Hindawi
Advances in Mathematical Physics
Volume 2020, Article ID 1320635, 15 pages
https://doi.org/10.1155/2020/1320635

Research Article

Variational Approach for the Variable-Order Fractional Magnetic Schrödinger Equation with Variable Growth and Steep Potential in $\mathbb{R}^{N*}$

Jianwen Zhou,[1] Bianxiang Zhou,[1] Liping Tian,[1] and Yanning Wang[2]

[1]*Department of Mathematics, Yunnan University, Kunming, Yunnan 650091, China*
[2]*School of Basic Medical Science, Kunming Medical University, Kunming, Yunnan 650500, China*

Correspondence should be addressed to Yanning Wang; wangscanf2004@126.com

Received 4 June 2020; Revised 1 September 2020; Accepted 28 September 2020; Published 28 October 2020

Academic Editor: Zhi-Yuan Sun

In this paper, we show the existence of solutions for an indefinite fractional Schrödinger equation driven by the variable-order fractional magnetic Laplace operator involving variable exponents and steep potential. By using the decomposition of the Nehari manifold and variational method, we obtain the existence results of nontrivial solutions to the equation under suitable conditions.

1. Introduction

In this paper, we investigate the existence of solutions of the following concave-convex fractional elliptic equation driven by the variable-order fractional magnetic Laplace operator involving variable exponents:

$$(-\Delta)_A^{s(\cdot)}u + V_\lambda(x)u = f(x)\left|u\right|^{q(x)-2}u + g(x)\left|u\right|^{p(x)-2}u \text{ in } \mathbb{R}^N, \quad (1)$$

where $N \geq 1, s(\cdot)$: $\mathbb{R}^N \times \mathbb{R}^N \to (0,1)$, is a continuous function, $(-\Delta)_A^{s(\cdot)}$ is the variable-order fractional magnetic Laplace operator, the potential $V_\lambda(x) = \lambda V^+(x) - V^-(x)$ with $V^\pm = \max\{\pm V, 0\}$, $\lambda > 0$ is a parameter, and the magnetic field is $A \in C^{0,\alpha}(\mathbb{R}^N, \mathbb{R}^N)$ with $\alpha \in (0,1]$,p, $q \in C(\mathbb{R}^N)$ and $u : \mathbb{R}^N \to \mathbb{C}$. In [1], the fractional magnetic Laplacian has been defined as

$$(-\Delta)_A^s u(x) = \lim_{r\to 0}\int_{B_r^c(x)} \frac{u(x) - e^{i(x-y)\cdot A(x+y/2)}u(y)}{|x-y|^{N+2s}}dy, \quad (2)$$

for $x \in \mathbb{R}^N$. In [2], the variable-order fractional Laplace $(-\Delta)^{s(\cdot)}$ is defined as for each $x \in \mathbb{R}^N$,

$$(-\Delta)^{s(\cdot)}u(x) = 2P.V\int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s(x,y)}}dy, \quad (3)$$

along any $u \in C_0^\infty(\Omega)$. Inspired by them, we define the variable-order fractional magnetic Laplacian $(-\Delta)_A^{s(\cdot)}$ as for each $x \in \mathbb{R}^N$,

$$(-\Delta)_A^{s(\cdot)}u(x) = \lim_{r\to 0}\int_{B_r^c(x)} \frac{u(x) - e^{i(x-y)\cdot A(x+y/2)}u(y)}{|x-y|^{N+2s(x,y)}}dy. \quad (4)$$

Since $s(\cdot)$ is a function, magnetic field $A \in C^{0,\alpha}(\mathbb{R}^N, \mathbb{R}^N)$ with $\alpha \in (0,1]$, we see that operator $(-\Delta)_A^{s(\cdot)}$ is a variable-order fractional magnetic Laplace operator. Especially, when $s(\cdot) \equiv constant$,$(-\Delta)_A^{s(\cdot)}$ reduces to the usual fractional magnetic Laplace operator. When $s(\cdot) \equiv constant$, $A = 0$, $(-\Delta)_A^{s(\cdot)}$ reduces to the usual fractional Laplace operator. Very recently, when $A = 0$, $V^-(x) = 0$ and $f(x), g(x) \equiv constant$, authors in [2] are given some sufficient conditions to ensure the existence of two different weak solutions, and used the variational method and the mountain pass theorem to obtain the two weak solutions of problem (5) which converge to two solutions of its limit problems, and the existence of infinitely many solutions to its limit problem:

$$\begin{cases} (-\Delta)^{s(\cdot)}u + \lambda V(x)u = \alpha|u|^{p(x)-2}u + \beta|u|^{q(x)-2}u \text{ in } \Omega, \\ u = 0 \text{ in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{5}$$

In addition, authors studied the multiplicity and concentration of solutions for a Hamiltonian system driven by the fractional Laplace operator with variable-order derivative in [3]. For $s(\cdot) = 1$, $p(x)$, $q(x) \equiv constant$, and $A = 0$, in [4], authors obtained the multiplicity and concentration of the positive solution of the following indefinite semilinear elliptic equations involving concave-convex nonlinearities by the variational method:

$$\begin{cases} -\Delta u + V_\lambda(x)u = f(x)|u|^{q-2}u + g(x)|u|^{p-2}u \text{ in } \mathbb{R}^N, \\ u \geq 0 \text{ in } \mathbb{R}^N. \end{cases} \tag{6}$$

For $s(\cdot)$, $p(x)$, $q(x) \equiv constant$, and $A = 0$, in [5], under appropriate assumptions, Peng et al. obtained the existence, multiplicity, and concentration of nontrivial solutions for the following indefinite fractional elliptic equation by using the Nehari manifold decomposition:

$$\begin{cases} (-\Delta)^{\alpha}u + V_\lambda(x)u = a(x)|u|^{q-2}u + b(x)|u|^{p-2}u \text{ in } \mathbb{R}^N, \\ u \geq 0 \text{ in } \mathbb{R}^N. \end{cases} \tag{7}$$

In [1], by using the Nehari manifold decomposition, authors studied the concave-convex elliptic equation involving the fractional order nonlinear Schrödinger equation:

$$(-\Delta)^s_A u + V_\lambda(x)u = f(x)|u|^{q-2}u + g(x)|u|^{p-2}u \text{ in } \mathbb{R}^N. \tag{8}$$

Some sufficient conditions for the existence of nontrivial solutions of equation (8) are obtained. Nevertheless, only a few papers see [6–12] deal with the existence and multiplicity of fractional magnetic problems. Some papers see [8, 13–16] deal with the solvability of Kirchhoff problems. Inspired by above, we are interested in the existence and multiplicity of solutions to problem (1) with variable growth and steep potential in $\mathbb{R}^N$. As far as we know, this is the first time to study the multiplicity of nontrivial solutions of the indefinite fractional elliptic equation driven by the variable-order fractional magnetic Laplace operator with variable exponents and steep potential in $\mathbb{R}^N$. This result was improved in the recent paper [1].

It is worth mentioning that in this paper, we not only obtain the existence and multiplicity results of nontrivial solutions of the variable-order fractional magnetic Schrodinger equation with variable growth and steep well potential in $\mathbb{R}^N$ but also our main results are based on the study for the decomposition of Nehari manifolds. On the one hand, relative to [1], we extend the exponent to variable exponent, thus introducing the variable exponent Lebesgue space. In addition, compared with [2], we extend the range of $p(x)$ to $(2, \infty)$ and the research range from the bounded region Ω to the whole space $\mathbb{R}^N$. On the other hand, if we want to find the nontrivial solution of the equation (1) by the variational method, we need some geometry, such as a mountain structure and a link structure. However, the energy functional of equation (1) does not have the mountain structure. In order to overcome this obstacle, we seek another method, the Nehari manifold. By decomposing the Nehari manifold into three parts, we obtain the existence of nontrivial solutions of each part.

Inspired by the above works, we assume

(S_1) There exist two constants $0 < s_0 < s_1 < 1$ such that $s_0 < s(x,y) < s_1$ for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$.

(S_2) $s(\cdot)$ is symmetric, that is, $s(x,y) = s(y,x)$ for all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$.

(V_1) V^+ is a continuous function on $\mathbb{R}^N$ and $V^- \in L^{N/2}(\mathbb{R}^N)$.

(V_2) There exists $k > 0$ such that the set $\{V^+ < k\} = \{x \in \mathbb{R}^N : V^+(x) < k\}$ is a nonempty and has finite measure. In addition, $M^2 |\{V^+ < k\}| < 1$, where $|\cdot|$ is the Lebesgue measure and M is the best Sobolev constant (see Lemma 9).

(V_3) $\Omega = \{x \in \mathbb{R}^N, V^+(x) = 0\}$ is nonempty and has a smooth boundary with $\bar{\Omega} = \text{int}\,\{x \in \mathbb{R}^N, V^+(x) = 0\}$.

(V_4) There exists a constant $\vartheta_0 > 1$ such that

$$\inf_{u \in D_A^{s(\cdot)}(\mathbb{R}^N,\mathbb{C})\setminus\{0\}} \frac{\int_{\mathbb{R}^{2N}} \left|u(x) - e^{i(x-y)\cdot A(x+y/2)}u(y)\right|^2 / |x-y|^{N+2s(x,y)}\,dx\,dy + \lambda\int_{\mathbb{R}^N} V^+u^2\,dx}{\int_{\mathbb{R}^N} V^-u^2\,dx} \geq \vartheta_0, \tag{9}$$

for all $\lambda > 0$, where $D_A^{s(\cdot)}(\mathbb{R}^N, \mathbb{C})$ is the Hilbert space related to the magnetic field A (see Section 2).

(V_5) $|\{V^+ < k\}| > \max\{A_1, A_2, A_3, A_4\} > 0$ where

$$A_1 = \left(\frac{(p^- - 2)q^-}{2(p^- - q^-)\|f\|_{L\wedge(2/(2-q(x)))(\mathbb{R}^N)}}\right)^{\frac{2}{q^+-2}} \left(\frac{2-q^+}{(p^+ - q^+)\|g\|_\infty}\right)^{\frac{2}{2-p^+}} \cdot \left(\frac{(\vartheta_0 - 1)\theta}{\vartheta_0}\right)^{\frac{2(p^+-q^+)}{(p^+-2)(q^+-2)}},$$

$$A_2 = \left(\frac{(p^- - 2)q^-}{2(p^- - q^-)\|f\|_{L\wedge(2/(2-q(x)))(\mathbb{R}^N)}}\right)^{\frac{2(p^--2)}{(2-p^+)(2-q^+)}} \left(\frac{\vartheta_0 - 1}{\vartheta_0}\right)^{\frac{2(p^--q^+)}{(2-p^+)(2-q^+)}} \cdot \left(\frac{2-q^+}{(p^+ - q^+)\|g\|_\infty}\right)^{\frac{2}{2-p^+}} \cdot \theta^{\frac{2p^+-q^+p^++q^+p^--2q^+}{(2-p^+)(2-q^+)}} M^{\frac{2(p^+-p^-)}{2-p^+}},$$

$$A_3 = \left(\frac{(p^- - 2)q^-}{2(p^- - q^-)\|f\|_{L\wedge(2/(2-q(x)))(\mathbb{R}^N)}}\right)^{\frac{2}{q^--2}} \left(\frac{(\vartheta_0 - 1)\theta}{\vartheta_0}\right)^{\frac{2(p^+-q^-)}{(p^+-2)(q^--2)}} \cdot \left(\frac{2-q^+}{(p^+ - q^+)\|g\|_\infty}\right)^{\frac{2}{2-p^+}},$$

$$A_4=\left(\frac{(p^- -2)q^-}{2(p^- -q^-)\|f\|_{L\wedge(2/(2-q(x)))}(\mathbb{R}^N)}\right)^{\frac{2(p^- -2)}{(2-p^+)(2-q^-)}}\left(\frac{2-q^+}{(p^+ -q^+)\|g\|_\infty}\right)^{\frac{2}{2-p^+}} \cdot\left(\frac{\vartheta_0-1}{\vartheta_0}\right)^{\frac{2(p^- -q^-)}{(2-p^+)(2-q^-)}}\cdot\theta^{\frac{2p^+ -q^-p^+ +q^-p^- -2q^-}{(2-p^+)(2-q^-)}}M^{\frac{2(p^+ -p^-)}{2-p^+}}. \tag{10}$$

To the best of our knowledge, this type of hypothesis is the first introduced by Bartsch and Wang in [17]. In addition, we recall the potential V_λ satisfied the conditions $(V_1)-(V_3)$ as the steep well potential.

Concerning $p(x)$, $q(x)$ and $f(x)$, $g(x)$, we suppose

(H_1) A measurable function $p:\mathbb{R}^N\to(2,+\infty)$ satisfy

$$2<p^-:=essinf_{x\in\mathbb{R}^N}p(x)\le p^+=\operatorname{esssup}_{x\in\mathbb{R}^N}p(x)<\infty. \tag{11}$$

(H_2) A measurable function $q:\mathbb{R}^N\to(1,2)$ satisfy

$$1<q^-:=\operatorname{essinf}_{x\in\mathbb{R}^N}q(x)\le q^+=\operatorname{esssup}_{x\in\mathbb{R}^N}q(x)<2. \tag{12}$$

(H_3) $f\in L^{2/2-q(x)}(\mathbb{R}^N,\mathbb{C})$ and $\|f\|_{L^{2/2-q(x)}(\mathbb{R}^N,\mathbb{C})}>0$, where $L^{p(x)}(\mathbb{R}^N,\mathbb{C})$ will be given in Section 2.

(H_4) $g\in L^\infty(\mathbb{R}^N,\mathbb{C})$ and $\|g\|_\infty:=\|g\|_{L^\infty(\mathbb{R}^N,\mathbb{C})}>0.$(13)

In what follows, it will always be assumed that the hypothesis (S_2) holds. Then, we will give the following definition of weak solutions for problem (1).

Definition 1. We say that $u\in X_\lambda$ is a weak solution of equation (1), if

$$\begin{aligned}&\mathfrak{R}\int_{\mathbb{R}^{2N}}\frac{(u(x)-e^{i(x-y)}\cdot A(x+y/2)u(y))\left(v(x)e^{i(xy)}\right)\cdot\overline{A}(x+y/2)v(y))}{|x-y|^{N+2s(x,y)}}dxdy\\&+\lambda\mathfrak{R}\int_{\mathbb{R}^N}V^+u\bar{v}dx-\mathfrak{R}\int_{\mathbb{R}^N}V^-u\bar{v}dx-\mathfrak{R}\int_{\mathbb{R}^N}\left(f(x)|u|^{q(x)-2}u\right.\\&\left.+g(x)|u|^{p(x)-2}u\right)\bar{v}dx=0,\end{aligned} \tag{13}$$

for any $v\in X_\lambda$, where X_λ will be given in Section 2.

Our main results are as follows.

Theorem 2. *Under (V_1)–(V_4), (H_1)–(H_4), and (H_5), there exists a nonempty open set $\Omega_g\subset\Omega$ such that $g(x)>0$ in Ω_g. Then, equation (1) allows at least a nontrivial solution for all $\lambda>1/kM\,|\,\{V^+<k\}\,|$.*

Theorem 3. *Suppose that (S_1), (S_2), (V_1)–(V_2), and (H_1)–(H_4) are satisfied. Then, there exists $\lambda^*\ge 0$ such that for every $\lambda>\lambda^*$, equation (1) has at least two nontrivial solutions.*

Remark 4. Generally speaking, if $s(\cdot)$: $\mathbb{R}^N\times\mathbb{R}^N\to(0,1)$ is a continuous function, magnetic field $A\in C^{0,\alpha}(\mathbb{R}^N,\mathbb{R}^N)$ with $\alpha\in(0,1]$, then the variable-order fractional magnetic Laplacian can be defined as for each given $u\in C_0^\infty(\mathbb{R}^N,\mathbb{C})$,

$$\begin{aligned}&\left\langle(-\Delta)_A^{s(\cdot)}u,v\right\rangle\\&=\mathfrak{R}\int_{\mathbb{R}^{2N}}\frac{\left(u(x)-e^{i(x-y)\cdot A(x+y/2)}u(y)\right)\left(v(x)e^{i(xy)\cdot\overline{A}(x+y/2)}v(y)\right)}{|x-y|^{N+2s(x,y)}}dxdy,\end{aligned} \tag{14}$$

along any $v\in C_0^\infty(\mathbb{R}^N,\mathbb{C})$.

2. Preliminaries and Notations

For the reader's convenience, we first review some necessary definitions that we are later going to use of variable exponent Lebesgue spaces. We refer the reader to [2, 3, 18–20] for details. Furthermore, we give the variational setting for equation (1) and some preliminary results.

Denote

$$p^+=\operatorname{esssup}_{x\in\mathbb{R}^N}p(x),\ p^-=\operatorname{essinf}_{x\in\mathbb{R}^N}p(x). \tag{15}$$

If $p^+<\infty$, then p is said to be bounded. If $(1/p(x))+(1/p'(x))=1$, then $p'(x)=p(x)/p(x)-1$ is called the dual variable exponent of $p(x)$. The variable exponent Lebesgue space can be defined as

$$\begin{aligned}L^{p(x)}(\mathbb{R}^N,\mathbb{C})=&\left\{u:\mathbb{R}^N\to\mathbb{C}\text{ is a measurable function };\rho_{p(x)}(u)\right.\\&\left.=\int_{\mathbb{R}^N}|u(x)|^{p(x)}dx<\infty\right\}\end{aligned} \tag{16}$$

with the norm

$$\|u\|_{L^{p(\cdot)}(\mathbb{R}^N,\mathbb{C})}=\inf\left\{\mu>0:\rho_{p(x)}\left(\mu^{-1}u\right)\le 1\right\}, \tag{17}$$

then $L^{p(x)}(\mathbb{R}^N,\mathbb{C})$ is a Banach space. When p is bounded, we have

$$\begin{aligned}&\min\left\{\|u\|^{p^-}_{L^{p(x)}(\mathbb{R}^N,\mathbb{C})},\|u\|^{p^+}_{L^{p(x)}(\mathbb{R}^N,\mathbb{C})}\right\}\\&\le\int_{\mathbb{R}^N}|u(x)|^{p(x)}dx\le\max\left\{\|u\|^{p^-}_{L^{p(x)}(\mathbb{R}^N,\mathbb{C})},\|u\|^{p^+}_{L^{p(x)}(\mathbb{R}^N,\mathbb{C})}\right\}.\end{aligned} \tag{18}$$

For bounded exponent, the dual space $(L^{p(x)}(\mathbb{R}^N,\mathbb{C}))'$ can be identified with $L^{p'(x)}(\mathbb{R}^N,\mathbb{C})$, where $p'(x)$ is called the dual variable exponent of $p(x)$. Especially,

$$L^2(\mathbb{R}^N,\mathbb{C})=\left\{u:\mathbb{R}^N\to\mathbb{C}\text{ is a measurable function };\int_{\mathbb{R}^N}|u(x)|^2dx<\infty\right\} \tag{19}$$

with the real scalar product $\langle u,v\rangle_{L^2(\mathbb{R}^N,\mathbb{C})}:=\mathfrak{R}\int_{\mathbb{R}^N}u\bar{v}dx$, for all $u,v\in L^2(\mathbb{R}^N,\mathbb{C})$. By Lemma 11, 20 of [20] and $\|\cdot$

$\|_{L^{p(x)}(\mathbb{R}^N,\mathbb{C})} = \| |\cdot| \|_{L^{p(x)}(\mathbb{R}^N,\mathbb{R})}$, we know that in the variable exponent Lebesgue space, the Hölder inequality is still valid. For all $u \in L^{p(x)}(\mathbb{R}^N,\mathbb{C}), v \in L^{p'(x)}(\mathbb{R}^N,\mathbb{C})$ with $p(x) \in (1,\infty)$, the following inequality holds

$$\int_{\mathbb{R}^N} |u||v| dx \le \left(\frac{1}{p^-} + \frac{1}{\left(p'\right)^-}\right) \|u\|_{L^{p(x)}(\mathbb{R}^N,\mathbb{C})} \|v\|_{L^{p'(x)}(\mathbb{R}^N,\mathbb{C})} \le 2\|u\|_{L^{p(x)}(\mathbb{R}^N,\mathbb{C})} \|v\|_{L^{p'(x)}(\mathbb{R}^N,\mathbb{C})}. \tag{20}$$

Define

$$D^{s(\cdot)}(\mathbb{R}^N,\mathbb{C}) = \left\{ u \in L^2(\mathbb{R}^N,\mathbb{C}) : \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s(x,y)}} dxdy < \infty \right\}. \tag{21}$$

Equip $D^{s(\cdot)}(\mathbb{R}^N,\mathbb{C})$ with the inner product

$$\langle u, v\rangle_{s(\cdot)} = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s(x,y)}} dxdy + \int_{\mathbb{R}^N} u(x)v(x)dx, \tag{22}$$

and the corresponding norm $\|u\|^2_{s(\cdot)} = \langle u, u\rangle_{s(\cdot)}$. Especially, if $s(\cdot) \equiv$ constant, then the space $D^{s(\cdot)}(\mathbb{R}^N,\mathbb{C})$ is the usual fractional Sobolev space $D^s(\mathbb{R}^N,\mathbb{C})$.

Lemma 5 (see [3] Lemma 5). *Let $p \in [2, 2^*_{s_0}]$, $2^*_{s_0} = 2N/N - 2s_0$, if $N > 2$; $2^*_{s_0} = \infty$ if $N \le 2$. The embedding $D^{s_1}(\mathbb{R}^N,\mathbb{R}) \hookrightarrow D^{s(\cdot)}(\mathbb{R}^N,\mathbb{R}) \hookrightarrow D^{s_0}(\mathbb{R}^N,\mathbb{R}) \hookrightarrow L^p(\mathbb{R}^N,\mathbb{R})$ are continuous.*

For each function $u : \mathbb{R}^N \to \mathbb{C}$, set

$$[u]^2_{s(\cdot),A} := \int_{\mathbb{R}^{2N}} \frac{\left|u(x) - e^{i(x-y)\cdot A(x+y/2)} u(y)\right|^2}{|x-y|^{N+2s(x,y)}} dxdy, \tag{23}$$

and the corresponding norm is defined as $\|u\|^2_{s(\cdot),A} = \|u\|^2_{L^2(\mathbb{R}^N,\mathbb{C})} + [u]^2_{s(\cdot),A}$. Set D be the space of measurable functions $u : \mathbb{R}^N \to \mathbb{C}$ such that $\|u\|_{s(\cdot),A} < \infty$; then, $(D, \langle \cdot, \cdot \rangle_{s(\cdot),A})$ is a Hilbert space. If we let $D^{s(\cdot)}_A(\mathbb{R}^N,\mathbb{C})$ as the closure of $C^\infty_c(\mathbb{R}^N,\mathbb{C})$ in D, then $D^{s(\cdot)}_A(\mathbb{R}^N,\mathbb{C})$ is a Hilbert space.

Lemma 6. *For each compact subset $W \subset \mathbb{R}^N$, the embedding $D^{s(\cdot)}_A(\mathbb{R}^N,\mathbb{C}) \hookrightarrow D^{s(\cdot)}(W,\mathbb{C})$ is continuous.*

Proof. Fixed any compact subset $W \subset \mathbb{R}^N$, for any $u \in D^{s(\cdot)}(W,\mathbb{C})$, we have

$$\begin{aligned}
\|u\|^2_{D^{s(\cdot)}(W,\mathbb{C})} &= \int_W |u(x)|^2 dx + \int_W\int_W \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s(x,y)}} dxdy \\
&\le \int_{\mathbb{R}^N} |u(x)|^2 dx + 2\int_W\int_W \frac{\left|u(x) - e^{i(x-y)\cdot A(x+y/2)}u(y)\right|^2}{|x-y|^{N+2s(x,y)}} dxdy \\
&\quad + 2\int_W\int_W \frac{|u(y)|^2 \left|e^{i(x-y)\cdot A((x+y)/2)} - 1\right|^2}{|x-y|^{N+2s(x,y)}} dxdy \\
&\le 2\int_{\mathbb{R}^N} |u(x)|^2 dx + 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\left|u(x) - e^{i(x-y)\cdot A(x+y/2)}u(y)\right|^2}{|x-y|^{N+2s(x,y)}} dxdy \\
&\quad + 2\int_W\int_W \frac{|u(y)|^2 \left|e^{i(x-y)\cdot A(x+y/2)} - 1\right|^2}{|x-y|^{N+2s(x,y)}} dxdy \le 2\|u\|^2_{s(\cdot),A} + 2J,
\end{aligned} \tag{24}$$

where

$$\begin{aligned}
J &:= \int_W\int_W \frac{|u(y)|^2 \left|e^{i(x-y)\cdot A(x+y/2)} - 1\right|^2}{|x-y|^{N+2s(x,y)}} dxdy \\
&= \int_W |u(y)|^2 \left(\int_{W\cap\{|x-y|>1\}} \frac{\left|e^{i(x-y)\cdot A(x+y/2)} - 1\right|^2}{|x-y|^{N+2s(x,y)}} dx\right) dy \\
&\quad + \int_W |u(y)|^2 \left(\int_{W\cap\{|x-y|\le 1\}} \frac{\left|e^{i(x-y)\cdot A(x+y/2)} - 1\right|^2}{|x-y|^{N+2s(x,y)}} dx\right) dy \\
&= J_1 + J_2.
\end{aligned} \tag{25}$$

Since $|e^{it} - 1| \le 2$, we have

$$\begin{aligned}
J_1 &\le 4\int_W |u(y)|^2 \left(\int_{W\cap\{|x-y|>1\}} \frac{1}{|x-y|^{N+2s(x,y)}} dx\right) dy \\
&\le 4\int_W |u(y)|^2 \left(\int_{W\cap\{|x-y|>1\}} \frac{1}{|x-y|^{N+2s_0}} dx\right) d \\
&= 4\int_W |u(y)|^2 \left(\int_{W\cap\{|z|>1\}} \frac{1}{|z|^{N+2s_0}} dz\right) dy \le C_1 \int_W |u(y)|^2 dy \\
&\le C_1 \int_{\mathbb{R}^N} |u(y)|^2 dy = C_1 \|u\|^2_{L^2(\mathbb{R}^N,\mathbb{C})}.
\end{aligned} \tag{26}$$

By Lemma 6 of [21], we know that A is locally bounded, and $W \subset \mathbb{R}^N$ is compact, $\left|e^{i(x-y)\cdot A(x+y/2)} - 1\right|^2 \le$

$C_2|x-y|^2$, for $|x-y|\le 1$, $x,y\in W$. Thus, we obtain

$$\begin{aligned} J_2 &= \int_W |u(y)|^2 \left(\int_{W\cap\{|x-y|\le 1\}} \frac{\left|e^{i(x-y)\cdot A(x+y/2)}-1\right|^2}{|x-y|^{N+2s(x,y)}} dx \right) dy \\ &\le \int_W |u(y)|^2 \left(\int_{W\cap\{|x-y|\le 1\}} \frac{C_2}{|x-y|^{N+2s(x,y)-2}} dx \right) dy \\ &\le \int_W |u(y)|^2 \left(\int_{W\cap\{|x-y|\le 1\}} \frac{C_2}{|x-y|^{N+2s_1-2}} dx \right) dy \\ &\le C_2 \int_W |u(y)|^2 \left(\int_{W\cap\{|z|\le 1\}} \frac{1}{|z|^{N+2s_1-2}} dz \right) dy \\ &\le C_3 \int_W |u(y)|^2 dy \le C_3 \int_{\mathbb{R}^N} |u(y)|^2 dy = C_3 \|u\|^2_{L^2(\mathbb{R}^N,\mathbb{C})}. \end{aligned} \tag{27}$$

By (24)-(27), we can easily get that

$$\|u\|^2_{D^{s(\cdot)}(W,\mathbb{C})} \le 2\|u\|^2_{s(\cdot),A} + 2C_1\|u\|^2_{L^2(\mathbb{R}^N,\mathbb{C})} + 2C_3\|u\|^2_{L^2(\mathbb{R}^N,\mathbb{C})} \le C_4\|u\|^2_{s(\cdot),A}, \tag{28}$$

which implies that the embedding $D_A^{s(\cdot)}(\mathbb{R}^N,\mathbb{C})$ is continuously embedded into $D^{s(\cdot)}(W,\mathbb{C})$.

Through the above lemma, we know that $D_A^{s(\cdot)}(\mathbb{R}^N,\mathbb{C}) \hookrightarrow D^{s(\cdot)}(W,\mathbb{C})$, and from Theorem 2.1 of [2], we know that for Ω be a bounded subset of $\mathbb{R}^N$ and $p:\bar{\Omega}\to[1,\infty)$ is continuous functions, $D^{s(\cdot)}(\Omega,\mathbb{C})$ is continuously embedded into $L^{p(x)}(\Omega,\mathbb{C})$, so we seek another method to prove the size relationship between $\int_{\mathbb{R}^N}|u(x)|^{p(x)}dx$, $\int_{\mathbb{R}^N}|u(x)|^{q(x)}dx$, and $\|u\|_\lambda$.

Lemma 7 (see [6] Lemma 10). *For every $u\in D_A^{s(\cdot)}(\mathbb{R}^N,\mathbb{C})$, it holds $|u|\in D^{s(\cdot)}(\mathbb{R}^N,\mathbb{R})$. More precisely, °Í*

$$\||u|\|_{s(\cdot)} \le \|u\|_{s(\cdot),A}, \text{for every } u\in D_A^{s(\cdot)}(\mathbb{R}^N,\mathbb{C}). \tag{29}$$

Remark 8 (see [6] Remark 9). There holds

$$|u(x)-e^{i(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)| \ge ||u(x)|-|u(y)||, \text{for } a.e. x, y\in\mathbb{R}^N. \tag{30}$$

Lemma 9. *Let $p\in[2,2^*_{s_0}]$, where $2^*_{s_0}=2N/N-2s_0$ if $N>2$; $2^*_{s_0}=\infty$ if $N\le 2$. $D_A^{s(\cdot)}(\mathbb{R}^N,\mathbb{C})$ is continuously embedded into $L^p(\mathbb{R}^N,\mathbb{C})$. Moreover, if $s_0\in(1/2,1)$, then $D_A^{s(\cdot)}(\mathbb{R}^N,\mathbb{C})$ can be continuously embedded into $L^\infty(\mathbb{R}^N,\mathbb{C})$; that is, there exists a constant $M>0$ such that*

$$\|u\|_\infty \le M\|u\|_{s(\cdot),A}. \tag{31}$$

Proof. By Lemma 7, we know that for every $u\in D_A^{s(\cdot)}(\mathbb{R}^N,\mathbb{C})$, it holds $|u|\in D^{s(\cdot)}(\mathbb{R}^N,\mathbb{R})$. By Lemma 5, we know that for $D^{s(\cdot)}(\mathbb{R}^N,\mathbb{R})\hookrightarrow L^p(\mathbb{R}^N,\mathbb{R})$ is continuous. In light of Remark 8, one has

$$\begin{aligned} \|u\|_{L^p(\mathbb{R}^N,\mathbb{C})} &= \||u|\|_{L^p(\mathbb{R}^N,\mathbb{R})} \le \tilde{c}\||u|\|_{s(\cdot)} \\ &= \tilde{c}\left(\int_{\mathbb{R}^N}\|u\|^2dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\||u(x)|-|u(y)|\|^2}{|x-y|^{N+2s(x,y)}}dxdy\right)^{\frac{1}{2}} \\ &\le \tilde{c}\left(\int_{\mathbb{R}^N}|u|^2dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\left|u(x)-e^{i(x-y)\cdot A(x+y/2)}u(y)\right|^2}{|x-y|^{N+2s(x,y)}}dxdy\right)^{\frac{1}{2}} \\ &= \tilde{c}\|u\|_{s(\cdot),A}. \end{aligned} \tag{32}$$

From the above inequality, we immediately obtain the embedding $D_A^{s(\cdot)}(\mathbb{R}^N,\mathbb{C})\hookrightarrow L^p(\mathbb{R}^N,\mathbb{C})$ which is continuous.

For $\lambda>0$, define

$$\begin{aligned} \langle u,v\rangle_\lambda &:= \Re\int_{\mathbb{R}^{2N}}\frac{\left(u(x)-e^{i(x-y)\cdot A(x+y/2)}u(y)\right)\left(v(x)e^{i(xy)\cdot\bar{A}(x+y/2)}v(y)\right)}{|x-y|^{N+2s(x,y)}}dxdy \\ &\quad + \Re\lambda\int_{\mathbb{R}^N}V^+u\bar{v}dx, \\ \|u\|_\lambda &:= \langle u,u\rangle_\lambda^{\frac{1}{2}}. \end{aligned} \tag{33}$$

Set $E=\{u\in D_A^{s(\cdot)}(\mathbb{R}^N,\mathbb{C}):\int_{\mathbb{R}^N}V^+u^2dx<\infty\}$ be equipped with the inner product $\langle u,v\rangle_E=\langle u,v\rangle_1$(i.e., $\lambda=1$ in $\langle u,v\rangle_\lambda$). Obviously, $\|u\|_E\le\|u\|_\lambda$ for $\lambda\ge 1$. Set $X_\lambda=(X,\|\cdot\|_\lambda)$. Combining condition (V_4) and fractional Sobolev inequality, we could get

$$\begin{aligned} \int_{\mathbb{R}^N}|u(x)|^2dx &= \int_{\{V^+<k\}}|u(x)|^2dx + \int_{\{V^+\ge k\}}|u(x)|^2dx \\ &\le \|u\|^2_\infty|\{V^+<k\}| + \frac{1}{k}\int_{\mathbb{R}^N}V^+(x)|u(x)|^2dx \\ &\le M^2\|u\|^2_{s(\cdot),A}|\{V^+<k\}| + \frac{1}{k}\int_{\mathbb{R}^N}V^+u^2dx \\ &= M^2|\{V^+<k\}|\left(\int_{\mathbb{R}^N}|u(x)|^2dx + [u]^2_{s(\cdot),A}\right) \\ &\quad + \frac{1}{k}\int_{\mathbb{R}^N}V^+u^2dx, \end{aligned} \tag{34}$$

which shows that

$$\begin{aligned} \int_{\mathbb{R}^N}|u(x)|^2dx &\le \frac{1}{1-M^2|\{V^+<k\}|}\left[M^2|\{V^+<k\}|[u]^2_{s(\cdot),A}\right. \\ &\quad \left.+\frac{1}{k}\int_{\mathbb{R}^N}V^+u^2dx\right] \\ &\le \frac{\max\{M^2|\{V^+<k\}|,(1/k)\}}{1-M^2|\{V^+<k\}|}\left[[u]^2_{s(\cdot),A}+\int_{\mathbb{R}^N}V^+u^2dx\right] \\ &= \frac{\max\{M^2|\{V^+<k\}|,(1/k)\}}{1-M^2|\{V^+<k\}|}\|u\|^2_X. \end{aligned} \tag{35}$$

From the above inequality, it holds that

$$[u]^2_{s(\cdot),A}+\int_{\mathbb{R}^N}|u(x)|^2dx\le\left(1+\frac{\max\left\{M^2\mid\{V^+<k\}|,(1/k)\right\}}{1-M^2\mid\{V^+<k\}\mid}\right)\|u\|^2_X, \tag{36}$$

which shows that X is continuously embedded into $D_A^{s(\cdot)}(\mathbb{R}^N,\mathbb{C})$. Similarly, for all $\lambda\ge 1/kM^2\mid\{V^+<k\}\mid$, there holds

$$\begin{aligned}\int_{\mathbb{R}^N}|u(x)|^2dx&\le\frac{1}{1-M^2\mid\{V^+<k\}\mid}\\&\quad\cdot\left[M^2\mid\{V^+<k\}\mid[u]^2_{s(\cdot),A}+\lambda M^2\mid\{V^+<k\}\mid\int_{\mathbb{R}^N}V^+u^2dx\right]\\&=\frac{M^2\mid\{V^+<k\}\mid}{1-M^2\mid\{V^+<k\}\mid}\left[[u]^2_{s(\cdot),A}+\lambda\int_{\mathbb{R}^N}V^+u^2dx\right]\\&=\frac{M^2\mid\{V^+<k\}\mid}{1-M^2\mid\{V^+<k\}\mid}\|u\|^2_\lambda=\frac{1}{\theta}\|u\|^2_\lambda,\end{aligned}\tag{37}$$

where $\theta=1-M^2|\{V^+<k\}|/\mathrm{M}^2|\{V^+<k\}|$. In addition, we have

$$\begin{aligned}\int_{\mathbb{R}^N}|u(x)|^{p(x)}dx&=\int_{\mathbb{R}^N}|u(x)|^{p(x)-2}\cdot|u(x)|^2dx\\&\le\max\left\{\|u\|_\infty^{p^+-2},\|u\|_\infty^{p^--2}\right\}\int_{\mathbb{R}^N}|u(x)|^2dx\\&\le\max\left\{M^{p^+-2}\left(\int_{\mathbb{R}^N}|u(x)|^2dx+[u]^2_{s(\cdot),A}\right)^{\frac{p^+-2}{2}},M^{p^--2}\right.\\&\quad\left.\cdot\left(\int_{\mathbb{R}^N}|u(x)|^2dx+[u]^2_{s(\cdot),A}\right)^{\frac{p^--2}{2}}\right\}\frac{M^2\mid\{V^+<k\}\mid}{1-M^2\mid\{V^+<k\}\mid}\|u\|^2_\lambda.\end{aligned}\tag{38}$$

This together $M^2\mid\{V^+<k\}\mid<1$ yields that

$$\begin{aligned}\int_{\mathbb{R}^N}|u(x)|^{p(x)}dx&\le\max\left\{M^{p^+-2}\left(\frac{M^2\mid\{V^+<k\}\mid}{1-M^2\mid\{V^+<k\}\mid}\|u\|^2_\lambda+\|u\|^2_\lambda\right)^{\frac{p^+-2}{2}},M^{p^--2}\right.\\&\quad\left.\cdot\left(\frac{M^2\mid\{V^+<k\}\mid}{1-M^2\mid\{V^+<k\}\mid}\|u\|^2_\lambda+\|u\|^2_\lambda\right)^{\frac{p^--2}{2}}\right\}\\&\quad\frac{M^2\mid\{V^+<k\}\mid}{1-M^2\mid\{V^+<k\}\mid}\|u\|^2_\lambda\le|\{V^+<k\}|\\&\quad\cdot\left(\frac{1}{1-M^2\mid\{V^+<k\}\mid}\right)^{\frac{p^+}{2}}\max\left\{M^{p^+}\|u\|_\lambda^{p^+},M^{p^-}\|u\|_\lambda^{p^-}\right\}\\&=\frac{1}{\theta^{p^+/2}M^{p^+}|\{V^+<k\}|^{p^+-2/2}}\max\left\{M^{p^+}\|u\|_\lambda^{p^+},M^{p^-}\|u\|_\lambda^{p^-}\right\}\\&=\frac{1}{\theta^{p^+/2}|\{V^+<k\}|^{p^+-2/2}}\max\left\{\|u\|_\lambda^{p^+},M^{p^--p^+}\|u\|_\lambda^{p^-}\right\}.\end{aligned}\tag{39}$$

For the sake of notational simplicity, we let $\|u\|^2_{\lambda,V}:=[u]^2_{s(\cdot),A}+\int_{\mathbb{R}^N}V_\lambda u^2dx$. Hence, by condition (V_4), we have

$$\|u\|^2_\lambda\ge\|u\|^2_{\lambda,V}\ge\frac{\vartheta_0-1}{\vartheta_0}\|u\|^2_\lambda,\quad\text{for } all\ \lambda\ge 0. \tag{40}$$

Related to equation (1), we think the functional $\Psi_\lambda:X_\lambda\to\mathbb{R}$,

$$\begin{aligned}\Psi_\lambda(u)&=\frac{1}{2}\|u\|^2_\lambda-\frac{1}{2}\int_{\mathbb{R}^N}V^-u^2dx-\int_{\mathbb{R}^N}\left(\frac{f(x)}{q(x)}|u|^{q(x)}+\frac{g(x)}{p(x)}|u|^{p(x)}\right)dx\\&=\frac{1}{2}\|u\|^2_{\lambda,V}-\int_{\mathbb{R}^N}\left(\frac{f(x)}{q(x)}|u|^{q(x)}+\frac{g(x)}{p(x)}|u|^{p(x)}\right)dx.\end{aligned}\tag{41}$$

In fact, we can easily verify that Ψ_λ is well-defined of class C^1 in X_λ and

$$\begin{aligned}\left\langle\Psi'_\lambda(u),v\right\rangle&=\langle u,v\rangle_\lambda-\Re\int_{\mathbb{R}^N}V^-u\bar{v}dx-\Re\int_{\mathbb{R}^N}\left(f(x)|u|^{q(x)-2}u\right.\\&\quad\left.+g(x)|u|^{p(x)-2}u\right)\bar{v}dx\\&=\langle u,v\rangle_{\lambda,V}-\Re\int_{\mathbb{R}^N}\left(f(x)|u|^{q(x)-2}u\right.\\&\quad\left.+g(x)|u|^{p(x)-2}u\right)\bar{v}dx,\end{aligned}\tag{42}$$

for all $u,v\in X_\lambda$. Therefore, if $u\in X_\lambda$ is a critical point of Ψ_λ, then u is a solution of equation (1). Since the energy functional Ψ_λ is unbounded below on X_λ, in order to overcome this problem, we use the Nehari manifold $\mathcal{N}_\lambda=\{u\in X_\lambda\setminus\{0\}:\langle\Psi'_\lambda(u),u\rangle=0\}$ to study the energy functional. In addition, we also note that $\mathcal{N}_\lambda$ contains every nonzero solution of equation (1). Especially, all critical points of must be located in $\mathcal{N}_\lambda$, and the local minimizers on $\mathcal{N}_\lambda$ are usually critical points of Ψ_λ.

3. Main Results

To start with, we can get an estimate of Ψ_λ. Then, we will discuss some basic properties of $\mathcal{N}_\lambda$. Finally, we prove Theorem 2 and Theorem 3 using the variational methods.

Lemma 10. *Ψ_λ is coercive and bounded below on $\mathcal{N}_\lambda$. Furthermore, one has*

$$\begin{aligned}\Psi_\lambda(u)\ge\max&\left\{-\frac{2-q^+}{2p^-}\left(\frac{\vartheta_0q^+}{\theta(\vartheta_0-1)(p^--2)}\right)^{\frac{q^+}{2-q^+}}\right.\\&\cdot\left(\frac{(p^--q^-)\|f\|_{L^{2/2-q(x)}}}{q^-}\right)^{\frac{2}{2-q^+}},-\frac{2-q^-}{2p^-q^-}\\&\left.\cdot\left(\frac{\vartheta_0}{\theta(\vartheta_0-1)(p^--2)}\right)^{\frac{q^-}{2-q^-}}\left((p^--q^-)\|f\|_{L^{\frac{2}{2-q(x)}}}\right)^{\frac{2}{2-q^-}}\right\}.\end{aligned}\tag{43}$$

Proof. If $u\in\mathcal{N}_\lambda$, in view of (37), (40), and Hölder inequality, it gains

$$\begin{aligned}
\Psi_\lambda(u) &= \Psi_\lambda(u) - \frac{1}{p^-}\left\langle \Psi'_\lambda(u), u\right\rangle \\
&= \left(\frac{1}{2}-\frac{1}{p^-}\right)\|u\|^2_{\lambda,V} - \int_{\mathbb{R}^N}\left(\frac{1}{q(x)}-\frac{1}{p^-}\right)f(x)|u|^{q(x)}dx \\
&\quad - \int_{\mathbb{R}^N}\left(\frac{1}{p(x)}-\frac{1}{p^-}\right)g(x)|u|^{p(x)}dx \geq \left(\frac{1}{2}-\frac{1}{p^-}\right)\|u\|^2_{\lambda,V} \\
&\quad - \left(\frac{1}{q^-}-\frac{1}{p^-}\right)\int_{\mathbb{R}^N} f(x)|u|^{q(x)}dx \geq \left(\frac{1}{2}-\frac{1}{p^-}\right)\|u\|^2_{\lambda,V} \\
&\quad - \left(\frac{1}{q^-}-\frac{1}{p^-}\right)\|f\|_{L^{\frac{2}{2-q(x)}}(\mathbb{R}^N)}\left(\int_{\mathbb{R}^N}|u|^2dx\right)^{\frac{q(x)}{2}} \\
&\geq \left(\frac{1}{2}-\frac{1}{p^-}\right)\|u\|^2_{\lambda,V} - \left(\frac{1}{q^-}-\frac{1}{p^-}\right)\|f\|_{L^{\frac{2}{2-q(x)}}(\mathbb{R}^N)}\frac{1}{\theta^{q(x)/2}}\|u\|^{q(x)}_\lambda \\
&\geq \frac{p^- - 2}{2p^-}\|u\|^2_{\lambda,V} - \left(\frac{1}{q^-}-\frac{1}{p^-}\right)\|f\|_{L^{\frac{2}{2-q(x)}}(\mathbb{R}^N)}\max\left\{\frac{1}{\theta^{q^+/2}}\|u\|^{q^+}_\lambda,\right. \\
&\quad \left.\frac{1}{\theta^{q^-/2}}\|u\|^{q^-}_\lambda\right\} \geq \frac{p^- - 2}{2p^-}\frac{\vartheta_0 - 1}{\vartheta_0}\|u\|^2_\lambda \\
&\quad - \left(\frac{1}{q^-}-\frac{1}{p^-}\right)\|f\|_{L^{\frac{2}{2-q(x)}}(\mathbb{R}^N)}\max\left\{\frac{1}{\theta^{q^+/2}}\|u\|^{q^+}_\lambda, \frac{1}{\theta^{q^-/2}}\|u\|^{q^-}_\lambda\right\} \\
&\geq \max\left\{-\frac{2-q^+}{2p^-}\left(\frac{\vartheta_0 q^+}{\theta(\vartheta_0-1)(p^- - 2)}\right)^{\frac{q^+}{2-q^+}}\right. \\
&\quad \cdot\left(\frac{(p^- - q^-)\|f\|_{L^{2/2-q(x)}(\mathbb{R}^N)}}{q^-}\right)^{\frac{2}{2-q^+}}, -\frac{2-q^-}{2p^-q^-} \\
&\quad \left.\cdot\left(\frac{\vartheta_0}{\theta(\vartheta_0-1)(p^- - 2)}\right)^{\frac{q^-}{2-q^-}}\left((p^- - q^-)\|f\|_{L^{\frac{2}{2-q(x)}}(\mathbb{R}^N)}\right)^{\frac{2}{2-q^-}}\right\}.
\end{aligned} \tag{44}$$

Therefore, Ψ_λ is coercive and bounded below on $\mathcal{N}_\lambda$.

We know that $\mathcal{N}_\lambda$ is linked to the behavior of the function of the form $L_u(t)$: $t \rightarrow \Psi_\lambda(tu)$ for $t > 0$. This map is called as the fibering map which can be traced back to basic works [1, 22, 23]. If $u \in X_\lambda$, then

$$L_u(t) = \frac{t^2}{2}\|u\|^2_{\lambda,V} - \int_{\mathbb{R}^N}\frac{f(x)}{q(x)}|tu|^{q(x)}dx - \int_{\mathbb{R}^N}\frac{g(x)}{p(x)}|tu|^{p(x)}dx,$$

$$\begin{aligned}
L'_u(t) &= t\|u\|^2_{\lambda,V} - \int_{\mathbb{R}^N} f(x)t^{q(x)-1}|u|^{q(x)}dx - \int_{\mathbb{R}^N} g(x)t^{p(x)-1}|u|^{p(x)}dx; \\
L''_u(t) &= \|u\|^2_{\lambda,V} - \int_{\mathbb{R}^N} f(x)(q(x)-1)t^{q(x)-2}|u|^{q(x)}dx \\
&\quad - \int_{\mathbb{R}^N} g(x)(p(x)-1)t^{p(x)-2}|u|^{p(x)}dx.
\end{aligned} \tag{45}$$

After observation, we can get that

$$tL'_u(t) = \|tu\|^2_{\lambda,V} - \int_{\mathbb{R}^N} f(x)|tu|^{q(x)}dx - \int_{\mathbb{R}^N} g(x)|tu|^{p(x)}dx \tag{46}$$

and thus, for $u \in X_\lambda \setminus \{0\}$ and $t > 0$, $L'_u(t) = 0$ if and only if $t u \in \mathcal{N}_\lambda$, i.e., positive critical points of L_u correspond points on the Nehari manifold. Especially, $L'_u(1) = 0$ if and only if $u \in \mathcal{N}_\lambda$. We found that $\mathcal{N}_\lambda$ can be divided into three parts corresponding local minimal, local maximum, and points of inflection. Based on the above, we can define

$$\begin{aligned}
\mathcal{N}^+_\lambda &= \left\{u \in \mathcal{N}_\lambda : L''_u(1) > 0\right\}; \\
\mathcal{N}^0_\lambda &= \left\{u \in \mathcal{N}_\lambda : L''_u(1) = 0\right\}; \\
\mathcal{N}^-_\lambda &= \left\{u \in \mathcal{N}_\lambda : L''_u(1) < 0\right\}.
\end{aligned} \tag{47}$$

For each $u \in \mathcal{N}_\lambda$, we can find that

$$\begin{aligned}
L''_u(1) &= \|u\|^2_{\lambda,V} - \int_{\mathbb{R}^N}(q(x)-1)f(x)|u|^{q(x)}dx \\
&\quad - \int_{\mathbb{R}^N}(p(x)-1)g(x)|u|^{p(x)}dx.
\end{aligned} \tag{48}$$

Now, we will deduce some results of $\mathcal{N}^+_\lambda$, $\mathcal{N}^0_\lambda$, and $\mathcal{N}^-_\lambda$.

Lemma 11. *Assume u_0 is a local minimizer of Ψ_λ on $\mathcal{N}_\lambda$ and $u_0 \notin \mathcal{N}^0_\lambda$, then $\Psi'_\lambda(u_0) = 0$ in X^{-1}_λ.*

Proof. If u_0 is a local minimizer of Ψ_λ on $\mathcal{N}_\lambda$, then u_0 is a solution of the optimization problem

$$\text{minimizer}\,\Psi_\lambda(u)\text{ subject to }K(u) = 0, \tag{49}$$

where $K(u) = \|u\|^2_{\lambda,V} - \int_{\mathbb{R}^N} f(x)|u|^{q(x)}dx - \int_{\mathbb{R}^N} g(x)|u|^{p(x)}dx$. Consequently, by the theory of Lagrange multipliers, there exists $\nu \in \mathbb{R}$ such that $\Psi'_\lambda(u_0) = \nu K'(u_0)$. Therefore,

$$\left\langle \Psi'_\lambda(u_0), u_0\right\rangle = \nu\left\langle K'(u_0), u_0\right\rangle. \tag{50}$$

It follows from $u_0 \in \mathcal{N}_\lambda$ that

$$\|u_0\|^2_{\lambda,V} - \int_{\mathbb{R}^N} f(x)|u_0|^{q(x)}dx - \int_{\mathbb{R}^N} g(x)|u_0|^{p(x)}dx = 0. \tag{51}$$

Thus,

$$\begin{aligned}
\left\langle K'(u_0), u_0\right\rangle &= 2\|u_0\|^2_{\lambda,V} - \int_{\mathbb{R}^N} q(x)f(x)|u_0|^{q(x)}dx \\
&\quad - \int_{\mathbb{R}^N} p(x)g(x)|u_0|^{p(x)}dx = \|u_0\|^2_{\lambda,V} \\
&\quad - \int_{\mathbb{R}^N}(q(x)-1)f(x)|u_0|^{q(x)}dx - \int_{\mathbb{R}^N}(p(x) \\
&\quad - 1)g(x)|u_0|^{p(x)}dx.
\end{aligned} \tag{52}$$

If $u_0 \notin \mathcal{N}^0_\lambda$, then $\langle K'(u_0), u_0\rangle \neq 0$. In view of (50), it gains $\nu = 0$.

Lemma 12.

(1) $\forall u \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0$, *one has* $\int_{\mathbb{R}^N} f(x)|u|^{q(x)}dx > 0$

(2) $\forall u \in \mathcal{N}_\lambda^-$, *one has* $\int_{\mathbb{R}^N} g(x)|u|^{p(x)}dx > 0$

Proof. By the definitions of $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^0$, we can obtain

$$\begin{aligned} 0 \le L_u''(1) &= \|u\|_{\lambda,V}^2 - \int_{\mathbb{R}^N} (q(x)-1) f(x)|u|^{q(x)}dx \\ &\quad - \int_{\mathbb{R}^N} (p(x)-1) g(x)|u|^{p(x)}dx = \|u\|_{\lambda,V}^2 \\ &\quad - \int_{\mathbb{R}^N} (q^- -1) f(x)|u|^{q(x)}dx \\ &\quad - \int_{\mathbb{R}^N} (p^- -1) g(x)|u|^{p(x)}dx < (2-p^-)\|u\|_{\lambda,V}^2 \\ &\quad - (q^- - p^-)\int_{\mathbb{R}^N} f(x)|u|^{q(x)}dx. \end{aligned} \tag{53}$$

It is easy to get that $\int_{\mathbb{R}^N} f(x)|u|^{q(x)}dx > p^- - 2/p^- - q^- \|u\|_{\lambda,V}^2 \ge 0$. It follows from the definition of $\mathcal{N}_\lambda^-$ that

$$\begin{aligned} 0 > L_u''(1) &= \|u\|_{\lambda,V}^2 - \int_{\mathbb{R}^N} (q(x)-1) f(x)|u|^{q(x)}dx \\ &\quad - \int_{\mathbb{R}^N} (\mathrm{p}(x)-1) g(x)|u|^{p(x)}dx \ge \|u\|_{\lambda,V}^2 \\ &\quad - \int_{\mathbb{R}^N} (q^+ -1) f(x)|u|^{q(x)}dx \\ &\quad - \int_{\mathbb{R}^N} (p^+ -1) g(x)|u|^{p(x)}dx \\ &= (2-q^+)\|u\|_{\lambda,V}^2 - (p^+ - q^+)\int_{\mathbb{R}^N} g(x)|u|^{p(x)}dx, \end{aligned} \tag{54}$$

which implies that $\int_{\mathbb{R}^N} g(x)|u|^{p(x)}dx > 2 - q^+/p^+ - q^+\|u\|_{\lambda,V}^2 \ge 0$.

Lemma 13. *Let the condition* (H_3), (H_4), *and* (V_1)-(V_5) *are satisfied. Then, for all* $\lambda \ge 1/kM^2 \mid \{V^+ < k\} \mid$, *one has* $\mathcal{N}_\lambda^0 = \varnothing$.

Proof. If the conclusion does not hold, then there exists $\lambda \ge 1/kM^2 \mid \{V^+ < k\} \mid$, such that $\mathcal{N}_\lambda^0 \neq \varnothing$. Then, for $u \in \mathcal{N}_\lambda^0$, by (40), (48), and the Hölder inequality, we have

$$\begin{aligned} 0 = L_u''(1) &= \|u\|_{\lambda,V}^2 - \int_{\mathbb{R}^N} (q(x)-1) f(x)|u|^{q(x)}dx \\ &\quad - \int_{\mathbb{R}^N} (p(x)-1) g(x)|u|^{p(x)}dx < (2-p^-)\|u\|_{\lambda,V}^2 \\ &\quad - (q^- - p^-)\int_{\mathbb{R}^N} f(x)|u|^{q(x)}dx. \end{aligned} \tag{55}$$

This means that

$$\begin{aligned} \frac{q^-}{2}\frac{\vartheta_0-1}{\vartheta_0}\|u\|_\lambda^2 &\le \frac{\vartheta_0-1}{\vartheta_0}\|u\|_\lambda^2 \le \|u\|_{\lambda,V}^2 < \frac{p^- - q^-}{p^- - 2}\int_{\mathbb{R}^N} f(x)|u|^{q(x)}dx \\ &< \frac{p^- - q^-}{p^- - 2}\|f\|_{L^{\frac{2}{2-q(x)}}(\mathbb{R}^N)}\int_{\mathbb{R}^N}|u|^{q(x)}dx \\ &< \frac{p^- - q^-}{p^- - 2}\|f\|_{L^{\frac{2}{2-q(x)}}(\mathbb{R}^N)} \max\left\{\frac{1}{\theta^{q^+/2}}\|u\|_\lambda^{q^+}, \frac{1}{\theta^{q^-/2}}\|u\|_\lambda^{q^-}\right\}. \end{aligned} \tag{56}$$

Thus, we have

$$\|u\|_\lambda \le \min\left\{\left(\frac{2\vartheta_0(p^- - q^-)\|f\|_{L^{2/2-q(x)}(\mathbb{R}^N)}}{q^-(\vartheta_0-1)(p^- - 2)\theta^{q^+/2}}\right)^{\frac{1}{2-q^+}}, \cdot \left(\frac{2\vartheta_0(p^- - q^-)\|f\|_{L^{2/2-q(x)}(\mathbb{R}^N)}}{q^-(\vartheta_0-1)(p^- - 2)\theta^{q^-/2}}\right)^{\frac{1}{2-q^-}}\right\}. \tag{57}$$

From (48), we seem to easily get that

$$\begin{aligned} &(2-q^+)\|u\|_{\lambda,V}^2 - (p^+ - q^+)\int_{\mathbb{R}^N} g(x)|u|^{p(x)}dx \le \|u\|_{\lambda,V}^2 \\ &\quad - \int_{\mathbb{R}^N} (q(x)-1) f(x)|u|^{q(x)}dx - \int_{\mathbb{R}^N} (p(x)-1) g(x)|u|^{p(x)}dx \\ &\quad = L_u''(1) = 0, \end{aligned} \tag{58}$$

which implies that

$$\frac{2-q^+}{p^+ - q^+}\|u\|_{\lambda,V}^2 \le \int_{\mathbb{R}^N} g(x)|u|^{p(x)}dx. \tag{59}$$

Combining (39) and (40) with the Sobolev inequality, we have

$$\begin{aligned} \frac{(\vartheta_0-1)(2-q^+)}{\vartheta_0(p^+ - q^+)}\|u\|_\lambda^2 &\le \frac{2-q^+}{p^+ - q^+}\|u\|_{\lambda,V}^2 \le \int_{\mathbb{R}^N} g(x)|u|^{p(x)}dx \\ &\le \frac{\|g\|_\infty}{\theta^{p^+/2}|\{V^+ < k\}|^{p^+-2/2}} \max \\ &\quad \cdot \left\{\|u\|_\lambda^{p^+}, M^{p^- - p^+}\|u\|_\lambda^{p^-}\right\}. \end{aligned} \tag{60}$$

This means that

$$\|u\|_\lambda \ge \max\left\{\left[\frac{(\vartheta_0-1)(2-q^+)\theta \wedge (p \wedge +/2) \mid \{V^+ < k\} \mid \wedge((p \wedge + -2)/2)}{\vartheta_0(p^+ - q^+)\|g\|_\infty}\right]^{\frac{1}{p^+-2}}, \cdot \left[\frac{(\vartheta_0-1)(2-q^+)\theta \wedge (p \wedge +/2) \mid \{V^+ < k\} \mid \wedge((p \wedge + -2)/2)}{\vartheta_0(p^+ - q^+)\|g\|_\infty M^{p^- - p^+}}\right]^{\frac{1}{p^--2}}\right\}. \tag{61}$$

Hence, combining (57) and (61), we have

$$|\{V^+ < k\}| < \min\{A_1, A_2, A_3, A_4\} \le \max\{A_1, A_2, A_3, A_4\}, \tag{62}$$

which is a contradictive with (V_5). Therefore, for all $\lambda \ge 1/k$ $M^2 \mid \{V^+ < k\} \mid$, one has $\mathcal{N}_\lambda^0 = \varnothing$.

By Lemma 13, $\lambda \ge 1/kM^2 \mid \{V^+ < k\} \mid$, we can easily get that $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$ and define

$$c_\lambda^+ = \inf_{u\in\mathcal{N}_\lambda^+} \Psi_\lambda(u) \text{ and } c_\lambda^- = \inf_{u\in\mathcal{N}_\lambda^-} \Psi_\lambda(u). \tag{63}$$

Furthermore, we derive the following results.

Lemma 14. *Under the condition (H_3), (H_4), and (V_1)-(V_5). Then, for all $\lambda \ge 1/kM^2 \mid \{V^+ < k\} \mid$, there exists C_5 such that $c_\lambda^+ < 0 < C_5 < c_\lambda^-$. Particularly, $c_\lambda^+ = \inf_{u\in\mathcal{N}_\lambda} \Psi_\lambda(u)$.*

Proof. Our proof is decoupled in the following two steps:

Step 1. We claim that there exist $u \in \mathcal{N}_\lambda^+$ such that $\Psi_\lambda(u) < 0$. Indeed, let $tv_0 \in \mathcal{N}_\lambda^+ \subset \mathcal{N}_\lambda$, where $t \in (0,1)$ small enough. It is follows from (48) that

$$\begin{aligned} 0 < L''_{tv_0}(1) &= \|tv_0\|_{\lambda,V}^2 - \int_{\mathbb{R}^N} (q(x)-1) f(x)|tv_0|^{q(x)} dx \\ &\quad - \int_{\mathbb{R}^N} (p(x)-1) g(x)|tv_0|^{p(x)} dx < (2-p^-)\|tv_0\|_{\lambda,V}^2 \\ &\quad - (q^- - p^-)\int_{\mathbb{R}^N} f(x)|tv_0|^{q(x)} dx. \end{aligned} \tag{64}$$

This shows that

$$0 \le \frac{p^- - 2}{p^- - q^-}\|tv_0\|_{\lambda,V}^2 < \int_{\mathbb{R}^N} f(x)\left|tv_0\right|^{q(x)} dx < t^{q^-}\int_{\mathbb{R}^N} f(x)\left|v_0\right|^{q(x)} dx. \tag{65}$$

This yields at once that

$$\int_{\mathbb{R}^N} f(x)|v_0|^{q(x)} dx > 0. \tag{66}$$

Similarly,

$$\begin{aligned} 0 < L''_{tv_0}(1) &= \|tv_0\|_{\lambda,V}^2 - \int_{\mathbb{R}^N} (q(x)-1) f(x)|tv_0|^{q(x)} dx \\ &\quad - \int_{\mathbb{R}^N} (p(x)-1) g(x)|tv_0|^{p(x)} dx < (2-q^-)\|tv_0\|_{\lambda,V}^2 \\ &\quad - (p^- - q^-)\int_{\mathbb{R}^N} g(x)|tv_0|^{p(x)} dx. \end{aligned} \tag{67}$$

From (67), we can easily get that

$$\frac{2-q^-}{p^- - q^-}\|tv_0\|_{\lambda,V}^2 > \int_{\mathbb{R}^N} g(x)|tv_0|^{p(x)} \mathrm{d}x. \tag{68}$$

Consequently, it derives from (67) and (68) that

$$\begin{aligned} \Psi_\lambda(tv_0) &= \Psi_\lambda(tv_0) - \frac{1}{p^-}\left\langle \Psi_\lambda'(tv_0), tv_0 \right\rangle < \frac{1}{2}\|tv_0\|_{\lambda,V}^2 \\ &\quad - \frac{1}{q^+}\int_{\mathbb{R}^N} f(x)|tv_0|^{q(x)} dx - \frac{1}{p^+}\int_{\mathbb{R}^N} g(x)|tv_0|^{p(x)} dx \\ &\quad - \frac{1}{p^-}\|tv_0\|_{\lambda,V}^2 + \frac{1}{p^-}\int_{\mathbb{R}^N} f(x)|tv_0|^{q(x)} dx \\ &\quad + \frac{1}{p^-}\int_{\mathbb{R}^N} g(x)|tv_0|^{p(x)} dx = \left(\frac{1}{2} - \frac{1}{p^-}\right)\|tv_0\|_{\lambda,V}^2 \\ &\quad - \int_{\mathbb{R}^N} \left(\frac{1}{q^+} - \frac{1}{p^-}\right) f(x)|tv_0|^{q(x)} dx \\ &\quad + \left(\frac{1}{p^-} - \frac{1}{p^+}\right)\int_{\mathbb{R}^N} g(x)|tv_0|^{p(x)} dx \\ &< \left(\frac{1}{2} - \frac{1}{p^-}\right)\|tv_0\|_{\lambda,V}^2 - \int_{\mathbb{R}^N}\left(\frac{1}{q^+} - \frac{1}{p^-}\right) f(x)|tv_0|^{q(x)} dx \\ &\quad + \left(\frac{1}{p^-} - \frac{1}{p^+}\right)\frac{2-q^-}{p^- - q^-}\|tv_0\|_{\lambda,V}^2 = \left[\left(\frac{1}{2} - \frac{1}{p^-}\right)\right. \\ &\quad \left. + \left(\frac{1}{p^-} - \frac{1}{p^+}\right)\frac{2-q^-}{p^- - q^-}\right]\|tv_0\|_{\lambda,V}^2 \\ &\quad - \int_{\mathbb{R}^N}\left(\frac{1}{q^+} - \frac{1}{p^-}\right) f(x)|tv_0|^{q(x)} dx < t^2\left[\left(\frac{1}{2} - \frac{1}{p^-}\right)\right. \\ &\quad \left. + \left(\frac{1}{p^-} - \frac{1}{p^+}\right)\frac{2-q^-}{p^- - q^-}\right]\|v_0\|_{\lambda,V}^2 - t^{q^+}\int_{\mathbb{R}^N} \\ &\quad \cdot \left(\frac{1}{q^+} - \frac{1}{p^-}\right) f(x)|v_0|^{q(x)} dx. \end{aligned} \tag{69}$$

Hence, $c_\lambda^+ < 0$.

Step 2. We assert that there exist $u \in \mathcal{N}_\lambda^-$ such that $\Psi_\lambda(u) > 0$. In fact, let $u \in \mathcal{N}_\lambda^- \subset \mathcal{N}_\lambda$. From (48), we seem to easily get that

$$\begin{aligned} &(2-q^+)\|u\|_{\lambda,V}^2 - (p^+ - q^+)\int_{\mathbb{R}^N} g(x)|u|^{p(x)} dx \le \|u\|_{\lambda,V}^2 \\ &\quad - \int_{\mathbb{R}^N} (q(x)-1) f(x)|u|^{q(x)} dx - \int_{\mathbb{R}^N} (p(x)-1) g(x)|u|^{p(x)} dx \\ &\quad = L''_u(1) < 0, \end{aligned} \tag{70}$$

which implies that

$$\frac{2-q^+}{p^+ - q^+}\|u\|_{\lambda,V}^2 < \int_{\mathbb{R}^N} g(x)|u|^{p(x)} dx. \tag{71}$$

Combining (39), (40), and (71) with Sobolev inequality, we have

$$\frac{(\vartheta_0-1)(2-q^+)}{\vartheta_0(p^+-q^+)}\|u\|_\lambda^2 \le \frac{2-q^+}{p^+-q^+}\|u\|_{\lambda,V}^2 < \int_{\mathbb{R}^N} g(x)|u|^{p(x)}dx \le \frac{\|g\|_\infty}{\theta^{p^+/2}|\{V^+<k\}|^{p^+-2/2}} \max\left\{\|u\|_\lambda^{p^+}, M^{p^--p^+}\|u\|_\lambda^{p^-}\right\} \tag{72}$$

and so

$$\|u\|_\lambda \ge \max\left\{\left[\frac{(\vartheta_0-1)(2-q^+)\theta\wedge(p\wedge+/2)\,|\,\{V^+<k\}\,|\wedge((p\wedge+-2)/2)}{\vartheta_0(p^+-q^+)\|g\|_\infty}\right]^{\frac{1}{p^+-2}}, \cdot\left[\frac{(\vartheta_0-1)(2-q^+)\theta\wedge(p\wedge+/2)\,|\,\{V^+<k\}\,|\wedge((p\wedge+-2)/2)}{\vartheta_0(p^+-q^+)\|g\|_\infty M^{p^--p^+}}\right]^{\frac{1}{p^--2}}\right\} := C_6. \tag{73}$$

It follows from (44) that

$$\Psi_\lambda(u) \ge \frac{p^--2}{2p^-}\frac{\vartheta_0-1}{\vartheta_0}\|u\|_\lambda^2 - \left(\frac{1}{q^-}-\frac{1}{p^-}\right)\|f\|_{L^{\frac{2}{2-q(x)}}(\mathbb{R}^N)} \max \cdot\left\{\frac{1}{\theta^{q^+/2}}\|u\|_\lambda^{q^+}, \frac{1}{\theta^{q^-/2}}\|u\|_\lambda^{q^-}\right\} > \max \cdot\left\{C_6^{q^+}\left[\frac{p^--2}{2p^-}\frac{\vartheta_0-1}{\vartheta_0}C_6^{2-q^+} - \left(\frac{1}{q^-}-\frac{1}{p^-}\right)\|f\|_{L^{\frac{2}{2-q(x)}}(\mathbb{R}^N)}\frac{1}{\theta^{q^+/2}}\right], C_6^{q^-} \cdot\left[\frac{p^--2}{2p^-}\frac{\vartheta_0-1}{\vartheta_0}C_6^{2-q^-} - \left(\frac{1}{q^-}-\frac{1}{p^-}\right)\|f\|_{L^{\frac{2}{2-q(x)}}(\mathbb{R}^N)}\frac{1}{\theta^{q^-/2}}\right]\right\}. \tag{74}$$

Consequently, if $\lambda > 1/kM^2\,|\,\{V^+<k\}\,|$, then $c_\lambda^- > C_5$ for some $C_5>0$.

We note that if f, g and V_λ satisfy the hypotheses in Theorem 3, we can choose $\varphi \in C_0^\infty(\Omega_g, \mathbb{C})$, such that $L_\varphi(t) = \Psi_\lambda(t\varphi) = t^2/2\|\varphi\|_{\lambda,V}^2 - \int_{\Omega_g}(f(x)/q(x))|t\varphi|^{q(x)}dx - \int_{\Omega_g}(g(x)/p(x))|t\varphi|^{p(x)}dx$ have $t_0>0$ and C_0 which are independent of λ that satisfy $t_0\varphi \in \mathcal{N}_\lambda^-$ for all $\lambda>\lambda^*$ and

$$\sup_{t\ge0} L_\varphi(t) = L_\varphi(t_0) = C_0 > 0, \tag{75}$$

which shows $c_\lambda^- \le C_0$ for all $\lambda > \lambda^*$.

Lemma 15. *Assume that the conditions (H_1)-(H_5) and (V_1)-(V_5) hold, then there exists $\lambda^* \ge 1/kM^2\,|\,\{V^+<k\}\,|$ such that Ψ_λ satisfies the $(PS)_c$ condition in X_λ for all $c<C_0$ and $\lambda>\lambda^*$.*

Proof. First, we assume $\{u_n\}$ is a $(PS)_c$ sequence with $c<C_0$. In view of Lemma 10, there exists a positive constant $\widehat{C}$ related to λ such that $\|u_n\|_\lambda \le \widehat{C}$. Consequently, there is a subsequence which is still denote as $\{u_n\}$ and u_0 in X_λ such that

$$u_n \rightharpoonup u_0 \text{ in } X_\lambda, \quad u_n \to u_0 \text{ in } L_{loc}^r(\mathbb{R}^N,\mathbb{C}), \text{ for } 2\le r\le\infty, \quad g(x)|u_n|^{p(x)-2}u_n \rightharpoonup g(x)|u_0|^{p(x)-2}u_0 \text{ in } L^{p'(x)}(\mathbb{R}^N,\mathbb{C}). \tag{76}$$

Besides, $\Psi_\lambda'(u_0)=0$. Let $v_n = u_n - u_0$. Making use of the Vitali theorem, it holds that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} f(x)|v_n|^{q(x)}dx = 0. \tag{77}$$

In fact, note that $f \in L^{2/2-q(x)}(\mathbb{R}^N,\mathbb{C})$, for any $0<\varepsilon<1$; then, there exists $r(\varepsilon)>0$ such that for $\zeta\in\mathbb{R}^N$ and $r>r(\varepsilon)$,

$$\int_{\mathbb{R}^N\setminus B_r(\zeta)} |f(x)|^{\frac{2}{2-q(x)}}dx < \varepsilon^{\frac{2}{2-q^+}}. \tag{78}$$

For each $\Omega_0 \subset B_r(\zeta)$, one has

$$\int_{\Omega_0} f(x)|v_n|^{q(x)}dx \le \max\left\{\left(\int_{\Omega_0}|f(x)|^{\frac{2}{2-q(x)}}dx\right)^{\frac{2-q^-}{2}}, \cdot\left(\int_{\Omega_0}|f(x)|^{\frac{2}{2-q(x)}}dx\right)^{\frac{2-q^+}{2}}\right\}\max \cdot\left\{\left(\int_{\Omega_0}|v_n|^2dx\right)^{\frac{q^-}{2}}, \left(\int_{\Omega_0}|v_n|^2dx\right)^{\frac{q^+}{2}}\right\} \le \widehat{C}\max\left\{\left(\int_{\Omega_0}|f(x)|^{\frac{2}{2-q(x)}}dx\right)^{\frac{2-q^-}{2}}, \cdot\left(\int_{\Omega_0}|f(x)|^{\frac{2}{2-q(x)}}dx\right)^{\frac{2-q^+}{2}}\right\}. \tag{79}$$

It is easy to get that $\{f(x)|v_n|^{q(x)}\}$ is a equi-integrable on $B_r(\zeta)$. Besides, $f(x)|v_n|^{q(x)}\to0$, a.e., in $B_r(\zeta)$. It follows from the Vitali theorem that

$$\lim_{n\to\infty}\int_{B_r(\zeta)} f(x)|v_n|^{q(x)}dx = 0. \tag{80}$$

Hence, there holds

$$\int_{\mathbb{R}^N} f(x)|v_n|^{q(x)}dx = \int_{B_r(\zeta)} f(x)|v_n|^{q(x)}dx + \int_{\mathbb{R}^N\setminus B_r(\zeta 0)} f(x)|v_n|^{q(x)}dx \le \widehat{C}\varepsilon + \int_{B_r(\zeta)} f(x)|v_n|^{q(x)}dx, \quad (81)$$

which implies that $\lim_{n\to\infty}\int_{\mathbb{R}^N} f(x)|v_n|^{q(x)}dx = 0$.

Next, we assert that $u_n \to u_0$ in X_λ. In fact, by (V_2), we obtain

$$\int_{\mathbb{R}^N} v_n^2 dx = \int_{\{V^+\ge k\}} v_n^2 dx + \int_{\{V^+<k\}} v_n^2 dx \le \frac{1}{\lambda k}\int_{\{V^+\ge k\}} \lambda V^+ v_n^2 dx + \int_{\{V^+<k\}} v_n^2 dx \le \frac{1}{\lambda k}\|v_n\|_\lambda^2 + o(1). \quad (82)$$

In light of the Hölder inequality with the Sobolev inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^N} |v_n|^{p(x)}dx &= \int_{\mathbb{R}^N} |v_n|^{p(x)-2}\cdot |v_n|^2 dx \le \max\cdot\left\{\|v_n\|_\infty^{p^+-2},\ \|v_n\|_\infty^{p^--2}\right\}\int_{\mathbb{R}^N}|v_n|^2dx \le \max\\ &\cdot\left\{M^{p^+-2}\|v_n\|_{s(\cdot),A}^{p^+-2},\ M^{p^--2}\|v_n\|_{s(\cdot),A}^{p^--2}\right\}\int_{\mathbb{R}^N}|v_n|^2dx\\ &\le \max\left\{M^{p^+-2}\left(\|v_n\|_{L^2(\mathbb{R}^N)}^2 + [v_n]_{s(\cdot),A}^2\right)^{\frac{p^+-2}{2}},\ M^{p^--2}\cdot\left(\|v_n\|_{L^2(\mathbb{R}^N)}^2 + [v_n]_{s(\cdot),A}^2\right)^{\frac{p^--2}{2}}\right\}\int_{\mathbb{R}^N}|v_n|^2 d \le \max\\ &\cdot\left\{M^{p^+-2}\left(\frac{1}{\lambda k}\|v_n\|_\lambda^2 + \|v_n\|_\lambda^2 + o(1)\right)^{\frac{p^+-2}{2}},\ M^{p^--2}\cdot\left(\frac{1}{\lambda k}\|v_n\|_\lambda^2 + \|v_n\|_\lambda^2 + o(1)\right)^{\frac{p^--2}{2}}\right\}\int_{\mathbb{R}^N}|v_n|^2dx \le \max\\ &\cdot\left\{M^{p^+-2}\left(\frac{\lambda k+1}{\lambda k}\right)^{\frac{p^+-2}{2}}\|v_n\|_\lambda^{p^+-2} + o(1), M^{p^--2}\cdot\left(\frac{\lambda k+1}{\lambda k}\right)^{\frac{p^--2}{2}}\|v_n\|_\lambda^{p^--2} + o(1)\right\}\int_{\mathbb{R}^N}|v_n|^2dx\\ &\le \frac{1}{\lambda k}\max\left\{M^{p^+-2}\|v_n\|_\lambda^{p^+-2},\ M^{p^--2}\|v_n\|_\lambda^{p^--2}\right\}\cdot\left(\frac{\lambda k+1}{\lambda k}\right)^{\frac{p^+-2}{2}}\|v_n\|_\lambda^2 + o(1).\end{aligned} \quad (83)$$

By Bre$'$is-Lieb Lemma, we have

$$\|u_n\|_{\lambda,V}^2 = \|u_n - u_0\|_{\lambda,V}^2 + \|u_0\|_{\lambda,V}^2 + o(1). \quad (84)$$

By applying a Bre$'$is-Lieb type result on variable exponent Lebesgue space (see [24]) and (H_3)-(H_4), it is easy to obtain that

$$\int_{\mathbb{R}^N} f(x)|u_n|^{q(x)}dx = \int_{\mathbb{R}^N} f(x)|u_n-u_0|^{q(x)}dx + \int_{\mathbb{R}^N} f(x)|u_0|^{q(x)}dx + o(1). \quad (85)$$

Similarly,

$$\int_{\mathbb{R}^N} g(x)|u_n|^{p(x)}dx = \int_{\mathbb{R}^N} g(x)|u_n-u_0|^{p(x)}dx + \int_{\mathbb{R}^N} g(x)|u_0|^{p(x)}dx + o(1). \quad (86)$$

Then, overall, we can get that $\Psi_\lambda(v_n) = \Psi_\lambda(u_n) - \Psi_\lambda(u_0) + o(1)$ and $\Psi'_\lambda(v_n) = o(1)$. Then, by virtue of (77) and Lemma 10, we get that

$$\begin{aligned}C_0 + C_7 + o(1) &> c - \Psi_\lambda(u_0) + o(1) = \Psi_\lambda(v_n) - \frac{1}{p^-}\left\langle \Psi'_\lambda(v_n), v_n\right\rangle\\ &+ o(1) \ge \frac{p^- - 2}{2p^-}\|v_n\|_{\lambda,V}^2 - \left(\frac{1}{q^-} - \frac{1}{p^-}\right)\int_{\mathbb{R}^N} f(x)|v_n|^{q(x)}dx + o(1)\\ &\ge \frac{p^- - 2}{2p^-}\|v_n\|_{\lambda,V}^2 - \left(\frac{1}{q^-} - \frac{1}{p^-}\right)\|f\|_{L^{\frac{2}{2-q(x)}}}\max\left\{\frac{1}{\theta^{q^+/2}}\|u\|_\lambda^{q^+},\ \frac{1}{\theta^{q^-/2}}\|u\|_\lambda^{q^-}\right\} + o(1) \ge \frac{p^- - 2}{2p^-}\frac{\vartheta_0 - 1}{\vartheta_0}\|v_n\|_\lambda^2\\ &- \left(\frac{1}{q^-} - \frac{1}{p^-}\right)\|f\|_{L^{\frac{2}{2-q(x)}}}\max\left\{\frac{1}{\theta^{q^+/2}}\|u\|_\lambda^{q^+}, \frac{1}{\theta^{q^-/2}}\|u\|_\lambda^{q^-}\right\} + o(1),\end{aligned} \quad (87)$$

where

$$\begin{aligned}C_7 = \min&\left\{\frac{2-q^+}{2p^-}\left(\frac{\vartheta_0 q^+}{\theta(\vartheta_0 - 1)(p^- - 2)}\right)^{\frac{q^+}{2-q^+}}\left(\frac{(p^- - q^-)\|f\|_{L^{2/2-q(x)}}}{q^-}\right)^{\frac{2}{2-q^+}},\right.\\ &\left.\frac{2-q^-}{2p^-q^-}\left(\frac{\vartheta_0}{\theta(\vartheta_0 - 1)(p^- - 2)}\right)^{\frac{q^-}{2-q^-}}\left((p^- - q^-)\|f\|_{L^{\frac{2}{2-q(x)}}}\right)^{\frac{2}{2-q^-}}\right\}.\end{aligned} \quad (88)$$

Suppose by contradiction that $\{v_n\}$ is not bounded in X_λ. Then, there exists a subsequence still denoted by $\{v_n\}$ such that $\|v_n\|_\lambda \to \infty$ as $n \to \infty$. Hence, by virtue of (87), we have

$$\begin{aligned}\frac{C_0 + C_7}{\|v_n\|_\lambda^2} + o(1)\frac{1}{\|v_n\|_\lambda^2} &\ge \frac{p^- - 2}{2p^-}\frac{\vartheta_0 - 1}{\vartheta_0} - \left(\frac{1}{q^-} - \frac{1}{p^-}\right)\|f\|_{L^{\frac{2}{2-q(x)}}}\max\\ &\cdot\left\{\frac{1}{\theta^{q^+/2}}\|v_n\|_\lambda^{q^+-2}, \frac{1}{\theta^{q^-/2}}\|v_n\|_\lambda^{q^--2}\right\} + o(1)\frac{1}{\|v_n\|_\lambda^2},\end{aligned} \quad (89)$$

which is contradictory since $1 < q^- \le q^+ < 2 < p^-$. Thus, $\{v_n\}$ is bounded in X_λ for all $\lambda > \lambda^* \ge 1/kM^2\,|\,\{V^+ < k\}\,|$. That is, there exist a constant $M_1 > 0$ such that $\|v_n\|_\lambda \le M_1$. From

(83), we can get that

$$\int_{\mathbb{R}^N} |v_n|^{p(x)} dx \le \frac{1}{\lambda k} \max\left\{ (MM_1)^{p^+-2}, (MM_1)^{p^--2} \right\} \cdot \left(\frac{\lambda k + 1}{\lambda k} \right)^{\frac{p^+-2}{2}} \|v_n\|_\lambda^2 + o(1). \tag{90}$$

Together $\langle \Psi_\lambda'(v_n), v_n \rangle = o(1)$ with (77)-(90), there holds

$$\begin{aligned} o(1) &= \left\langle \Psi_\lambda'(v_n), v_n \right\rangle = \|v_n\|_{\lambda,V}^2 - \int_{\mathbb{R}^N} g(x)|v_n|^{p(x)} dx \\ &\ge \frac{\vartheta_0 - 1}{\vartheta_0} \|v_n\|_\lambda^2 - \|g\|_\infty \int_{\mathbb{R}^N} |v_n|^{p(x)} dx \ge \frac{\vartheta_0 - 1}{\vartheta_0} \|v_n\|_\lambda^2 \\ &\quad - \frac{\|g\|_\infty}{\lambda k} \max\left\{ (MM_1)^{p^+-2}, (MM_1)^{p^--2} \right\} \\ &\quad \cdot \left(\frac{\lambda k + 1}{\lambda k} \right)^{\frac{p^+-2}{2}} \|v_n\|_\lambda^2 + o(1). \end{aligned} \tag{91}$$

We find that there exists $\lambda^* \ge 1/kM^2 \mid \{V^+ < k\} \mid$ large enough such that

$$\frac{\|g\|_\infty}{\lambda_k} \max\left\{ (MM_1)^{p^+-2}, (MM_1)^{p^--2} \right\} \left(\frac{\lambda k + 1}{\lambda k} \right)^{\frac{p^+-2}{2}} < \frac{\vartheta_0 - 1}{\vartheta_0}, \tag{92}$$

for all $\lambda > \lambda^*$. It follows from (91) that $v_n \to 0$ in X_λ for all $\lambda > \lambda^*$.

Theorem 16. *Assume that (H_1)-(H_5) and (V_1)-(V_5) hold, then there exists $\lambda^* \ge 0$ such that for every $\lambda > \lambda^*$, Ψ_λ has a minimizer u_λ^+ in $\mathcal{N}_\lambda^+$ satisfying that*

$$\Psi_\lambda(u_\lambda^+) = c_\lambda^+ = \inf_{u \in \mathcal{N}_\lambda^+} \Psi_\lambda(u), \tag{93}$$

(1) u_λ^+ is a nontrivial solution of equation (1).

Proof. Combining Lemma 14 and the Ekeland variational principle in [25], there exists $\{u_n\} \subset \mathcal{N}_\lambda^+$ such that $\{u_n\}$ is a $(PS)_{c_\lambda^+}$ sequence for Ψ_λ. Furthermore, using Lemma 10, we can get that $\{u_n\}$ is bounded in X_λ. Consequently, there exists a subsequence of $\{u_n\}$ (we still denote as $\{u_n\}$) and u_λ^+ in X_λ such that

$$\begin{gathered} u_n \rightharpoonup u_\lambda^+ \text{ in } X_\lambda, \\ u_n \to u_\lambda^+ \text{ in } L_{loc}^r(\mathbb{R}^N, \mathbb{C}), \text{ for } 2 \le r \le \infty. \end{gathered} \tag{94}$$

Besides, $\Psi_\lambda'(u_\lambda^+) = 0$. In view of Lemma 15, we know that $u_n \to u_\lambda^+$ in X_λ and $\Psi_\lambda(u_\lambda^+) = c_\lambda^+$. In other words, u_λ^+ is a solution of equation (1).

Now, we will check that $u_\lambda^+ \ne 0$. On the contrary, by combining (40), (H_3), the Egoroff theorem and the Hölder inequality, there holds $\int_{\mathbb{R}^N} f(x)|u_n|^{q(x)} dx \to 0$ as $n \to \infty$, which shows that

$$\|u_n\|_{\lambda,V}^2 = \int_{\mathbb{R}^N} g(x)|u_n|^{p(x)} dx + o(1)$$

$$\begin{aligned} \Psi_\lambda(u_n) &= \frac{1}{2}\|u_n\|_{\lambda,V}^2 - \int_{\mathbb{R}^N} \frac{f(x)}{q(x)} |u_n|^{q(x)} dx - \int_{\mathbb{R}^N} \frac{g(x)}{p(x)} |u_n|^{p(x)} dx \\ &\ge \frac{1}{2}\|u_n\|_{\lambda,V}^2 - \frac{1}{q^-} \int_{\mathbb{R}^N} f(x)|u_n|^{q(x)} dx - \frac{1}{p^-} \int_{\mathbb{R}^N} g(x)|u_n|^{p(x)} dx \\ &= \frac{1}{2}\|u_n\|_{\lambda,V}^2 - \frac{1}{p^-}\|u_n\|_{\lambda,V}^2 + o(1) = \frac{p^- - 2}{2p^-}\|u_n\|_{\lambda,V}^2 + o(1) \\ &\ge \frac{(p^- - 2)(\vartheta_0 - 1)}{2p^-\vartheta_0}\|u_n\|_\lambda^2 + o(1) \ge 0. \end{aligned} \tag{95}$$

This is contradictive with $\lim_{n\to\infty} \Psi_\lambda(u_n) = c_\lambda^+ < 0$. Hence, $u_\lambda^+ \ne 0$; that is, u_λ^+ is a nontrivial solution of equation (1).

Proof of Theorem 17. The result of Theorem 2 is immediately available from Theorem 20.

Theorem 18. *Assume that the conditions (H_3), (H_4), and (V_1)-(V_5) are satisfied, then there exists $\lambda^* \ge 0$ such that for every $\lambda > \lambda^*$, Ψ_λ has a minimizer u_λ^- in $\mathcal{N}_\lambda^-$ satisfying that*

$$\Psi_\lambda(u_\lambda^-) = c_\lambda^-, \tag{96}$$

(1) u_λ^- is a nontrivial solution of equation (1).

Proof. According to Lemma 14 and the Ekeland variational principle in [25], there exists $\{u_n\} \subset \mathcal{N}_\lambda^-$ such that $\{u_n\}$ is a $(PS)_{c_\lambda^-}$ sequence for Ψ_λ. Furthermore, using Lemma 10, we can get that $\{u_n\}$ is bounded in X_λ. Consequently, there exists a subsequence of $\{u_n\}$ (we still denote as $\{u_n\}$) and u_λ^- in X_λ such that

$$\begin{gathered} u_n \rightharpoonup u_\lambda^- \text{ in } X_\lambda, \\ u_n \to u_\lambda^- \text{ in } L_{loc}^r(\mathbb{R}^N), \text{ for } 2 \le r \le \infty. \end{gathered} \tag{97}$$

Besides, $\Psi_\lambda'(u_\lambda^-) = 0$. In view of Lemma 15, we know that $u_n \to u_\lambda^-$ in X_λ and $\Psi_\lambda(u_\lambda^-) = c_\lambda^-$. In other words, u_λ^- is a solution of equation (1).

Now, we will check that $u_\lambda^- \ne 0$. Suppose the contrary, combining (39), (40), Egoroff theorem, and (H_4), there holds $\int_{\mathbb{R}^N} g(x)|u_n|^{p(x)} dx \to 0$ as $n \to \infty$, which shows that

$$\|u_n\|_{\lambda,V}^2 = \int_{\mathbb{R}^N} f(x)|u_n|^{q(x)} dx + o(1)$$

$$\begin{aligned}\Psi_\lambda(u_n) &= \frac{1}{2}\|u_n\|_{\lambda,V}^2 - \int_{\mathbb{R}^N}\frac{f(x)}{q(x)}|u_n|^{q(x)}dx - \int_{\mathbb{R}^N}\frac{g(x)}{p(x)}|u_n|^{p(x)}dx \\ &\le \frac{1}{2}\|u_n\|_{\lambda,V}^2 - \frac{1}{q^+}\int_{\mathbb{R}^N} f(x)|u_n|^{q(x)}dx - \frac{1}{p^+}\int_{\mathbb{R}^N} g(x)|u_n|^{p(x)}dx \\ &= \frac{1}{2}\|u_n\|_{\lambda,V}^2 - \frac{1}{q^+}\|u_n\|_{\lambda,V}^2 + o(1) = \frac{q^+-2}{2q^+}\|u_n\|_{\lambda,V}^2 + o(1) \\ &\le \frac{(q^+-2)(\vartheta_0-1)}{2q^+\vartheta_0}\|u_n\|_\lambda^2 + o(1) \le 0.\end{aligned} \tag{98}$$

This is contradictive with $\lim_{n\to\infty}\Psi_\lambda(u_n) = c_\lambda^- > 0$. Hence, $u_\lambda^- \neq 0$; that is, u_λ^- is a nontrivial solution of equation (1).

Proof of Theorem 19. It derives from Theorem 20, Theorem 22, and Lemma 14 that equation (1) has two nontrivial solutions u_λ^+ and u_λ^- such that $u_\lambda^+ \in \mathcal{N}_\lambda^+$ and $u_\lambda^- \in \mathcal{N}_\lambda^-$ with $\Psi_\lambda(u_\lambda^+) = c_\lambda^+ < 0 < C_5 < \Psi_\lambda(u_\lambda^-) = c_\lambda^-$.

Theorem 20. *Assume that (H_3), (H_4), and (V_1)-(V_5) hold, then for all $\lambda \ge 1/kM^2|\{V^+ < k\}|$, Ψ_λ has a minimizer u_λ^+ in $\mathcal{N}_\lambda^+$ satisfying that*

$$\Psi_\lambda(u_\lambda^+) = c_\lambda^+ = \inf_{u\in\mathcal{N}_\lambda^+}\Psi_\lambda(u); \tag{99}$$

(1) u_λ^+ is a nontrivial solution of equation (1).

Proof. Combining Lemma 14 and the Ekeland variational principle in [25], there exists $\{u_n\} \subset \mathcal{N}_\lambda^+$ such that $\{u_n\}$ is a $(PS)_{c_\lambda^+}$ sequence for Ψ_λ. Furthermore, using Lemma 10, we can get that $\{u_n\}$ is bounded in X_λ. Consequently, there exists a subsequence of $\{u_n\}$ (we still denote as $\{u_n\}$) and u_λ^+ in X_λ such that

$$\begin{gathered} u_n \rightharpoonup u_\lambda^+ \text{ in } X_\lambda, \\ u_n \to u_\lambda^+ \text{ in } L_{loc}^r(\mathbb{R}^N), \text{ for } 2 \le r \le \infty. \end{gathered} \tag{100}$$

Besides, $\Psi_\lambda'(u_\lambda^+) = 0$. In view of Lemma 15, we know that $u_n \to u_\lambda^+$ in X_λ and $\Psi_\lambda(u_\lambda^+) = c_\lambda^+$. In other words, u_λ^+ is a solution of equation (1).

Now, we will check that $u_\lambda^+ \neq 0$. On the contrary, by combining (40), (H_3), the Egoroff theorem, and the Hölder inequality, there holds $\int_{\mathbb{R}^N} f(x)|u_n|^{q(x)}dx \to 0$ as $n\to\infty$, which shows that

$$\|u_n\|_{\lambda,V}^2 = \int_{\mathbb{R}^N} g(x)|u_n|^{p(x)}dx + o(1)$$

$$\begin{aligned}\Psi_\lambda(u_n) &= \frac{1}{2}\|u_n\|_{\lambda,V}^2 - \int_{\mathbb{R}^N}\frac{f(x)}{q(x)}|u_n|^{q(x)}dx - \int_{\mathbb{R}^N}\frac{g(x)}{p(x)}|u_n|^{p(x)}dx \\ &\ge \frac{1}{2}\|u_n\|_{\lambda,V}^2 - \frac{1}{q^-}\int_{\mathbb{R}^N} f(x)|u_n|^{q(x)}dx - \frac{1}{p^-}\int_{\mathbb{R}^N} g(x)|u_n|^{p(x)}dx \\ &= \frac{1}{2}\|u_n\|_{\lambda,V}^2 - \frac{1}{p^-}\|u_n\|_{\lambda,V}^2 + o(1) = \frac{p^--2}{2p^-}\|u_n\|_{\lambda,V}^2 + o(1) \\ &\ge \frac{(p^--2)(\vartheta_0-1)}{2p^-\vartheta_0}\|u_n\|_\lambda^2 + o(1) \ge 0.\end{aligned} \tag{101}$$

This is contradictive with $\lim_{n\to\infty}\Psi_\lambda(u_n) = c_\lambda^+ < 0$. Hence, $u_\lambda^+ \neq 0$; that is, u_λ^+ is a nontrivial solution of equation (1).

Proof of Theorem 21. The result of Theorem 2 is immediately available from Theorem 20.

We note that if f, g and V_λ satisfy the hypotheses in Theorem 3, we can choose $\varphi \in C_0^\infty(\Omega_g)$, such that $L_\varphi(t) = \Psi_\lambda(t\varphi) = t^2/2\|\varphi\|_{\lambda,V}^2 - \int_{\Omega_g}(f(x)/q(x))|t\varphi|^{q(x)}dx - \int_{\Omega_g}(g(x)/p(x))|t\varphi|^{p(x)}dx$ have $t_0 > 0$ and C_8 which are independent of λ that satisfy $t_0\varphi \in \mathcal{N}_\lambda^-$ for all $\lambda > \lambda^*$ and

$$\sup_{t\ge 0} L_\varphi(t) = L_\varphi(t_0) = C_8 > 0, \tag{102}$$

which shows $c_\lambda^- \le C_8$ for all $\lambda > \lambda^*$.

Theorem 22. *Assume that the conditions (H_3), (H_4), and (V_1)-(V_5) are satisfied, then for all $\lambda > 1/kM^2|\{V^+ < k\}|$, Ψ_λ has a minimizer u_λ^- in $\mathcal{N}_\lambda^-$ satisfying that*

$$\Psi_\lambda(u_\lambda^-) = c_\lambda^-; \tag{103}$$

(1) u_λ^- is a nontrivial solution of equation (1).

Proof. According to Lemma 14 and the Ekeland variational principle in [25], there exists $\{u_n\} \subset \mathcal{N}_\lambda^-$ such that $\{u_n\}$ is a $(PS)_{c_\lambda^-}$ sequence for Ψ_λ. Furthermore, using Lemma 10, we can get that $\{u_n\}$ is bounded in X_λ. Consequently, there exists a subsequence of $\{u_n\}$ (we still denote as $\{u_n\}$) and u_λ^- in X_λ such that

$$\begin{gathered} u_n \rightharpoonup u_\lambda^- \text{ in } X_\lambda, \\ u_n \to u_\lambda^- \text{ in } L_{loc}^r(\mathbb{R}^N), \text{ for } 2 \le r \le \infty. \end{gathered} \tag{104}$$

Besides, $\Psi_\lambda'(u_\lambda^-) = 0$. In view of Lemma 15, we know that $u_n \to u_\lambda^-$ in X_λ and $\Psi_\lambda(u_\lambda^-) = c_\lambda^-$. In other words, u_λ^- is a solution of equation (1).

Now, we will check that $u_\lambda^- \neq 0$. Suppose the contrary, combining (39), (40), Egoroff theorem, and (H_4), there holds

$\int_{\mathbb{R}^N} g(x)|u_n|^{p(x)}dx \to 0$ as $n \to \infty$, which shows that

$$\|u_n\|_{\lambda,V}^2 = \int_{\mathbb{R}^N} f(x)|u_n|^{q(x)}dx + o(1)$$

$$\begin{aligned}\Psi_\lambda(u_n) &= \frac{1}{2}\|u_n\|_{\lambda,V}^2 - \int_{\mathbb{R}^N}\frac{f(x)}{q(x)}|u_n|^{q(x)}dx - \int_{\mathbb{R}^N}\frac{g(x)}{p(x)}|u_n|^{p(x)}dx \\ &\le \frac{1}{2}\|u_n\|_{\lambda,V}^2 - \frac{1}{q^+}\int_{\mathbb{R}^N} f(x)|u_n|^{q(x)}dx - \frac{1}{p^+}\int_{\mathbb{R}^N} g(x)|u_n|^{p(x)}dx \\ &= \frac{1}{2}\|u_n\|_{\lambda,V}^2 - \frac{1}{q^+}\|u_n\|_{\lambda,V}^2 + o(1) = \frac{q^+ - 2}{2q^+}\|u_n\|_{\lambda,V}^2 + o(1) \\ &\le \frac{(q^+ - 2)(\vartheta_0 - 1)}{2q^+\vartheta_0}\|u_n\|_{\lambda,V}^2 + o(1) \le 0.\end{aligned} \tag{105}$$

This is contradictive with $\lim_{n\to\infty}\Psi_\lambda(u_n) = c_\lambda^- > 0$. Hence, $u_\lambda^- \neq 0$; that is, u_λ^- is a nontrivial solution of equation (1).

Proof of Theorem 23. It derives from Theorem 20, Theorem 22, and Lemma 14 that equation (1) has two nontrivial solutions u_λ^+ and u_λ^- such that $u_\lambda^+ \in \mathcal{N}_\lambda^+$ and $u_\lambda^- \in \mathcal{N}_\lambda^-$ with $\Psi_\lambda(u_\lambda^+) = c_\lambda^+ < 0 < C_5 < \Psi_\lambda(u_\lambda^-) = c_\lambda^-$.

Data Availability

Not applicable data and material.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

Acknowledgments

The authors expresses their gratitude to the referees for their valuable comments and suggestions which have led to a significant improvement on the presentation and quality of this paper. This work is supported by the Natural Science Foundation of Yunnan Province under grants 2018FE001(-136) and 2017zzx199, the National Natural Science Foundation of People's Republic of China under grants 11961078 and 11561072, the Yunnan Province, Young Academic and Technical Leaders Program (2015HB010), and the Natural Science Foundation of Yunnan Province under grant 2016FB011.

References

[1] S. Z. Mao and A. Xia, "Multiplicity results of nonlinear fractional magnetic Schrödinger equation with steep potential," *Applied Mathematics Letters*, vol. 97, pp. 73–80, 2019.

[2] M. Q. Xiang, B. L. Zhang, and D. Yang, "Multiplicity results for variable-order fractional Laplacian equations with variable growth," *Nonlinear Analysis*, vol. 178, pp. 190–204, 2019.

[3] M. Q. Xiang, D. Yang, and B. Zhang, "Homoclinic solutions for Hamiltonian systems with variable-order fractional derivatives," *Complex Variables and Elliptic Equations*, vol. 65, pp. 1412–1432, 2019.

[4] Y.-h. Cheng and T.-F. Wu, "Multiplicity and concentration of positive solutions for semilinear elliptic equations with steep potential," *Communications on Pure and Applied Analysis*, vol. 15, no. 6, pp. 2457–2473, 2016.

[5] S. Peng and A. Xia, "Multiplicity and concentration of solutions for nonlinear fractional elliptic equations with steep potential," *Communications on Pure & Applied Analysis*, vol. 17, no. 3, pp. 1201–1217, 2018.

[6] P. d'Avenia and M. Squassina, "Ground states for fractional magnetic operators," *ESAIM: Control, Optimisation and Calculus of Variations*, vol. 24, no. 1, pp. 1–24, 2018.

[7] V. Ambrosio and P. d'Avenia, "Nonlinear fractional magnetic Schrödinger equation: existence and multiplicity," *Journal of Differential Equations*, vol. 264, no. 5, pp. 3336–3368, 2018.

[8] M. Xiang, P. Pucci, M. Squassina, and B. Zhang, "Nonlocal Schrodinger-Kirchhoff equations with external magnetic field," *Discrete & Continuous Dynamical Systems - A*, vol. 37, pp. 1631–1649, 2017.

[9] A. Pinamonti, M. Squassina, and E. Vecchi, "The Maz'ya¨C-Shaposhnikova limit in the magnetic setting," *Journal of Mathematical Analysis and Applications*, vol. 449, no. 2, pp. 1152–1159, 2017.

[10] H.-M. Nguyen, A. Pinamonti, M. Squassina, and E. Vecchi, "Some characterizations of magnetic Sobolev spaces," *Complex Variables and Elliptic Equations*, vol. 65, no. 7, pp. 1104–1114, 2020.

[11] A. Pinamonti, M. Squassina, and E. Vecchi, "Magnetic BV-functions and the Bourgain-Brezis-Mironescu formula," *Advances in Calculus of Variations*, vol. 12, no. 3, pp. 225–252, 2019.

[12] H.-M. Nguyen, A. Pinamonti, M. Squassina, and E. Vecchi, "New characterizations of magnetic Sobolev spaces," *Advances in Nonlinear Analysis*, vol. 7, no. 2, pp. 227–245, 2018.

[13] X. Mingqi, V. D. Radulescu, and B. Zhang, "Fractional Kirchhoff problems with critical Trudinger Moser nonlinearity," *Calculus of Variations and Partial Differential Equations*, vol. 58, no. 2, p. 57, 2019.

[14] C. Ji, F. Fang, and B. L. Zhang, "A multiplicity result for asymptotically linear Kirchhoff equations," *Advances in Nonlinear Analysis*, vol. 8, pp. 267–277, 2019.

[15] X. Mingqi, V. D. Rădulescu, and B. Zhang, "Nonlocal Kirchhff problems with singular exponential nonlinearity," *Applied Mathematics and Optimization*, 2020.

[16] A. Fiscella, "A fractional Kirchhoff problem involving a singular term and a critical nonlinearity," *Advances in Nonlinear Analysis*, vol. 8, pp. 645–660, 2019.

[17] T. Bartsch and Z. Q. Wang, "Existence and multiplicity results for some superlinear elliptic problems on $\mathbb{R}^N$, Comm," *Partial Differential Equations*, vol. 20, no. 9-10, pp. 1725–1741, 1995.

[18] O. Kovacik and J. Bakosnik, "On spaces $L^{p(x)}$ and $W^{m,p(x)}$," *Czechoslovak Mathematical Journal*, vol. 41, pp. 592–618, 1991.

[19] X. Fan and D. Zhao, "On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$," *Journal of Mathematical Analysis and Applications*, vol. 263, pp. 424–446, 2001.

[20] L. Diening, P. Harjulehto, P. Hästö, and M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Springe-Verlag, Heidelberg, 2011.

[21] A. Fiscella, A. Pinamonti, and E. Vecchi, "Multiplicity results for magnetic fractional problems," *Journal of Differential Equations*, vol. 263, no. 8, pp. 4617–4633, 2017.

[22] S. I. Pohozaev, "An approach to nonlinear equations," *Proceedings of the USSR Academy of Sciences*, vol. 247, no. 6, pp. 1327–1331, 1979.

[23] P. Drabek and S. Pohozaev, "Positive solutions for thep-Laplacian: application of the fibrering method," *Proceedings of the Royal Society of Edinburgh*, vol. 127, no. 4, pp. 703–726, 1997.

[24] C. O. Alves and M. C. Ferreira, "Existence of solutions for a class of $p(x)$-Laplacian equations involving a concave-convex nonlinearity with critical growth in $\mathbb{R}^N$," *Topological Methods in Nonlinear Analysis*, vol. 45, no. 2, pp. 399–422, 2015.

[25] I. Ekeland, "On the variational principle," *Journal of Mathematical Analysis and Applications*, vol. 17, pp. 324–353, 1974.