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Research Article

Alpha Power Transformed Log-Logistic Distribution with Application to Breaking Stress Data

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In this paper, a new three-parameter lifetime distribution is introduced; the new model is a generalization of the log-logistic (LL) model, and it is called the alpha power transformed log-logistic (APTLL) distribution. The APTLL distribution is more flexible than some generalizations of log-logistic distribution. We derived some mathematical properties including moments, moment-generating function, quantile function, Rényi entropy, and order statistics of the new model. The model parameters are estimated using maximum likelihood method of estimation. The simulation study is performed to investigate the effectiveness of the estimates. Finally, we used one real-life dataset to show the flexibility of the APTLL distribution.

1. Introduction

Mahdavi and Kundu [1] introduced the alpha power transformation (APT) method to add an additional parameter to a family of distributions to increase flexibility (more applicable) in the given family. The cumulative distribution function (cdf) of an APT-G family is

$$F(x;\alpha)=\begin{cases}\dfrac{\alpha^{G(x)}-1}{\alpha-1} & \text{if } \alpha>0, \alpha\neq 1,\\ G(x) & \text{if } \alpha=1,\end{cases} \tag{1}$$

and the corresponding probability density function (pdf) is

$$f(x;\alpha)=\begin{cases}\dfrac{\ln\alpha}{\alpha-1}g(x)\alpha^{G(x)} & \text{if } \alpha>0, \alpha\neq 1,\\ g(x) & \text{if } \alpha=1.\end{cases} \tag{2}$$

In literature, many distributions are generalized using this generated family, for example, the APT Weibull distribution by Dey et al. [2], the APT generalized exponential distribution by Dey et al. [3], the APT extended exponential distribution by Hassan et al. [4], the alpha power inverted exponential distribution by Unal et al. [5], the APT inverse-Weibull distribution by Ramadan and Magdy [6], the APT Lindley distribution by Dey et al. [7], the APT inverse-Lindley distribution by Dey et al. [8], the APT power Lindley studied by Hassan et al. [9], and the APT Pareto distribution proposed in Ihtisham et al. [10].

The LL distribution is very popular and is used in many areas like survival analysis, economics, actuarial science, hydrology, geophysics (see [11]), and engineering. In some cases, LL distribution is better than the log-normal distribution. The pdf and cdf of the LL distribution is given by

$$\begin{aligned} g(x;a,b)&=\frac{b}{a^b}x^{b-1}\left(1+\left(\frac{x}{a}\right)^b\right)^{-2}, \quad x>0,\\ G(x;a,b)&=1-\left(1+\left(\frac{x}{a}\right)^b\right)^{-1}, \quad x>0, \end{aligned} \tag{3}$$

where a is a positive scale parameter and b is a positive shape parameter.

The main goal of this article is to introduce a new flexible and simple model called an APTLL distribution, and this model is more flexible than some generalizations of log-logistic distribution. The new model is discussed in Section 2. Various statistical properties of the APTLL distribution are derived in Section 3 along with more attractive expressions for quantile function, median, moments, Rényi entropy, and order statistics. The estimation of parameters of the new model using the maximum likelihood (ML) method of parameter estimation is discussed in Section 4. The simulation study is performed to investigate the effectiveness of the estimates in Section 5. In Section 6, we deal with one application to show the flexibility of the new model. Finally, conclusions are discussed in Section 7.

2. The New APTLL Distribution

The random variable (r.v.) X is said to have the APTLL model denoted by APTLL (a, b, α) with three parameters, if the pdf of X for $x \geq 0$ is

$$f(x)=\begin{cases}\dfrac{b\ \ln(\alpha)}{a^b(\alpha-1)}x^{b-1}\left(1+\left(\dfrac{x}{a}\right)^b\right)^{-2}\alpha^{1-\left(1+(x/a)^b\right)^{-1}} & \text{if } \alpha\neq 1, a, b, \alpha>0,\\ \dfrac{b}{a^b}x^{b-1}\left(1+\left(\dfrac{x}{a}\right)^b\right)^{-2} & \text{if } \alpha=1, a, b, \alpha>0,\end{cases} \tag{4}$$

$$F(x)=\begin{cases}\dfrac{\alpha^{1-\left(1+(x/a)^b\right)^{-1}}-1}{\alpha-1} & \text{if } \alpha\neq 1,\\ 1-\left(1+\left(\dfrac{x}{a}\right)^b\right)^{-1} & \text{if } \alpha=1.\end{cases} \tag{5}$$

The survival function (sf) and the hazard rate function (hrf) for the APTLL distribution for $x > 0$ are in the following forms:

$$\begin{aligned}S(x)&=\frac{\alpha-\alpha^{1-\left(1+(x/a)^b\right)^{-1}}}{\alpha-1},\\ h(x)&=\frac{a^{-b}b\ \ln(\alpha)x^{b-1}\left(1+(x/a)^b\right)^{-2}\alpha^{1-\left(1+(x/a)^b\right)^{-1}}}{\alpha-\alpha^{1-\left(1+(x/a)^b\right)^{-1}}}.\end{aligned} \tag{6}$$

The reversed and cumulative hazard rate functions are given by

$$\begin{aligned}\tau(x)&=\frac{a^{-b}b\ \ln(\alpha)x^{b-1}\left(1+(x/a)^b\right)^{-2}\alpha^{1-\left(1+(x/a)^b\right)^{-1}}}{\alpha^{1-\left(1+(x/a)^b\right)^{-1}}-1},\\ H(x)&=-\ln\left(\frac{\alpha-\alpha^{1-\left(1+(x/a)^b\right)^{-1}}}{\alpha-1}\right).\end{aligned} \tag{7}$$

Figures 1 and 2 demonstrate the plots of pdf and hrf of the APTLL model for different values of α, a, and b. Clearly, the pdf of the APTLL model is a decreasing function, unimodal, and right skewed. The hrf of the APTLL model can be decreasing, upside down, and J-shaped.

3. Fundamental Properties of the New APTLL Model

This section deals with some statistical properties of the APTLL distribution.

3.1. Quantile Function and Median. We can generate random samples from the APTLL model by inverting (5).

$$x_q=a\sqrt[b]{\frac{\ln(\alpha u-u+1)}{\ln\alpha-\ln(\alpha u-u+1)}}. \tag{8}$$

If $U \sim (0, 1)$, then $X \sim$ APTLL, the qth quantile function of APTLL is given by

$$x_q=a\sqrt[b]{\frac{\ln(\alpha u-u+1)}{\ln\alpha-\ln(\alpha u-u+1)}}. \tag{9}$$

and the median can be obtained as

$$x_{0.5}=a\sqrt[b]{\frac{\ln(0.5\alpha+0.5)}{\ln\alpha-\ln(0.5\alpha+0.5)}}. \tag{10}$$

3.2. Important Expansions. Here, in this subsection, an explicit expression for the APTLL pdf and cdf is given. By using the next exponential series representation

$$z^\beta=\sum_{i=0}^{\infty}\frac{(\log z)^i}{i!}\beta^i. \tag{11}$$

By inserting (11) in (4), we can rewrite (4) as

$$\begin{aligned}f(x)=&\frac{b\ \ln(\alpha)}{a^b(\alpha-1)}\sum_{i=0}^{\infty}\frac{(\log\alpha)^i}{i!}x^{b-1}\left(1+\left(\frac{x}{a}\right)^b\right)^{-1}\\ &\cdot\left\{1-\left(1+\left(\frac{x}{a}\right)^b\right)^{-1}\right\}^i.\end{aligned} \tag{12}$$

By applying the binomial expansion

$$(1-z)^{\beta-1}=\sum_{j=0}^{\infty}(-1)^j\binom{\beta-1}{j}z^j, \tag{13}$$

to the previous equation, we get

$$f(x)=\sum_{i,j=0}^{\infty}w_{i,j}\,x^{b-1}\left(1+\left(\frac{x}{a}\right)^b\right)^{-j-1}, \tag{14}$$

where

$$w_{i,j}=\frac{(-1)^j(i/j)b\ \ln(\alpha)(\log\alpha)^i}{a^b(\alpha-1)i!}. \tag{15}$$

The expansion for $[F(x)]^h$ is calculated, with h as the integer, by using the previous two expansions:

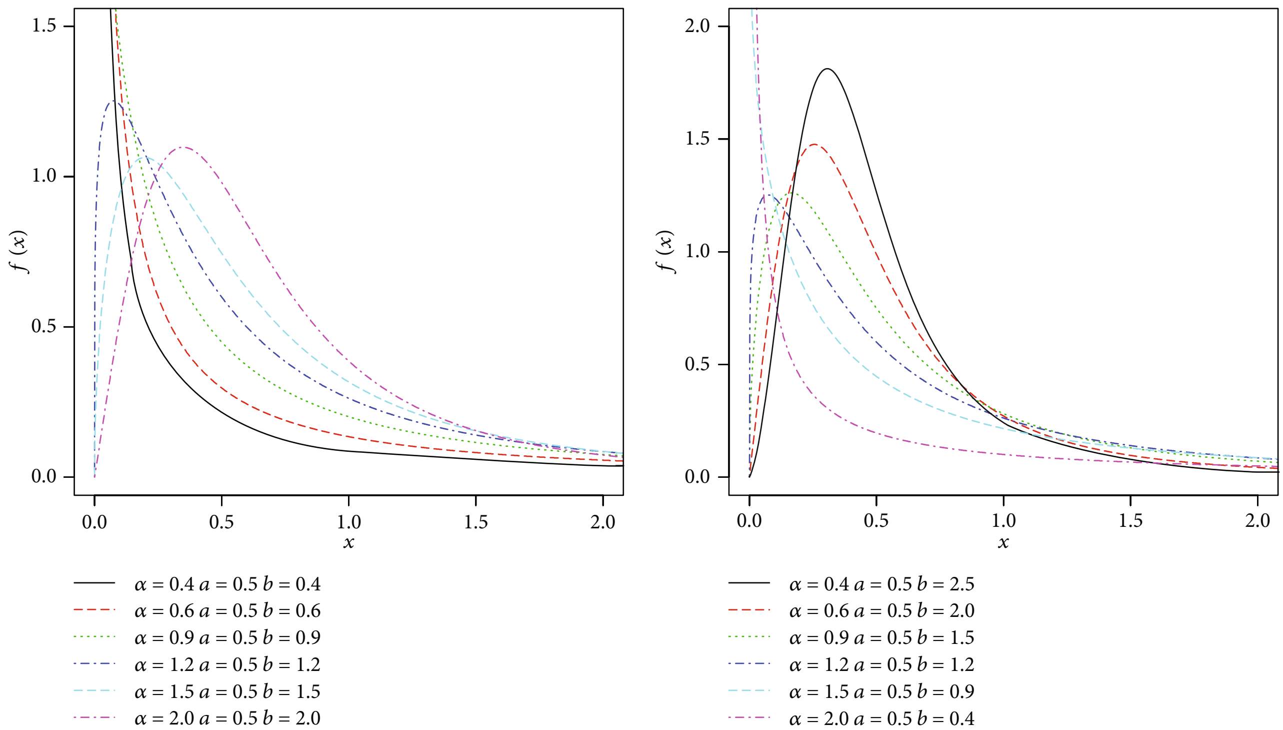

Figure 1: Plots of the pdf of the APTLL distribution.

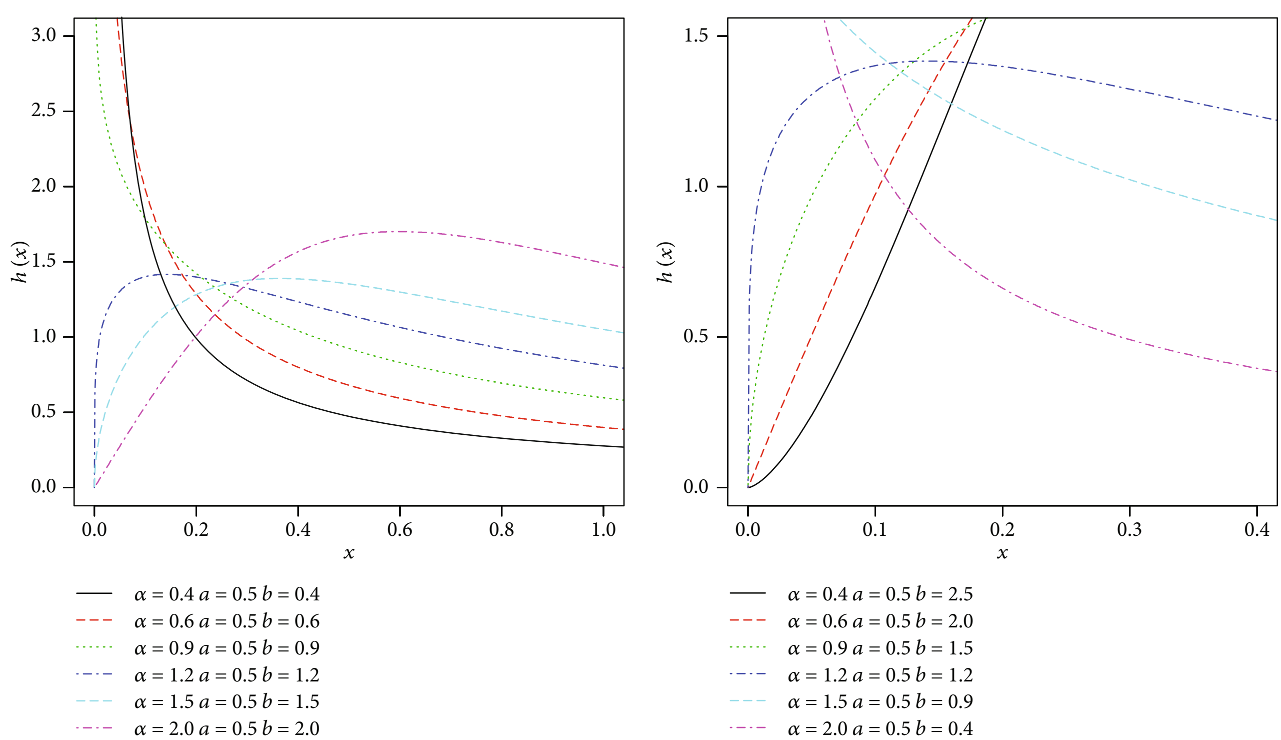

Figure 2: Plots of the hrf of the APTLL distribution.

$$[F(x)]^h \sum_{m=0}^{\infty} w_m \left(1+\left(\frac{x}{a}\right)^b\right)^{-m}, \quad (16)$$

where

$$w_m = (1-\alpha)^{-h} \sum_{i=0}^{h}\sum_{j=0}^{\infty} (-1)^{i+m} \binom{h}{i}\binom{j}{m} \frac{i^j (\log(\alpha))^j}{j!} \quad (17)$$

3.3. Moments

Theorem 1. *Let X be a r.v. from the APTLL distribution, then its k^{th} moment is*

$$\acute{\mu}_k = \sum_{i,j=0}^{\infty} w_{i,j} \frac{a^{k-b}}{b} B\left(\frac{k}{b}+1, j-\frac{k}{b}\right). \quad (18)$$

Proof. Let X be a r.v. with pdf (4). The k^{th} moments of the APTLL model are calculated as

$$\acute{\mu}_k = \int_0^{\infty} x^k f(x; a, b, \alpha) dx = \int_0^{\infty} \sum_{i,j=0}^{\infty} w_{i,j} x^{k+b-1} \left(1+\left(\frac{x}{a}\right)^b\right)^{-j-1} dx. \quad (19)$$

Let $y=(x/a)^b$, then

$$\acute{\mu}_k = \sum_{i,j=0}^{\infty} w_{i,j} \frac{a^{k+b}}{b} \int_0^{\infty} y^{k/b} (1+y)^{-j-1} dy. \quad (20)$$

Then,

$$\acute{\mu}_k = \sum_{i,j=0}^{\infty} w_{i,j} \frac{a^{k+b}}{b} B\left(\frac{k}{b}+1, j-\frac{k}{b}\right). \quad (21)$$

The mean of the APTLL distribution is easily obtained by putting $k=1$ as

$$\mu = \acute{\mu}_1 = E(X) = \sum_{i,j=0}^{\infty} w_{i,j} \frac{a^{1+b}}{b} B\left(\frac{1}{b}+1, j-\frac{1}{b}\right). \quad (22)$$

The variance of the APTLL distribution is given by

$$\sigma^2 = \acute{\mu}_2 - \left(\acute{\mu}_1\right)^2 = \sum_{i,j=0}^{\infty} w_{i,j} \frac{a^{2+b}}{b} B\left(\frac{1}{b}+1, j-\frac{2}{b}\right) - \left(\acute{\mu}_1\right)^2. \quad (23)$$

Numerical values of the first four moments, variance (σ^2), skewness (SK), and kurtosis (KU) of the APTLL distribution for $a=2$ and some choice values of b and α are as follows: (1) ($\alpha=0.2, b=5$), (2) ($\alpha=0.2, b=6$), (3) ($\alpha=0.2, b=10$), (4) ($\alpha=1.5, b=5$), (5) ($\alpha=1.5, b=6$), (6) ($\alpha=1.5, b=10$), (7) ($\alpha=0.5,$ ⊠$b=5$), (8) ($\alpha=0.5, b=6$), (9) ($\alpha=0.5, b=10$), (10) ($\alpha=2, b=5$), (11) ($\alpha=2, b=6$), and (12) ($\alpha=2, b=10$) are displayed in Tables 1 and 2.

The moment generating function of X is obtained as

$$M_X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \acute{\mu}_k = \sum_{k,i,j=0}^{\infty} \frac{t^k}{k!} w_{i,j} \frac{a^{k+b}}{b} B\left(\frac{k}{b}+1, j-\frac{k}{b}\right). \quad (24)$$

3.4. The Probability Weighted Moments.

For a r.v. X, the probability-weighted moments (PWMs) are given by

$$\tau_{k,s} = E\left(X^k F(x)^s\right) = \int_{-\infty}^{\infty} x^k f(x) F(x)^s dx. \quad (25)$$

The PWMs of APTLL are obtained by substituting (14) and (16) into (25) and replacing h with s, as follows:

$$\tau_{k,s} = \int_0^{\infty} \sum_{i,j,m=0}^{\infty} w_{i,j} w_m x^{k+b-1} \left(1+\left(\frac{x}{a}\right)^b\right)^{-j-1} dx. \quad (26)$$

Then,

$$\tau_{k,s} = \sum_{i,j,m=0}^{\infty} w_{i,j} w_m \frac{a^{k+b}}{b} B\left(\frac{1}{b}+1, m+j-\frac{1}{b}\right). \quad (27)$$

3.5. Rényi Entropy.

For a given pdf, the Rényi entropy is defined by

$$I_R(\delta) = \frac{1}{1-\delta} \log \int_0^{\infty} f(x)^{\delta} dx, \quad \delta > 0, \delta \neq 1. \quad (28)$$

The function $f(x)^{\delta}$ can be written as

$$f(x)^{\delta} = \left(\frac{b \ln \alpha}{a^b(\alpha-1)}\right)^{\delta} x^{\delta b-\delta} \left(1+\left(\frac{x}{a}\right)^b\right)^{-2\delta} \alpha^{\delta\left[1-\left(1+(x/a)^b\right)^{-1}\right]}. \quad (29)$$

By applying the expansions (11) and (13), then, we can rewrite the last equation as

$$f(x)^{\delta} = \sum_{i,j=0}^{\infty} t_{i,j} x^{\delta b-\delta} \left(1+\left(\frac{x}{a}\right)^b\right)^{-j-2\delta}, \quad (30)$$

where

$$t_{i,j} = \left(\frac{b \ln \alpha}{a^b(\alpha-1)}\right)^{\delta} \frac{\delta^i (\ln \alpha)^i (-1)^j}{i!} \binom{h}{j}. \quad (31)$$

Table 1: First four moments, σ^2, SK, and KU, of X for some choices of parameter values.

μ_k'	(1)	(2)	(3)	(4)	(5)	(6)
μ_1'	1.823	1.835	1.879	2.226	2.166	2.075
μ_2'	3.806	3.689	3.645	5.725	5.17	4.451
μ_3'	9.541	8.284	7.311	17.859	13.881	9.897
μ_4'	33.943	21.868	15.224	80.098	44.152	22.898
σ^2	0.484	0.323	0.113	0.771	0.48	0.147
SK	2.488	1.814	0.92	2.488	1.822	0.938
KU	30.457	15.17	6.673	29.638	14.806	6.527

Table 2: First four moments, σ^2, SK, and KU of X for some choices of parameter values.

μ_k'	(7)	(8)	(9)	(10)	(11)	(12)
μ_1'	1.994	1.977	1.964	2.289	2.217	2.104
μ_2'	4.59	4.303	3.988	6.047	5.414	4.577
μ_3'	12.799	10.524	8.39	19.365	14.858	10.316
μ_4'	51.228	30.469	18.361	89.074	48.284	24.19
σ^2	0.614	0.396	0.131	0.81	0.499	0.15
SK	2.49	1.823	0.937	2.495	1.828	0.942
KU	29.77	14.867	6.552	29.798	14.885	6.561

Now, we will investigate the integral

$$I = \int_0^\infty f(x)^\delta dx = \sum_{i,j=0}^{\infty} t_{i,j} \int_0^\infty x^{\delta b-\delta}\left(1+\left(\frac{x}{a}\right)^b\right)^{-j-2\delta} dx = \sum_{i,j=0}^{\infty} t_{i,j} \frac{a^{\delta b-\delta+1}}{b} B\left(\delta-\frac{\delta}{b}+\frac{1}{b}, j+\delta+\frac{\delta}{b}-\frac{1}{b}\right). \quad (32)$$

Then, the Rényi entropy is

$$I_R(R) = \frac{1}{1-\delta} \log \sum_{i,j=0}^{\infty} t_{i,j} \frac{a^{\delta b-\delta+1}}{b} B\left(\delta-\frac{\delta}{b}+\frac{1}{b}, j+\delta+\frac{\delta}{b}-\frac{1}{b}\right). \quad (33)$$

3.6. Order Statistics. Let $X_1, X_2, \cdots, X_n$ be the r.v. sample from the APTLL model with order statistics $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$. The pdf of r.v. $X_{(k)}$ is calculated as

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} F^{k-1}(x) f(x)(1-F(x))^{n-k}. \quad (34)$$

The pdf of $X_{(k)}$ can be expressed as

$$f_{X_{(k)}}(x) = \frac{n! b \ln(\alpha)}{(k-1)!(n-k)! a^b (\alpha-1)^n} x^{b-1} \Delta^{-2} \alpha^{1-\Delta^{-1}} \cdot \left(\alpha^{1-\Delta^{-1}}-1\right)^{k-1}\left(\alpha-\alpha^{1-\Delta^{-1}}\right)^{n-k}, \quad (35)$$

where $(1+(x/a)^b) = \Delta$. In particular, the pdf of the first and largest order statistics can be derived as

$$f_{X_{(1)}}(x) = \frac{nb \ln(\alpha)}{a^b(\alpha-1)^n} x^{b-1} \Delta^{-2} \alpha^{1-\Delta^{-1}} \left(\alpha-\alpha^{1-\Delta^{-1}}\right)^{n-1},$$
$$f_{X_{(n)}}(x) = \frac{nb \ln(\alpha)}{a^b(\alpha-1)^n} x^{b-1} \Delta^{-2} \alpha^{1-\Delta^{-1}} \left(\alpha^{1-\Delta^{-1}}-1\right)^{n-1}, \quad (36)$$

respectively.

4. ML Estimation

Let $X_1, \cdots, X_n$ be the observed values from the APTLL distribution. The maximum likelihood estimates (MLEs) of the proposed model parameters a, b, and α are derived using the log-likelihood function, say ℓ, which is given by

$$\ln L = n \ln b + n \ln(\ln \alpha) - n \ln(\alpha-1) - nb \ln a + (b-1)\sum_{i=1}^{n} \ln x_i - 2\sum_{i=1}^{n} \ln\left(1+\left(\frac{x_i}{a}\right)^b\right) + \ln \alpha \sum_{i=1}^{n}\left[1-\left(1+\left(\frac{x_i}{a}\right)^b\right)^{-1}\right]. \quad (37)$$

Table 3: The MLEs and MSEs of the APTLL model.

N	Set1		Set2		Set3	
	MLE	MSE	MLE	MSE	MLE	MSE
30	0.562	0.2020	0.585	0.1610	0.685	0.215
	0.529	8.23×10^{-3}	0.524	7.93×10^{-3}	0.531	7×10^{-3}
	0.784	0.0380	1.349	0.2090	1.766	0.576
50	0.516	0.0630	0.522	0.0580	0.681	0.191
	0.514	3.13×10^{-3}	0.517	5.88×10^{-3}	0.509	3.6×10^{-3}
	0.729	0.0280	1.346	0.1540	1.648	0.426
100	0.52	0.0480	0.508	0.0340	0.628	0.105
	0.506	1.6×10^{-3}	0.502	1.83×10^{-3}	0.497	1.86×10^{-3}
	0.74	0.0210	1.272	0.0670	1.621	0.127
200	0.489	0.0240	0.5	0.0160	0.571	0.03
	0.507	8.23×10^{-4}	0.501	1.05×10^{-3}	0.503	1.01×10^{-3}
	0.756	0.0210	1.233	0.0220	1.657	0.123
300	0.479	0.0120	0.494	9.12×10^{-3}	0.56	0.018
	0.503	6.61×10^{-4}	0.506	6.68×10^{-4}	0.5	5.42×10^{-4}
	0.756	0.0180	1.236	0.0170	1.61	0.076
500	0.495	0.0120	0.502	7.68×10^{-3}	0.577	0.016
	0.504	3.91×10^{-4}	0.501	3.71×10^{-4}	0.5	4.31×10^{-4}
	0.747	0.0170	1.21	9.69×10^{-3}	1.589	0.074

The ML equations of the APTLL distribution are given by

$$\frac{\partial \ln L}{\partial a} = \frac{-nb}{a} + \frac{2b}{a}\sum_{i=1}^{n}\frac{(x_i/a)^b}{1+(x_i/a)^b} - \frac{b\ln\alpha}{a}\sum_{i=1}^{n}\left(1+\left(\frac{x_i}{a}\right)^b\right)^{-2},$$

$$\frac{\partial \ln L}{\partial b} = \frac{n}{b} - n\ln a + \sum_{i=1}^{n}\ln x_i - 2\sum_{i=1}^{n}\frac{(x_i/a)^b\ln(x_i/a)}{1+(x_i/a)^b} + \ln\alpha\sum_{i=1}^{n}\left(1+\left(\frac{x_i}{a}\right)^b\right)^{-2}\left(\frac{x_i}{a}\right)^b\ln\left(\frac{x_i}{a}\right),$$

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha\ln\alpha} - \frac{n}{\alpha-1} + \frac{1}{\alpha}\sum_{i=1}^{n}\left[1-\left(1+\left(\frac{x_i}{a}\right)^b\right)^{-1}\right]. \quad (38)$$

Equating $\partial\ell/\partial a$, $\partial\ell/\partial b$, and $\partial\ell/\partial\alpha$ with zeros and solving simultaneously, we obtain the ML estimators of a, b, and α.

5. Simulation Study

The simulation study is required to compare the performances of the ML method of estimation mainly with respect to mean square errors. 3000 random samples from different sample sizes $n = 30$, 50, 100, 200, 300, and 500. Numerical results are performed using Mathematica 9 software. Six different sets of parameters are considered: set1($a = 0.5$, $b = 0.5$, $\alpha = 0.7$), set2($a = 0.5$, $b = 0.5$, $\alpha = 1.2$), set3($a = 0.5$, $b = 0.5$, $\alpha = 1.8$), set4($a = 0.5$, $b = 1$, $\alpha = 1.5$), set5($a = 0.5$, $b = 1.5$, $\alpha = 1.5$), and set6($a = 0.5$, $b = 1.8$, $\alpha = 1.5$).

The MLEs and MSEs of a, b, and α are calculated. Simulated outcomes are listed in Tables 3 and 4. We can notice from Tables 3 and 4 that the MSEs decreased when n increased.

6. Applications

In this section, we provide the effectiveness, importance, and flexibility of the APTLL model by using one data set. These data have been used by several authors to show the applicability of other competing models. The data set is given by Nichols and Padgett [12], and it is named the breaking stress data. This data set consists of 100 observations of breaking stress of carbon fibres (in Gba).

We also provide a formative evaluation of the goodness of fit of the models and make comparisons with other distributions. The measures of goodness of fit include the Akaike information criterion (AIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), and consistent Akaike information criterion (CAIC). More information about the goodness of fit for stress-related studies can be found in Misra and Li. In general, the smaller the values of these statistics, the better the fit to the data. For this data, we shall compare the fits of the APTLL model with other models like the BXII, Zografos-Balakrishnan BXII (ZBBXII), Marshall-Olkin BXII (MOBXII), five-parameter beta BXII (FBBXII), BBXII, beta exponentiated BXII (BEBXII), five-

Table 4: The MLEs and MSEs of the APTLL model.

n	Set4		Set5		Set6	
	MLE	MSE	MLE	MSE	MLE	MSE
30	0.58	0.044	0.519	0.014	0.503	5.57×10^{-3}
	1.028	0.027	1.556	0.056	1.848	0.094
	1.528	0.168	1.632	0.308	1.513	0.169
50	0.546	0.025	0.516	6.42×10^{-3}	0.505	3.85×10^{-3}
	1.057	0.024	1.544	0.033	1.854	0.07
	1.546	0.140	1.549	0.165	1.491	0.142
100	0.519	0.010	0.504	2.99×10^{-3}	0.513	2.29×10^{-3}
	1.015	6.91×10^{-3}	1.510	0.014	1.824	0.02
	1.489	0.061	1.578	0.093	1.468	0.035
200	0.51	4.5×10^{-3}	0.500	1.59×10^{-3}	0.51	9.274×10^{-4}
	1.019	3.435×10^{-3}	1.499	6.12×10^{-3}	1.833	0.013
	1.471	0.033	1.538	0.027	1.459	0.027
300	0.512	3.732×10^{-3}	0.504	1.174×10^{-3}	0.507	7.141×10^{-4}
	1	2.479×10^{-3}	1.508	5.473×10^{-3}	1.809	6.613×10^{-3}
	1.469	0.030	1.534	0.021	1.457	0.022
500	0.507	2.728×10^{-3}	0.496	9.028×10^{-4}	0.508	4.724×10^{-4}
	1.002	1.29×10^{-3}	1.503	3.339×10^{-3}	1.803	3.728×10^{-3}
	1.48	0.028	1.529	0.010	1.429	0.017

Table 5: Estimates and SE (in parentheses) for breaking stress data.

Distribution	Estimates and SE (in parentheses)				
APTLL (a, b,α)	2.485 (4.349)	4.145 (0.348)	1.015 (14.735)		
BXII (α, β)	5.941 (1.279)	0.187 (0.044)			
MOBXII (α, β, γ)	1.192 (0.952)	4.834 (4.896)	838.73 (229.34)		
TLBXII (α, β, γ)	1.350 (0.378)	1.061 (0.384)	13.728 (8.400)		
KwBXII (λ, θ, α, β)	48.103 (19.348)	79.516 (58.186)	0.351 (0.098)	2.730 (1.077)	
BBXII (λ, θ, α, β)	359.683 (57.941)	260.097 (132.213)	0.175 (0.013)	1.123 (0.243)	
BEBXII (λ, θ, α, β, γ)	0.381 (0.078)	11.949 (4.635)	0.937 (0.267)	33.402 (6.287)	1.705 (0.478)
FBBXII (λ, θ, α, β, γ)	0.421 (0.011)	0.834 (0.943)	6.111 (2.314)	1.674 (0.226)	3.450 (1.957)
FKwBXII (λ, θ, α, β, γ)	0.542 (0.137)	4.223 (1.882)	5.313 (2.318)	0.411 (0.497)	4.152 (1.995)

parameter Kumaraswamy BXII (FKumBXII), Topp Leone BXII (TLBXII), and KumBXII distributions (for more details about the competitive models, see [13–16]).

Table 5 shows the ML estimates and SEs for the competitive models. Also, Table 6 provides the numerical results of some measures of goodness of fit for all competitive models. Based on the values in Table 6, the APTLL model has the smallest values for BIC, AIC, CAIC, and HQIC.

According to these criteria, we found that the APTPL model is the best fitted model compared to the other competitive models. The estimated pdf, cdf, sf, and pp plots are displayed in Figure 3. It is clear from Figure 3 that the APTPL model provides a better fit to this data.

Table 6: Measures of goodness of fits for breaking stress data.

Distribution	AIC	BIC	CAIC	HQIC
APTLL	296.796	296.796	297.046	299.959
BXII	382.94	388.15	383.06	385.05
MOBXII	305.78	313.61	306.03	308.96
TLBXII	323.52	331.35	323.77	326.70
KwBXII	303.76	314.20	304.18	308.00
BBXII	305.64	316.06	306.06	309.85
BEBXII	305.82	318.84	306.46	311.09
FBBXII	304.26	317.31	304.89	309.56
FKwBXII	305.50	318.55	306.14	310.80

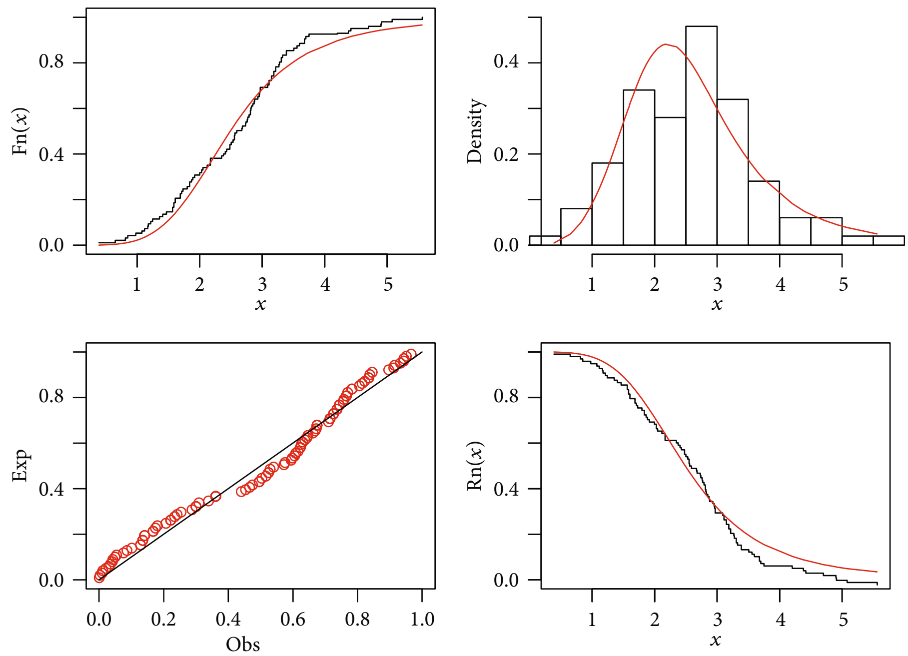

Figure 3: Plots of estimated pdf, cdf, sf, and pp for breaking stress data.

7. Conclusions

In this research, we introduced and studied the APTLL model. Some mathematical properties of the APTLL distribution are investigated. Estimation of the population parameters is done by using the ML method of estimation. The simulation study is performed to investigate the effectiveness of the estimates. A real data set is used for the application to show the flexibility of the APTLL model against the competitive models.

Data Availability

In order to obtain the numerical dataset used to carry out the analysis reported in the manuscript, please contact the author Maha Aldahlan.

Conflicts of Interest

The author declares no conflicts of interest.

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