Hindawi
Advances in Mathematical Physics
Volume 2020, Article ID 1361498, 9 pages
https://doi.org/10.1155/2020/1361498

Research Article

A Posteriori Error Estimates for Hughes Stabilized SUPG Technique and Adaptive Refinement for a Convection-Diffusion Problem

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Received 30 November 2019; Revised 24 March 2020; Accepted 13 April 2020; Published 16 May 2020

Academic Editor: Sergey Shmarev

The motive of the present work is to propose an adaptive numerical technique for singularly perturbed convection-diffusion problem in two dimensions. It has been observed that for small singular perturbation parameter, the problem under consideration displays sharp interior or boundary layers in the solution which cannot be captured by standard numerical techniques. In the present work, Hughes stabilization strategy along with the streamline upwind/Petrov-Galerkin (SUPG) method has been proposed to capture these boundary layers. Reliable a posteriori error estimates in energy norm on anisotropic meshes have been developed for the proposed scheme. But these estimates prove to be dependent on the singular perturbation parameter. Therefore, to overcome the difficulty of oscillations in the solution, an efficient adaptive mesh refinement algorithm has been proposed. Numerical experiments have been performed to test the efficiency of the proposed algorithm.

1. Introduction

Singularly perturbed problems occur frequently in various branches of applied science and engineering, e.g., fluid dynamics, aerodynamics, oceanography, quantum mechanics, chemical reactor theory, reaction-diffusion processes, and radiating flows. In general, it has been observed that singularly perturbed problems exhibit singularities as the singular perturbation parameter $\varepsilon \longrightarrow 0$. Therefore, it becomes essential to implement some robust numerical technique to capture these singularities. In literature, there exist various numerical techniques to handle these singularities. Adaptive mesh refinement techniques are one of such techniques. Very few researchers have proposed adaptive refinement strategies for singularly perturbed convection-diffusion problems. Generally, adaptive refinement techniques are based on two types of error estimates, namely, a priori error estimates and a posteriori error estimates. Nicaise [1] developed a posteriori residual error estimates for convection-diffusion-reaction problems using some cell-centered finite volume methods. Based on a posteriori error estimates, the author proposed an adaptive algorithm. John [2, 3] did numerical study of various a posteriori error estimates and indicators for convection-diffusion problems. On the basis of error estimates, the author proposed numerical solution of singularly perturbed convection-dominated problems on adaptive refined grid. Repin and Nicaise [4] derived a posteriori error estimates for linear convection-diffusion-reaction problems using functional arguments. Verfurth [5] derived a posteriori error estimates for convection-dominated stationary convection-diffusion equation using locally refined isotropic meshes. Zhao et al. [6] proposed adaptive numerical technique for convection-diffusion equations based on semirobust residual a posteriori error estimates for lower order nonconforming finite element approximations of streamline diffusion method.

From literature, we know that the classical finite element methods [7] fail to provide satisfactory results for small values of singular perturbation parameter, i.e., when $\varepsilon \longrightarrow 0$. It has also been observed that streamline upwind/Petrov-Galerkin (SUPG) method provides good approximate solution in the region where there is no sharp change in the solution but fails in the small subregions of sharp boundary layers. It has been observed that occurrence of these nonphysical oscillations in the region of sharp boundary layers in the discrete solution of SUPG method is based on the fact that this scheme is not monotonicity preserving. To overcome this difficulty, in the present work, we have proposed Hughes stabilization strategy [8] along with the SUPG method. It involves suitable addition of one more term which is multiple of a function in the direction where spurious oscillations were seen in approximate SUPG solution. This additional term is added on the left-hand side of SUPG discretization of convection-diffusion problem. The a posteriori error estimates have been derived for the proposed scheme. Based on these estimates, an anisotropic mesh refinement strategy has been proposed for singularly perturbed problems.

The outline of the paper is as follows.

In Section 2, the continuous problem under consideration and its streamline upwind Petrov-Galerkin finite element approximation have been presented. In Section 3, some auxiliary tools which are required for deriving reliable error bounds have been presented. In Section 4, we have discussed residual-based a posteriori error estimates and derived error bounds on anisotropic meshes. An adaptive refinement algorithm based on derived a posteriori error estimates has been proposed in Section 5. Section 6 deals with some numerical experiments which have been performed to analyze the robustness and efficiency of the proposed adaptive refinement strategy. In the last section, conclusion has been presented.

2. Continuous Problem

Consider the following convection-diffusion equation in two dimensions:

$$-\nabla \cdot \varepsilon \nabla u + a \cdot \nabla u + bu = f \text{ in } \Omega, \tag{1a}$$

$$u = 0 \quad \text{on} \quad \partial\Omega_D, \tag{1b}$$

$$\varepsilon \frac{\partial u}{\partial n} = g \quad \text{on} \quad \partial\Omega_N, \tag{1c}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with Lipschitz-continuous boundary $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$, ε $(0 < \varepsilon \ll 1)$ is singular perturbation parameter, and a, b, and f are sufficiently smooth. Here, $\partial\Omega_D$ and $\partial\Omega_N$ denote the Dirichlet and Neumann boundaries of the domain, respectively.

$W^{1,\infty}(\Omega)$ and $L^{\infty}(\Omega)$ represent the usual Sobolev and Lebesgue spaces, respectively. The notation $(.,.)$ has been used for inner product $(.,.)_{\Omega}$.

Throughout the paper, we assume that $-(1/2)\nabla \cdot a + b \geq b_0 > 0$.

For any open bounded subset $K \subset \bar{\Omega}$, let $H^1(K)$ be the standard Sobolev space. Further, we define

$$V_0 = \left\{ v \in H^1(\Omega), \quad v = 0 \text{ on } \partial\Omega_D \right\}. \tag{2}$$

Let

$$|||v|||_K^2 = \varepsilon \|\nabla v\|_K^2 + b_0 \|v\|_K^2 \tag{3}$$

be energy norm on bounded subset $K \subset \bar{\Omega}$. The weak formulation of equations (1a), (1b) and (1c) is given by the following.

Find $u \in H^1(\Omega)$ such that

$$\mathbf{B}(u, v) = \langle F, v \rangle \tag{4}$$

where

$$\mathbf{B}(u, v) = \varepsilon(\nabla u, \nabla v) + (a \cdot \nabla u, v) + (bu, v), \tag{5}$$

$$\langle F, v \rangle = (f, v) + (g, v)_{\partial\Omega_N} \quad \forall v \in V_0. \tag{6}$$

The existence and uniqueness of the solution of the above weak formulation (4) are guaranteed using Lax Milgram Lemma together with condition (1c). Let Γ_h be the admissible and shape-regular triangulation of domain $\bar{\Omega}$ consisting of triangles. Let L be any two-dimensional element with edge E. Let $n_{L,E} = (n_x, n_y)$ be unit outward normal vector to L along E (see Figure 1). Fixing one of the two normal vectors, let n_E be the normal vector for each edge E.

It has been observed that the solution of singularly perturbed problem displays boundary layers if the Peclet number, as discussed below, is large. Define local mesh Peclet number as

$$\mathrm{Pe}_K = \frac{\|a\|_{\infty,K} h_{\min}^K}{2\varepsilon}, \tag{7}$$

where $h_{\min}^K$ is the minimal length of element K as defined in the next section.

Let $V^h = \{v_h \in H^1(\Omega) \colon v_h|_K \in P_1(K)\}$, where $P_1(K)$ is the space of all linear polynomials over the element K and $V_0^h = \{v_h \in V^h : v_h|_{\partial\Omega_D} = 0\}$. Next, we discuss the SUPG method along with the Hughes stabilization technique for approximating the solution of problem (1a), (1b) and (1c). The SUPG method [9] for problem (1a), (1b) and (1c) is defined as follows.

Find $u_h \in V^h$ such that

$$\mathbf{B}_\rho(u_h, v_h) = \langle F, v_h \rangle \forall v_h \in V_0^h, \tag{8}$$

where $\mathbf{B}_\rho(u_h, v_h) = \mathbf{B}(u_h, v_h) + \langle R_h(u_h), \rho a \cdot \nabla_h v_h \rangle$, $R_h(u) = -\varepsilon \Delta_h u + a \cdot \nabla_h u + bu - f$, ρ is nonnegative stabilization parameter, and $\mathbf{B}(u, v)$ and $\langle F, v \rangle$ are defined in (5).

2.1. Hughes Stabilization Technique. It has been observed that the streamline upwind/Petrov-Galerkin (SUPG) method provides good approximate solution in the region where there is no sharp change in the solution but fails badly in the small subregions of sharp boundary layers. To overcome this difficulty, we use Hughes stabilization technique [10] to SUPG method. It involves an additional term $\langle R_h(u_h), \sigma a_h \cdot \nabla_h v_h \rangle$ in the left-hand side of SUPG

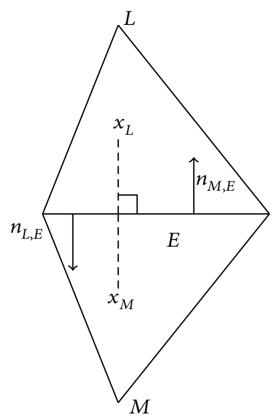

Figure 1: Orthogonality condition.

finite element discretization of convection-diffusion equation where

$$a_h = \begin{cases} \dfrac{(a\cdot\nabla u_h)\nabla u_h}{|\nabla u_h|^2}, & \text{if}\,|\nabla u_h|\neq 0, \\ 0, & \text{if}\,|\nabla u_h| = 0, \end{cases} \tag{9}$$

and σ is a nonnegative stabilization parameter. This additional term increases the robustness of SUPG method in the boundary layer region by controlling oscillations. Using Hughes stabilization technique to SUPG finite element method, equation (1a), (1b) and (1c) is discretized as follows.

Find $u_h \in V^h$ such that

$$\mathbf{B}_{\rho,\sigma}(u_h, v_h) = \langle F, v_h\rangle \forall v_h \in V_0^h, \tag{10}$$

where $\mathbf{B}_{\rho,\sigma}(u_h, v_h) = \mathbf{B}(u_h, v_h) + \langle R_h(u_h), \rho a\cdot\nabla_h v_h\rangle + \langle R_h(u_h), \sigma a_h\cdot\nabla_h v_h\rangle$ and $R_h(u) = -\varepsilon\Delta_h u + a\cdot\nabla_h u + bu - f$.

Let ρ_K be the stabilization parameter over each element K. Ross et al. [7] showed that the approximate solution u_h obtained using SUPG finite element discretization exists and is unique provided stabilization parameter ρ_K is small and satisfies

$$0 \le \rho_K \le \frac{1}{2}\min\left\{b_0\|b\|_{\infty,K}^{-2}, \left(h_{\min}^K\right)^2 \varepsilon^{-1}\nu^{-2}\right\}, \tag{11}$$

where $h_{\min}^K$ is the minimal length of element K and the constant ν satisfies the inequality

$$\|\nabla\cdot\nabla v_h\|_K \le \nu\left(h_{\min}^K\right)^{-1}\|\nabla v_h\|_K \forall v_h \in V_0^h. \tag{12}$$

From inequality (12), it can easily be observed that $\nu = 0$ for piecewise linear functions in V_0^h. Therefore, the above bounds reduce to $0 \le \rho_K \le b_0/2\|b\|_{\infty,K}^{-2}$. In order to simplify the calculations, we introduce the notation $c \lesssim d$ which means that there exists a positive constant A independent of c, d, Γ_h, and ε such that $c \le Ad$. Further, we assume that

$$\rho_K \lesssim h_{\min}^K \|a\|_{\infty,K}^{-1} \forall K \in \Gamma_h. \tag{13}$$

Also, for any mesh function $v_h \in V_0^h$, using (12) and scaling arguments, we can get

$$\|\nabla v_h\|_K \lesssim \left(h_{\min}^K\right)^{-1}\|v_h\|_K. \tag{14}$$

Using energy norm def. (3),

$$\|v_h\|_K \le b_0^{-1/2}|||v_h|||_K. \tag{15}$$

Thus, we have

$$\|\nabla v_h\|_K \lesssim \left(h_{\min}^K\right)^{-1} b_0^{-\frac{1}{2}}|||v_h|||_K. \tag{16}$$

Again, from energy norm, we have

$$\|\nabla v_h\|_K \le \varepsilon^{-1/2}|||v_h|||_K. \tag{17}$$

Using (16) and (17), we get

$$\|\nabla v_h\|_K \lesssim \min\left\{\left(h_{\min}^K\right)^{-1} b_0^{-1/2}, \varepsilon^{-1/2}\right\}|||v_h|||_K. \tag{18}$$

3. Some Important Notations and Tools

Since singularly perturbed convection-diffusion problems exhibit sharp boundary layers when Peclet number becomes large or the singular perturbation parameter becomes smaller, in such situations, elements with large aspect ratio (anisotropic meshes) are recommended. In this section, we will discuss some important results on anisotropic meshes.

3.1. Notations. Consider an arbitrary triangle $K \in \Gamma_h$ with Q_0Q_1 as the longest edge (see Figure 2). Denote two orthogonal vectors $q_{i,K}$ with length $h_{i,K} = |q_{i,K}|$, $i = 1, 2$, where $q_{1,K}$ is taken along the largest edge Q_0Q_1. From Figure 2, it can be verified that $h_{1,K} \ge h_{2,K}$. Define $h_{\min}^K = h_{2,K}$. These $q_{i,K}$'s correspond to two anisotropic directions. Further, define an orthogonal matrix $C_K = (q_{1,K}, q_{2,K}) \in \mathbb{R}^{2\times 2}$. Let α_K be the scaling factor defined as

$$\alpha_K = \min\left\{b_0^{-1/2}, \varepsilon^{-1/2}\cdot h_{\min}^K\right\}. \tag{19}$$

We represent triangles by K or K' or K_i and its edges by E. Further define its height over edge E as

$$h_{E,K} = 2\cdot\frac{|K|}{|E|}, \tag{20}$$

where $|K|$ represents the area of triangle K. Let w_E be the bounded domain formed by using two triangles having common edge E. Further, define w_K to be the domain consisting of triangle K and its edge neighboring triangles.

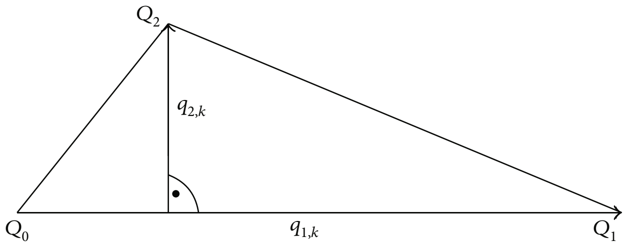

FIGURE 2: Triangle K.

Let

$$\mathrm{Pe}_{w_K} = \max_{K'\subset w_K} \mathrm{Pe}_{K'} \tag{21}$$

be mesh Peclet number on the domain w_K where Pe_K is defined in (7). For an interior edge $E = K_1 \cap K_2$, define parameters $h_E = (h_{E,K_1} + h_{E,K_2})/2$, $h_{\min}^E = (h_{\min}^{K_1} + h_{\min}^{K_2})/2$, and $\alpha_E = (\alpha_{K_1} + \alpha_{K_2})/2$.

For boundary edge $E \subset \partial K \cap \partial\Omega$, we define $h_E = h_{E,K}$, $h_{\min}^E = h_{\min}^K$, and $\alpha_E = \alpha_K$.

Since the mesh considered is assumed to be shape-regular and admissible, along with these requirements, we take

$$h_{i,K} \sim h_{i,K'} \forall K, K' \text{ with } K \cap K' \neq \emptyset, i = 1, 2, \tag{22}$$

and the number of triangles with node y_j is bounded uniformly.

3.2. Interpolation. In order to obtain reliable error upper bounds, we define a suitable matching function [11, 12] to measure the alignment of anisotropic mesh Γ_h and anisotropic function.

Definition 1 (matching function). Let $u \in H^1(\Omega)$ and $\Gamma_h \in F$ be the triangulation of Ω. We define $M_1 : H^1(\Omega) \times F \longrightarrow \mathbb{R}$ by

$$M_1(u, \Gamma_h) := \frac{\left(\sum_{K\in\Gamma_h} \left(h_{\min}^K\right)^{-2} \cdot \left\|C_K^T \nabla u\right\|_K^2\right)^{1/2}}{\|\nabla u\|}, \tag{23}$$

where $C_K \in \mathbb{R}^{2\times 2}$ as defined earlier.

We can easily verify that $M_1(u, \Gamma_h) \sim 1$ for isotropic meshes. Similarly, it can easily be observed that $M_1(u, \Gamma_h) \sim 1$ for anisotropic meshes suitably aligned with anisotropic function u. Therefore, $M_1(u, \Gamma_h) \approx C$ for anisotropic meshes.

To propose reliable error estimates in energy norm, we will use Clément interpolation operator R_C [13] for $u \in H^1(\Omega)$ as standard Lagrange interpolation cannot be defined for these functions.

Lemma 2. *Let $u \in H_0^1(\Omega)$ and α_K be the scaling factor defined by (19). Then, the Clément interpolation operator $R_C : H_0^1(\Omega) \mapsto V_0^h$ satisfies*

$$\begin{aligned} \sum_{K\in\Gamma_h} \alpha_K^{-2} \cdot \|u - R_c u\|_K^2 &\lesssim C^2 |||u|||^2, \\ \varepsilon^{1/2} \sum_{E\subset\Omega\setminus\partial\Omega_D} \alpha_E^{-1} \cdot \|u - R_c u\|_E^2 &\lesssim C^2 |||u|||^2. \end{aligned} \tag{24}$$

Proof. The proof is discussed in [14].

4. Residual Error Estimates

In this section, firstly we discuss exact and approximate residuals. Further, we will develop reliable error upper bounds for Hughes stabilized SUPG finite element solution on anisotropic meshes. It is shown that the error bounds obtained depend on anisotropic interpolation.

4.1. Exact Residuals. We define exact element residual R_K and exact edge residual R_E as

$$\begin{aligned} R_K &= f - (-\varepsilon\Delta v_h + a\cdot\nabla v_h + b v_h) \quad \text{on} \quad K, \\ R_E(x) &= \begin{cases} \varepsilon \cdot \lim\limits_{s\to+0} \left[\partial_{n_E} v_h(x + s n_E) - \partial_{n_E} v_h(x - s n_E)\right] & \text{if} \quad E \subset \Omega\setminus\partial\Omega, \\ g - \varepsilon\partial_n v_h & \text{if} \quad E \subset \partial\Omega_N, \\ 0 & \text{if} \quad E \subset \partial\Omega_D, \end{cases} \end{aligned} \tag{25}$$

where $n_E \perp E \subset \Omega\setminus\partial\Omega$ is the unitary normal vector and $n \perp E \subset \partial\Omega_N$ is the outer unitary normal vector.

4.2. Approximate Residuals. Let Q be the approximation operator used to approximate the element residual and the face residual, i.e.,

$$\begin{aligned} r_K &= Q(R_K) \in P^0(K) \quad \forall K \in \Gamma_h, \\ r_E &= Q(R_E) \in P^0(E) \quad \forall E, \end{aligned} \tag{26}$$

where we have denoted (approximate) element residual by r_K and the (approximate) face residual by r_E. Since the finite element solution v_h is linear, thus

$$r_E = R_E \quad \forall E \subset \Omega\setminus\partial\Omega_N. \tag{27}$$

4.3. Residual Error Estimator. Residual error estimator η_K and the approximation term ζ_K over triangle K are given by

$$\begin{aligned} \eta_K^2 &= a_K^2 \cdot \|r_K\|_K^2 + \varepsilon^{-1/2} \cdot \alpha_K \cdot \sum_{E\subset\partial K\setminus\partial\Omega_D} \|r_E\|_E^2, \\ \zeta_K^2 &= a_K^2 \cdot \|r_K - R_K\|_{w_K}^2 + \varepsilon^{-1/2} \cdot \alpha_K \cdot \sum_{E\subset\partial K\cap\partial\Omega_N} \|r_E - R_E\|_E^2, \end{aligned} \tag{28}$$

where α_K is the scaling factor defined earlier and $a_K^2 = 3\alpha_K^2$. Further, we define global error estimators as

$$\eta^2 = \sum_{K\in\Gamma_h} \eta_K^2,$$
$$\zeta^2 = \sum_{K\in\Gamma_h} \zeta_K^2. \tag{29}$$

Next, we derive reliable upper error bounds on anisotropic meshes.

Theorem 1 (residual error estimation). *Let $v \in H_0^1(\Omega)$ be the exact solution and $v_h \in V_0^h$ be the finite element solution obtained by the proposed scheme. Then, the error in energy norm is bounded above globally by*

$$|||v - v_h||| \lesssim C\left[\sum_{K\in\Gamma_h} 3\alpha_K^2 \left(\|r_K - R_K\|_K^2 + \|r_K\|_K^2\right) + \sum_{E\subset\partial K\backslash\partial\Omega_D} \varepsilon^{-1/2}\alpha_E\left(\|r_E - R_E\|_E^2 + \|r_E\|_E^2\right)\right]^{1/2}. \tag{30}$$

Proof. We know that $\mathbf{B}(v, v) \geq |||v|||^2 \forall v \in H_0^1(\Omega)$.

Using this result, we get

$$|||v - v_h||| \leq \frac{\mathbf{B}(v - v_h, u)}{|||v - v_h|||} = \frac{\mathbf{B}(v - v_h, u)}{|||u|||}, \tag{31}$$

where $u = v - v_h$. Introducing Clément interpolation operator R_C, we can write the bilinear form $\mathbf{B}(.,.)$ as

$$\mathbf{B}(v - v_h, u) = \mathbf{B}(v - v_h, u - R_C u) + \mathbf{B}(v - v_h, R_C u). \tag{32}$$

Now, using the error equation and integration by parts, we have

$$\mathbf{B}(v - v_h, w) = \sum_{K\in\Gamma_h} (R_K, w)_K + \sum_{E\subset\Omega\backslash\partial\Omega_D} (R_E, w)_E \forall w \in H_0^1(\Omega). \tag{33}$$

Using equation (33), the middle term of equation (32) can be written as

$$\mathbf{B}(v - v_h, u - R_C u) = \sum_{K\in\Gamma_h} (R_K, u - R_C u)_K + \sum_{E\subset\Omega\backslash\partial\Omega_D} (R_E, u - R_C u)_E. \tag{34}$$

Using Cauchy Schwarz inequality, we get

$$\sum_{K\in\Gamma_h} (R_K, u - R_C u)_K \leq \left(\sum_{K\in\Gamma_h} \alpha_K^2 \|R_K\|_K^2\right)^{1/2} \cdot \left(\sum_{K\in\Gamma_h} \alpha_K^{-2} \|u - R_C u\|_K^2\right)^{1/2},$$
$$\sum_{E\subset\Omega\backslash\partial\Omega_D} (R_E, u - R_C u)_E \leq \left(\sum_{E\subset\Omega\backslash\partial\Omega_D} \varepsilon^{-1/2}\alpha_E \|R_E\|_E^2\right)^{1/2} \cdot \left(\sum_{E\subset\Omega\backslash\partial\Omega_D} \varepsilon^{1/2}\alpha_E^{-1} \|u - R_C u\|_E^2\right)^{1/2}. \tag{35}$$

Further, using Lemma 2, we get

$$\sum_{K\in\Gamma_h} (R_K, u - R_C u)_K \lesssim \left(\sum_{K\in\Gamma_h} \alpha_K^2 \|R_K\|_K^2\right)^{1/2} C|||u|||,$$
$$\sum_{E\subset\Omega\backslash\partial\Omega_D} (R_E, u - R_C u)_E \lesssim \left(\sum_{E\subset\Omega\backslash\partial\Omega_D} \varepsilon^{-1/2}\alpha_E \|R_E\|_E^2\right)^{1/2} C|||u|||. \tag{36}$$

Therefore, the term $\mathbf{B}(v - v_h, u - R_C u)$ is bounded above by

$$\mathbf{B}(v - v_h, u - R_C u) \lesssim \left(\sum_{K\in\Gamma_h} \alpha_K^2 \|R_K\|_K^2 + \sum_{E\subset\Omega\backslash\partial\Omega_D} \varepsilon^{-1/2}\alpha_E \|R_E\|_E^2\right)^{1/2} C|||u|||. \tag{37}$$

From (18) and (19), we have

$$\|\nabla u_h\|_K \lesssim \min\left\{\left(h_{\min}^K\right)^{-1} b_0^{-1/2}, \varepsilon^{-1/2}\right\} |||u_h|||_K = \left(h_{\min}^K\right)^{-1} \alpha_K |||u_h|||_K \quad \text{for} \quad u_h \in V_0^h. \tag{38}$$

Next, we will find the bounds on the second term $\mathbf{B}(v - v_h, R_C u)$ of equation (32).

$$\begin{aligned}\mathbf{B}(v - v_h, R_C u) &= \langle \varepsilon\nabla v, \nabla R_C u\rangle + \langle a\cdot\nabla v, R_C u\rangle + \langle bv, R_C u\rangle \\ &\quad - \Big\{\langle \varepsilon\nabla v_h, \nabla R_C u\rangle + \langle a\cdot\nabla v_h, R_C u\rangle + \langle bv_h, R_C u\rangle \\ &\quad + \sum_K \rho_K (R_K, a\cdot\nabla R_C u) + \sum_K \sigma_K (R_K, a_h\cdot\nabla R_C u)\Big\}.\end{aligned} \tag{39}$$

Using the standard Galerkin orthogonality condition and standard scaling results, the above equation reduces to

$$\begin{aligned}\mathbf{B}(v-v_h,R_Cu) &= -\sum_K \rho_K(R_K,a\cdot\nabla R_Cu)-\sum_K \sigma_K(R_K,a_h\cdot\nabla R_Cu)\\ &\le \sum_{K\in\Gamma_h}\rho_K\|R_K\|_K\|a\|_{\infty,K}\|\nabla R_Cu\|_K\\ &\quad+\sum_{K\in\Gamma_h}\sigma_K\|R_K\|_K\|a_h\|_{\infty,K}\|\nabla R_Cu\|_K\\ &\le \sum_{K\in\Gamma_h}\rho_K\|R_K\|_K\|a\|_{\infty,K}\left(h^K_{\min}\right)^{-1}\alpha_K|||R_Cu|||_K\\ &\quad+\sum_{K\in\Gamma_h}\sigma_K\|R_K\|_K\|a_h\|_{\infty,K}\left(h^K_{\min}\right)^{-1}\alpha_K|||R_Cu|||_K.\end{aligned} \tag{40}$$

We know that for Clément operator [11]

$$|||R_Cu||| \lesssim C|||u|||\forall u\in H^1_0(\Omega). \tag{41}$$

Thus, we have

$$\begin{aligned}\mathbf{B}(v-v_h,R_Cu) \le{}& \left(\sum_{K\in\Gamma_h}\rho_K^2\|R_K\|_K^2\|a\|^2_{\infty,K}\left(h^K_{\min}\right)^{-2}\alpha_K^2\right)^{1/2}\\ &\cdot C|||u|||+\left(\sum_{K\in\Gamma_h}\sigma_K^2\|R_K\|_K^2\|a_h\|^2_{\infty,K}\left(h^K_{\min}\right)^{-2}\alpha_K^2\right)^{1/2}C|||u|||.\end{aligned} \tag{42}$$

It may be noted that the effect of nonlinear term in the L_∞ norm will be bounded by that of the term $\|a\|_\infty$ as shown below, i.e.,

$$\begin{aligned} a_h &= \frac{(a\cdot\nabla u_h)\nabla u_h}{|\nabla u_h|^2}, |\nabla u_h|\ne 0,\\ a_h &\le \frac{\|a\|\|\nabla u_h\|\nabla u_h}{\|\nabla u_h\|^2}\{\text{Using Cauchy} - \text{Schwarz inequality}\},\\ \|a_h\| &\le \frac{\|a\|\|\nabla u_h\|\|\nabla u_h\|}{\|\nabla u_h\|^2},\\ &\le \|a\|.\end{aligned} \tag{43}$$

Since a, the convection coefficient, is assumed to be smooth in the domain under consideration, it is bounded above. Hence, the nonlinear term $\|a_h\|_{\infty,K}$ is taken as bounded above by some constant and is absorbed in the constant term.

We know that

$$\begin{aligned}\rho_K &\lesssim h^K_{\min}/\|a\|_{\infty,K}\forall K\in\Gamma_h,\\ \sigma_K &\lesssim h^K_{\min}/\|a_h\|_{\infty,K}\forall K\in\Gamma_h.\end{aligned} \tag{44}$$

Therefore,

$$\begin{aligned}\mathbf{B}(v-v_h,R_Cu) \lesssim{}& \left(\sum_{K\in\Gamma_h}\alpha_K^2\|R_K\|_K^2+\sum_{K\in\Gamma_h}\alpha_K^2\|R_K\|_K^2\right)^{1/2}\\ &\cdot C|||u|||\cdot\{M_1(u,\Gamma_h)\approx C\}.\end{aligned} \tag{45}$$

Since

$$|||v-v_h|||\le\frac{B(v-v_h,u)}{|||u|||}, \tag{46}$$

using equation (45) and equation (37) in equation (32), we get

$$|||v-v_h|||\lesssim\left(\sum_{K\in\Gamma_h}3\alpha_K^2\|R_K\|_K^2+\sum_{E\subset\Omega\backslash\partial\Omega_D}\varepsilon^{-1/2}\alpha_E\|R_E\|_E^2\right)^{1/2}C. \tag{47}$$

Using triangle inequalities,

$$\begin{aligned}\|R_K\|_K^2 &\le \|r_K-R_K\|_K^2+\|r_K\|_K^2,\\ \|R_E\|_E^2 &\le \|r_E-R_E\|_E^2+\|r_E\|_E^2,\end{aligned} \tag{48}$$

we get

$$\begin{aligned}|||v-v_h|||\lesssim C\Bigg[&\sum_{K\in\Gamma_h}3\alpha_K^2\left(\|r_K-R_K\|_K^2+\|r_K\|_K^2\right)\\ &+\sum_{E\subset\partial K\backslash\partial\Omega_D}\varepsilon^{-1/2}\alpha_E\left(\|r_E-R_E\|_E^2+\|r_E\|_E^2\right)\Bigg]^{1/2}.\end{aligned} \tag{49}$$

5. Adaptive Refinement Algorithm

In this section, we propose an adaptive refinement strategy based on the a posteriori error estimates obtained in the last section. We propose the following adaptive refinement algorithm:

(1) Discretize the domain using triangular elements. Triangulation is being carried out using red refinement

(2) Solve the problem using the proposed scheme described in Section 2

(3) Over each element K, the residual error estimates η_K have been calculated as defined in Section 4

(4) Mark the elements $\{K_{e_i}\}_{e_i=1}^{M}$ satisfying $\eta_{K_{e_i}} > C \max_{L'} \eta_{L}{}'$, where C is a user chosen constant from $(0, 1)$, for refinement

(5) Refine these marked elements using green refinement procedure

(6) Refine all those elements having hanging nodes also to avoid the discontinuity in the solution

(7) Solve the problem again on the new adapted mesh

(8) Repeat the process of grid refinement until the solution has been obtained up to a given desired accuracy

6. Numerical Results

In this section, numerical experiments have been carried out in order to test the efficiency and robustness of the proposed adaptive refinement technique based on the derived error estimates.

Example 4. Consider the following singularly perturbed convection-diffusion problem:

$$-\nabla \cdot \varepsilon \nabla u + 2u_x + 3u_y + u = f \text{ in } \Omega = (0,1)^2, \qquad u = 0 \text{ on } \partial\Omega. \tag{50}$$

The right-hand side function f is so chosen to satisfy the exact solution

$$u = \sin(x)\left(1 - e^{-2(1-x)/\varepsilon}\right)y^2\left(1 - e^{-3(1-y)/\varepsilon}\right). \tag{51}$$

The solution of the above problem exhibits exponential boundary layers along the lines $x = 1$ and $y = 1$. For adaptive refinement, an anisotropic triangular mesh has been taken into consideration. In Figures 3(a) and 3(b), we present a portion of adaptive triangular mesh for $\varepsilon = 2^{-3}$ with different degrees of freedom. Figures 4(a) and 4(b) present adaptive refined meshes for $\varepsilon = 2^{-5}$ with different degrees of freedoms.

In Figures 5 and 6, the numerical solution obtained using the proposed refinement algorithm for different values of the singular perturbation parameter $\varepsilon = 2^{-3}$ and $\varepsilon = 2^{-5}$ has been plotted. It can be easily seen that even very sharp boundary layers have been efficiently captured using the proposed refinement algorithm. From the solutions, it can also be observed that the problem is very sensitive to the singular perturbation parameter, i.e., even for $\varepsilon = 2^{-5}$, very sharp boundary layers appear in the solution. In Figure 7, energy norm errors for $\varepsilon = 2^{-5}$ have been presented. The behavior of effectivity index $\psi = |||v - v_h|||/\text{ErrorEstimator}$ which is used to measure reliability of the estimator is shown in Figure 8.

7. Conclusion

In the presented work, an adaptive numerical technique has been proposed for singularly perturbed convection-diffusion problems in two dimensions. The singularly perturbed problem under consideration has been solved using Hughes stabilization technique under SUPG finite element framework. Anisotropic meshes have been considered for the domain discretization. Reliable a posteriori error estimates have been developed in energy norm on anisotropic meshes. Based on these a posteriori error estimates, an

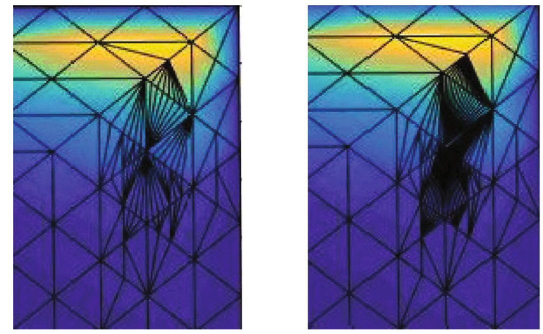

Figure 3: Portion of adaptive mesh for $\varepsilon = 2^{-3}$ with DOF = 187,224.

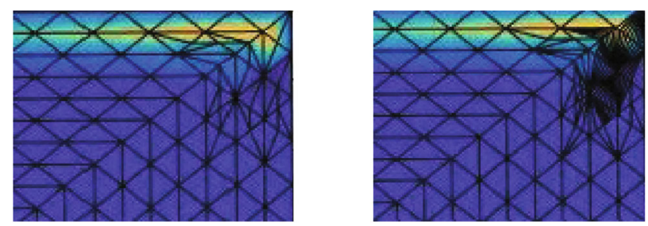

Figure 4: Portion of adaptive mesh for $\varepsilon = 2^{-5}$ with DOF = 559,587.

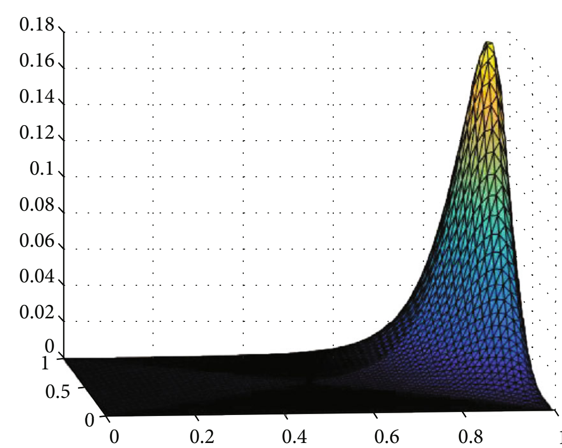

Figure 5: Numerical solution profile for $\varepsilon = 2^{-3}$.

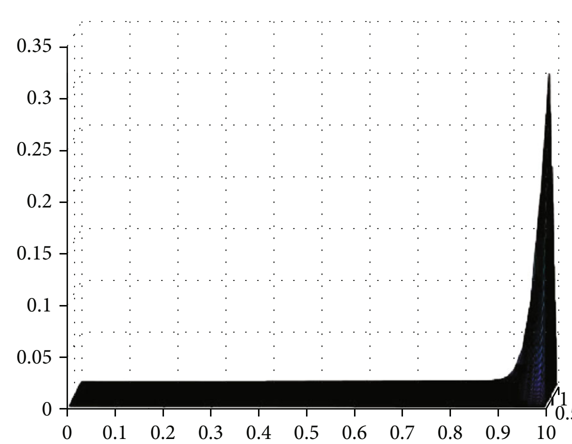

Figure 6: Numerical solution profile for $\varepsilon = 2^{-5}$.

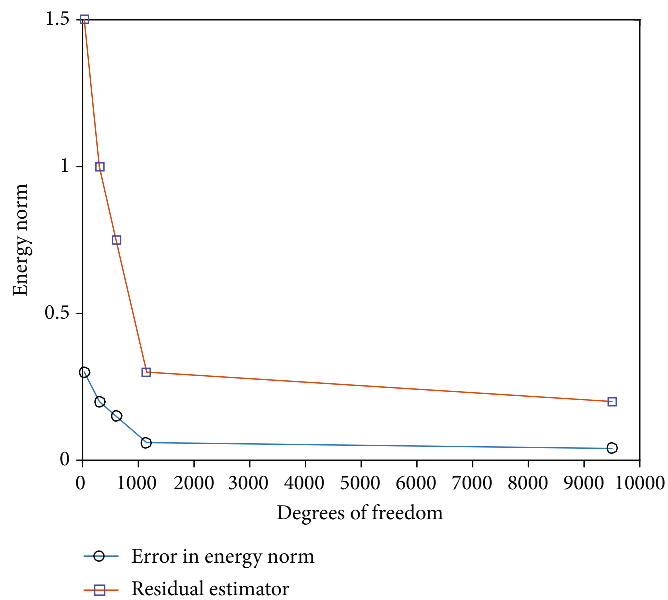

FIGURE 7: Error $|||v - v_h|||$ and error estimator for $\varepsilon = 2^{-5}$.

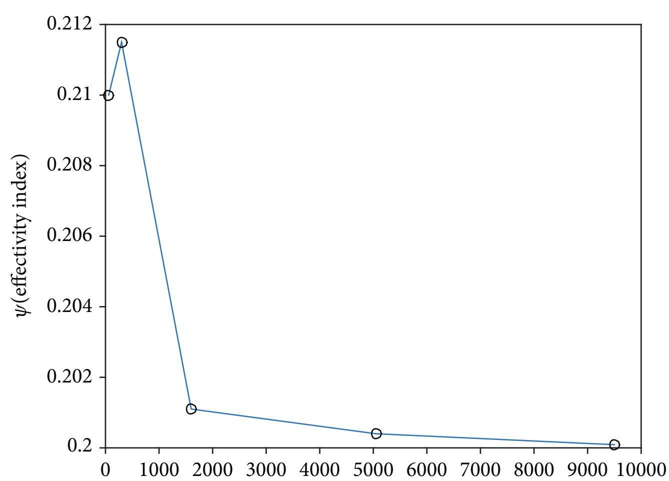

FIGURE 8: ψ (effectivity index) for $\varepsilon = 2^{-5}$.

adaptive mesh refinement strategy has been proposed. It has been depicted through numerical experiments that the proposed adaptive refinement strategy is very much efficient in capturing sharp boundary layers as the singular perturbation parameter ε approaches to 0.

Data Availability

There is no data were used to support this study

Conflicts of Interest

The authors declare that they have no competing interest.

Authors' Contributions

All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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