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Research Article

Application of Local Fractional Homotopy Perturbation Method in Physical Problems

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Nonlinear phenomena have important effects on applied mathematics, physics, and issues related to engineering. Most physical phenomena are modeled according to partial differential equations. It is difficult for nonlinear models to obtain the closed form of the solution, and in many cases, only an approximation of the real solution can be obtained. The perturbation method is a wave equation solution using HPM compared with the Fourier series method, and both methods results are good agreement. The percentage of error of $u(x, t)$ with $\alpha = 1$ and 0.33, $t = 0.1$ sec, between the present research and Yong-Ju Yang study for $x \geq 0.6$ is less than 10. Also, the % error for $x \geq 0.5$ in $\alpha = 1$ and 0.33, $t = 0.3$ sec, is less than 5, whereas for $\alpha = 1$ and 0.33, $t = 0.8$ and 0.7 sec, the % error for $x \geq 0.4$ is less than 8.

1. Introduction

The homotopy perturbation (HP) method is an analytical method that can solve linear, nonlinear, and random operator equations [1, 2]. The HP method was defined by He [3–5] as a symmetrical structure using combining HP methods in topology and was presented to be used to solve various nonlinear problems in science and engineering, including the argued nonlinear equation in fluid mechanics [4]. He [5] provided a solution for the specific functional integral equations. Khan and Wu [6] have evolved a solution for homogenous and nonhomogeneous advection equation. Also, Xu [7] solved the boundary layer equation in unlimited domains. This method makes very difficult problems extremely easy, which do not need to be solved to convert nonlinear terms. The performance of nonlinear HPM has been demonstrated by many researchers. In recent years, most attention has been paid to the use of this method to solve various scientific models [6–9].

Various analytical methods are used to solve equations such as nonlinear wave equations. The variance repetition method is able to reduce the size of the calculations [10]. This method is used in problems related to chaos and super chaos and hard equations [11]. The HP method is a quick technique to produce convergent responses [12].

Recently, using HPM for local fractional solving of nonlinear problems in mathematical physics successfully achieved contains the following: diffusion equations on cantor space-time [13], linear and nonlinear local fractional equation of Korteweg-de Vries [14], fractal heat conduction problem [15, 16], wave equation in fractal strings [17], and the Laplace equation [18]. Finding solutions for solving differential equations of local fractional is an interesting topic. There are some approximate and analysis methods: the local fractional functional iteration method [13–17], variationally iteration method within the Yang-Laplace transforms [15], Yang-Laplace transform method [16], local fractional Adomian decomposition and function decomposition methods [18], local fractional iteration of the continuously nondifferentiable functions [19], and Sumudu transform method [20].

The formulation is applied to the generalized thermoelasticity based on the fractional time derivatives under the effect of diffusion. Herzallah [21] criticized the research of Ref [*a*] and showed that their results are incorrect. New methodologies in fractional and fractal derivatives modeling in scientific and engineering were studied by Chen and Liang [22].

Uchaikin and Sibatov [23] studied fractional derivatives on cosmic scales. The numerical calculation results demonstrated in this paper do speak well for the NoRD-model as compared with the traditional one based on integer-order operators.

Chaos in a 5D hyperchaotic system with four wings in the light of nonlocal and nonsingular fractional derivatives was investigated by Bonyah [24]. The numerical simulation results depict a new chaotic behaviors with the ABC numerical scheme.

Goufo and Toudjeu [25] evaluated the analysis of recent fractional evolution equations and applications. The numerical approximations of a second-order nonhomogeneous fractional Cauchy problem are performed and show regularity in the dynamics.

In the study of Turkyilmazoglu [26], it is theoretically shown that under a special constraint, the HPM converges to the exact solution of sought solution of nonlinear ordinary or partial differential equation. Examples (the classical Blasius flat-plate flow problem of fluid mechanics, fourth-order parabolic partial differential equation arising in the study of the transverse vibrations of a uniform flexible beam) clearly demonstrate, why and on what interval, the corresponding homotopy series generated by the HPM converges to the exact solution. The physical problems involving stronger nonlinearities can also be dealt with the homotopy method under the assurance of convergence provided by the presented theorems. This method (HPM) is a powerful device for solving a wide variety of problems. Using the HPM, it is possible to find the exact solution or an approximate solution of the problem. Some examples such as Burgers', Schrodinger's, and fourth-order parabolic partial differential equations are presented [26].

For this reason, the method of homeopathic disorder has been selected for some important physical issues that have received less attention in previous studies. A review of previous studies suggests that local fractional homotopy perturbation method is less commonly used for physical problems. The aim of this research is expansion of an HP method for the definition of local fractional derivation order that can solve a local fractional differential equation in usages of physical mathematics. Therefore, first, the HP method will be explained, and then the local damped wave equation, local fractional wave equation, and heat diffusion equation with several initial values are solved using this method. Also, the obtained results in applied examples are compared with the Fourier series method [27]. In the present research, the homotopy perturbation method (HPM) was applied for real physical problems in engineering which has been less studied in previous studies.

2. Mathematical Fundamentals

Definition 1. Suppose that the function $f(x) \in C_\alpha(a, b)$, local fractional derivative of $f(x)$ of order α, $0 < \alpha \leq 1$, at the point $x = x_0$ is defined as follows [28–41]:

$$f^{(\alpha)}(x_0) = \mathrm{D}_x^\alpha f(x_0) = \frac{d^\alpha f(x)}{dx^\alpha}\Big|_{x \to x_0} = \lim_{x \to x_0} \frac{\Delta^\alpha \{f(x) - f(x_0)\}}{(x - x_0)^\alpha}, \tag{1}$$

where $\Delta^\alpha\{f(x) - f(x_0)\} \cong \Gamma(1+\alpha)\Delta\{f(x) - f(x_0)\}$ and $f(x)$ are satisfied with the condition [22] $|f(x) - f(x_0)| \lessapprox \tau^\alpha |x - x_0|^\alpha$, so that [28, 36] $|f(x) - f(x_0)| < \varepsilon^\alpha$ with $U : |x - x_0| < \delta$ for $\varepsilon, \delta \epsilon (0, 1)$ and $\varepsilon, \delta \in R$.

However, the partial of local fractional is defined as follows [13–18]:

$$\frac{\partial^{k\alpha}}{\partial x^{k\alpha}} f(x.y) = \frac{\partial^\alpha}{\partial x^\alpha} \cdots \frac{\partial^\alpha}{\partial x^\alpha} f(x.y). \tag{2}$$

If there exists $f^{(k+1)\alpha}(x) = D_x^k \cdots D_x^{\alpha \overbrace{}^{k+1 \text{ times}}} f(x)$ for any $x \in I \subseteq R$, then we denote $f \in D_{(k+1)\alpha}(I)$ where $k = 0, 1, 2, \cdots$ [28].

Definition 2. The diffusion equation on the Cantor sets (called local fractional diffusion equation) was recently described [29] as

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = a^{2\alpha} \frac{\partial^\alpha u(x,y)}{\partial x^{2\alpha}}, \tag{3}$$

where $a^{2\alpha}$ denotes the fractional diffusion constant which is, in essence, a measure of the efficiency of the spreading of the underlying substance, while the local fractional wave equation is written in the following form [30, 31]:

$$\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} = a^{2\alpha} \frac{\partial^{2\alpha} u(x,y)}{\partial x^{2\alpha}}. \tag{4}$$

The local fractional Laplace operator is given [27, 42] as follows:

$$\nabla^{2\alpha} = \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha}}{\partial y^{2\alpha}} + \frac{\partial^{2\alpha}}{\partial z^{2\alpha}}. \tag{5}$$

We notice that the local fractional diffusion equation yields

$$\nabla^{2\alpha} u = \frac{1}{a^{2\alpha}} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha}. \tag{6}$$

And the local fractional wave equation has the following form:

$$\nabla^{2\alpha} u = \frac{1}{a^{2\alpha}} \frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}}. \tag{7}$$

where $1/a^{2\alpha}$ is a constant. This equation describes vibrations in a fractal medium.

The Helmholtz equation with local fractional derivative operators in the two-dimensional case was suggested in [43, 44] as follows:

$$\frac{\partial^{2\alpha}H}{\partial x^{2\alpha}}+\frac{\partial^{2\alpha}H}{\partial y^{2\alpha}}+w^{2\alpha}H=f(x,y),0\prec\alpha\leq 1,\tag{8}$$

with the initial value conditions as follows:

$$H(0,y)=\varphi(y),\ \frac{\partial^{2\alpha}H(0,y)}{\partial x^{2\alpha}}=\Psi(y),\tag{9}$$

where $H(x,y)$ is the unknown function and $f(x,y)$ is a source term.

Definition 3. The formulas of the local fractional of special functions used in the present research are as follows [28, 30]:

$$\begin{aligned}E_\alpha(x^\alpha)&=\sum_{k=0}^{\infty}\frac{x^{\alpha k}}{\Gamma(1+\alpha k)},\\ \sin_\alpha(x^\alpha)&=\sum_{k=0}^{\infty}(-1)^k\frac{x^{(2k+1)\alpha}}{\Gamma(1+(2k+1)\alpha)},\\ \cos_\alpha(x^\alpha)&=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k\alpha}}{\Gamma(1+2\alpha k)},\\ \sin h_\alpha(x^\alpha)&=\sum_{k=0}^{\infty}\frac{x^{(2k+1)\alpha}}{\Gamma(1+(2k+1)\alpha)},\\ \cos h_\alpha(x^\alpha)&=\sum_{k=0}^{\infty}\frac{x^{2k\alpha}}{\Gamma(1+2\alpha k)}.\end{aligned}\tag{10}$$

Definition 4. The properties of local fractional derivatives and local fractional integrals of nondifferentiable functions are as follows [28, 30]:

$$\begin{aligned}\frac{d^\alpha}{dx^\alpha}\left\{\frac{x^{n\alpha}}{\Gamma(1+(n-1)\alpha)}\right\}&=\frac{x^{(n-1)\alpha}}{\Gamma(1+(n-1)\alpha)},\\ \frac{d^\alpha}{dx^\alpha}\{E_\alpha(x^\alpha)\}&=E_\alpha(x^\alpha),\\ \frac{d^\alpha}{dx^\alpha}\{\sin_\alpha(x^\alpha)\}&=\cos_\alpha(x^\alpha),\\ \frac{d^\alpha}{dx^\alpha}\{\cos_\alpha(x^\alpha)\}&=-\sin_\alpha(x^\alpha),\\ \frac{d^\alpha}{dx^\alpha}\{\sin h_\alpha(x^\alpha)\}&=\cos h_\alpha(x^\alpha),\\ \frac{d^\alpha}{dx^\alpha}\{\cos h_\alpha(x^\alpha)\}&=-\sin h_\alpha(x^\alpha).\end{aligned}\tag{11}$$

3. Analysis of HP Method

To show the ability of local fractional HP, the local fractional differential equation is written as follows:

$$L_\alpha(u^\alpha)=0,u\in R,\tag{12}$$

where L_α is a local fractional differential operator. Convex nondifferentiable homotopy $H_\alpha(u,p)$ is defined as follows:

$$\begin{aligned}H_\alpha(u,p)=(1-p^\alpha)&(L_\alpha(u^\alpha)-L_\alpha(u_0{}^\alpha))\\ &+p^\alpha L_\alpha(u^\alpha),u\in R,p\in[0.1],\end{aligned}\tag{13}$$

that is equivalent to relationship (14):

$$H_\alpha(u,p)=L_\alpha(u^\alpha)-L_\alpha(u_0{}^\alpha)+p^\alpha L_\alpha(u^\alpha),u\in R,p\in[0.1],\tag{14}$$

where p is an imbedding parameter and u_0 is an initial approximation for relation (13). By substituting $H_\alpha(u,p)$ equal to zero, it can be written as

$$H_\alpha(u,p)=L_\alpha(u^\alpha)-L_\alpha(u_0{}^\alpha)=0,\tag{15}$$

$$H_\alpha(u,1)=L_\alpha(u_0{}^\alpha)=0.\tag{16}$$

In nondistinguishable homotopy, these relations are called nondistinguishable deformation, and $L_\alpha(u^\alpha)-L_\alpha(u_0^\alpha)$, $L_\alpha(u_0^\alpha)$ are called nondistinguishable homotopy. By applying nondistinguishable perturbation method, relationship (15) is written as follows:

$$u^\alpha=u_0{}^\alpha+P^\alpha u_1{}^\alpha+P^{2\alpha}u_2{}^\alpha+P^{3\alpha}u_3{}^\alpha+\cdots=\sum_{i=0}^{n}P^{i\alpha}u_i{}^\alpha.\tag{17}$$

Substituting relation (17) in relationship (13) could write

$$\begin{aligned}H_\alpha\left(\sum_{i=0}^{n}P^iu_i,P\right)=(1-P^\alpha)&\left(L_\alpha\left(\sum_{i=0}^{n}P^iu_i\right)-L_\alpha(u_0)\right)\\ &+P^\alpha L_\alpha\left(\sum_{i=0}^{n}P^iu_i\right).\end{aligned}\tag{18}$$

In the other words, for the calculation of the approximated solution of relation (12), relationship $L_\alpha(u)$ is expanded as a local fractional Taylor series. Therefore,

$$\begin{aligned} L_\alpha(u^\alpha) = L_\alpha\left(\sum_{i=0}^{n} P^{i\alpha} {u_i}^\alpha\right) &= L_\alpha({u_0}^\alpha) \\ &+ \frac{d^\alpha(L_\alpha({u_0}^\alpha))}{du^\alpha}\frac{\left(\sum_{i=0}^{n} P^i u_i - u_0\right)^\alpha}{\Gamma(1+\alpha)} \\ &+ \frac{d^{2\alpha}(L_\alpha({u_0}^\alpha))}{du^{2\alpha}}\frac{\left(\sum_{i=0}^{n} P^i u_i - u_0\right)^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots \\ &+ \frac{d^{n\alpha}(L_\alpha({u_0}^\alpha))}{du^\alpha}\frac{\left(\sum_{i=0}^{n} P^i u_i - u_0\right)^{n\alpha}}{\Gamma(1+n\alpha)} + \cdots. \end{aligned} \tag{19}$$

Now, substituting relation (17) in relationship (18) and then in equation (13), we can write

$$\begin{aligned} H_\alpha\left(\sum_{i=0}^{n} P^i u_i, P\right) = (1-P^\alpha)\Bigg(&\frac{d^\alpha(L_\alpha({u_0}^\alpha))}{du^\alpha}\frac{\left(\sum_{i=0}^{n} P^i u_i - u_0\right)^\alpha}{\Gamma(1+\alpha)} \\ &+ \frac{d^{2\alpha}(L_\alpha({u_0}^\alpha))}{du^{2\alpha}}\frac{\left(\sum_{i=0}^{n} P^i u_i - u_0\right)^{2\alpha}}{\Gamma(1+2\alpha)} \\ &+ \frac{d^{n\alpha}(L_\alpha({u_0}^\alpha))}{du^\alpha}\frac{\left(\sum_{i=0}^{n} P^i u_i - u_0\right)^{n\alpha}}{\Gamma(1+n\alpha)}\Bigg) \\ + P^\alpha\Bigg(&\frac{d^\alpha(L_\alpha({u_0}^\alpha))}{du^\alpha}\frac{\left(\sum_{i=0}^{n} P^i u_i - u_0\right)^\alpha}{\Gamma(1+\alpha)} \\ &+ \frac{d^{2\alpha}(L_\alpha({u_0}^\alpha))}{du^{2\alpha}}\frac{\left(\sum_{i=0}^{n} P^i u_i - u_0\right)^{2\alpha}}{\Gamma(1+2\alpha)} \\ &+ \frac{d^{n\alpha}(L_\alpha({u_0}^\alpha))}{du^\alpha}\frac{\left(\sum_{i=0}^{n} P^i u_i - u_0\right)^{n\alpha}}{\Gamma(1+n\alpha)} + \cdots\Bigg), \end{aligned} \tag{20}$$

that is reduceable to relations (21) and (22).

$$\begin{aligned} H_\alpha(u,1) = L_\alpha(u^\alpha) - L_\alpha({u_0}^\alpha) &= \frac{d^\alpha(L_\alpha({u_0}^\alpha))}{du^\alpha}\frac{\left(\sum_{i=0}^{n} P^i u_i - u_0\right)^\alpha}{\Gamma(1+\alpha)} \\ &+ \frac{d^{2\alpha}(L_\alpha({u_0}^\alpha))}{du^{2\alpha}}\frac{\left(\sum_{i=0}^{n} P^i u_i - u_0\right)^{2\alpha}}{\Gamma(1+2\alpha)} \\ &+ \frac{d^{n\alpha}(L_\alpha({u_0}^\alpha))}{du^\alpha}\frac{\left(\sum_{i=0}^{n} P^i u_i - u_0\right)^{n\alpha}}{\Gamma(1+n\alpha)} + \cdots = 0, \end{aligned} \tag{21}$$

$$\begin{aligned} H_\alpha(u,1) = L_\alpha(u^\alpha) = L_\alpha({u_0}^\alpha) &+ \frac{d^\alpha(L_\alpha({u_0}^\alpha))}{du^\alpha}\frac{\left(\sum_{i=0}^{n} P^i u_i - u_0\right)^\alpha}{\Gamma(1+\alpha)} \\ &+ \frac{d^{2\alpha}(L_\alpha({u_0}^\alpha))}{du^{2\alpha}}\frac{\left(\sum_{i=0}^{n} P^i u_i - u_0\right)^{2\alpha}}{\Gamma(1+2\alpha)} \\ &+ \frac{d^{n\alpha}(L_\alpha({u_0}^\alpha))}{du^\alpha}\frac{\left(\sum_{i=0}^{n} P^i u_i - u_0\right)^{n\alpha}}{\Gamma(1+n\alpha)} + \cdots = 0. \end{aligned} \tag{22}$$

Using relation (21) and simplification, we can write

$$p^{0\alpha} : L_\alpha(u^\alpha) - L_\alpha({u_0}^\alpha) = 0, \tag{23}$$

$$p^{1\alpha} : \frac{d^\alpha(L_\alpha({u_0}^\alpha))}{du^\alpha}\frac{{u_1}^\alpha}{\Gamma(1+\alpha)} + L_\alpha({u_0}^\alpha) = 0, \tag{24}$$

$$p^{2\alpha} : \frac{d^\alpha(L_\alpha({u_0}^\alpha))}{du^\alpha}\frac{{u_2}^\alpha}{\Gamma(1+\alpha)} + \frac{d^{2\alpha}(L_\alpha({u_0}^\alpha))}{du^{2\alpha}}\frac{{u_1}^{2\alpha}}{\Gamma(1+2\alpha)}. \tag{25}$$

It can be written according to relation (24):

$${u_1}^\alpha = -\frac{\Gamma(1+\alpha)L_\alpha({u_0}^\alpha)}{d^\alpha(L_\alpha({u_0}^\alpha))/du^\alpha}. \tag{26}$$

Therefore, by considering u^α as the first approximation, it results to $u^\alpha = u_0^\alpha + p^\alpha u_1^\alpha$; consequently,

$$u^\alpha = {u_0}^\alpha - p^\alpha\frac{\Gamma(1+\alpha)L_\alpha({u_0}^\alpha)}{d^\alpha(L_\alpha({u_0}^\alpha))/du^\alpha}, \tag{27}$$

When $p = 1$, applying relation (27), reversible relation (28) is concluded.

$${u_{n+1}}^\alpha = {u_n}^\alpha - \frac{\Gamma(1+\alpha)L_\alpha({u_n}^\alpha)}{d^\alpha(L_\alpha({u_n}^\alpha))/du^\alpha}. \tag{28}$$

This equation is a local fractional iteration Newton famous formula [3–11], and this relation is converged. Using relationship (28), we can calculate the relation of a local fractional iteration semi-Newton with second-order approximation parameters as follows:

$$\begin{aligned} {u_{n+1}}^\alpha = {u_n}^\alpha &- \frac{\Gamma(1+\alpha)L_\alpha({u_n}^\alpha)}{d^\alpha(L_\alpha({u_n}^\alpha))/du^\alpha} - \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{d^{2\alpha}(L_\alpha({u_n}^\alpha))/du^{2\alpha}}{d^\alpha(L_\alpha({u_n}^\alpha))/du^\alpha} \\ &\cdot \left(\frac{\Gamma(1+\alpha)L_\alpha({u_n}^\alpha)}{d^\alpha(L_\alpha({u_n}^\alpha))/du^\alpha}\right)^{2\alpha}. \end{aligned} \tag{29}$$

When $p \longrightarrow 1$, then the approximation solution is as follows:

$$U = \lim_{P\to 1}\sum_{i=0}^{n} P^{i\alpha} u_i = \sum_{i=0}^{n} u_i. \tag{30}$$

It should be noted that the old perturbation method in [1–7] is significant in the case of the fractional domain of α equal to 1.

4. Illustrative Applied Examples

In this section, local damped wave equation, wave equation in the fractal strings, and heat diffusion equation in the case of local fractional using HPM were solved.

Example 1. The wave equation in the fractal strings.

At first, the wave equation in the fractal strings is also intended that is given the initial conditions [17].

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}} - \frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}} = 0, t > 0, x\epsilon R, 0 < \alpha < 1, \tag{31}$$

considering the boundary conditions:

$$u(x,0) = 0, \frac{\partial^{\alpha}u(x,0)}{\partial t^{\alpha}} = E_{\alpha}(x^{\alpha}). \tag{32}$$

According to the local fractional HPM, the unrecognizable homotopy structure can be written as follows:

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}} = p^{\alpha}\left(\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}} - \frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}}\right), 0 \le p \le 1. \tag{33}$$

On the other hand, according to equation (13), we can write

$$u(x,t) = \sum_{i=0}^{n} p^{i\alpha}u_i(x,t). \tag{34}$$

By substituting (17) in (16), we can say

$$\begin{aligned}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\left[\sum_{i=0}^{n} p^{i\alpha}u_i(x,t)\right] = p^{\alpha}\Bigg[\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\left[\sum_{i=0}^{n} p^{i\alpha}u_i(x,t)\right] \\ - \frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\left[\sum_{i=0}^{n} p^{i\alpha}u_i(x,t)\right]\Bigg].\end{aligned} \tag{35}$$

Now, for separating the sentences of relationship (18), just this relationship can be expanded and separately written to various powers p; therefore, linear fractional derivations of differential equations are adjusted as follows:

$$p^{0\alpha}: \frac{\partial^{2\alpha}u_0(x,t)}{\partial t^{2\alpha}} = 0, u_0(x,0) = 0, \frac{\partial^{\alpha}u_0(x,0)}{\partial t^{\alpha}} = E_{\alpha}(x^{\alpha}), \tag{36}$$

$$p^{1\alpha}: \frac{\partial^{2\alpha}u_1(x,t)}{\partial t^{2\alpha}} = \frac{\partial^{2\alpha}u_0(x,t)}{\partial x^{2\alpha}}, u_1(x,0) = 0, \frac{\partial^{\alpha}u_1(x,0)}{\partial t^{\alpha}} = 0, \tag{37}$$

$$p^{2\alpha}: \frac{\partial^{2\alpha}u_2(x,t)}{\partial t^{2\alpha}} = \frac{\partial^{2\alpha}u_1(x,t)}{\partial x^{2\alpha}}, u_2(x,0) = 0, \frac{\partial^{\alpha}u_2(x,0)}{\partial t^{\alpha}} = 0, \tag{38}$$

$$p^{3\alpha}: \frac{\partial^{2\alpha}u_3(x,t)}{\partial t^{2\alpha}} = \frac{\partial^{2\alpha}u_2(x,t)}{\partial x^{2\alpha}}, u_3(x,0) = 0, \frac{\partial^{\alpha}u_3(x,0)}{\partial t^{\alpha}} = 0, \tag{39}$$

$$p^{4\alpha}: \frac{\partial^{2\alpha}u_4(x,t)}{\partial t^{2\alpha}} = \frac{\partial^{2\alpha}u_3(x,t)}{\partial x^{2\alpha}}, u_4(x,0) = 0, \frac{\partial^{\alpha}u_4(x,0)}{\partial t^{\alpha}} = 0. \tag{40}$$

Finally,:

$$p^{t\alpha}: \frac{\partial^{2\alpha}\mathrm{u}_{\mathrm{i}}(x,t)}{\partial t^{2\alpha}} = \frac{\partial^{2\alpha}u_{i-1}(x,t)}{\partial x^{2\alpha}}, u_{i-1}(x,0) = 0, \frac{\partial^{\alpha}u_{i-1}(x,0)}{\partial t^{\alpha}} = 0. \tag{41}$$

Evaluation of equations (36) to (41) for u_0, u_1, u_2, u_3, u_4, u_i, some components of the solution of local fractional HP are as follows:

$$\begin{aligned}u_0(x,t) &= \frac{t^{\alpha}}{\Gamma(1+\alpha)}E_{\alpha}(x^{\alpha}), \\ u_1(x,t) &= \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}E_{\alpha}(x^{\alpha}), \\ u_2(x,t) &= \frac{t^{5\alpha}}{\Gamma(1+5\alpha)}E_{\alpha}(x^{\alpha}), \\ u_3(x,t) &= \frac{t^{7\alpha}}{\Gamma(1+7\alpha)}E_{\alpha}(x^{\alpha}), \\ u_4(x,t) &= \frac{t^{9\alpha}}{\Gamma(1+9\alpha)}E_{\alpha}(x^{\alpha}).\end{aligned} \tag{42}$$

Finally,

$$u_i(x,t) = \frac{t^{(2i+1)\alpha}}{\Gamma(1+(2i+1)\alpha)}E_{\alpha}(x^{\alpha}). \tag{43}$$

Therefore, almost unrecognizable for relation (14) will be as follows:

$$u(x,t) = \sum_{i=0}^{n} \frac{t^{(2i+1)\alpha}}{\Gamma(1+(2i+1)\alpha)}E_{\alpha}(x^{\alpha}) = \sinh_{\alpha}(t^{\alpha})E_{\alpha}(x^{\alpha}). \tag{44}$$

According to equation (15), $E_{\alpha}(x^{\alpha})$ is considered as boundary condition for computations as follows:

$$E_{\alpha}(x^{\alpha}) = \sum_{i=0}^{N} \frac{x^{i\alpha}}{\Gamma(1+i\alpha)}. \tag{45}$$

With these conditions using the computer programming in the MATLAB software, $u(x,t)$ can be plotted as shown in Figure 1. For Figures 1 and 2, $\alpha = 1$ and $\alpha = \ln 2/\ln 3$, respectively.

Since most references in their calculations α equal to $\boldsymbol{\alpha} = \mathbf{ln\ 2/ln\ 3}$ are considered, therefore, in this study, for better understanding of local fractional wave equation, $\boldsymbol{\alpha} = \mathbf{ln\ 2/ln\ 3}$ is considered and for three different time versus x plotted in Figure 3, and it is remarkable.

Example 2. Heat diffusion equation in the case of local fractional.

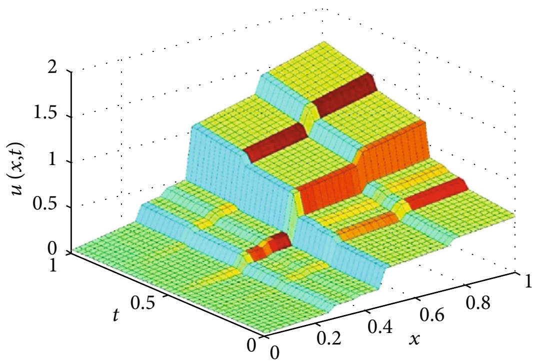

Figure 1: Variation of $u(x,t)$ versus x, t and $\alpha = 1$.

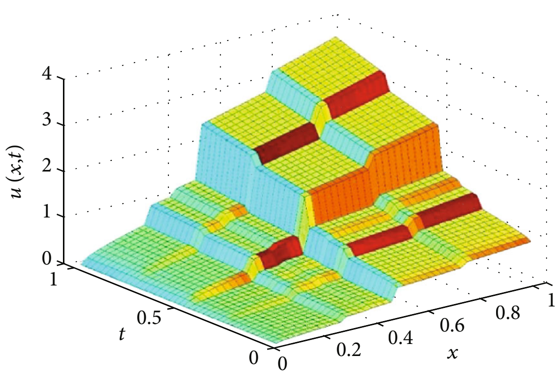

Figure 2: Variation of $u(x,t)$ versus x, t and $\alpha = \ln 2/\ln 3$.

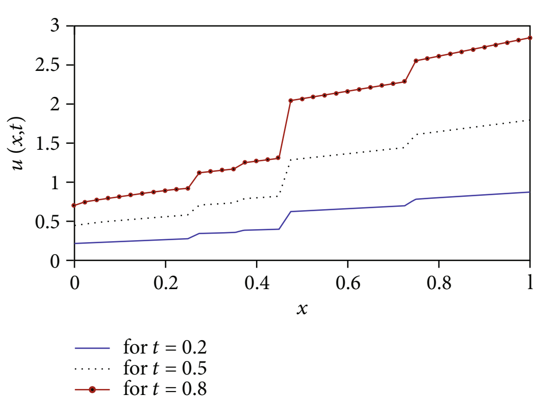

Figure 3: Variation of $u(x,t)$ versus x, with $\alpha = \ln 2/\ln 3$, $t = 0.2$, $0.5, 0.8$ sec.

The next example, heat diffusion equation in the case of local fractional, using the HPM, is solved as follows:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}} = -\cos_\alpha(x^\alpha). \tag{46}$$

In this case, the initial condition is considered as follows:

$$u(x,0) = \sin_\alpha(x^\alpha). \tag{47}$$

Now, HPM can be used, and the above equation can be written as follows:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^\alpha u_0(x,t)}{\partial t^\alpha} = p^\alpha \left[\frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}} - \cos_\alpha(x^\alpha) - \frac{\partial^\alpha u_0(x,t)}{\partial t^\alpha}\right]. \tag{48}$$

According to relationship $u(x,t) = \sum_{i=0}^{n} p^{i\alpha} u_i(x,t)$ and substitution in equation (48), consequently

$$\frac{\partial^\alpha}{\partial t^\alpha}\left[\sum_{i=0}^{n} p^{i\alpha} u_i(x,t)\right] - \frac{\partial^\alpha u_0(x,t)}{\partial t^\alpha} = p^\alpha \left[\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\left[\sum_{i=0}^{n} p^{i\alpha} u_i(x,t)\right] - \cos_\alpha(x^\alpha) - \frac{\partial^\alpha u_0(x,t)}{\partial t^\alpha}\right]. \tag{49}$$

Therefore, linear fractional derivations of differential equations are as follows:

$$p^{0\alpha}: \frac{\partial^\alpha u_0(x,t)}{\partial t^\alpha} - \frac{\partial^\alpha u_0(x,t)}{\partial t^\alpha} = 0, u_0(x,0) = \sin_\alpha(x^\alpha), \tag{50}$$

$$p^{1\alpha}: \frac{\partial^\alpha u_1(x,t)}{\partial t^\alpha} = \frac{\partial^{2\alpha} u_0(x,t)}{\partial x^{2\alpha}} - \cos_\alpha(x^\alpha) - \frac{\partial^\alpha u_0(x,t)}{\partial t^\alpha}, u_1(x,0) = 0, \tag{51}$$

$$p^{2\alpha}: \frac{\partial^\alpha u_2(x,t)}{\partial t^\alpha} = \frac{\partial^{2\alpha} u_1(x,t)}{\partial x^{2\alpha}}, u_2(x,0) = 0, \tag{52}$$

$$p^{3\alpha}: \frac{\partial^\alpha u_3(x,t)}{\partial t^\alpha} = \frac{\partial^{2\alpha} u_2(x,t)}{\partial x^{2\alpha}}, u_3(x,0) = 0, \tag{53}$$

$$p^{4\alpha}: \frac{\partial^\alpha u_4(x,t)}{\partial t^\alpha} = \frac{\partial^{2\alpha} u_3(x,t)}{\partial x^{2\alpha}}, u_4(x,0) = 0, \tag{54}$$

and so on. By solving equations (50) to (54), it becomes clear that

$$\begin{aligned}
u_0(x,0) &= \sin_\alpha(x^\alpha),\\
u_1(x,t) &= -\frac{t^\alpha}{\Gamma(1+\alpha)}\left[\sin_\alpha(x^\alpha) + \cos_\alpha(x^\alpha)\right],\\
u_2(x,t) &= \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\left[\sin_\alpha(x^\alpha) + \cos_\alpha(x^\alpha)\right],\\
cu_3(x,t) &= -\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\left[\sin_\alpha(x^\alpha) + \cos_\alpha(x^\alpha)\right],\\
u_4(x,t) &= \frac{t^{4\alpha}}{\Gamma(1+4\alpha)}\left[\sin_\alpha(x^\alpha) + \cos_\alpha(x^\alpha)\right].
\end{aligned} \tag{55}$$

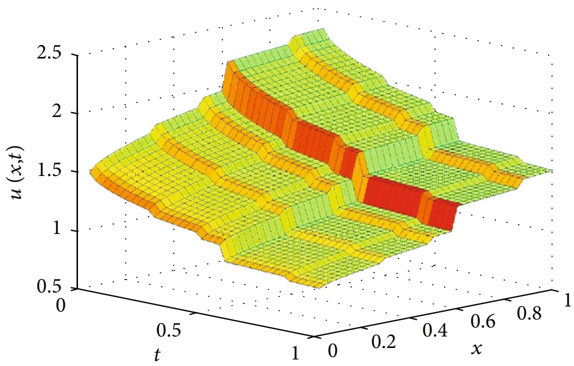

FIGURE 4: Variation of $u(x,t)$versus x, t, with $\alpha = \ln 2/\ln 3$.

When $p \longrightarrow 1$, then $u(x,t) = \sum_{i=0}^{n} u_i(x,t)$; therefore,

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots,$$
$$u(x,t) = \sin_\alpha(x^\alpha) + \sum_{j=1}^{\infty} \frac{(-1)^j t^{j\alpha}}{\Gamma(1+j\alpha)} [\sin_\alpha(x^\alpha) + \cos_\alpha(x^\alpha)]. \tag{56}$$

The equation can be rewritten as

$$u(x,t) = \sum_{j=0}^{\infty} \frac{(-1)^j t^{j\alpha}}{\Gamma(1+j\alpha)} [\sin_\alpha(x^\alpha) + \cos_\alpha(x^\alpha)] - \cos_\alpha(x^\alpha),$$
$$u(x,t) = E_\alpha(-t^\alpha)[\sin_\alpha(x^\alpha) + \cos_\alpha(x^\alpha)] - \cos_\alpha(x^\alpha). \tag{57}$$

Considering $\alpha = \ln 2/\ln 3$, $u(x,t)$ is plotted in Figure 4. Now, equation (48) is considered as follows:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}} = 0, \tag{58}$$

that the initial condition of the form $u(x,0) = E_\alpha(x^\alpha)$ is considered; applying the above operation is determined answering the following:

$$u(x,t) = \sum_{i=1}^{n} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} E_\alpha(x^\alpha) = E_\alpha(t^\alpha) E_\alpha(x^\alpha). \tag{59}$$

By placing $\alpha = \ln 2/\ln 3$, $u(x,t)$ is shown in Figure 5.

Example 3. Local damped wave equation.

The second example, local damped wave equation (60), using the HPM solved [45].

$$\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} - \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}} = 0. \tag{60}$$

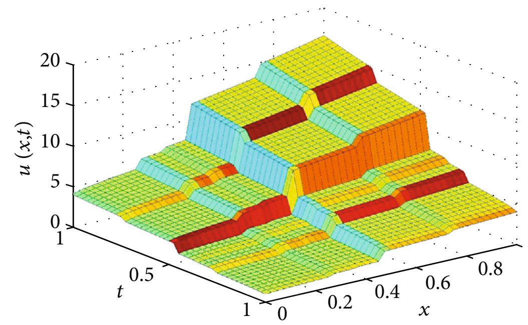

FIGURE 5: Variation of $u(x,t)$ versus x, t with $\alpha = \ln 2/\ln 3$, equation (59).

In this case, the initial condition is considered as follows:

$$u(x,0) = E_\alpha(x^\alpha) = \sum_{i=0}^{N} \frac{x^{i\alpha}}{\Gamma(1+i\alpha)}. \tag{61}$$

Now, HPM can be used, and the above equation can be written as follows:

$$\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} - \frac{\partial^{2\alpha} u_0(x,t)}{\partial t^{2\alpha}} = p^\alpha \left[\frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}} + \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^{2\alpha} u_0(x,t)}{\partial t^{2\alpha}} \right]. \tag{62}$$

According to relationship $u(x,t) = \sum_{i=0}^{n} p^{i\alpha} u_i(x,t)$ and substitution in equation (62), it results to the following:

$$\begin{aligned}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}} \left[\sum_{i=0}^{n} p^{i\alpha} u_i(x,t) \right] - \frac{\partial^{2\alpha} u_0(x,t)}{\partial t^{2\alpha}} = p^\alpha \Bigg[& \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} \left[\sum_{i=0}^{n} p^{i\alpha} u_i(x,t) \right] \\ & + \frac{\partial^\alpha}{\partial t^\alpha} \left[\sum_{i=0}^{n} p^{i\alpha} u_i(x,t) \right] \\ & - \frac{\partial^{2\alpha} u_0(x,t)}{\partial t^{2\alpha}} \Bigg].\end{aligned} \tag{63}$$

Therefore, linear fractional derivations of differential equations are as follows:

$$p^{0\alpha} : \frac{\partial^{2\alpha} u_0(x,t)}{\partial t^{2\alpha}} - \frac{\partial^{2\alpha} u_0(x,t)}{\partial t^{2\alpha}} = 0, u_0(x,0) = E_\alpha(x^\alpha), \tag{64}$$

$$p^{1\alpha} : \frac{\partial^{2\alpha} u_1(x,t)}{\partial t^{2\alpha}} = \frac{\partial^{2\alpha} u_0(x,t)}{\partial x^{2\alpha}} + \frac{\partial^\alpha u_0(x,t)}{\partial t^\alpha}, u_1(x,0) = 0, \tag{65}$$

$$p^{2\alpha} : \frac{\partial^{2\alpha} u_2(x,t)}{\partial t^{2\alpha}} = \frac{\partial^{2\alpha} u_1(x,t)}{\partial x^{2\alpha}} + \frac{\partial^\alpha u_1(x,t)}{\partial t^\alpha}, u_2(x,0) = 0, \tag{66}$$

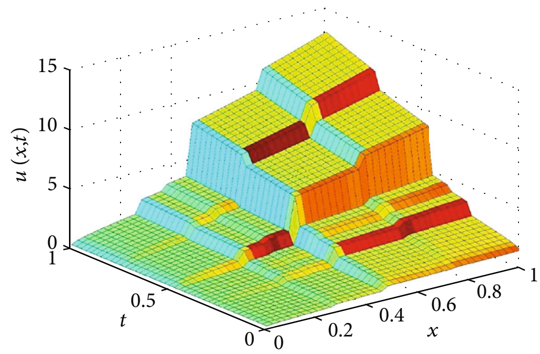

FIGURE 6: Variation of $u(x,t)$ versus x, t with $\alpha = \ln 2/\ln 3$, equation (71).

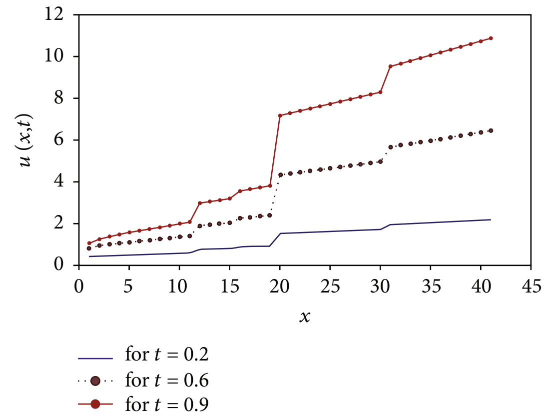

FIGURE 7: Variation of $u(x,t)$ versus x with $\alpha = \ln 2/\ln 3$, $t = 0.2$, $0.6, 0.9$ sec, relation (71).

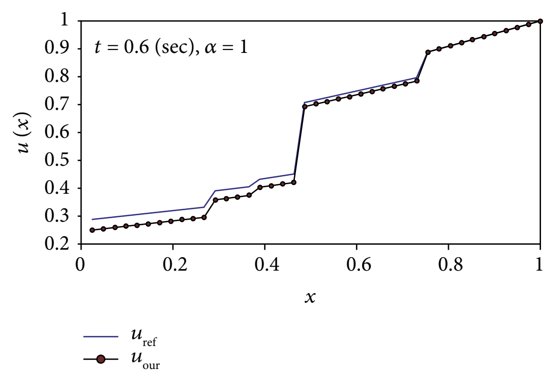

FIGURE 8: Variation of $u(x,t)$ versus x with $\alpha = 1$, $t = 0.6$ sec.

$$p^{3\alpha}: \frac{\partial^{2\alpha} u_3(x,t)}{\partial t^{2\alpha}} = \frac{\partial^{2\alpha} u_2(x,t)}{\partial x^{2\alpha}} + \frac{\partial^{\alpha} u_2(x,t)}{\partial t^{\alpha}}, u_3(x,0) = 0, \tag{67}$$

$$p^{4\alpha}: \frac{\partial^{2\alpha} u_4(x,t)}{\partial t^{2\alpha}} = \frac{\partial^{2\alpha} u_3(x,t)}{\partial x^{2\alpha}} + \frac{\partial^{\alpha} u_3(x,t)}{\partial t^{\alpha}}, u_4(x,0) = 0. \tag{68}$$

By solving the equations (64) to (68), it becomes clear that

$$\begin{aligned}
u_0(x,0) &= E_\alpha(x^\alpha),\\
u_0(x,t) &= \int u_0(x,0)dt^\alpha = \frac{t^\alpha}{\Gamma(1+\alpha)} E_\alpha(x^\alpha),\\
\frac{\partial^{2\alpha} u_1(x,t)}{\partial t^{2\alpha}} &= \frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\left[\frac{t^\alpha}{\Gamma(1+\alpha)} E_\alpha(x^\alpha)\right] + \frac{\partial^{\alpha}}{\partial t^{\alpha}}\left[\frac{t^\alpha}{\Gamma(1+\alpha)} E_\alpha(x^\alpha)\right],\\
u_1(x,t) &= \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} E_\alpha(x^\alpha) + \frac{t^{\alpha}}{\Gamma(1+\alpha)} E_\alpha(x^\alpha),\\
u_2(x,t) &= \frac{t^{5\alpha}}{\Gamma(1+5\alpha)} E_\alpha(x^\alpha) + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} E_\alpha(x^\alpha),\\
u_3(x,t) &= \frac{t^{7\alpha}}{\Gamma(1+7\alpha)} E_\alpha(x^\alpha) + \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} E_\alpha(x^\alpha),\\
u_4(x,t) &= \frac{t^{9\alpha}}{\Gamma(1+9\alpha)} E_\alpha(x^\alpha) + \frac{t^{4\alpha}}{\Gamma(1+4\alpha)} E_\alpha(x^\alpha).
\end{aligned} \tag{69}$$

Finally,

$$u_m(x,t) = \frac{t^{(m+1)\alpha}}{\Gamma(1+(m+1)\alpha)} E_\alpha(x^\alpha) + \frac{t^{m\alpha}}{\Gamma(1+m\alpha)} E_\alpha(x^\alpha). \tag{70}$$

When $p \longrightarrow 1$, then $u(x,t) = \sum_{i=0}^{n} u_i(x,t)$; therefore,

$$\begin{aligned}
u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots u(x,t)\\
&= \frac{t^\alpha}{\Gamma(1+\alpha)} E_\alpha(x^\alpha) + \sum_{j=1}^{\infty}\left[\frac{t^{(j+1)\alpha}}{\Gamma(1+(j+1)\alpha)}\right.\\
&\quad \left. + \frac{t^{j\alpha}}{\Gamma(1+j\alpha)}\right] E_\alpha(x^\alpha).
\end{aligned} \tag{71}$$

By placing $\alpha = \ln 2/\ln 3$, $u(x,t)$ versus x, t is illustrated in Figure 6. Also, Figure 7 shows that for three different times, the variation of $u(x,t)$ versus x is considerable.

5. Comparison of Results of HP and Fourier Series Methods

In this section, for the validation and benchmarking of the present method, equation (60) means that the local damped wave equation by this method and the results of the Fourier series method for this equation are compared and discussed, according to the results for the following equation with initial condition:

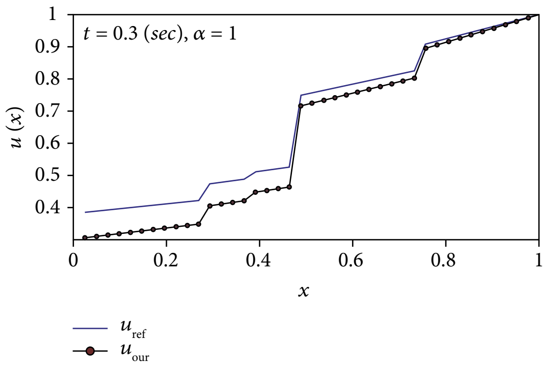

FIGURE 9: Variation of $u(x, t)$ versus x with $\alpha = 1$, $t = 0.3$ sec.

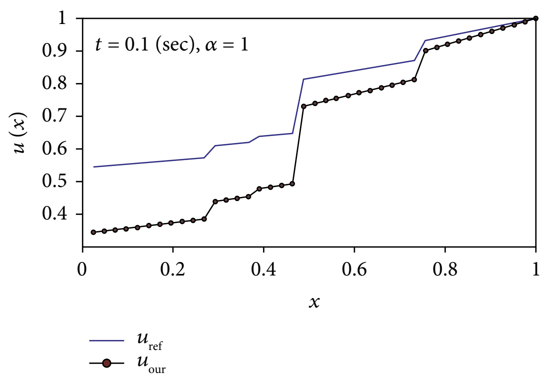

FIGURE 10: Variation of $u(x, t)$ versus x with $\alpha = 1$, $t = 0.1$ sec.

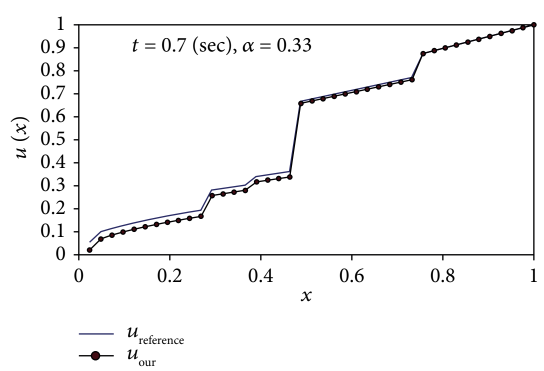

FIGURE 11: Variation of $u(x, t)$ versus x, with $\alpha = 0.33$, $t = 0.7$ sec.

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}} - \frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} - \frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}} = 0\, u(x,0) = E_\alpha(x^\alpha) = \sum_{i=0}^{N} \frac{x^{i\alpha}}{\Gamma(1+i\alpha)}. \tag{72}$$

Using HPM, the solution is determined as follows:

$$u_{\text{homotopy}}(x,t) = \frac{t^\alpha}{\Gamma(1+\alpha)} E_\alpha(x^\alpha) + \sum_{j=1}^{\infty}\left[\frac{t^{(j+1)\alpha}}{\Gamma(1+(j+1)\alpha)} + \frac{t^{j\alpha}}{\Gamma(1+j\alpha)}\right] E_\alpha(x^\alpha). \tag{73}$$

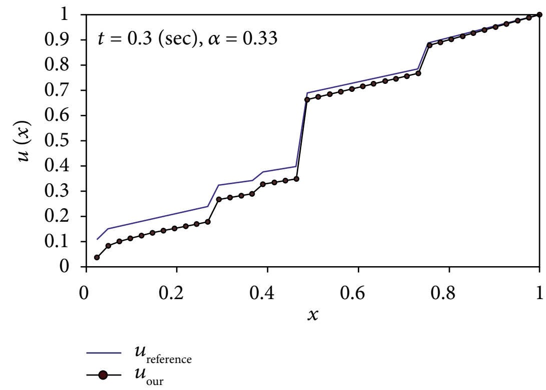

FIGURE 12: Variation of $u(x, t)$ versus x, with $\alpha = 0.33$, $t = 0.3$ sec.

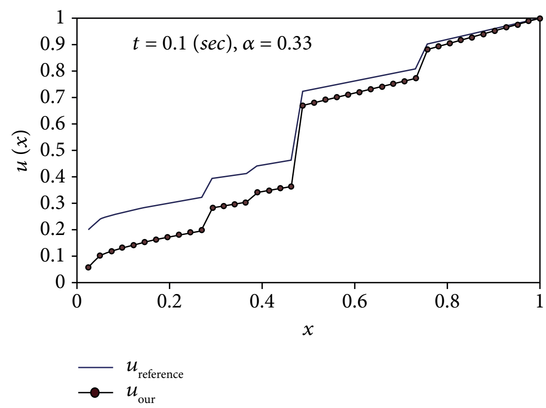

FIGURE 13: Variation of $u(x, t)$ versus x with $\alpha = 0.33$, $t = 0.1$ sec.

In reference paper [36], the local fractional wave equation is solved:

$$\begin{aligned} u_{\text{series}}(x,t) &= \sum_{n=1}^{\infty} E_\alpha\left(-\frac{t^\alpha}{2}\right)\left[A_n \cos_\alpha(\rho t^\alpha) + B_n \sin_\alpha(\rho t^\alpha)\right] \sin_\alpha n^\alpha \left(\frac{\pi x}{l}\right)^\alpha, \\ A_n &= \frac{2\Gamma(1+\alpha)}{(n\pi)^\alpha}\left\{\frac{l^\alpha}{\Gamma(1+\alpha)} \sin_\alpha n^\alpha\left(\frac{\pi x}{l}\right)^\alpha - \left(\frac{l}{\pi x}\right)^\alpha \cdot \left[\cos_\alpha n^\alpha \left(\frac{\pi x}{l}\right)^\alpha - 1\right]\right\}, \\ B_n &= -\frac{\Gamma(1+\alpha)}{\rho(n\pi)^\alpha}\left\{\frac{l^\alpha}{\Gamma(1+\alpha)} \cos_\alpha n^\alpha\left(\frac{\pi x}{l}\right)^\alpha - \left(\frac{l}{\pi x}\right)^\alpha \sin_\alpha n^\alpha \left(\frac{\pi x}{l}\right)^\alpha\right\}, \end{aligned} \tag{74}$$

where $\rho = 2\sqrt{\pi x/l} - 1/2$ and $E_\alpha(x^\alpha) = \sum_{i=0}^{N}(x^{i\alpha}/\Gamma(1+i\alpha))$.

In this section, for several times and $\alpha = 1$ and 0.33, the variations of $u(x,t)$ are obtained using the Fourier series and HP methods. In these graphs, the u_{our} show that the HP method is considered in the present research and u_{ref} illustrate the obtained results of the Fourier series method. Comparison of two graphs of $u(x)$ for time equal to 0.1,

Table 1: The variation of $u(x, t)$ versus x with $\alpha = 1$, $t = 0.1, 0.3,$ and 0.8 sec.

x	$u(x,t)$, $\alpha=1$; t (sec) = 0.8			t (sec) = 0.3			t (sec) = 0.1		
	Present work	Yong-Ju Yang [36]	Percentage of difference	Present work	Yong-Ju Yang [36]	Percentage of difference	Present work	Yong-Ju Yang [36]	Percentage of difference
0.1	0.2623	0.2975	13.4198	0.3181	0.3912	22.9810	0.3588	0.5503	53.3724
0.2	0.2795	0.3187	14.0250	0.3229	0.4079	26.3239	0.3746	0.5631	50.3203
0.3	0.3574	0.3902	9.1774	0.4068	0.4726	16.1750	0.4383	0.6093	39.0144
0.4	0.4023	0.4339	7.8548	0.4463	0.5122	14.7659	0.4796	0.6344	32.2769
0.5	0.6972	0.7189	3.1125	0.7157	0.7509	4.9217	0.7313	0.8150	14.5789
0.6	0.7352	0.7435	1.1289	0.7519	0.7826	4.0830	0.7694	0.8374	8.8381
0.7	0.7732	0.7873	1.8236	0.7815	0.8150	4.2863	0.8012	0.8629	7.7010
0.8	0.9084	0.9126	0.4623	0.9120	0.9252	1.4474	0.9174	0.9415	2.6270
0.9	0.9516	0.9632	1.2190	0.9525	0.9651	1.3228	0.9565	0.9728	1.7041
1.0	1.0	1.0	0	1.0	1.0	0	1.0	1.0	0

Table 2: The variation of $u(x, t)$ versus x with $\alpha = 0.33$, $t = 0.1, 0.3,$ and 0.7 sec.

x	$u(x,t)$, $\alpha=0.33$; t (sec) = 0.7			t (sec) = 0.3			t (sec) = 0.1		
	Present work	Yong-Ju Yang [36]	Percentage of difference	Present work	Yong-Ju Yang [36]	Percentage of difference	Present work	Yong-Ju Yang [36]	Percentage of difference
0.1	0.0986	0.1301	31.9473	0.1228	0.1834	49.3485	0.1414	0.2643	86.9165
0.2	0.1353	0.1713	26.6075	0.1584	0.2913	38.4470	0.1771	0.2996	69.1670
0.3	0.2523	0.2854	11.5978	0.2678	0.3244	21.1352	0.2855	0.3921	37.3380
0.4	0.3142	0.3381	7.6066	0.3302	0.3780	14.4761	0.3388	0.4317	27.4203
0.5	0.6606	0.6754	2.2404	0.6725	0.6958	3.4647	0.6784	0.7269	7.1492
0.6	0.7064	0.7166	1.4439	0.7123	0.7293	2.3866	0.7195	0.7577	5.3092
0.7	0.7477	0.7602	1.6718	0.7523	0.7651	1.7015	0.7630	0.7929	3.9187
0.8	0.8965	0.9016	0.5689	0.8980	0.9075	1.0579	0.9154	0.9163	0.0098
0.9	0.9449	0.9542	0.9842	0.9443	0.9508	0.6883	0.9533	0.9603	0.0073
1.0	1.0	1.0	0	1.0	1.0	0	1.0	1.0	0

0.3, and 0.6 sec for two methods with $\alpha = 1, 0.33$ is illustrated in Figures 8–13.

According to Figures 8–10 ($\alpha = 1$) and Figures 11–13 ($\alpha = 0.33$), it is observed that for $x > 0.5$, the value of error for the results is very less, and by increasing time $t = 0.1$ to 0.7 sec for $\alpha = 1, 0.33$, the ln 2/ln3 values of convergence increased. In wave equation, by increasing time, due to damping, the wave domain decreased. Therefore, the best of case is equal to $\alpha = 0.33$ and $t = 0.6, 0.7$ sec.

Variations of $u(x, t)$ versus x with $\alpha = 1$, $t = 0.1, 0.3,$ and 0.8 sec are illustrated in Tables 1 and 2. According to Table 1, the percentage of error of $u(x, t)$ between the present study and Ref [36] with $\alpha = 1$, $t = 0.1$ sec, and $x \geq 0.6$, is less than 10. Also, the % error for $x \geq 0.5$, in $\alpha = 1$, $t =$ 0.3 sec, is less than 5, whereas for $\alpha = 1$, $t = 0.8$ sec, the % error for $x \geq 0.3$ is less than 10. This comparison shows that by increasing x from 0.5 to 1 for all of the three times, the % error between the two methods of homotopy perturbation and the Fourier series decreases sharply and converges to 1.

According to Table 1, the percentage of error of $u(x, t)$ between two methods with $\alpha = 0.33$, $t = 0.1$ sec, and $x \geq 0.5$, is less than 8. Also, the % error for $x \geq 0.5$ in $\alpha = 0.33$, $t =$ 0.3 sec, is less than 4, whereas for $\alpha = 0.33$, $t = 0.7$ sec, the % error for $x \geq 0.4$ is less than 8. It can also be concluded that by increasing x from 0.5 to 1 for the three times, the percentage of difference between the two methods decreases sharply and converges to 1.

6. Conclusion

The relationship between mathematics and engineering sciences (especially mechanical engineering, chemistry, and civil engineering) and the application of high-precision mathematical theoretical methods have received much attention in recent years, but less homotopy has been considered for real physical problems in engineering. Defining common projects between mathematicians and engineers can fill the gap.

The obvious advantage of the method is that it can be applied to various nonlinear problems. The main disadvantage is that we should suitably choose an initial guess, or infinite iterations are required.

In the present study, the analysis of HP method was conducted. This method was applied for three examples, heat diffusion equation in the case of local fractional, the wave equation in the fractional strings, the local damped wave equation. Also, the variation of $u(x, t)$ for several x, t in $\alpha = 1$ and $\alpha = \ln 2/\ln 3$ is plotted. Therefore, the local fractional for solving of partial differential equations using the HP method is down, and the obtained results in applied examples are compared to the Fourier series method (Ref. [36]); the results of two methods are in good agreement. In these examples, for $x > 0.5$, $\alpha = 0.33$, a least error was concluded. According to Tables 1 and 2, the values of $u(x, t)$ for $\alpha = 1$ and 0.33, $t = 0.1$ sec, the percentage of error between present research and Ref [36] for $x \geq 0.6$ is less than 10. Also, the % error for $x \geq 0.5$ in $\alpha = 1$ and 0.33, $t = 0.3$ sec, is less than 5, whereas for $\alpha = 1$ and 0.33, $t = 0.8$ and 0.7 sec, the % error for $x \geq 0.4$ is less than 8. Also, by increasing x, the field variable in the two methods is converged.

Data Availability

No data were used to support this study.

Additional Points

We are currently implementing this method on equations governing an FGM thick-walled cylindrical pressure vessel subjected to autofrettage pressure. In future researches, we intend to use this method to solve the equations governing thick-walled spherical pressure vessels by considering creep and damage made of functionally graded, nanocomposite, and smart materials.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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