Hindawi
Advances in Mathematical Physics
Volume 2020, Article ID 1825235, 14 pages
https://doi.org/10.1155/2020/1825235

Research Article

Inverse Source Problem for a Multiterm Time-Fractional Diffusion Equation with Nonhomogeneous Boundary Condition

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Received 7 July 2020; Revised 29 August 2020; Accepted 2 September 2020; Published 19 September 2020

Academic Editor: Nicola L. Rizzi

This paper is devoted to identify a space-dependent source function in a multiterm time-fractional diffusion equation with nonhomogeneous boundary condition from a part of noisy boundary data. The well-posedness of a weak solution for the corresponding direct problem is proved by the variational method. We firstly investigate the uniqueness of an inverse initial problem by the analytic continuation technique and the Laplace transformation. Then, the uniqueness of the inverse source problem is derived by employing the fractional Duhamel principle. The inverse problem is solved by the Levenberg-Marquardt regularization method, and an approximate source function is found. Numerical examples are provided to show the effectiveness of the proposed method in one- and two-dimensional cases.

1. Introduction

It is well known that the standard diffusion equation has been used to describe the Gaussian process of particle motion. Time-fractional diffusion equations (TFDEs in short) are deduced by replacing the standard time derivative with a time-fractional derivative and can be used to describe anomalous diffusion phenomena. Anomalous diffusion deviates from the standard Fickian description of Brownian motion, the main character of which is that its mean squared displacement is a nonlinear growth with respect to time, such as $\langle x^2(t)\rangle \sim t^\alpha$. However, as the research problems become more and more complex, the differential order of a time-fractional diffusion equation is no longer limited to a fixed number but distributed over the unit interval. This creates TFDEs of distributed order of which a particular case is the multiterm time-fractional diffusion equations (MTFDEs in short).

Direct problems, i.e., initial value problems and initial boundary value problems for MTFDEs, have attracted much more attentions in recent years. For example, the maximum principle in [1, 2] and the well-posedness and long-time asymptotic behavior in [3] for general multiterm time-fractional diffusion equations are investigated. The numerical solutions are shown in [4, 5] by the high-order space-time spectral method and the Galerkin finite element method.

However, in most cases, the parameters that characterize the diffusion process cannot be measured directly or easily. The inverse source problem of diffusion process is aimed at detecting the source function of a physical field from some indirect measurements (such as final time information or boundary measurement) and is of great importance in engineering. There are many ripe regularization theories on the inverse source problems at present, e.g., the reproducing kernel Hilbert space method [6], the Fourier truncation method [7], the Tikhonov regularization method [8], and the modified quasiboundary value method and quasireversibility method [9, 10].

In this paper, we investigate an inverse space-dependent source problem in a multiterm time-fractional diffusion equation with nonhomogeneous boundary condition. Let Ω be a bounded domain in $\mathbb{R}^d$ with sufficient smooth boundary $\partial\Omega$. For a fixed positive integer m, let q_j and $\alpha = \{\alpha_j : j = 1, 2,\cdots,m\}$ be the positive constants such that $0 < \alpha_m < \cdots < \alpha_1$

< 1. Consider the following initial boundary value problem (IBVP) for a time-fractional diffusion equation

$$\begin{cases} \sum_{j=1}^{m} q_j \partial_{0+}^{\alpha_j} u(x,t) - \Delta u(x,t) = f(x) r(t), \quad (x,t) \in \Omega_T := \Omega \times I, \\ u(x,0) = \phi(x), x \in \bar{\Omega}, \\ \frac{\partial u}{\partial n}(x,t) = b(x,t), \quad (x,t) \in \partial\Omega \times I, \end{cases} \tag{1}$$

where $I = (0, T)$ and n is the unitary outer normal vector of $\partial\Omega$. We can assume $q_1 = 1$ without loss of generality.

Here, $\partial_{0+}^{\alpha_j}$ denotes the Caputo derivative defined by

$$\partial_{0+}^{\alpha_j} u(x,t) = \frac{1}{\Gamma(1-\alpha_j)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^{\alpha_j}}, \quad t > 0, \tag{2}$$

where $\Gamma(\cdot)$ is the Gamma function.

If all functions $f(x)$, $r(t)$, $\phi(x)$, and $b(x,t)$, and the parameters m, q_j and α_j are given appropriately, problem (1) is a direct problem. The inverse problem here is to determine the source term $f(x)$ based on problem (1) and additional data

$$u(x,t)|_\Gamma = h(x,t), \quad t \in I, \tag{3}$$

where Γ is a nonempty open part of $\partial\Omega$.

To our best knowledge, it has made significant progress in determining the source problem for (1) with part of boundary data. For a single term (i.e. $m = 1$), Wei et al. identified a time-dependent source term in a multidimensional TFDE from boundary Cauchy data. Zhang et al. [11] proved a uniqueness result to inverse the space-dependent source term in one-dimensional case by using one-point Cauchy data and provided an efficient numerical method. For a general area of high dimension, Wei et al. [12] proved the uniqueness of inverse space-dependent source but no numerical computation. Yan et al. [13] studied an inverse spatial-dependent source problem by noisy boundary data in a time-fractional diffusion-wave equation and carried out the numerical inversion experiments by a nonstationary iterative Tikhonov regularization method. For a multiterm case, Jiang et al. [14] established a weak unique continuation property for time-fractional diffusion-advection equations and studied an inverse problem on determining the spatial component in the source term by interior measurements. Very recently, Li et al. [15] investigated an inverse time-dependent source term problem in a MTFDE from the boundary Cauchy data and employed the conjugate gradient method to find the approximate source term.

Nevertheless, as far as the authors know, the most of the existing literature only treated the inverse source problems with homogeneous boundary condition, in which the series expression provides convenience for the argument. The case of nonhomogeneous boundary is also significant from the practical point of view but difficult whether the forward problem or the inverse problem. Although the weak unique continuation property in [14] is also valid for the boundary measurements only taking a zero extension, however, it does not hold true for the nonhomogeneous boundary case. So far, there is no publication on inversion space-dependent source term by the Cauchy data in a MTFDE with nonhomogeneous boundary condition.

Generally speaking, the problems of recovering spatial information from data along a time trace are notoriously severely ill posed. In this paper, we focus on an inverse space-dependent source problem of (1) from the Cauchy data in a general domain. This is an extension and improvement of [12], which dealt with an inverse space-dependent source problem by the boundary measured data in an infinite time interval for a single-term equation with homogeneous boundary. In this study, however, only part of the boundary Cauchy data in a finite time interval $t \in (0, T)(T<\infty)$ is enough to obtain the uniqueness of inverse problem. We firstly obtain the well-posedness for the direct problem by the variational method for studying the inverse problem. For inverse problem, roughly speaking, we first transfer the original forward problems into three IBVPs by the superposition principle and turn the inverse source problem into an inverse initial problem by employing the fractional Duhamel principle. Then, we prove the uniqueness of the inverse initial problem by the analytic continuation technique and the Laplace transformation. Finally, we employ the Levenberg-Marquardt regularization method to solve numerically the inverse source problem. Here, we point out that the Levenberg-Marquardt method is modified to a more efficient algorithm without bringing in a differential step in the present paper because of the linearity of the operator. The numerical results for three examples in one- and two-dimensional cases are provided, and the numerical implementation shows the effectiveness and robustness of the proposed methods.

The main result in this paper is the following uniqueness result for the inverse space-dependent source problem.

Theorem 1. *Assume that $f \in \mathcal{D}((-\Delta+1)^\gamma)$ for $\gamma > \max\{0, d/2-1\}$ and $r \in C^1[0,T]$ with $r(0) \neq 0$, the initial function $\phi \in H^2(\Omega)$, and the boundary function $b \in L^2(\partial\Omega \times (0,T))$. Suppose $u_i(x,t), (i=1,2)$ is the solutions of (1) with $f = f_i$, then $u_1(x,t) = u_2(x,t)$ for $(x,t) \in \Gamma \times I$ implies $f_1(x) = f_2(x)$.*

The remainder of this paper is organized as follows. Some preliminaries are presented in Section 2. The well-posedness for the direct problem is proved in Section 3. In Section 4, we present the uniqueness result of inverse space-dependent source problem. In Section 5, we use the Levenberg-Marquardt regularization method to find the approximate space-dependent source function. Numerical results for three examples are provided to illustrate the efficiency of our method in Section 6. Finally, we give a brief conclusion in Section 7.

2. Preliminaries

We firstly introduce some preliminaries as follows in this section.

Definition 2 (see [16, 17]). If $f(t) \in L^1(0, T)$, then for $\alpha > 0$, the Riemann-Liouville fractional left-sided integral $I_{0+}^{\alpha} f$ and right-sided integral $I_{T-}^{\alpha} f$ are defined by

$$\begin{aligned} I_{0+}^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)ds}{(t-s)^{1-\alpha}}, \quad 0 < t \le T, \\ I_{T-}^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_t^T \frac{f(s)ds}{(s-t)^{1-\alpha}}, \quad 0 \le t < T. \end{aligned} \tag{4}$$

Definition 3. (see [16, 17]). Let $z(t) \in AC(0, T)$, then for $0 < \alpha < 1$, the Riemann-Liouville fractional left-sided derivative $D_{0+}^{\alpha} y(t)$ and right-sided derivative $D_{T-}^{\alpha} y(t)$ of order α are defined by

$$\begin{aligned} D_{0+}^{\alpha} y(t) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{y(s)ds}{(t-s)^{\alpha}} =: \frac{d}{dt} \left(I_{0+}^{1-\alpha} y\right)(t), \quad 0 < t \le T, \\ D_{T-}^{\alpha} y(t) &= -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T \frac{y(s)ds}{(s-t)^{\alpha}} =: -\frac{d}{dt} \left(I_{T-}^{1-\alpha} y\right)(t), \quad 0 \le t < T. \end{aligned} \tag{5}$$

Definition 4 (see [16, 18]). The multinomial Mittag-Leffler function is defined by

$$E_{(\theta_1,\cdots,\theta_m),\theta_0}(z_1,\cdots,z_m) := \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_m=k} \frac{(k\,;k_1,\cdots,k_m)\Pi_{j=1}^m z_j^{k_j}}{\Gamma\left(\theta_0 + \sum_{j=1}^m \theta_j k_j\right)}, \tag{6}$$

where $\theta_0, \theta_j \in \mathbb{R}$, and $z_j \in \mathbb{C}(j = 1,\cdots,m)$, and $(k\,;k_1,\cdots,k_m)$ denotes the multinomial coefficient

$$(k\,;k_1,\cdots,k_m) := \frac{k!}{k_1!\cdots k_m!} \text{ with } k = \sum_{j=1}^m k_j. \tag{7}$$

For later use, we adopt the abbreviation

$$E_{\alpha',\beta}^{(\lambda_n)}(t) := E_{(\alpha_1,\alpha_1-\alpha_2,\cdots,\alpha_1-\alpha_m),\beta}(-\lambda_n t^{\alpha_1}, -q_2 t^{\alpha_1-\alpha_2}, \cdots, -q_m t^{\alpha_1-\alpha_m}), \quad t > 0. \tag{8}$$

We have the following three results on the multinomial Mittag-Leffler function [3].

Lemma 5. *Let $\theta_0 > 0$, $0 < \theta_j < 1$ $(j = 1,\cdots,m)$ and $z_j \in \mathbb{C}(j = 1,\cdots,m)$ be fixed.*

Then,

$$\frac{1}{\Gamma(\theta_0)} + \sum_{j=1}^m z_j E_{(\theta_1,\cdots,\theta_m),\theta_0+\theta_j}(z_1,\cdots,z_m) = E_{(\theta_1,\cdots,\theta_m),\theta_0}(z_1,\cdots,z_m). \tag{9}$$

Lemma 6. *Let $\theta_0 > 0, 1 > \alpha_1 > \cdots > \alpha_m > 0$ be given. Assume that $\alpha_1\pi/2 < \mu < \alpha_1\pi$, $\mu \le |\arg z_1| \le \pi$ and $\pi - \epsilon \le |\arg z_j| \le \pi$, $j = 2, \cdots, m$ with ϵ small enough. Then, there exists a constant $C > 0$ depending only μ, K, $\alpha_j (j=1, \cdots, m)$ and β such that*

$$\left|E_{(\alpha_1,\alpha_1-\alpha_2,\cdots,\alpha_1-\alpha_m),\beta}(z_1,\cdots,z_m)\right| \le \frac{C}{1+|z_1|}. \tag{10}$$

Lemma 7. *Let $1 > \alpha_1 > \cdots > \alpha_m > 0$. Then,*

$$\frac{d}{dt}\left\{t^{\alpha_1} E_{\alpha',1+\alpha_1}^{(\lambda_n)}(t)\right\} = t^{\alpha_1-1} E_{\alpha',\alpha_1}^{(\lambda_n)}(t), \quad t > 0. \tag{11}$$

Lemma 8. *Assume $r \in L^{\infty}(0, T)$ denote*

$$r_n(t) = \int_0^t r(s)(t-s)^{\alpha_1-1} E_{\alpha',\alpha_1}^{(\lambda_n)}(t-s)ds, t \in (0, T], n = 1, 2, \cdots, \tag{12}$$

then $r_n(t) \in C[0, T]$ and $r_n(0) = 0$.

Proof. Let $e_n(t) = t^{\alpha_1-1} E_{\alpha',\alpha_1}^{(\lambda_n)}(t)$. Then, $r_n(t) = r * e_n(t)$. By Lemma 6, we have

$$\|e_n\|_{L^1(0,T)} = \int_0^T \left|t^{\alpha_1-1} E_{\alpha',\alpha_1}^{(\lambda_n)}(t)\right| dt \le C \int_0^T t^{\alpha_1-1} dt < \infty. \tag{13}$$

Therefore, we arrive at $e_n \in L^1(0, T)$.

Using the Young theorem, we obtain $r_n(t) \in C[0, T]$. Combining Lemma 7, we have

$$|r_n(0)| = \lim_{t\to 0} r_n(t) \le \lim_{t\to 0} \|r\|_{L^{\infty}(0,t)} t^{\alpha_1} E_{\alpha',1+\alpha_1}^{(\lambda_n)}(t), \quad \forall t \in [0, T]. \tag{14}$$

Thus, we have $r_n(0) = 0$.

Lemma 9. *Denote $a_n(t) = 1 - \lambda_n t^{\alpha_1} E_{\alpha',1+\alpha_1}^{(\lambda_n)}(t)$, $t \in (0,+\infty)$, $n = 1, 2, \cdots$, then*

$$\mathscr{L}(a_n)(s) = \frac{\sum_{j=1}^m q_j s^{\alpha_j-1}}{\sum_{j=1}^m q_j s^{\alpha_j} + \lambda_n}, \quad s \in \mathbb{C} \text{ with } \operatorname{Re} s > 0, \tag{15}$$

where $\mathscr{L}(a_n)$ denotes the Laplace transform of function $a_n(t)$.

Proof. First, we can take the Laplace transforms termwise for $\int_0^{\infty} e^{-zt} \lambda_n t^{\alpha_1} E_{\alpha',1+\alpha_1}^{(\lambda_n)}(t)dt$ with $\operatorname{Re} z > \lambda_n^{1/\alpha_1}$ by its definition (8). Since $\lambda_n t^{\alpha_1} E_{\alpha',1+\alpha_1}^{(\lambda_n)}(t) \le C$ by Lemma 6, we see that $\int_0^{\infty} e^{-zt} \lambda_n t^{\alpha_1} E_{\alpha',1+\alpha_1}^{(\lambda_n)}(t)$ is analytic with respect to z in $\operatorname{Re} z > 0$. Therefore, the analytic continuation can be done for $\operatorname{Re} z > 0$. By Theorem 4.1 in [18], $a_n(t)$ satisfies an initial value

problem for a fractional ODE

$$\begin{cases} \sum_{j=1}^{m} q_j \partial_{0+}^{\alpha_j} a_n(t) + \lambda_n a_n(t) = 0, \quad t > 0, \\ a_n(0) = 1. \end{cases} \tag{16}$$

Taking the Laplace transform to (16) and using the formula

$$\mathscr{L}(\partial_{0+}^{\alpha} f)(s) = s^{\alpha} \mathscr{L}(f)(s) = s^{\alpha-1} f(0+), \tag{17}$$

we obtain

$$\mathscr{L}(a_n)(s) = \frac{\sum_{j=1}^{m} q_j s^{\alpha_j - 1}}{\sum_{j=1}^{m} q_j s^{\alpha_j} + \lambda_n}. \tag{18}$$

3. Existence and Uniqueness of a Weak Solution for the Direct Problem

In this section, we will obtain the existence and uniqueness of a weak solution for direct problem (1). First, we define a Hilbert scale space $D((-\Delta+1)^{\gamma})$ in $L^2(\Omega)$. Denote the eigenvalues of $-\Delta$ with homogeneous Neumann boundary condition as λ_n and the corresponding eigenfunctions as $\varphi_n \in \{\psi \in H^2(\Omega); \partial\psi/\partial n|_{\partial\Omega} = 0\}$ such that $\Delta\varphi_n = -\lambda_n \varphi_n$.

We know that $0 = \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \le \lambda_n \le \cdots$ counting according to the multiplicities, and $\{\varphi_n\}$ is an orthonormal basis in $L^2(\Omega)$. Define the Hilbert scale space

$$\mathscr{D}((-\Delta+1)^{\gamma}) = \left\{ \psi \in L^2(\Omega); \sum_{n=1}^{\infty} (\lambda_n + 1)^{2\gamma} |(\psi, \varphi_n)|^2 < \infty \right\}, \tag{19}$$

where $(\cdot, \cdot)$ is the inner product in $L^2(\Omega)$ and define its norm

$$\|\psi\|_{\mathscr{D}((-\Delta+1)^{\gamma})} = \left(\sum_{n=1}^{\infty} (\lambda_n + 1)^{2\gamma} |(\psi, \varphi_n)|^2 \right)^{1/2}. \tag{20}$$

Here, one points out that the number 1 in the operator $-\Delta + 1$ is not necessary. It may be any nonzero positive number.

In the following, we assume that the initial data $\phi(x) \in H^2(\Omega)$, the boundary data $b(x,t) \epsilon L^2(\partial\Omega \times I)$, and the source functions, $f(x) \epsilon L^2(\Omega) r(t) \in L^2(0, T)$. We give a weak formulation for direct problem (1) and prove that its solution exists uniquely.

Let $u(x,t)$ be the smooth solution of (1). Denote $v(x,t) = u(x,t) - \phi(x)$, then v solves the following IBVP

$$\begin{cases} \sum_{j=1}^{m} q_j \partial_{0+}^{\alpha_j} v(x,t) - \Delta v(x,t) = \bar{f}(x,t), \quad (x,t) \in \Omega_T, \\ v(x,0) = 0, \quad x \in \bar{\Omega}, \\ \dfrac{\partial v}{\partial n}(x,t) = b - \dfrac{\partial \phi}{\partial n}, \quad (x,t) \in \partial\Omega \times I, \end{cases} \tag{21}$$

where $\bar{f}(x,t) = f(x) r(t) + \Delta\phi(x)$.

Let $H^{\alpha_1}(0, T)$ be the fractional Sobolev space on the interval $(0,T)$ (see, e.g., Adams [19]).

We set

$${}_0H^{a_1}(0, T) = \{u \in H^{a_1}(0, T) \colon u(0) = 0\}, \tag{22}$$

if $1/2 < \alpha_1 \le 1$ and we identify ${}_0H^{a_1}(0, T)$ with $H^{\alpha_1}(0, T)$ for $0 < \alpha_1 < 1/2$. For $0 < \alpha_1 < 1$, we define a space

$$B^{\alpha_1}(\Omega_T) := {}_0H^{\alpha_1}\left(0, T; L^2(\Omega)\right) \cap L^2\left(0, T; H^1(\Omega)\right), \tag{23}$$

equipped with the norm

$$\|u\|_{B^{\alpha_1}(\Omega_T)} = \left\{ \|u\|^2_{H^{\alpha_1}\left(0,T;L^2(\Omega)\right)} + \|u\|^2_{L^2\left(0,T;H^1(\Omega)\right)} \right\}^{1/2}. \tag{24}$$

Referring to [20], we know that $B^{\alpha_1}(\Omega_T)$ is a Hilbert space, and for $\alpha_1 \in (0, 1/2)$, we have $\|u\|^2_{H^{\alpha}(0,T;L^2(\Omega))} \cong \|u\|^2_{L^2(0,T;L^2(\Omega))} + \|D^{\alpha}_{0+} u\|^2_{L^2(0,T;L^2(\Omega))} \cong \|u\|^2_{L^2(0,T;L^2(\Omega))} + \|D^{\alpha}_{T-} u\|^2_{L^2(0,T;L^2(\Omega))}$, where $D^{\alpha}_{0+} u$ and $D^{\alpha}_{T-} u$ denote the Riemann-Liouville fractional left derivative and right derivative of order α (see Definition 3). Based on [20–22], we can deduce a weak formulation for problem (21) as follows.

Find $v \in B^{\alpha_1/2}(\Omega_T)$ such that

$$A(v, w) = F(w), \forall w \in B^{\alpha_1/2}(\Omega_T), \tag{25}$$

where

$$\begin{aligned} A(v, w) &:= \sum_{j=1}^{m} \left(q_j D^{\alpha_j/2}_{0+} v, D^{\alpha_j/2}_{T-} w \right)_{\Omega_T} + (\nabla v, \nabla w)_{\Omega_T}, \\ F(w) &:= \left(\bar{f}, w\right)_{\Omega_T} + \left(b - \frac{\partial \phi}{\partial n}, w \right)_{\partial\Omega \times I}. \end{aligned} \tag{26}$$

Now, we could obtain the following theorem.

Theorem 10. *Let $f \in L^2(\Omega)$, $r(t) \in L^2(0, T)$, and $\phi \in H^2(\Omega)$. Then, there exists a unique solution $v(x,t)$ to (25) and the solution satisfies*

$$\|v\|_{B^{\alpha_1/2}(\Omega_T)} \le C\left(\alpha_1, T, q_j\right)\left(\|f\|_{L^2(\Omega)}\|r\|_{L^2(0,T)} + \|\phi\|_{H^2(\Omega)} + \|b\|_{L^2(\partial\Omega\times I)}\right), \tag{27}$$

where $C(\alpha_1, T, q_j) > 0$ is a constant independent of v.

Proof. The well-posed of problem (25) is guaranteed by the well-known Lax-Milgram theorem. The continuity of the functional F is obvious. Here, we give the continuity and coercivity of the bilinear form A. By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}|A(v,w)| &\le \sum_{j=1}^{m} q_j \left\|D_{0+}^{\alpha_j/2} v\right\|_{L^2(\Omega_T)} \left\|D_{T-}^{\alpha_j/2} w\right\|_{L^2(\Omega_T)} + \|\nabla v\|_{L^2(\Omega_T)}\|\nabla w\|_{L^2(\Omega_T)} \\ &\le Cm \max_{1\le j\le m}\left\{q_j\right\}\|v\|_{B^{\alpha_1/2}(\Omega_T)}\|w\|_{B^{\alpha_1/2}(\Omega_T)} + \|v\|_{B^{\alpha_1/2}(\Omega_T)}\|w\|_{B^{\alpha_1/2}(\Omega_T)} \\ &\le \left(1 + Cm \max_{1\le j\le m}\left\{q_j\right\}\right)\|v\|_{B^{\alpha_1/2}(\Omega_T)}\|w\|_{B^{\alpha_1/2}(\Omega_T)}.\end{aligned} \tag{28}$$

On the other hand, we arrive at

$$\begin{aligned}A(w,w) &\ge \sum_{j=1}^{m} q_j \cos\left(\pi\alpha_j/2\right)\left\|D_{0+}^{\alpha_1/2} w\right\|^2_{L^2(\Omega_T)} + \|\nabla w\|^2_{L^2(\Omega_T)} \\ &\ge \frac{\cos\left(\pi\alpha_1/2\right)\left\|D_{0+}^{\alpha_1/2} w\right\|^2_{L^2(\Omega_T)}}{2} + \frac{\cos\left(\pi\alpha_1/2\right)\|w\|^2_{L^2(\Omega_T)}}{2} + \|\nabla w\|^2_{L^2(\Omega_T)} \\ &\ge \frac{c\cos\left(\pi\alpha_1/2\right)}{2}\|w\|^2_{B^{\alpha_1/2}(\Omega_T)},\end{aligned} \tag{29}$$

where we use Lemma 2.2 of [20] in the first inequality and the embedding theorem in the second inequality. In order to obtain the stability of (27), we can take $w = v$ in (25) and use (29) to obtain

$$\|v\|_{B^{\alpha_1/2}(\Omega_T)} \le \frac{C}{\cos\left(\pi\alpha_1/2\right)}\left(\|f\|_{L^2(\Omega)}\|r\|_{L^2(0,T)} + \|\phi\|_{H^2(\Omega)} + \|b\|_{L^2(\partial\Omega\times I)}\right). \tag{30}$$

4. Uniqueness for the Inverse Problem

In order to prove our main result, we firstly introduce the following famous fractional Duhamel principle.

Assume u solves equation (1) with $\phi = 0$ and $b = 0$, and let v be the solution of the following problem

$$\begin{cases} \displaystyle\sum_{j=1}^{m} q_j \partial_{0+}^{\alpha_j} v(x,t) - \Delta v(x,t) = 0, & (x,t) \in \Omega_T, \\ v(x,0) = f(x), & x \in \bar{\Omega}, \\ \dfrac{\partial v}{\partial n}(x,t) = 0, & (x,t) \in \partial\Omega \times I. \end{cases} \tag{31}$$

Lemma 11. (see Lemma 4.2 of [2]). *Let u be the solution to (1) with $\phi = 0$ and $b = 0$, where $r \in C^1[0,T]$ and $f \in \mathcal{D}\left((-\Delta+1)^{\epsilon}\right)$ with some $\epsilon > 0$. Then, u allows the representation*

$$u(\cdot,t) = \int_0^t \mu(t-s) v_f(\cdot,s)\,ds,\ 0 < t \le T, \tag{32}$$

where v_f solves homogeneous problem (31) with f as the initial data, and μ satisfies

$$\sum_{j=1}^{m} q_j I_{0+}^{1-\alpha_j}\mu(t) = r(t), \quad 0 < t \le T. \tag{33}$$

Moreover, there exists a unique $\mu \in L^1(0,T)$ satisfying (33), and there is a constant $C > 0$ independent of t such that

$$|\mu(t)| \le Ct^{\alpha_1 - 1}\ a.e. t \in (0,T). \tag{34}$$

Based on the method of separation of variables, we could obtain a formal solution to the homogeneous boundary problem of (1) given by

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n\varphi_n(x) + \sum_{n=1}^{\infty} r_n(t) f_n \varphi_n(x), \tag{35}$$

where $a_n(t) = (1 - \lambda_n t^{\alpha_1} E^{(\lambda_n)}_{\alpha',1+\alpha_1}(t))$, $r_n(t) = \int_0^t r(s)(t-s)^{\alpha_1-1} E^{(\lambda_n)}_{\alpha',1+\alpha_1}(t-s)ds$, $\phi_n = (\phi, \varphi_n)$ *and* $f_n = (f, \varphi_n)$.

Now, we proceed to the proof of the uniqueness of the inverse initial problem for the homogeneous initial-boundary value problem (31).

Theorem 12. *Assume that f and $\tilde{f} \in \mathcal{D}\left((-\Delta+1)^{\gamma}\right)$ for $\gamma > \max\{0, d/2 - 1\}$. Suppose $v(x,t)$ and $\tilde{v}(x,t)$ are the weak solutions of (31) with respect to initial conditions f and $\tilde{f}$, respectively, then $v(x,t) = \tilde{v}(x,t)$ for $(x,t) \in \Gamma \times I$ implies $f(x) = \tilde{f}(x)$ in $L^2(\Omega)$.*

Proof. By (35), the expressions $v(x,t)$ and $\tilde{v}(x,t)$ are given by

$$v(x,t) = \sum_{n=1}^{\infty} a_n(t) f_n \varphi_n(x), \quad \tilde{v}(x,t) = \sum_{n=1}^{\infty} a_n(t)\tilde{f}_n\varphi_n(x), \tag{36}$$

where $\tilde{f}_n = (\tilde{f}, \varphi_n)$. Next, we prove the uniform convergence of above series. By the Sobolev embedding theorem, we have

$$\|\varphi_n\|_{C(\bar{\Omega})} \le C\|\varphi_n\|_{H^{2m}(\Omega)}, \tag{37}$$

for $m > d/4$. Moreover, we have

$$\|\varphi_n\|_{H^{2m}(\Omega)} \le C\|(-\Delta+1)^m \varphi_n\|_{L^2(\Omega)} = C(\lambda_n + 1)^m. \tag{38}$$

On the one hand, when $z \in S_1 := \{z \in \mathbb{C}; |\arg z| \le \min\{(\pi-\mu)/\alpha_1, \epsilon/(\alpha_1 - \alpha_m), \pi\}$, by Lemma 5 and Lemma 6, combining (37) and (38) and the Cauchy-Schwarz inequality, we arrive at

$$\begin{aligned}
&\sum_{n=2}^{\infty}\max_{x\in\bar{\Omega}}\left|\left(1-\lambda_n z^{\alpha_1}E^{(\lambda_n)}_{\alpha',1+\alpha_1}(z)\right)f_n\varphi_n(x)\right| \\
&\quad=\sum_{n=2}^{\infty}\max_{x\in\bar{\Omega}}\left|E^{(\lambda_n)}_{\alpha',1}(z)+\sum_{j=2}^{m}q_j z^{\alpha_1-\alpha_j}E^{(\lambda_n)}_{\boldsymbol{\alpha}',1+\alpha_1-\alpha_2}(z)\right||f_n\varphi_n(x)| \\
&\quad\le C\sum_{n=2}^{\infty}\sum_{j=1}^{m}\left|\frac{z^{\alpha_1-\alpha_j}}{1+\lambda_n z^{\alpha_1}}\right|\|f_n\varphi_n(x)\|_{L^\infty(\bar{\Omega})}\le C\sum_{n=2}^{\infty}|f_n|(\lambda_n+1)^{m-1} \\
&\quad\le C\left(\sum_{n=2}^{\infty}\frac{1}{(\lambda_n+1)^{2(\gamma+1-m)}}\right)^{1/2}\left(\sum_{n=2}^{\infty}(\lambda_n+1)^{2\gamma}|f_n|\right)^{1/2}.
\end{aligned}\tag{39}$$

By the book [23], we know that $\lambda_n \ge Cn^{2/d}$, $n\in\mathbb{N}$. If we choose $\gamma > m+(d/4)-1$, i.e.,$\gamma > (d/2)-1$, then the first series of the last term is convergent. By $f\in\mathscr{D}((-\Delta+1)^\gamma)$, we know that the above series is convergent uniformly over $z\in S_1$ for $x\in\bar{\Omega}$. On the other hand, we know that the multinomial Mittag-Leffler function $E^{(\lambda_n)}_{\alpha',1+\alpha_1}(z)$ is analytic over the domain $S_2 := \{z\in\mathbb{C}\,;|z|>0, |\arg z|<\pi\}$ based on its definition. Thus, the expressions $v(x,t)$ and $\tilde{v}(x,t)$ are analytic over the domain $z\in S_1\cap S_2$ for $x\in\bar{\Omega}$ based on the Weierstrass theorem. Especially, $u(x,t)$ and $v(x,t)$ can be extended to $\bar{\Omega}\times(0,\infty)$.

Therefore, by the analytic continuation, the condition $v(x,t)=\tilde{v}(x,t)$ for $(x,t)\in\Gamma\times I$ gives

$$\sum_{n=1}^{\infty}a_n(t)f_n\varphi_n(x)=\sum_{n=1}^{\infty}a_n(t)\tilde{f}_n\varphi_n(x),\quad x\in\Gamma,\ t>0.\tag{40}$$

We set $\sigma(-\Delta)=\{\mu_n\}_{n\in\mathbb{N}}$ with $0=\mu_1<\mu_2<\mu_3<\cdots$ and denote by $\{\varphi_{n_k}\}1\le k\le m_n$ an orthonormal basis of Ker $(\mu_n+\Delta)$. Now, we consider $\sigma(-\Delta)$ as a set, not as a sequence with multiplicities. Then, we can rewrite (40) by

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{m_n}f_{n_k}\varphi_{n_k}(x)\right)\tilde{a}_n(t)=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{m_n}\tilde{f}_{n_k}\varphi_{n_k}(x)\right)\tilde{a}_n(t),\ x\in\Gamma,\ t>0,\tag{41}$$

where $\tilde{a}_n(t)=(1-\mu_n t^{\alpha_1}E^{(\mu_n)}_{\alpha',1+\alpha_1}(t))$ and $f_{n_k}=(f,\varphi_{n_k})$. From (39), one obtains

$$\left|e^{-t\ \mathrm{Re}\ z}\sum_{n=2}^{\infty}f_n a_n(t)\varphi_n(x)\right|\le Ce^{-t\ \mathrm{Re}\ z}\|f\|_{D((-\Delta+1)^\gamma)},\tag{42}$$

and $e^{-t\ \mathrm{Re}\ z}$ is integrable in $t\in(0,+\infty)$ for fixed z satisfying Re $z>0$. By the Lebesgue convergence theorem, we can take the Laplace transform for (41). By Lemma 9, we have

$$\mathscr{L}(a_n)(s)=\frac{\sum_{j=1}^{m}q_j s^{\alpha_j-1}}{\sum_{j=1}^{m}q_j s^{\alpha j}+\lambda_n},\ \mathrm{Re}\ s>0.\tag{43}$$

Therefore, we arrive at

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(\sum_{k=1}^{m_n}f_{n_k}\varphi_{n_k}(x)\right)\frac{\sum_{j=1}^{m}q_j s^{\alpha_{j-1}}}{\sum_{j=1}^{m}q_j s^{\alpha_j}+\mu_n} \\
&\quad=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{m_n}\tilde{f}_{n_k}\varphi_{n_k}(x)\right)\frac{\sum_{j=1}^{m}q_j s^{\alpha_j+1}}{\sum_{j=1}^{m}q_j s^{\alpha_j}+\mu_n},\quad x\in\Gamma,\ \mathrm{Re}\ s>0,
\end{aligned}\tag{44}$$

which implies

$$\sum_{n=1}^{\infty}\sum_{k=1}^{m_n}f_{n_k}\frac{\varphi_{n_k}(x)\eta}{\eta+\mu_\eta}=\sum_{n=1}^{\infty}\sum_{k=1}^{m_n}\tilde{f}_{n_k}\frac{\varphi_{n_k}(x)\eta}{\eta+\mu_n},\quad x\in\Gamma,\eta\in S_3\supset\mathbb{R}^+,\tag{45}$$

where $\eta=\sum_{j=1}^{m}q_j s^{\alpha_j}$. As

$$\sum_{n=1}^{\infty}\sum_{k=1}^{m_n}\left|\left(f,\varphi_{n_k}\right)\frac{\eta}{\eta+\mu_n}\varphi_{n_k}(x)\right|\le\sum_{n=1}^{\infty}\|(f,\varphi_n)\varphi_n\|_{L^\infty(\bar{\Omega})}\left|\frac{\eta}{\eta+\lambda_n}\right|,\tag{46}$$

one can see that the above series is internally closed uniform convergence in $\eta\in\mathbb{C}\setminus\{-\mu_n\}_{n=1}^{\infty}$ from (39). Using the Weierstrass theorem, both sides of (45) are analytic in $\eta\in\mathbb{C}\setminus\{-\mu_n\}_{n=1}^{\infty}$. Therefore, one can analytically continue η such that (45) holds for $\eta\in\mathbb{C}\setminus\{-\mu_n\}_{n=1}^{\infty}$.

Now, we can take a suitable disk which only includes μ_l. Using the Cauchy integral theorem, integrating (45) along this disk, we obtain that $\mu_l(x)=\sum_{k=1}^{m_l}(f-\tilde{f}\varphi_{l_k})\varphi_{l_k}(x)=0, x\in\Gamma$. Since $(\Delta+\mu_l)u_l=0$ in Ω, and $u_l=\partial_{u_l}/\partial n=0$, on Γ the uniqueness of the Cauchy problem for elliptic equations (e.g., see Theorem 3.3.1 of Isakov [24], p. 58) implies $u_l=0$ in Ω for each $l\in\mathbb{N}$. Combining the linearly independence of $\{\varphi_{l_k}\}_{1\le k\le m_l}$ in Ω, we obtain that $(f-\tilde{f},\varphi_{l_k})=0$ for $1\le k\le m_l$, $l\in\mathbb{N}$. Therefore $f=\tilde{f}$ in $L^2(\Omega)$.

Remark 13. Assume that $\phi=0$ and $b=0$, $f\in\mathscr{D}((-\Delta+1)^\gamma)$ for $\gamma>\max\{0,d/2-1\}$ and $r\in L^\infty(0,\infty)$, then the solution of (1) $u\in C(\bar{\Omega}\times(0,+\infty))$.

Proof. From (35), the solution $u(x,t)$ is given by

$$u(x,t)=\sum_{n=1}^{\infty}r_n(t)f_n\varphi_n(x).\tag{47}$$

Let $t_0>0$ be arbitrarily fixed. For $t\ge t_0$, by Lemma 7 and Lemma 6, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}\max_{x\in\bar{\Omega}}\left|\int_0^t r(s)(t-s)^{a_1-1}E_{\alpha',a_1}^{(\lambda_n)}(t-s)dsf_n\varphi_n(x)\right| \\
&\quad\le \|r\|_\infty\sum_{n=2}^{\infty}\|f_n\varphi_n\|_{L^\infty(\bar{\Omega})}\frac{Ct^{\alpha_1}}{1+\lambda_n t^{\alpha_1}}\le C\|r\|_\infty\sum_{n=2}^{\infty}|f_n|(\lambda_n+1)^{m-1} \\
&\quad\le C\|r\|_\infty\left(\sum_{n=2}^{\infty}\frac{1}{(\lambda_n+1)^{2(\gamma+1-m)}}\right)^{1/2}\left(\sum_{n=2}^{\infty}(\lambda_n+1)^{2\gamma}|f_n|^2\right)^{1/2} \\
&\quad\le C\|r\|_\infty\|f\|_{\mathscr{D}((-\Delta+1)^\gamma)},
\end{aligned} \tag{48}$$

where $\gamma > \max\{0, d/2-1\}$. Thus, the series (48) is also convergent on $\bar{\Omega}\times[t_0,\infty)$ uniformly. By Lemma 8, we know u in (48) is continuous on $\bar{\Omega}\times[t0,\infty)$.

Finally, we can proceed to prove the main result Theorem 1 raised at the end of the introduction.

Proof of Theorem 1. As we know, by the principle of linear superposition, we have $u = u_f + u_\phi + u_b$ and they solve the following three IBVPs:

$$\begin{cases}
\sum_{j=1}^{m} q_j\partial_{0+}^{\alpha_j}u_f(x,t)-\Delta_f(x,t)=f(x)r(t), & (x,t)\in\Omega_T,\\
u_f(x,0)=0, & x\in\bar{\Omega},\\
\dfrac{\partial u_f}{\partial n_n}(x,t)=0, & (x,t)\in\partial\Omega\times I,
\end{cases} \tag{49}$$

$$\begin{cases}
\sum_{j=1}^{m} q_j\partial_{0+}^{\alpha_j}u_\phi(x,t)-\Delta u_\phi(x,t)=0 & (x,t)\in\Omega_T,\\
u_\phi(x,0)=\phi(x), & x\in\bar{\Omega},\\
\dfrac{\partial u_\phi}{\partial n}(x,t)=0, & (x,t)\in\partial\Omega\times I,
\end{cases} \tag{50}$$

$$\begin{cases}
\sum_{j=1}^{m} q_j\partial_{0+}^{\alpha_j}u_b(x,t)-\Delta u_b(x,t)=0, & (x,t)\in\Omega_T,\\
u_b(x,0)=0, & x\in\bar{\Omega},\\
\dfrac{\partial u_b}{\partial n}(x,t)=b(x,t), & (x,t)\in\partial\Omega\times I.
\end{cases} \tag{51}$$

It is not hard to find that the solutions of IBVPs (50) and (51) are independent of the source term f, and then, the inverse source problem becomes more precise, i.e., can we determine $f(x)$ from $u_f(x,t)$ for $(x,t)\in\Gamma\times I$ uniquely. According to Remark 13, we know that $u_f\in C(\bar{\Omega}\times(0,\infty))$. By Lemma 11, we know that $u_f(x,t)=0$, $x\in\Gamma$, $0<t\le T$ deduce

$$\int_0^t \mu(t-s)v_f(x,s)ds=0, x\in\Gamma,\quad 0<t\le T. \tag{52}$$

By the Fibini theorem, we obtain from (33) that

$$\begin{aligned}
&\sum_{j=1}^{m} q_j I_{0+}^{1-\alpha_j}\int_0^t \mu(t-s)v_f(x,s)ds \\
&\quad=\sum_{j=1}^{m}\frac{q_j}{\Gamma(1-\alpha_j)}\int_0^t\frac{d\tau}{(t-\tau)^{\alpha_j}}\int_0^\tau \mu(\tau-s)v_f(x,s)ds, \\
&\quad=\sum_{j=1}^{m}\frac{q_j}{\Gamma(1-\alpha_j)}\int_0^t v_f(x,s)ds\int_0^t\frac{\mu(\tau-s)}{(t-\tau)^{\alpha_j}}d\tau, \\
&\quad=\int_0^t r(t-s)v_f(x,s)ds.
\end{aligned} \tag{53}$$

So, we have $\int_0^t r(t-s)v_f(x,s)ds=0$, $x\in\Gamma$, $0<t\le T$. Differentiating the above equality with respect to t, we arrive at $r(0)v_f(x,t)+\int_0^t r'(t-s)v_f(x,s)ds=0$. As the assumption $r(0)\ne 0$, we have by the generalized Minkowski inequality that

$$\begin{aligned}
\|v_f(\cdot,t)\|_{L^2(\Gamma)} &\le \frac{1}{r(0)}\int_0^t r'(t-s)\|v_f(\cdot,s)\|_{L^2(\Gamma)}ds \\
&\le \frac{\|r\|_{C^1[0,T]}}{r(0)}\int_0^t\|v_f(\cdot,s)\|_{L^2(\Gamma)}ds.
\end{aligned} \tag{54}$$

Using the Gronwall inequality, we have $v_f(x,t)=0$, $x\in\Gamma$, $0<t\le T$. Therefore, we know $f(x)=0$ by Theorem 12. Thus, we complete the proof.

5. Levenberg-Marquardt Regularization Method

In this section, we solve numerically the inverse problem of identifying the space-dependent source $f(x)$. As we know, most of inversion algorithms are based on regularization strategies so as to overcome ill-posedness of inverse problems, and different kinds of inverse problems may need different approximate methods on the basis of conditional well-posedness analysis. In this paper, we employ the Levenberg-Marquardt regularization technique to obtain an approximation to the source term.

Based on Theorem 10, we can define a forward linear operator

$$\mathscr{F}:\quad f(x)\in L^2(\Omega)\to u(x,t;f)\in L^2(\Gamma\times(0,T)). \tag{55}$$

Thus, the inverse problem is translated into solving the following abstract operator equation

$$\mathscr{F}f=u(x,t)|_{\Gamma}\triangleq h(x,t),\quad t\in I. \tag{56}$$

From Theorem 10, we know that $u(x,t;f)\in B^{\alpha_1/2}(\Omega_T)$. By the trace theorem in [25], we know $u(x,t;f)|_{\Gamma\times I}\in L^2(0,T;H^{1/2}(\Gamma))\cap {}_0H^{a_1/4}(0,T;L^2(\Gamma))\hookrightarrow L^2(\Gamma\times(0,T))$

compactly, then the operator $\mathscr{F}: L^2(\Omega) \to L^2(\Gamma \times (0,T))$ is compact and thus the inverse source problem is ill-posed. In order to ensure a stable numerical reconstruction of $f(x)$, we give the following minimization problem with the Tikhonov-type regularization term

$$\min_{f \in \mathscr{D}((-\Delta+1)^\gamma)} J_\mu(f) = \left\| \mathscr{F}f - h^\delta \right\|^2_{L^2(\Gamma\times(0,T))} + \mu \| f - f^* \|^2_{\mathscr{D}((-\Delta+1)^\gamma)}, \tag{57}$$

where $\gamma > d/2 - 1$, $\mu > 0$ is a regularization parameter, $f^* \in \mathscr{D}((-\Delta+1)^\gamma)$ is a suitable guess of f, and h^δ is the noisy function of h.

Lemma 14. (see [26]). *Assume that E is a uniformly convex Banach space. Let (x_n) be a sequence in E such that $x_n \rightharpoonup x$ weakly $\sigma(E, E')$ and* $\limsup \|x_n\| \le \|x\|$. *Then, $x_n \rightharpoonup x$ strongly.*

By the canonical book [27], we know that with above minimization problem exists a unique stable solution of regularization. Moreover, the convergence result can be obtained in the following theorem.

Theorem 15. *Suppose that $\mathscr{F}f = h$ and the noisy data $h^\delta \in L^2(\Gamma \times (0,T))$ satisfying $\|h^\delta - h\|_{L^2(\Gamma\times(0,T))} \le \delta$ and let $\mu(\delta)$ satisfy $\mu(\delta) \to 0$ and $\delta^2/\mu(\delta) \to 0$ as $\delta \to 0$; then, the minimizer f^δ_μ of variational problem (57) is convergent, i.e., $f^\delta_\mu \to f$ in $\mathscr{D}((-\Delta+1)^\gamma)$ as $\delta \to 0$.*

Proof. Let δ_k be any sequence such that $\delta_k \to 0$ and denote $\mu_k = \mu(\delta_k)$. Since

$$\begin{aligned} &\left\| \mathscr{F}f^{\delta_k}_{\mu_k} - h^{\delta_k} \right\|^2_{L^2(\Gamma\times(0,T))} + \mu_k \left\| f^{\delta_k}_{\mu_k} - f^* \right\|^2_{\mathscr{D}((-\Delta+1)^\gamma)} \\ &\quad \le \left\| \mathscr{F}f - h^{\delta_k} \right\|^2_{L^2(\Gamma\times(0,T))} + \mu_k \| f - f^* \|^2_{\mathscr{D}((-\Delta+1)^\gamma)} \\ &\quad \le \delta_k^2 - \mu_k \| f - f^* \|^2_{\mathscr{D}((-\Delta+1)^\gamma)}, \end{aligned} \tag{58}$$

we have $\|f^{\delta_k}_{\mu_k} - f^*\|^2_{D((-\Delta+1)^\gamma)} \le \delta_k^2/\mu_k + \|f - f^*\|^2_{\mathscr{D}((-\Delta+1)^\gamma)}$ and $\limsup_{\delta_k \to 0} \|f^{\delta_k}_{\mu_k} - f^*\|_{\mathscr{D}((-\Delta+1)^\gamma)} \le \|f - f^*\|_{\mathscr{D}((-\Delta+1)^\gamma)}$ by the condition for μ. So we know that $\|f^{\delta_k}_{\mu_k}\|_{\mathscr{D}((\Delta+1)^\gamma)}$ is bounded and has a weak convergence subsequence by the reflexivity of $\mathscr{D}((-\Delta+1)^\gamma)$ and also denotes $f^{\delta_k}_{\mu_k}$ such that $f^{\delta_k}_{\mu_k} \rightharpoonup z$ in $\mathscr{D}((-\Delta+1)^\gamma)$.

Since the linear operator $\mathscr{F}$ is bounded, we know that $\mathscr{F}f^{\delta_k}_{\mu_k} \rightharpoonup \mathscr{F}_z$ in $L^2(\Gamma \times (0,T))$.

From (58), we have $\|\mathscr{F}f^{\delta_k}_{\mu_k} - h^{\delta_k}\|_{L^2(\Gamma\times(0,T))} \le \delta_k^2 + \mu_k \|f - f^*\|^2_{\mathscr{D}((-\Delta+1)^\gamma)}$, and thus $\lim_{\delta_k \to 0} f^{\delta_k}_{\mu_k} = h$ and $\mathscr{F}z = h$. By the uniqueness of $\mathscr{F}f = h$, we know that $z = f$.

By the weak lower semicontinuity of the norm in the Hilbert space, we have $\|f - f^*\|_{\mathscr{D}((-\Delta+1)^\gamma)} \liminf_{\delta_k \to 0} \|f^{\delta_k}_{\mu_k} - f^*\|_{\mathscr{D}((-\Delta+1)^\gamma)} \le \limsup_{\delta_k \to 0} \|f^{\delta_k}_{\mu_k} - f^*\|_{\mathscr{D}((-\Delta+1)^\gamma)} \le \|f - f^*\|_{\mathscr{D}((-\Delta+1)^\gamma)}$, and hence, we know that $\lim_{\delta_k \to 0} \|f^{\delta_k}_{\mu_k} - f^*\|_{\mathscr{D}((-\Delta+1)^\gamma)} = \|f - f^*\|_{\mathscr{D}((-\Delta+1)^\gamma)}$. Combing with the weak convergence of $f^{\delta_k}_{\mu_k}$, we have by Lemma 14 that $f^{\delta_k}_{\mu_k} \to f$. From the uniqueness of $\mathscr{F}f = h$, we have that $f^\delta_{\mu(\delta)} \to f$ in $\mathscr{D}((-\Delta+1)^\gamma)$.

In the following, we use the Levenberg-Marquardt method to minimize problem (57). The Levenberg-Marquardt method was first introduced by [28, 29], and it is a kind of the Newton-type method. Moreover, it has been well applied to fractional equations, for example, see [30–32]. From physical considerations, we know that h^δ is a reasonably close approximation of some ideal $h = \mathscr{F}f$ in the range of $\mathscr{F}$. Let f^* be an approximation of f. By the linearity expansion for linear operator $\mathscr{F}$ at f^*, we can obtain

$$\mathscr{F}f = \mathscr{F}f^* + \left(\mathscr{F}'_f f^*\right)(f - f^*). \tag{59}$$

Then, the inverse problem $\mathscr{F}f = h^\delta$ can be transformed into

$$\left(\mathscr{F}'_f f^*\right)(f - f^*) = h^\delta - \mathscr{F}f^*. \tag{60}$$

Therefore, it is easily seen that the problem of minimizing (57) is equivalent to minimizing

$$\begin{aligned} \min_{\delta f \in \mathscr{D}((-\Delta+1)^\gamma)} J_\mu(f) = &\left\| \left(\mathscr{F}'_f f^*\right)\delta f - \left(h^\delta - \mathscr{F}f^*\right) \right\|^2_{L^2(\Gamma\times(0,T))} \\ &+ \mu \|\delta f\|^2_{\mathscr{D}((-\Delta+1)^\gamma)}, \end{aligned} \tag{61}$$

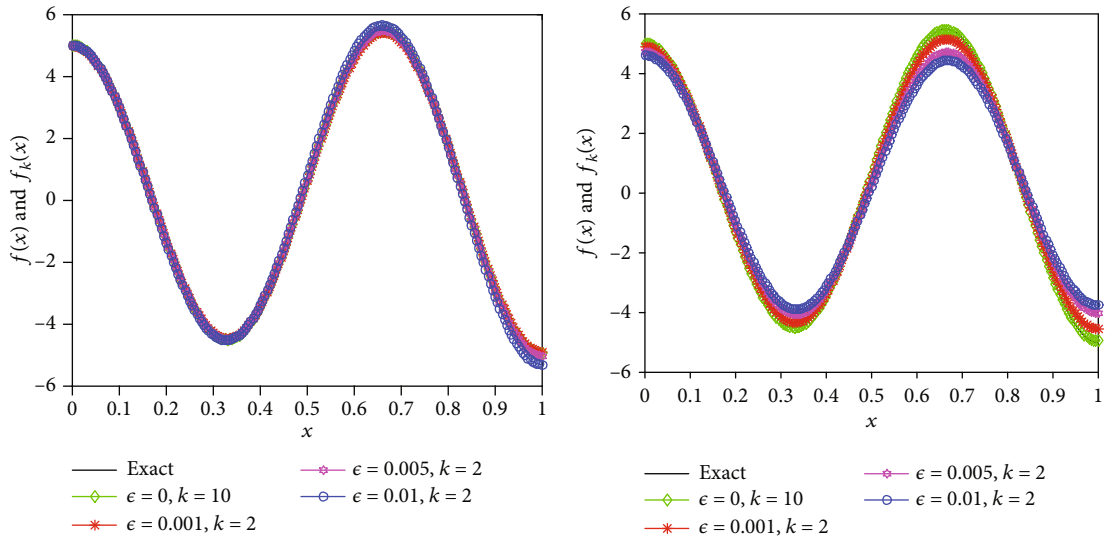
where $\delta f = f - f^*$.

Now, we consider the discretization of the minimization problem. We define an admissible set of unknown source $f(x)$. Suppose that $\{\varphi_s(x), s = 1, 2, \cdots, \infty\}$ is a basis in $\mathscr{D}((-\Delta+1)^\gamma)$, let

$$\begin{aligned} f(x) &\approx \hat{f}(x) = \sum_{s=1}^{s} a_s \varphi_s(x), \\ f^*(x) &\approx \widehat{f^*}(x) = \sum_{s=1}^{s} a_s^* \varphi_s(x), \end{aligned} \tag{62}$$

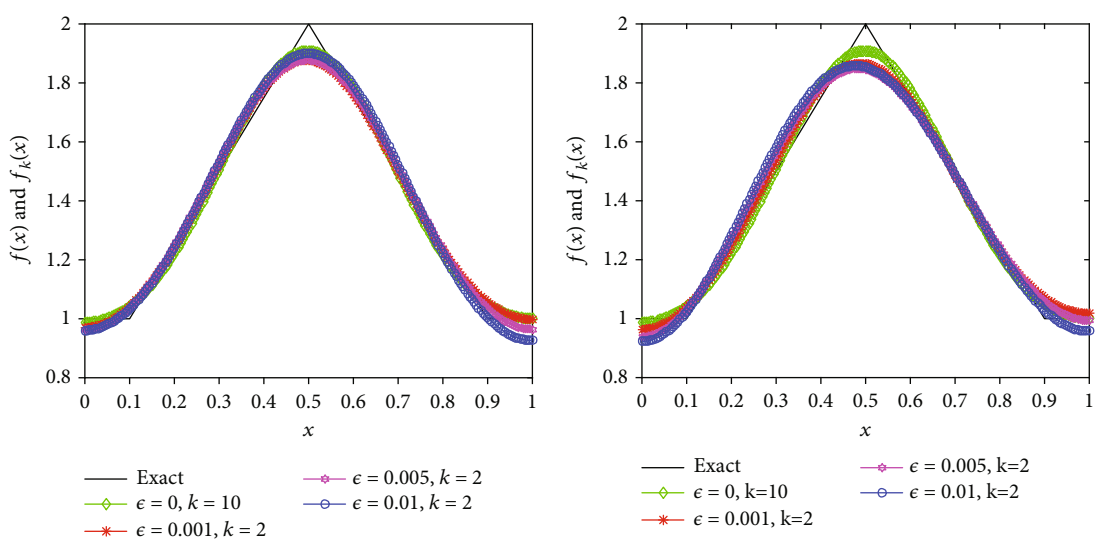
where $\hat{f}$ is the S-dimensional approximation solution to f and $S \in \mathbb{N}$ is a truncated level of f, and $a_s, s = 1, 2, \cdots, S$ are the expansion coefficients. It is convenient to set a finite dimensional space as

$$\Phi^S = \operatorname{span}\{\varphi_1, \varphi_2, \cdots, \varphi_S\}, \tag{63}$$



(a) $\alpha = (0.9, 0.3, 0.1)$ (b) $\alpha = (0.7, 0.6, 0.2)$

Figure 1: The numerical results for Example 16 for various noise levels with $\mu = 10^{-6}\delta^{3/4}$ and $\gamma = 0$.



(a) $\alpha = (0.9, 0.6, 0.1)$ (b) $\alpha = (0.8, 0.7, 0.6)$

Figure 2: The numerical results for Example 17 for various noise levels with $\mu = 10^{-5}\delta^{3/4}$ and $\gamma = 1/10$.

Table 1: The relative error re_k with different noise levels ϵ and fractional orders α for Example 16.

α / re_k / ϵ	0	0.001	0.005	0.01
(0.9,0.3,0.2)	0.0059	0.0135	0.0126	0.0484
(0.8,0.7,0.6)	0.0060	0.0082	0.0357	0.0941
(0.7,0.5,0.2)	0.0059	0.0900	0.2258	0.3104
(0.6,0.3,0.2)	0.0059	0.1217	0.6046	0.7569
(0.5,0.3,0.2)	0.0058	0.2947	0.7656	0.8595

and S-dimensional vector $\mathbf{a} = (a_1, a_2, \cdots, a_S) \in \mathbb{R}^S$. We identify an approximation $\widehat{f}(x) \in \Phi^S$ with a vector $\mathbf{a} \in \mathbb{R}^S$.

Based on the above discussions, by setting $u(x, t; \mathbf{a}) = u(x, t; \widehat{f})$ as a unique solution of the forward problem, a feasible way for numerical solution to solve the following minimization problem

$$\min_{\delta\mathbf{a}\in\mathbb{R}^S} \left\{ \left\| \delta\mathbf{a}\nabla_{\mathbf{a}}^T u(x, t; \mathbf{a}^*) - \left(h^\delta - u(x, t; \mathbf{a}^*)\right) \right\|_{L^2(\Gamma\times(0,T))}^2 + \mu\delta\mathbf{a}A\delta\mathbf{a}^T \right\}, \tag{64}$$

where $A = ((\lambda_i^\gamma \varphi_i, \lambda_j^\gamma \varphi_j)_{L^2(\Omega)})S \times S$, $\mathbf{a}^* = (a_1^*, a_2^*, \cdots, a_S^*)$, $\delta\mathbf{a} = \mathbf{a} - \mathbf{a}^*$, and $\mathbf{a}^T$ denotes the transpose of $\mathbf{a}$.

Table 2: The relative error re_k with different noise levels ϵ and index of penalty term γ for $\alpha = (0.8, 0.6, 0.1)$ in Example 16.

γ / re_k / ϵ	0	0.001	0.005	0.01
0	0.0059	0.0257	0.0348	0.0297
1/10	0.0059	0.0101	0.1851	0.2344
1/3	0.0059	0.0361	0.2462	0.4812
1/2	0.0059	0.0939	0.4363	0.5727

Table 3: The relative error re_k with different noise levels ϵ and index of penalty term γ for $\alpha = (0.9, 0.5, 0.2)$ in Example 17.

α / re_k / ϵ	0	0.001	0.005	0.01
−1/10	0.0194	0.0695	0.1372	0.1889
0	0.0194	0.0274	0.0532	0.0801
−1/10	0.0194	0.0231	0.0261	0.0272
1/4	0.0194	0.0238	0.0687	0.0704
1/2	0.0194	0.0333	0.0760	0.0772

Next, we give an inversion algorithm for determining coefficient $\mathbf{a}$. Given the initial $\mathbf{a}^0$ and let $\mathbf{a}^* = \mathbf{a}^0$. Suppose $\mathbf{a}^k \in \mathbb{R}^S$ is the kth step iteration value, then we obtain $(k+1)$th step approximation by solving

$$\mathbf{a}^{k+1} = \mathbf{a}^k + \delta\mathbf{a}^k, k = 0, 1, \cdots, \tag{65}$$

where $\delta\mathbf{a}^k$ denotes a perturbation of $\mathbf{a}^k$ for each k and k is the number of iterations. For convenience of writing, $\mathbf{a}^k$ and $\delta\mathbf{a}^k$ are also abbreviated as $\mathbf{a}$ and $\delta\mathbf{a}$, respectively.

Now, we compute $\delta\mathbf{a}$ by the variational theory. As we know,

$$\nabla_{\mathbf{a}} u\left(x, t; \mathbf{a}^k\right) = \left(\frac{u\left(x, t; \mathbf{a}^k + \tau\mathbf{e}_1\right) - u\left(x, t; \mathbf{a}^k\right)}{\tau}, \cdots, \frac{u\left(x, t; \mathbf{a}^k + \tau\mathbf{e}_S\right) - u\left(x, t; \mathbf{a}^k\right)}{\tau}\right), \tag{66}$$

where $\mathbf{e}_i = (0, \cdots, 0, 1, 0, \cdots, 0)$, $i = 1, \cdots, S$. Since operator $\mathcal{F}$ is linear respect to the source f, we have

$$\nabla_{\mathbf{a}} u\left(x, t; \mathbf{a}^k\right) = \left(u(x, t; \mathbf{e}_1), \cdots, u(x, t; \mathbf{e}_S)\right) =: (b_1, \cdots, b_S). \tag{67}$$

By the variational theory, problem (64) is equivalent to the following normal equations

$$(\mu A + B)\delta\mathbf{a}^T = V, \tag{68}$$

where $B = ((b_1, b_j)_{L^2(\Gamma\times(0,T))})S \times S$ and $V = ((h^\delta(x, t) - u(x, t; \mathbf{a}^k), b_i)_{L^2(\Gamma\times(0,T))})S \times 1$.

6. Numerical Experiments

In this section, we present the numerical results for three examples in one-dimensional and two-dimensional cases to show the effectiveness of the Levenberg-Marquardt method.

The noisy data is generated by adding a random perturbation, i.e.,

$$h^\delta = h + \epsilon h \cdot (2 \cdot \text{rand}\ (\text{size}(h)) - 1). \tag{69}$$

The corresponding noise level is calculated by $\delta = \|h^\delta - h\|_{L^2(0,T;L^2(\Gamma)\times(0,zt))}$.

To show the accuracy of numerical solution, we compute the approximate L^2 error denoted by

$$re_k = \frac{\|f_k(x) - f(x)\|_{L^2(\Omega)}}{\|f_k(x)\|_{L^2(\Omega)}}, \tag{70}$$

where $f_k(x)$ is the coefficient term reconstructed at the kth iteration, and $f(x)$ is the exact solution.

The residual E_k at the kth iteration is given by

$$E_k = \left\|u(x, t; f_k) - h^\delta(x, t)\right\|_{L^2(\Gamma\times(0,T))}. \tag{71}$$

In an iteration algorithm, the most important work is to find a suitable stopping rule. In this study, we use the well-known discrepancy principle [33], i.e., we choose k satisfying the following inequality

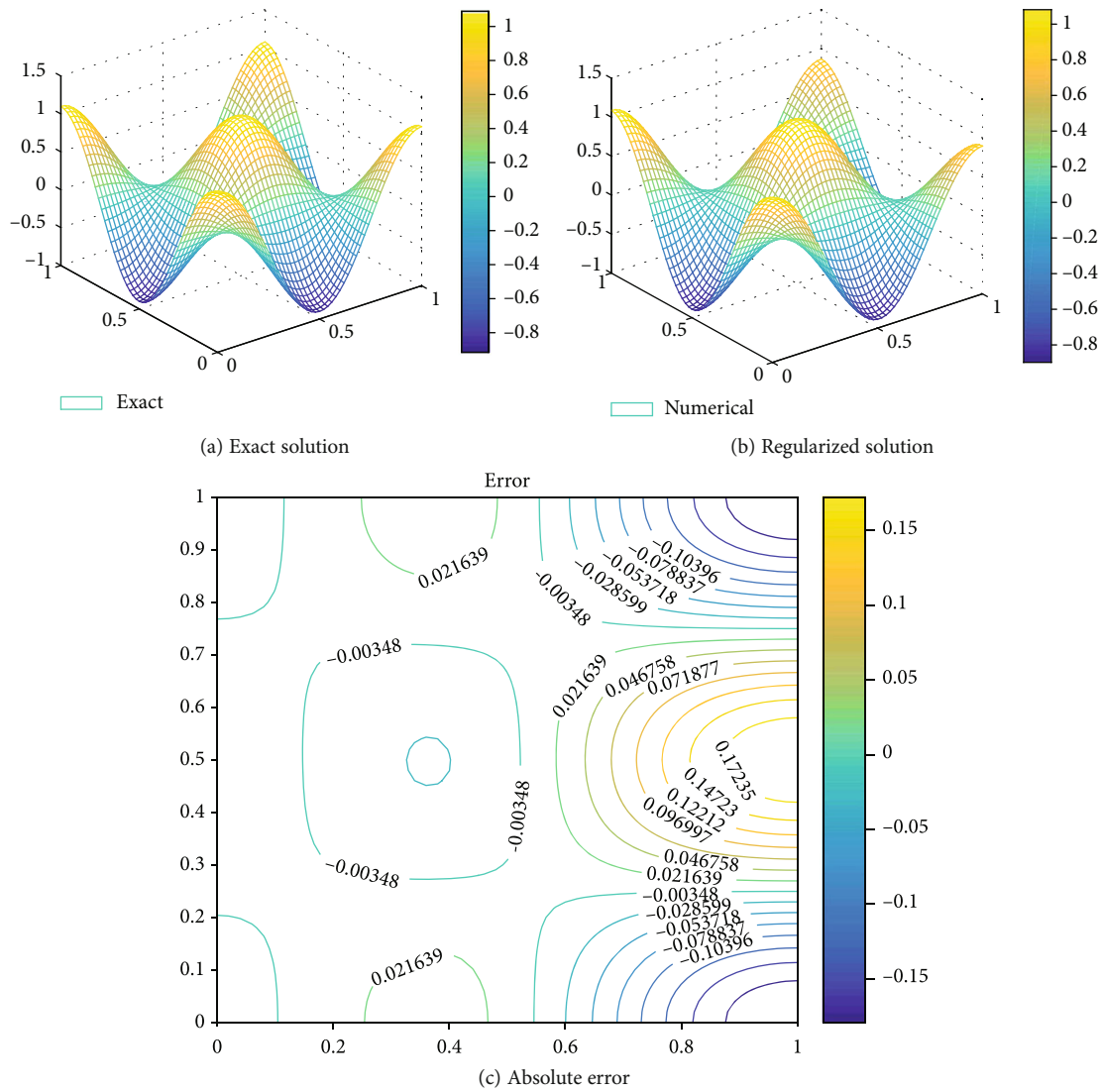
$$E_k \le \zeta\delta < E_{k-1}, \tag{72}$$

where $\zeta > 1$ is a constant and can be taken heuristically to be 1.01, as suggested by Hanke and Hansen [34].

6.1. One-Dimensional Case. Without loss of generality, we set $m = 3$, $q_j = 1 (j = 1, 2, 3)$, and $\Omega_T = (0, 1) \times (0, T)$. The grid points on [0,1] and $[0, T]$ are both 201 when solving the direct problem by finite difference method in [35]. In this case, we take Γ as a point $x = 0$ and Φ_S is chosen as a subspace of eigenfunctions

$$\Phi_S = \text{span}\left\{1, \sqrt{2}\cos(\pi x), \cdots, \sqrt{2}\cos((S-1)\pi x)\right\}. \tag{73}$$

Example 16. We firstly test a smooth solution. Take an unknown source $f(x) = 5\cos(3\pi x) + 10x^2(1-x)^2$, the temporal source $r(t) = \exp(t)$, the initial $\phi(x) = \cos(\pi x)$, and the boundary $b(0, t) = 1 + t$, $b(1, t) = -1 + t$. The boundary data $u(0, t)$ is obtained by solving the direct problem (1) by using the finite difference method. The numerical results for Example 16 by using the discrepancy principle for various noise levels in the cases of $\alpha = (0.9, 0.3, 0, 1)$ and $\alpha = (0.7, 0.6, 0.2)$ are shown in Figures 1(a) and 1(b), respectively.



(a) Exact solution (b) Regularized solution

(c) Absolute error

Figure 3: The numerical results for Example 18 for $\alpha = (0.9, 0.5, 0.1)$ with $\mu = 0.01\delta^{3/4}$.

We choose the initial guess as $f_0 = f^* = 0$, the truncated level $S = 4$.

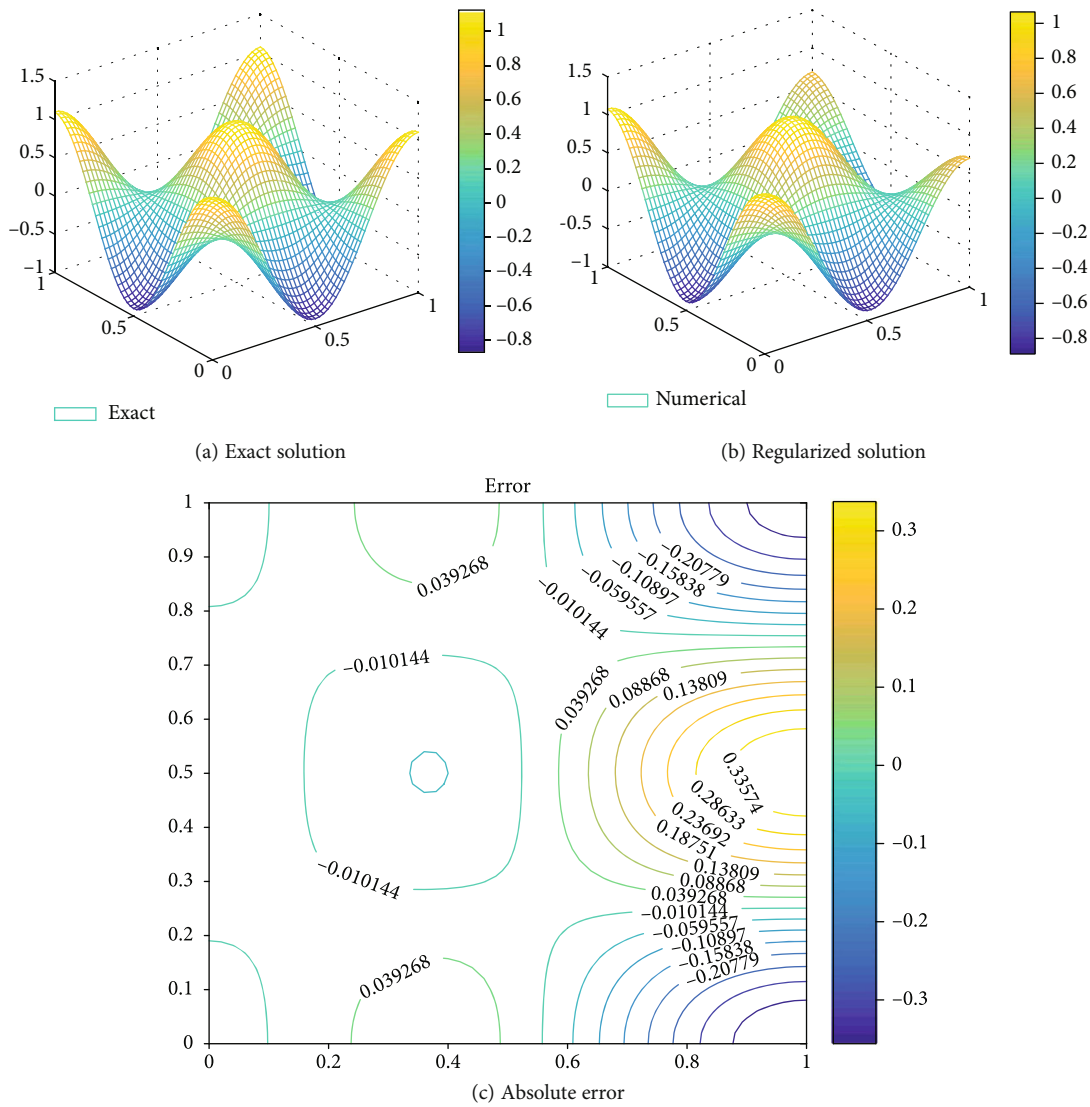
Example 17. In the second example, we test a nonsmooth solution with a cusp. Take an unknown source

$$f(x) = \begin{cases} 1, & 0 \le x < 0.1, \\ 1 + 2.5(x - 0.1), & 0.5 \le x < 0.5, \\ 2 + 2.5(0.5 - x), & 0.5 \le x < 0.9, \\ 1, & 0.9 \le x \le 1, \end{cases} \tag{74}$$

and the rest of definite conditions are the same as Example 16. The boundary data $u(0, t)$ is obtained by solving the direct problem (1) by using the finite difference method. The numerical results for Example 17 by using the discrepancy principle for various noise levels in the cases of $\alpha = (0.9, 0.6, 0, 1)$ and $\alpha = (0.8, 0.7, 0.6)$ are shown in Figures 2(a) and 2(b), respectively. We choose the initial guess as $f_0 = 0$ and $f^* = 0$, the truncated level $S = 5$.

From Tables 1 and 2, we can come to the following conclusions. First, it is observed that the dominate fractional order α_1 has a greater influence on the accuracy of the numerical inversion results. For each fixed ϵ, we can see that the numerical results become worse as the dominate fractional order α_1 becomes smaller, but it is not sensitive to the rest of the orders α_2 or α_3. On the other hand, the numerical results become worse as the index of penalty term γ becomes larger for each fixed ϵ. Since here we test a smooth function, the penalty term with high index may not have positive effect to the numerical inversion. So we choose $\gamma = 0$ in the first example.

From Table 3, we can see that the regularity of the penalty term can do modify the computation result well for a nonsmooth numerical test. But this regularity should not be too strong, or it will have the opposite effect. We find that $\gamma = 1/10$ is a better index by the numerical test, so we choose $\gamma = 1/10$ in the second example.



(a) Exact solution (b) Regularized solution

(c) Absolute error

FIGURE 4: The numerical results for Example 18 for $\alpha = (0.6, 0.5, 0.4)$ with $\mu = 0.01\delta^{3/4}$.

6.2. Two-Dimensional Case. In a two-dimensional case, we present one numerical example to show the effectiveness of our method. We also assume $T = 1$, $m = 3$, $q_j = 1 (j = 1, 2, 3)$ without loss of generality. The space domain Ω is taken as $(0, 1) \times (0, 1)$. The grid points on $[0,1]$ and $[0, T]$ are both 51 when solving the direct problem by finite difference method in [35]. We take $\Gamma = \{(x, y) | x = 0, 0 \le y \le 1\}$. In this case, we take $\Phi^{S_1 \times S_2} = \text{span}\{\varphi_i(x)\varphi_j(y)\}$, where

$$\varphi_i(x) = \begin{cases} 1, & i = 1, \\ \sqrt{2}\cos((i-1)\pi x), & i > 1, \end{cases} \quad i = 1, 2, \cdots, S_1,$$

$$\varphi_j(y) = \begin{cases} 1, & j = 1, \\ \sqrt{2}\cos((i-1)\pi y), & j > 1, \end{cases} \quad j = 1, 2, \cdots, S_2. \tag{75}$$

Example 18. Assume $f(x, y) = \cos(2\pi x)\cos(2\pi y) + 1/10 \exp(-1/3(x^3 - 3x^2)(2y^3 - 3y^2))$ and $r(t) = \exp(t)$, $\phi(x, y) = 0$, $b = 0$. The boundary data $u(0, y, t)$ is obtained by solving direct problem (1). The exact source solutions, reconstructed source terms, and the absolute errors between the exact source functions and the numerical solutions for $\alpha = (0.9, 0.5, 0, 1)$ and $\alpha = (0.6, 0.5, 0, 4)$ are shown in Figures 3 and 4 by taking noise level $\epsilon = 0.01$ and the regularization parameter $\mu = 0.01\delta^{3/4}$, respectively. Here, we choose the initial guess as $f_0 = 0$ and $f^* = 0$. The truncated levels are $S_1 = S_2 = 3$.

From Figures 3 and 4, we can see the proposed method still yields relatively accurate numerical solutions for a two-dimensional example.

Remark 19. As we know, we need to compute the gradient of an abstract operator by the Levenberg-Marquardt method, and we usually use numerical differential method to approximately substitute the gradient when it is hard to obtain the exact expression for the gradient. So, we need to introduce a numerical differentiation step τ. However, a lot of

numerical examples are sensitive to τ. In this computation, we have successfully avoided the problem of selection of the differential step. This is because that the operator $\mathscr{F}$ is linear to f and the gradient in (66) can be replaced by (67) where the gradient is not dependent to τ. But when we calculate (66) by a finite difference, the initial and boundary conditions need to be homogeneous.

7. Conclusions

In this paper, we investigate an inverse space-dependent source problem of a multiterm time-fractional diffusion equation with nonhomogeneous boundary condition in a general domain. The existence and uniqueness of a weak solution for the direct problem are obtained by the Lax-Milgram theorem. Then, the uniqueness for the inverse problem is provided by using the analytic continuation and the Laplace transformation. Finally, we employ the Levenberg-Marquardt regularization method to find an approximation of the source function.

In fact, we have obtained the well-posedness of the direct problem and also proved the uniqueness of the inverse problem for the case of nonhomogeneous boundary in the present paper. But there are many other important problems still remain open. For theoretical aspect, the conditional stability of the inverse source problem is still not obtained. For numerical computation, on the one hand, we have indeed improved the original M-L algorithm. However, some nonsmooth source functions, e.g., discontinuous function, are not reconstructed well although the uniqueness holds true theoretically.

Data Availability

The data that support the findings of this study are available from the corresponding author (Liangliang Sun, Email: sunll0321@163.com), upon reasonable request.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Acknowledgments

This work is supported by the Innovation Capacity Improvement Project for Colleges and Universities of Gansu Province (no. 2020B-088), the Young Teachers' Scientific Research Ability Promotion Project of NWNU (no. NWNU-LKQN-18-31), and the Doctoral Scientific Research Foundation of NWNU (no. 6014/0002020204).

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