sense of L^2 were considered under the strong dissipative condition. It showed that $\|v\|_{L^2} \leq Ct^{-(1/(p-1)-1/2)q}$ for $t > 1$, where $0 \le q \le 2/3$. In the case of $\Im\lambda > 0$, the a priori L^2-bound for v is not satisfied, since

$$\frac{1}{2}\|v\|_{L^2}^2 - \Im\lambda \int_0^t \int_{\mathbb{R}} |v|^{p+1}\, dx d\tau = \frac{1}{2}\|v_0\|_{L^2}^2. \quad (4)$$

There are some results about the lifespan of small solutions to (3) with $\Im\lambda > 0$ (see, e.g., [7]). When optical pluses propagate inside a fiber, nonlinearities in (1) affect optical pluses' shapes. Some related nonlinear Schrödinger equations have been studied (see, e.g., [8, 9]). In the case of $\mu(t) \equiv -a < 0$ and $\lambda = -1$, (1) becomes

$$\begin{aligned} i\partial_t v + \frac{1}{2}\partial_x^2 v &= -|v|^{p-1} v - iav, \\ v(0,x) &= v_0(x) \end{aligned} \quad (5)$$

in one space dimension, where $t \ge 0$, $a > 0$, and $p > 1$. The question (5) has been studied by some mathematicians from the mathematical point (see, e.g., [10, 11]). Let $E_1(t) = (1/2)\|\partial_x v\|_{L^2}^2 - (2/(p+1))\|v\|_{L^{p+1}}^{p+1}$. If $E_1(0) \le 0$, nonexistence of global solutions to (5) was studied under some assumptions in [10]. It showed that the damping term in (5) cannot prevent blowing up of solutions. In [11], some blow up and global existence of solutions to (5) was investigated. The authors showed that the size of the damping coefficient a affected the solutions. As far as we know, there are not any results about the time decay estimates of solutions to (1) for large initial data. Our question is how the term $i\mu(t)v$ and the nonlinearity $\lambda|v|^{p-1}v$ with strong dissipative condition affect solutions to (1) for large initial data.

Let $L^q(\mathbb{R})$ denote the usual Lebesgue space with the norm

$$\|f\|_{L^q(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x)|^q\, dx\right)^{1/q} \quad (6)$$

if $1 \le q < \infty$ and

$$\|f\|_{L^\infty(\mathbb{R})} = \operatorname*{ess.sup}_{x\in\mathbb{R}} |f(x)|. \quad (7)$$

For $m, s \in \mathbb{R}$, weighted Sobolev space $H^{m,s}$ is defined by

$$\begin{aligned} H^{m,s} = \Big\{ f \in L^2(\mathbb{R}); \|f\|_{H^{m,s}} \\ = \left\| \left(1+|x|^2\right)^{s/2} \left(1-\partial_x^2\right)^{m/2} f \right\|_{L^2(\mathbb{R})} < \infty \Big\}. \end{aligned} \quad (8)$$

We write $L^q(\mathbb{R}) = L^q$ for $1 \le q \le \infty$ and $H^{m,0} = H^m$ for simplicity.

Let us introduce some notations. We define the dilation operator by

$$(D_t\phi)(x) = \frac{1}{(it)^{1/2}} \phi\left(\frac{x}{t}\right) \quad (9)$$

and define $M = e^{(i/2t)x^2}$ for $t \ne 0$. Evolution operator $U(t)$ is written as

$$U(t) = M D_t \mathcal{F} M, \quad (10)$$

where the Fourier transform of f is

$$(\mathcal{F} f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx. \quad (11)$$

We also have

$$U(-t) = M^{-1} \mathcal{F}^{-1} D_t^{-1} M^{-1}, \quad (12)$$

where the inverse Fourier transform of f is

$$\left(\mathcal{F}^{-1} f\right)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} f(\xi)\, d\xi. \quad (13)$$

We denote by the same letter C various positive constants. And we write $v(t)$ for the spatial function $v(t,\cdot)$.

The standard generator of Galilei transformations is given as

$$J(t) = U(t)\, x U(-t) = x + it\partial_x. \quad (14)$$

We have

$$J^2(t) = U(t)\, x^2 U(-t) = M\left(-t^2\partial_x^2\right) M^{-1}. \quad (15)$$

We also have commutation relations with J^β and $L = i\partial_t + (1/2)\partial_x^2$ such that

$$\left[L, J^\beta\right] = 0, \quad (16)$$

where $\beta = 1, 2$.

Before stating our main theorem, we introduce the function space

$$\begin{aligned} X_{1,T} = \Big\{ y; U(-t)\, y \in C\left([0,T); H^1 \cap H^{0,1}\right), \|y\|_{X_{1,T}} \\ < \infty \Big\}, \end{aligned} \quad (17)$$

where $\|y\|_{X_{1,T}} = \sup_{0\le t<T} \|U(-t)y\|_{H^1\cap H^{0,1}}$ and $T > 1$. We have the following global existence of solutions to (1) for large initial data.

Theorem 1. *Let $p > 1$, $\Im\lambda < 0$, $|\Im\lambda| > ((p-1)/2\sqrt{p})|\Re\lambda|$, and $\mu(t) \in C[0,\infty)$. We assume that $v_0(x) \in H^{0,1} \cap H^1$. Then (1) has a unique global solution $v(t,x) \in C([0,\infty); H^{0,1} \cap H^1)$ satisfying $v(t,x) = e^{\int_0^t \mu(\tau)d\tau} u(t,x)$, where $u(t,x) \in X_{1,\infty}$ is the solution to (19).*

Let $E_0(t) = \|v(t)\|_{L^2}^2$. Multiplying both sides of (1) by $\overline{v}$, integrating over $\mathbb{R}$, and taking the imaginary parts, we have

$$\frac{d}{dt} E_0(t) = 2\Im\lambda \|v(t)\|_{L^{p+1}}^{p+1} + 2\mu(t) E_0(t). \quad (18)$$

We could not ensure the sign of $(d/dt)E_0(t)$ by the assumptions in Theorem 1. This case is interesting. We have the equation

$$\begin{aligned} i\partial_t u + \frac{1}{2}\partial_x^2 u &= \lambda e^{(p-1)\int_0^t \mu(\tau)d\tau} |u|^{p-1} u, \\ u(0,x) &= v_0(x) \end{aligned} \quad (19)$$

from (1) by using the transformation $u = e^{-\int_0^t \mu(\tau)d\tau} v$. A straightforward calculation shows that $v(t,x)$ solves (1) if and only if $u(t,x)$ solves (19). Thus the transformation $u = e^{-\int_0^t \mu(\tau)d\tau} v$ provides an effective tool to study the global existence of solutions to (1).

Remark 2. Let $p > 1$, $\Im\lambda < 0$, $|\Im\lambda| > ((p-1)/2\sqrt{p})|\Re\lambda|$, $\mu(t) \in C[0,\infty)$, and $e^{\int_0^t \mu(\tau)d\tau} \le C_0$ for all $t \ge 0$, where $C_0 > 0$. We assume that $v_0(x) \in H^{0,1} \cap H^1$. Then (1) has a unique global solution $v(t,x) \in X_{1,\infty}$.

Example 3. We consider

$$i\partial_t v + \frac{1}{2}\partial_x^2 v = -i|v|^{p-1} v + i(1+t)^{-2} v, \quad v(0,x) = v_0(x) \tag{20}$$

in one space dimension, where $t \ge 0$ and $p > 1$. Since $|\Im(-i)| = 1 > 0 = ((p-1)/2\sqrt{p})|\Re(-i)|$, and $e^{\int_0^t (1+\tau)^{-2} d\tau} \le e$ for $t \ge 0$, (20) has a unique global solution $v(t,x) \in X_{1,\infty}$ if $v_0(x) \in H^{0,1} \cap H^1$.

If $\mu(t) \le 0$ for $t \ge 0$ and $\Im\lambda < 0$, we have the result about (1):

$$\frac{d}{dt} E_0(t) = 2\Im\lambda \|v(t)\|_{L^{p+1}}^{p+1} + 2\mu(t)\|v(t)\|_{L^2}^2 \le 0, \tag{21}$$

where $E_0(t) = \|v(t)\|_{L^2}^2$. Time decay estimates of solutions to (1) for large initial data are shown as follows.

Theorem 4. *Let $p > 1$, $\mu(t) \le 0$ for $t \ge 0$, $\mu(t) \in C[0,\infty)$ and the strong dissipative condition $\Im\lambda < 0$, $|\Im\lambda| > ((p-1)/2\sqrt{p})|\Re\lambda|$ hold. We assume that $v_0(x) \in H^{0,1} \cap H^1$. Then (1) has a unique global solution $v(t,x) \in X_{1,\infty}$ satisfying the following time decay estimates:*

$$\|v(t)\|_{L^\infty} \le Ct^{-1/2} e^{\int_0^t \mu(\tau)d\tau} \tag{22}$$

and

$$\|v(t)\|_{H^1} \le Ce^{\int_0^t \mu(\tau)d\tau} \tag{23}$$

for $t > 1$.

From Theorem 4, we obtain that the solution $v(t,x)$ to (1) is global and bounded in $X_{1,\infty}$ for large initial data. Moreover, we show that $\mu(t)$ determines the time decay rate of the solution, when $\mu(t)$ satisfies the assumptions in Theorem 4. Then we consider a special situation of Theorem 4. Let $\mu(t) \equiv \mu \le 0$ in (1); we have the following equation:

$$i\partial_t v + \frac{1}{2}\partial_x^2 v = \lambda|v|^{p-1} v + i\mu v, \quad v(0,x) = v_0(x), \tag{24}$$

where $t \ge 0$, $p > 1$, $x \in \mathbb{R}$, $\mu \le 0$, and $\lambda \in \mathbb{C}$. By Theorem 4, we have the time decay estimates to (24).

Corollary 5. *Let $\mu \le 0$ and $p > 1$. We assume that $v_0(x) \in H^{0,1} \cap H^1$ and the strong dissipative condition $\Im\lambda < 0$ and $|\Im\lambda| > ((p-1)/2\sqrt{p})|\Re\lambda|$ hold. Then (24) has a unique global solution $v(t,x) \in X_{1,\infty}$ satisfying the following time decay estimates:*

$$\|v(t)\|_{L^\infty} \le Ct^{-1/2} e^{\mu t} \tag{25}$$

and

$$\|v(t)\|_{H^1} \le Ce^{\mu t} \tag{26}$$

for $t > 1$.

In the last section, we consider the equation

$$i\partial_t v + \frac{1}{2}\partial_x^2 v = \lambda|v|^{p-1} v + i\frac{a}{(1+t)(p-1)} v, \quad v(0,x) = v_0(x), \tag{27}$$

where $a \le 0$, $p > 1$, $\Im\lambda < 0$, $|\Im\lambda| > ((p-1)/2\sqrt{p})|\Re\lambda|$, and $t \ge 0$. From (27), we have

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = \lambda(1+t)^a |u|^{p-1} u, \quad u(0,x) = v_0(x), \tag{28}$$

by taking the variable change $v(t,x) = (1+t)^{a/(p-1)} u(t,x)$. A similar nonlinear equation $i\partial_t u + \Delta u = \lambda(1+bt)^{(n\alpha-4)/2}|u|^\alpha u$, where $x \in \mathbb{R}^n$, $b \in \mathbb{R}$, $\alpha > 0$, and $\lambda \in \mathbb{R}$, was derived to study the rapid decay solutions and scattering properties of the equation $i\partial_t v + \Delta v = \lambda|v|^\alpha v$ by letting $u(t,x) = (1+bt)^{-n/2} v(t/(1+bt), x/(1+bt)) e^{i(b|x|^2/4(1+bt))}$ in [12].

Let $\Im\lambda < 0$ and $|\Im\lambda| > ((p-1)/2\sqrt{p})|\Re\lambda|$. Multiplying both sides of (27) by $\overline{v}$, integrating over $\mathbb{R}$, and taking the imaginary parts, we have

$$\frac{d}{dt} E_0(t) = 2\Im\lambda \|v(t)\|_{L^{p+1}}^{p+1} + \frac{2a}{(p-1)(1+t)} \|v(t)\|_{L^2}^2 \le 0, \tag{29}$$

where $E_0(t) = \|v(t)\|_{L^2}^2$. We have global existence and time decay estimates of solutions to (27) for large initial data by Theorem 4. We show a better decay rate of solutions to (27) inspired by the papers [4, 5].

Theorem 6. *Let $a \le 0$ and $p > 1$. We assume that $v_0(x) \in H^{0,1} \cap H^1$ and the strong dissipative condition $\Im\lambda < 0$ and $|\Im\lambda| > ((p-1)/2\sqrt{p})|\Re\lambda|$ hold. Then (27) has a unique global solution $v(t,x) \in X_{1,\infty}$ satisfying the following time decay estimates:*

$$\|v(t)\|_{L^\infty} \le Ct^{-1/2+a/(p-1)} \tag{30}$$

for $t > 1$. Moreover, let $p_*(a) := a + 5/4 + (1/4)\sqrt{16a^2 + 40a + 33}$, and if $p_*(a) \le p < 2a + 3$ and $-1 < a \le 0$, we have

$$\|v(t)\|_{L^\infty} \le Ct^{-1/2+a/(p-1)-(2a+3-p)/2(p-1)} \tag{31}$$

for $t > 1$.

When $a = 0$, the exponent $p_*(a)$ coincides with $(5+\sqrt{33})/4$, which is the lower bound given by Kita-Shimomura [3] and Jin-Jin-Li [4]. Let $p_{**}(a) := a + 19/12 + (1/12)\sqrt{144a^2 + 264a + 145}$. Since $p_{**}(a) < p_*(a)$ for $-1/2 < a \le 0$, the lower bound $p_*(a)$ in Theorem 6 can be improved by $p_{**}(a)$ in Theorem 7. The operator J^2 plays an important role in achieving the lower bound $p_{**}(a)$.

Theorem 7. *Let $-1/2 < a \le 0$, $p_{**}(a) := a + 19/12 + (1/12)\sqrt{144a^2 + 264a + 145}$, and $p_{**}(a) \le p < 2a + 3$. We assume that $v_0(x) \in H^{0,2} \cap H^1$ and the strong dissipative condition $\Im\lambda < 0$ and $|\Im\lambda| > ((p-1)/2\sqrt{p})|\Re\lambda|$ hold. Then (27) has a unique global solution $v(t,x) \in X_{1,\infty}$ satisfying the following time decay estimates:*

$$\|v(t)\|_{L^\infty} \le Ct^{-1/2+a/(p-1)-(2a+3-p)/2(p-1)} \tag{32}$$

for $t > 1$.

If $a = 0$, we have $p_{**}(a) = (19+\sqrt{145})/12$, which is a lower bound of p shown in [4]. Theorems 6 and 7 say how the strong dissipative nonlinearity and gain coefficient of the nonlinear Schrödinger equation (27) affect decay estimates of solutions under different initial conditions. The rest of this paper is organized as follows. In Section 2, we give proofs of Theorems 1 and 4. Theorems 6 and 7 are proven in Section 3.

2. Proofs of Theorems 1 and 4

2.1. Proof of Theorem 1. We have the equation

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = \lambda e^{(p-1)\int_0^t \mu(\tau)d\tau} |u|^{p-1} u \tag{33}$$

from (1) by using the variable change $u = e^{-\int_0^t \mu(\tau)d\tau} v$ and $u(0,x) = v(0,x) = v_0(x)$.

Local existence of solutions $u \in X_{1,T}$ to (33) can be shown by the standard contraction mapping principle (see, e.g., [13]). Therefore, we have local existence of solutions $v \in X_{1,T}$ to (1).

First, we consider the equation

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = \lambda F(t) |u|^{p-1} u, \tag{34}$$

where $F(t) \ge 0$, $F(t) \in C^1[0,\infty)$, $\lambda \in \mathbb{C}$, and $p > 1$. The following lemma is useful to study global existence and time decay of solutions to (34).

Lemma 8. *Let $p > 1$, $F(t) \ge 0$, $F(t) \in C^1[0,\infty)$, and $u(t,x) \in X_{1,T}$ be the local solution of (34), where $T > 1$. And let the strong dissipative condition $\Im\lambda < 0$ and $|\Im\lambda| > ((p-1)/2\sqrt{p})|\Re\lambda|$ be satisfied. Then we have*

$$\partial_t \left(\|\partial_x u\|_{L^2} + \|Ju\|_{L^2} + \|u\|_{L^2}\right) \le 0. \tag{35}$$

Proof. Multiplying both sides of (34) by $\overline{\partial_x u}$ and taking the imaginary parts, we have, by $|u|^{p-1}u = u^{(p+1)/2}\overline{u}^{(p-1)/2}$ and the assumptions of $F(t)$,

$$\begin{aligned}
\frac{1}{2}\partial_t \|\partial_x u\|_{L^2}^2 &= \Im\left(\int_{\mathbb{R}} \partial_x \left(\lambda F(t)|u|^{p-1}u\right)\cdot\overline{\partial_x u}dx\right) \\
&= \Im\left(\lambda\frac{p+1}{2}F(t)\int_{\mathbb{R}} |u|^{p-1}|\partial_x u|^2 dx\right) \\
&\quad + \Im\left(\lambda\frac{p-1}{2}F(t)\int_{\mathbb{R}} |u|^{p-3}u^2\left(\overline{\partial_x u}\right)^2 dx\right) \\
&\le F(t)\left(\Im\lambda\frac{p+1}{2} + |\lambda|\frac{p-1}{2}\right)\int_{\mathbb{R}} |u|^{p-1}|\partial_x u|^2 dx.
\end{aligned} \tag{36}$$

By (36) and the strong dissipative condition $\Im\lambda < 0$ and $|\Im\lambda| > ((p-1)/2\sqrt{p})|\Re\lambda|$, we obtain

$$\begin{aligned}
\frac{1}{2}\partial_t \|\partial_x u\|_{L^2}^2 &= \Im\left(\int_{\mathbb{R}} \partial_x \left(\lambda F(t)|u|^{p-1}u\right)\cdot\overline{\partial_x u}dx\right) \\
&\le 0.
\end{aligned} \tag{37}$$

We note that

$$\begin{aligned}
J\left(|u|^{p-1}u\right) &= \frac{p+1}{2}|u|^{p-1}Ju - \frac{p-1}{2}|u|^{p-3}u^2\overline{Ju}, \\
J\left(F(t)|u|^{p-1}u\right) &= \left(x + it\partial_x\right)\left(F(t)|u|^{p-1}u\right) \\
&= F(t)J\left(|u|^{p-1}u\right),
\end{aligned} \tag{38}$$

and

$$\frac{1}{2}\partial_t \|Ju\|_{L^2}^2 = \Im\left(\int_{\mathbb{R}} J\left(\lambda F(t)|u|^{p-1}u\right)\cdot\overline{Ju}dx\right). \tag{39}$$

Calculating the right part of (39), we obtain

$$\begin{aligned}
&\Im\left(\int_{\mathbb{R}} J\left(\lambda F(t)|u|^{p-1}u\right)\cdot\overline{Ju}dx\right) \\
&\quad = \Im\left(\lambda\frac{p+1}{2}F(t)\int_{\mathbb{R}} |u|^{p-1}|Ju|^2 dx\right) \\
&\qquad - \frac{p-1}{2}\Im\left(\lambda F(t)\int_{\mathbb{R}} |u|^{p-3}u^2\left(\overline{Ju}\right)^2 dx\right) \\
&\quad \le F(t)\left(\Im\lambda\frac{p+1}{2} + |\lambda|\frac{p-1}{2}\right)\int_{\mathbb{R}} |u|^{p-1}|Ju|^2 dx \\
&\quad \le 0
\end{aligned} \tag{40}$$

under the strong dissipative condition.

Multiplying both sides of (34) by $\overline{u}$, we obtain

$$\frac{1}{2}\partial_t \|u\|_{L^2}^2 = \Im\lambda F(t)\int_{\mathbb{R}} |u|^{p+1} dx \le 0 \tag{41}$$

by the assumptions of λ and $F(t)$. Using (37), (40), and (41), we have

$$\partial_t \left(\|\partial_x u\|_{L^2} + \|Ju\|_{L^2} + \|u\|_{L^2}\right) \le 0. \tag{42}$$

□

By Lemma 8, we have

$$\|u\|_{L^2} + \|\partial_x u\|_{L^2} + \|Ju\|_{L^2} \le \|v_0(x)\|_{L^2} + \|\partial_x v_0(x)\|_{L^2} + \|x v_0(x)\|_{L^2}. \quad (43)$$

From (43), we obtain a unique global solution $u \in X_{1,\infty}$ to (33). Since $u = e^{-\int_0^t \mu(\tau)d\tau} v$, we have a unique global solution $v(t,x) \in C([0,\infty); H^{0,1} \cap H^1)$ to (1).

2.2. Proof of Theorem 4. Using the transform $u = e^{-\int_0^t \mu(\tau)d\tau} v$, we get the equation about u from (1):

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = \lambda e^{(p-1)\int_0^t \mu(\tau)d\tau} |u|^{p-1} u, \quad (44)$$

where $u(0,x) = v(0,x) = v_0(x)$. We have local existence of solutions $u \in X_{1,T}$ to (44) and $v \in X_{1,T}$ to (1), respectively (see, e.g., [13]).

By Lemma 8 and the assumptions in Theorem 4, we have

$$\|u\|_{L^2} + \|\partial_x u\|_{L^2} + \|Ju\|_{L^2} \le \|v_0(x)\|_{L^2} + \|\partial_x v_0(x)\|_{L^2} + \|x v_0(x)\|_{L^2}. \quad (45)$$

And

$$\|v\|_{L^2} + \|\partial_x v\|_{L^2} + \|Jv\|_{L^2} \le e^{\int_0^t \mu(\tau)d\tau} \left(\|v_0(x)\|_{L^2} + \|\partial_x v_0(x)\|_{L^2} + \|x v_0(x)\|_{L^2}\right). \quad (46)$$

Thus, we obtain global-in-time existence of solutions $u \in X_{1,\infty}$ to (44) by (45) and $v \in X_{1,\infty}$ to (1) by (46), respectively.

We are now in a position to prove time decay estimates of solutions to (1). By the Sobolev inequality

$$\|u\|_{L^\infty} \le C \|u\|_{L^2}^{1/2} \|\partial_x u\|_{L^2}^{1/2}, \quad (47)$$

the factorization formula $U(t) = M D_t \mathcal{F} M$, and (45), we have

$$\begin{aligned}\|u\|_{L^\infty} &\le C t^{-1/2} \|\mathcal{F} M U(-t) u\|_{L^\infty} \\ &\le C t^{-1/2} \|\mathcal{F} M U(-t) u\|_{L^2}^{1/2} \|\partial_x \mathcal{F} M U(-t) u\|_{L^2}^{1/2} \\ &= C t^{-1/2} \|u\|_{L^2}^{1/2} \|Ju\|_{L^2}^{1/2} \le C t^{-1/2}\end{aligned} \quad (48)$$

for $t > 1$. Using the transform $u = e^{-\int_0^t \mu(\tau)d\tau} v$, we obtain

$$\|v\|_{L^\infty} \le C t^{-1/2} e^{\int_0^t \mu(\tau)d\tau} \quad (49)$$

for $t > 1$. By (46), we have

$$\|v\|_{H^1} \le C e^{\int_0^t \mu(\tau)d\tau} \quad (50)$$

for $t > 1$.

3. Proofs of Theorems 6 and 7

3.1. Proof of Theorem 6. From Theorem 4, we have global existence of solutions $v \in X_{1,\infty}$ to (27) and the time decay estimates (30). To get a better decay estimates of solutions to (27), we use the method of [4, 5]. Changing a variable such as $u = (1+t)^{-a/(p-1)} v$, we have

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = \lambda (1+t)^a |u|^{p-1} u, \quad (51)$$

and $u(0,x) = v(0,x) = v_0(x)$.

Multiplying both sides of (51) by $\mathcal{F}U(-t)$, we get

$$\begin{aligned}&\mathcal{F}U(-t)\left(\lambda(1+t)^a |u|^{p-1} u\right) \\ &= \lambda (1+t)^a \mathcal{F} M^{-1} \mathcal{F}^{-1} D_t^{-1} M^{-1} |u|^{p-1} u \\ &= \lambda (1+t)^a t^{-(p-1)/2} |\mathcal{F}U(-t) u|^{p-1} \mathcal{F}U(-t) u \\ &\quad + \lambda R(t),\end{aligned} \quad (52)$$

where

$$\begin{aligned}R(t) &= (1+t)^a t^{-(p-1)/2} \\ &\times \left(|\mathcal{F} M U(-t) u|^{p-1} \mathcal{F} M U(-t) u \right. \\ &\left. - |\mathcal{F} U(-t) u|^{p-1} \mathcal{F} U(-t) u\right) + (1+t)^a \\ &\cdot t^{-(p-1)/2} \mathcal{F}\left(M^{-1} - 1\right) \mathcal{F}^{-1} |\mathcal{F} M U(-t) u|^{p-1} \\ &\cdot \mathcal{F} M U(-t) u.\end{aligned} \quad (53)$$

Therefore, we have

$$\begin{aligned}&i\partial_t \mathcal{F}U(-t) u \\ &= \lambda (1+t)^a t^{-(p-1)/2} |\mathcal{F}U(-t) u|^{p-1} \mathcal{F}U(-t) u \\ &\quad + \lambda R(t),\end{aligned} \quad (54)$$

where the remainder term $R(t)$ is given in (53). Substituting $\mathcal{F}U(-t)u$ by f, we have the following equation about f:

$$i\partial_t f(t) = \lambda (1+t)^a t^{-(p-1)/2} |f(t)|^{p-1} f(t) + \lambda R(t). \quad (55)$$

We have the following estimates of $R(t)$. Since the proof of these estimates is similar to that in [4, 6], we omit the proof.

Lemma 9. *Let $u(t,x)$ be a solution of (51) in the function space $X_{1,\infty}$. Then, we have*

$$\|R(t)\|_{L^\infty} \le C t^{-(p/2-1/4-a)} \left(\|\mathcal{F} M U(-t) u\|_{L^\infty}^{p-1} + \|\mathcal{F} U(-t) u\|_{L^\infty}^{p-1}\right) \|Ju\|_{L^2} \quad (56)$$

and

$$\|R(t)\|_{L^2} \le C t^{-p/2+a} \|\mathcal{F}U(-t) u\|_{L^2}^{(p-1)/2} \|Ju\|_{L^2}^{(p+1)/2} \quad (57)$$

for $t > 1$.

From (55), we obtain

$$\partial_t \left|f(t)\right|^2 = 2\Im\lambda (1+t)^a t^{-(p-1)/2} \left|f(t)\right|^{p+1} + 2\Im\left(\lambda R(t)\overline{f(t)}\right). \quad (58)$$

Hence, we get

$$\partial_t \left|f(t)\right| \le 2^a \Im\lambda t^{-(p-1)/2+a} \left|f(t)\right|^p + C\|R(t)\|_{L^\infty} \quad (59)$$

for $t > 1$. By Lemmas 8 and 9, we obtain

$$\partial_t \left|f(t)\right| \le 2^a \Im\lambda t^{-p/2+1/2+a} \left|f(t)\right|^p + Ct^{-p/2+1/4+a} \quad (60)$$

for $t > 1$.

Multiplying both sides of the above by t^δ, where $\delta > 0$, we have

$$t^\delta \partial_t \left|f(t)\right| \le 2^a \Im\lambda t^{\delta-p/2+1/2+a} \left|f(t)\right|^p + Ct^{\delta-p/2+1/4+a}. \quad (61)$$

By the Young inequality, we have

$$\begin{aligned}\partial_t \left(t^\delta \left|f(t)\right|\right) &\le \delta t^{\delta-1}\left|f(t)\right| - 2^a \left|\Im\lambda\right| t^{\delta-p/2+1/2+a}\left|f(t)\right|^p + Ct^{\delta-p/2+1/4+a} \\ &\le \frac{p-1}{p^{p/(p-1)}}\cdot\frac{1}{\left|2^a\Im\lambda\right|^{1/(p-1)}} \cdot \delta^{p/(p-1)} t^{\delta-(1+2a+p)/2(p-1)} + Ct^{\delta-p/2+1/4+a}\end{aligned} \quad (62)$$

for $t > 1$. Let $\delta = (2a+3-p)/2(p-1)$ for $1 < p < 2a+3$ and $-1 < a \le 0$. Integrating in time from 1 to t, we get

$$\left|f(t)\right| \le Ct^{-(2a+3-p)/2(p-1)} + Ct^{-(p/2-5/4-a)} \quad (63)$$

for $1 < p < 2a+3$, $p > 5/2+2a$, $-1 < a \le 0$, and $t > 1$. To guarantee that

$$0 < \frac{2a+3-p}{2(p-1)} \le \frac{p}{2} - \frac{5}{4} - a, \quad (64)$$

we need

$$\begin{aligned}&\frac{5}{2} + 2a < p, \\ &1 < p < 2a+3, \\ &1 \le p(2p-4a-5).\end{aligned} \quad (65)$$

Let $p_*(a) = a+5/4+(1/4)\sqrt{16a^2+40a+33}$. Then, from (63), we get

$$\left|f(t)\right| \le Ct^{-(2a+3-p)/2(p-1)} \quad (66)$$

for $p_*(a) \le p < 2a+3$, $-1 < a \le 0$, and $t > 1$.

Since $p_*(a) \le p < 2a+3$ and $-1 < a \le 0$, we have $(2a+3-p)/2(p-1) < 1/4$. By (66), we have

$$\begin{aligned}\|u\|_{L^\infty} &\le Ct^{-1/2}\|\mathcal{F}U(-t)u\|_{L^\infty} + ct^{-3/4}\|xU(-t)u\|_{L^\infty} \\ &\le Ct^{-1/2-(2a+3-p)/2(p-1)}\end{aligned} \quad (67)$$

for $t > 1$. Using the transform $v(t,x) = (1+t)^{a/(p-1)}u(t,x)$, we have our desired result.

3.2. Proof of Theorem 7. We have global existence of solutions $v \in X_{1,\infty}$ to (27) by Theorem 4. We consider the decay estimates of solutions to (27) by using the method of [4, 5] in the following steps.

Since $[J^2, i\partial_t + (1/2)\partial_x^2] = 0$ holds, from (51), we have

$$\frac{1}{2}\partial_t \left\|J^2 u\right\|_{L^2}^2 = \Im\left(\lambda \int_{\mathbb{R}} J^2\left((1+t)^a |u|^{p-1} u\right)\cdot \overline{J^2 u}dx\right). \quad (68)$$

By the strong dissipative condition $\Im\lambda < 0$ and $|\Im\lambda| > ((p-1)/2\sqrt{p})|\Re\lambda|$ and using $\|u\|_{L^\infty} \le Ct^{-1/2}\|u\|_{L^2}^{1/2}\|Ju\|_{L^2}^{1/2}$ for $t > 0$, we have

$$\begin{aligned}&\Im\left(\lambda \int_{\mathbb{R}} J^2\left((1+t)^a |u|^{p-1} u\right)\cdot\overline{J^2 u}dx\right) \\ &\le \Im\left(\lambda \frac{p+1}{2}\int_{\mathbb{R}} (1+t)^a |u|^{p-1}\left|J^2 u\right|^2 dx\right) \\ &+ \Im\left(\lambda\frac{p-1}{2}\int_{\mathbb{R}} (1+t)^a |u|^{p-3} u^2 \left(\overline{J^2 u}\right)^2 dx\right) \\ &+ C\int_{\mathbb{R}} (1+t)^a |u|^{p-2}|Ju|^2\left|J^2 u\right| dx \le (1+t)^a \\ &\cdot\left(\Im\lambda\frac{p+1}{2} + |\Im\lambda|\frac{p-1}{2}\right)\int_{\mathbb{R}} |u|^{p-1}\left|J^2 u\right|^2 dx \\ &+ C(1+t)^a \|u\|_{L^\infty}^{p-2}\|Ju\|_{L^\infty}\left\|J^2 u\right\|_{L^2}\|Ju\|_{L^2} \\ &\le Ct^{-(p-1)/2+a}\|u\|_{L^2}^{(p-2)/2}\|Ju\|_{L^2}^{3/2+(p-2)/2}\left\|J^2 u\right\|_{L^2}^{3/2}\end{aligned} \quad (69)$$

for $t > 0$. Thus, we obtain

$$\left\|J^2 u\right\|_{L^2} \le \|v_0\|_{H^{0,2}} + Ct^{-p+3+2a} \quad (70)$$

for $t > 0$.

By using a similar method to that in [4], we have the estimate of $R(t)$ as follows. Here we omit the proof.

Lemma 10. *Let u be a solution of (51) in the function space $X_{1,\infty}$. Then, we have*

$$\|R(t)\|_{L^\infty} \le Ct^{-(p/2+1/4-a)}\|v_0\|_{H^{0,1}}^{p-1}\left\|J^2 u\right\|_{L^2} \quad (71)$$

for $t > 1$.

(58) shows

$$\partial_t |f(t)|^2 = 2\Im\lambda (1+t)^a t^{-(p-1)/2} |f(t)|^{p+1} + 2\Im\left(\lambda R(t)\overline{f(t)}\right), \quad (72)$$

where $f = \mathscr{F}U(-t)u$. Let $3 - p + 2a > 0$. By Lemma 10 and (58), we get

$$\partial_t |f(t)| \le 2^a \Im\lambda t^{-(p-1)/2+a} |f(t)|^p + Ct^{-3p/2+11/4+3a} \quad (73)$$

for $t > 1$. Multiplying both sides of the above by t^ρ, where $\rho > 0$, we have

$$t^\rho \partial_t |f(t)| \le 2^a \Im\lambda t^{\rho-p/2+1/2+a} |f(t)|^p + Ct^{\rho-3p/2+11/4+3a}. \quad (74)$$

By the Young inequality, we have

$$\begin{aligned}\partial_t (t^\rho |f(t)|) &\le \rho t^{\rho-1} |f(t)| - 2^a |\Im\lambda| t^{\rho-p/2+1/2+a} |f(t)|^p + Ct^{\rho-3p/2+11/4+3a} \\ &\le \frac{p-1}{p^{p/(p-1)}} \cdot \frac{1}{|2^a\Im\lambda|^{1/(p-1)}} \cdot \rho^{p/(p-1)} t^{\rho-(1+2a+p)/2(p-1)} + Ct^{\rho-3p/2+11/4+3a}\end{aligned} \quad (75)$$

for $t > 1$. Let $\rho = (2a+3-p)/2(p-1)$ for $1 < p < 2a+3$ and $-1 < a \le 0$. Integrating in time from 1 to t, we get

$$|f(t)| \le Ct^{-(2a+3-p)/2(p-1)} + Ct^{-(3p/2-15/4-3a)} \quad (76)$$

for $1 < p < 2a+3$, $p > 2a + 13/6$, $-1 < a \le 0$, and $t > 1$. To obtain

$$0 < \frac{2a+3-p}{2(p-1)} \le \frac{3p}{2} - \frac{15}{4} - 3a, \quad (77)$$

we need

$$\begin{aligned} &1 < p < 2a+3, \\ &p > 2a + \frac{13}{6}, \\ &6p^2 - (12a+19)p + (8a+9) \ge 0. \end{aligned} \quad (78)$$

Let $p_{**}(a) =: a + 19/12 + (1/12)\sqrt{144a^2 + 264a + 145}$. Then, from (76), we get

$$|f(t)| \le Ct^{-(2a+3-p)/2(p-1)} \quad (79)$$

for $p_{**}(a) \le p < 2a+3$, $-1 < a \le 0$, and $t > 1$.

When $p_{**}(a) \le p < 2a+3$ and $-1/2 < a \le 0$, we have $(2a+3-p)/2(p-1) < 1/4$. By (79), we have

$$\begin{aligned}\|u\|_{L^\infty} &\le Ct^{-1/2} \|\mathscr{F}U(-t)u\|_{L^\infty} + ct^{-3/4} \|xU(-t)u\|_{L^\infty} \\ &\le Ct^{-1/2-((2a+3-p)/2(p-1))}\end{aligned} \quad (80)$$

for $t > 1$. By using the relation $v(t,x) = (1+t)^{a/(p-1)} u(t,x)$, we have our desired result.

Data Availability

All data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors would like to thank Professor Hideaki Sunagawa for some helpful comments. The work is partially supported by the Education Department of Jilin Province (JJKH20180892KJ) and NNSFC (Grant no. 11461074).

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