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Research Article

Soliton Molecules and Some Novel Types of Hybrid Solutions to (2+1)-Dimensional Variable-Coefficient Caudrey-Dodd-Gibbon-Kotera-Sawada Equation

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Soliton molecules of the $(2+1)$-dimensional variable-coefficient Caudrey-Dodd-Gibbon-Kotera-Sawada equation are derived by N-soliton solutions and a new velocity resonance condition. Moreover, soliton molecules can become asymmetric solitons when the distance between two solitons of the molecule is small enough. Finally, we obtained some novel types of hybrid solutions which are components of soliton molecules, lump waves, and breather waves by applying velocity resonance, module resonance of wave number, and long wave limit method. Some figures are presented to demonstrate clearly dynamics features of these solutions.

1. Introduction

Solitons as localized nonlinear waves exhibit many interesting properties [1]. In particular, solitons can form stable bound states known as soliton molecules, which have been observed experimentally in some fields [2–8]. For the first time, soliton molecules were experimentally observed in dispersion-managed optical fibers [2]. In 2017, the authors in [5] resolved the evolution of femtosecond soliton molecules in the cavity of a few-cycle mode-locked laser by means of an emerging timestretch technique. In 2018, Liu et al. have experimentally observed the real-time dynamics of the entire buildup process of stable soliton molecules for the first time [6]. These soliton compounds may find important applications in fiber optic communication systems to enhance their data-carrying capacity [9]. The existence of two-soliton bound states in Bose-Einstein condensates with contact atomic interactions and some dynamic phenomena involving soliton molecules was reported in [10, 11].

As we all know, exact solutions for some local or nonlocal nonlinear evolution equations (NLEEs) have been applied in the nonlinear science fields. Solitons and rational solutions of many NLEEs have been investigated by researches [12–16]. Among these rational solutions, lump solutions, breather wave solutions, and rogue wave solutions are hot point all the time. Recently, the hybrid solutions of lump solutions with other types of solutions draw a lot of attention, which include lump-soliton [17–19], lump-kink solution [20], resonance stripe solitons [21–23], and some hybrid solutions [24–26]. Very recently, Lou [27] introduced a new possible mechanism, the velocity resonant, to form soliton molecules and asymmetric solitons of three $(1+1)$-dimensional fluid models: fifth-order KdV, SK equation, and KK equation. To the best of our knowledge, soliton molecules interacting with lump waves and breather waves has not been studied yet. In this paper, we will extend the velocity resonant method to soliton molecules of $(2+1)$-dimensional systems and further explore some novel type hybrid solutions among soliton molecules, lump waves, and breather waves.

In many physical situations, with the inhomogeneities of media and boundaries taken into account, the variable coefficient nonlinear evolution equations might be more realistic than the constant coefficient ones [28–33]. In the past decades, the study of the nonlinear evolution of variable-coefficient has attracted the attention of mathematicians and physicists. In modeling, a variety of complex nonlinear phenomena of physical and engineering fields and many physical and mechanical situations are governed by variable-coefficient equations. Therefore, seeking for soliton molecules and hybrid solitons of variable-coefficient equations are also of great significance.

In this paper, we will focus on a generalized $(2+1)$-dimensional variable-coefficient Caudrey-Dodd-Gibbon-Kotera-Sawada (CDGKS) equation such as the one given below:

$$\begin{aligned} &u_t + a_1 u_{xxxxx} + a_2 u_x u_{xx} + a_3 u u_{xxx} + a_4 u^2 u_x \\ &\quad + a_5 u_{xxy} + a_6 \int u_{yy} dx + a_7 u_x \int u_y dx \\ &\quad + a_8 u u_y + a_9 u = 0, \end{aligned} \tag{1}$$

where u is a function of $\{x, y, t\}$ and $a_i = a_i(t)$, $i = 1, \cdots, 9$, are analytic functions with respect to t. If the parameters are specially chosen, a series of equations can be obtained, which can be integrable [34, 35] or used to describe physical phenomena as the interaction between a water and a floating ice cover the gravity-capillary waves [36].

Referring to [37], the bilinear form of Eq. (1) as follows:

$$\left(D_x D_t + a_1 D_x^6 + 5c_1 a_1 D_x^3 D_y - 5c_1^2 a_1 D_y^2\right) f \cdot f = 0, \tag{2}$$

under the following transformation:

$$u = 2c_0 \exp\left(-\int a_9 dt\right) (\ln f)_{xx}, \tag{3}$$

where D is the Hirota's bilinear differential operator, c_0 and c_1 are arbitrary constants, $f = f(x, y, t)$ is a real function of variables $\{x, y, t\}$, and $a_i (i = 1,\cdots,9)$ satisfies the following conditions:

$$\begin{aligned} a_2 &= a_3 = \frac{15a_1}{c_0} \exp\left(-\int a_9 dt\right), \\ a_4 &= \frac{45a_1}{c_0^2} \exp\left(-\int a_9 dt\right), \\ a_5 &= 5c_1 a_1, \\ a_6 &= -5c_1^2 a_1, \\ a_7 &= a_8 = \frac{15c_1 a_1}{c_0} \exp\left(-\int a_9 dt\right), \end{aligned} \tag{4}$$

with $\{a_1, a_9\}$ being the arbitrary functions of t and $\{c_0 \neq 0, c_1\}$ the arbitrary constants.

It is well known that the bilinear equation (2) includes several other forms: the usual $(2+1)$-dimensional fifth-order KdV equation for selections $\{a_1, c_1\}_{\{\mathrm{KdV}\}} = \{(1/36), 1\}$, the $(2+1)$-dimensional B-type Kadomtsev-Petviashvili equation (BKP) model by taking $\{a_1, c_1\}_{\{\mathrm{BKP}\}} = \{1,-1\}$, and the $(2+1)$-dimensional Sawada-Kotera equation (KP) model by taking $\{a_1, c_1\}_{\{\mathrm{KP}\}} = \{-1, 1\}$.

Based on the Hirota's bilinear theory, the N-soliton solutions for Eq. (1) can be constructed as

$$\begin{aligned} u &= 2c_0 \exp\left(-\int a_9 dt\right) \\ &\quad \cdot \left\{ \ln \left[\sum_{\mu=0,1} \exp\left(\sum_{1\le i<j}^{N} \mu_i \mu_j A_{ij} + \sum_{j=1}^{N} \mu_j \eta_j \right) \right] \right\}_{xx}, \end{aligned} \tag{5}$$

with

$$\begin{aligned} \eta_j &= k_j x + p_j y + \omega_j(t) + \phi_j, \\ \omega_j(t) &= -\frac{k_j^6 + 5c_1 k_j^3 p_j - 5c_1^2 p_j^2}{k_j} \int a_1 dt, \\ e^{A_{ij}} &= \frac{(k_i - k_j)\left[c_1 k_i k_j^2 p_i (2k_i - k_j) + c_1 k_i^2 k_j p_j (k_i - 2k_j) + k_i^2 k_j^2 \left(k_i^2 - k_i k_j + k_j^2\right)(k_i - k_j)\right] + c_1^2 \left(k_i p_j - k_j p_i\right)^2}{(k_i + k_j)\left[c_1 k_i k_j^2 p_i (2k_i + k_j) + c_1 k_i^2 k_j p_j (k_i + 2k_j) + k_i^2 k_j^2 \left(k_i^2 - k_i k_j + k_j^2\right)(k_i + k_j)\right] + c_1^2 \left(k_i p_j - k_j p_i\right)^2}, \end{aligned} \tag{6}$$

where k_j, p_j and $\phi_j (j = 1, 2, \cdots, N)$ being arbitrary constants, $\sum_{\mu=0,1}$ indicates a summation over all possible combinations of $\mu_j = 0, 1 (j = 1, 2, \cdots, N)$.

The remainder of this paper is organized as follows. First, we aim to introduce a new velocity resonant condition in Section 2, then soliton molecules are obtained based on N-soliton formula with applying the velocity resonant condition, and further we explore their fascinating dynamical behaviors. In Section 3, partial parameters are handled with the velocity resonant condition, module resonance of wave number and long wave limit method, and some novel types of interaction solutions including soliton molecules, lump waves, and breather waves are derived. Finally, the conclusions are summarized in Section 4.

2. Soliton Molecules and Asymmetric Solitons

To find nonsingular analytical resonant excitation from Eq. (5), we apply a novel type of resonant conditions ($k_i \neq \pm k_j, p_i \neq \pm p_j$), the velocity resonance,

$$\frac{k_i}{k_j} = \frac{p_i}{p_j} = \frac{\omega_i{}'(t)}{\omega_j{}'(t)}. \tag{7}$$

Combining Eq. (6), we can get the following expressions:

$$\begin{aligned} k_i &= \sqrt{\frac{-k_j^3 + 5c_1 p_j}{k_j}}, \\ p_i &= \frac{p_j}{k_j}\sqrt{\frac{-k_j^3 + 5c_1 p_j}{k_j}}, \end{aligned} \tag{8}$$

or

$$\begin{aligned} k_i &= -\sqrt{\frac{-k_j^3 + 5c_1 p_j}{k_j}}, \\ p_i &= -\frac{p_j}{k_j}\sqrt{\frac{-k_j^3 + 5c_1 p_j}{k_j}}. \end{aligned} \tag{9}$$

It can be known under the resonance conditions (Eq. (8) or Eq. (9)) that two solitons for $N = 2$ in Eq. (5) are bounded to generate a soliton molecule. From Eqs. (5)–(7), we can deduce that when a_1 is an arbitrary constant and $a_9 = 0$, the amplitudes and the velocities of solitons molecules all remain unchanged during the evolutions. While a_1 is a function of t and $a_9 \neq 0$, the amplitudes and the velocities of them vary with time. So, we will explore some dynamics of soliton molecules from the above two cases.

Figure 1 displays the molecule structure with the parameter selections

$$\begin{aligned} p_1 &= 1, \\ k_1 &= -\frac{5}{6}, \\ k_2 &= -\frac{\sqrt{191}}{6}, \\ p_2 &= \frac{\sqrt{191}}{5}, \\ \phi_1 &= 0, \\ \phi_2 &= 6, \\ c_0 &= c_1 = a_1 = 1, \\ a_9 &= 0. \end{aligned} \tag{10}$$

From Figure 1, one can find that two solitons in the molecule are different because $k_1 \neq k_2$ though their velocities are same.

If we change values ϕ_1 and ϕ_2, the distance between two solitons of the molecule will change, respectively. When the distance of two solitons is close enough to have an interaction with each other, the soliton molecule will become an asymmetric soliton. Figure 2 is the plots of the asymmetric soliton solutions with the parameters (10) except for $\phi_2 = -2$. From Figure 2, one can see that the soliton molecule keeps its asymmetric shape and velocity during the evolution with times. Because asymmetric solitons are just special case of soliton molecules, so we will not investigate these asymmetric solitons in the following paper.

Figure 3 shows the propagation of one-soliton molecule via the parameter selections (10) except for $\{a_1 = \cos(t), a_9 = 0.01t, \phi_2 = 16\}$. From Figures 3(b)–3(e), one can see that (i) the amplitudes of two solitons in the soliton molecules are changed with the function $\exp(-\int a_9 dt) = \exp(-0.005t^2)$; (ii) the velocities of two solitons are periodically changed with the function $a_1 = \cos(t)$ at the same time.

Two-soliton molecules can be generated from four solitons, k_1, p_1, w_1 and k_2, p_2, w_2 satisfy Eq. (8) and k_3, p_3, w_3 and k_4, p_4, w_4 satisfy Eq. (8) or Eq. (9) at the same time. Figure 4 displays the elastic interaction property for the solution (5) with $N = 4$ and with parameter selections

$$\begin{aligned} k_1 &= -\frac{5}{4}, \\ k_2 &= \frac{\sqrt{103}}{4}, \\ k_3 &= -\frac{2}{3}, \\ k_4 &= -\frac{\sqrt{41}}{3}, \\ p_1 &= 2, \\ p_2 &= -\frac{2\sqrt{103}}{5}, \\ p_3 &= -\frac{2}{3}, \\ p_4 &= \frac{\sqrt{41}}{3}, \\ \phi_1 &= 10, \\ \phi_2 &= 0, \\ \phi_3 &= -3, \\ \phi_4 &= 12, \\ a_1 &= c_0 = c_1 = 1, \\ a_9 &= 0. \end{aligned} \tag{11}$$

As can be seen from Figure 4, the height of wave peaks and the velocities of wave peaks do not change except for the phase after the collision of one-soliton molecule and another soliton molecule. It is necessary to point out that if taking a_1 as a function of t and $a_9 \neq 0$, the heights and velocities of two-soliton molecules will change with time t.

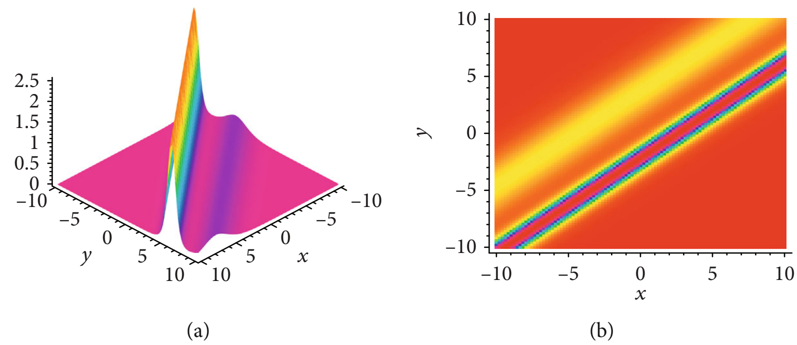


Figure 1: (Color online) soliton molecule structure for Eq. (1) with the parameter selections (10) at $t = 0$. (a) Three-dimensional plot. (b) Density plot.

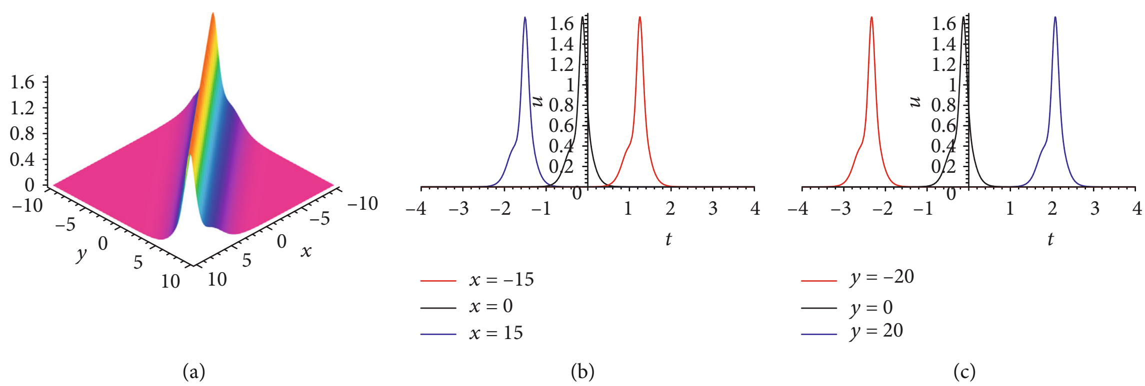


Figure 2: (Color online) asymmetric soliton for Eq. (1) with the parameter selections (9)) except for $\phi_2 = -2$. (a) Three-dimensional plot at $t = 0$. (b) Two-dimensional plot when $y = 0$ at $x = -15, 0, 15$. (c) Two-dimensional plot when $x = 0$ at $y = -20, 0, 20$.

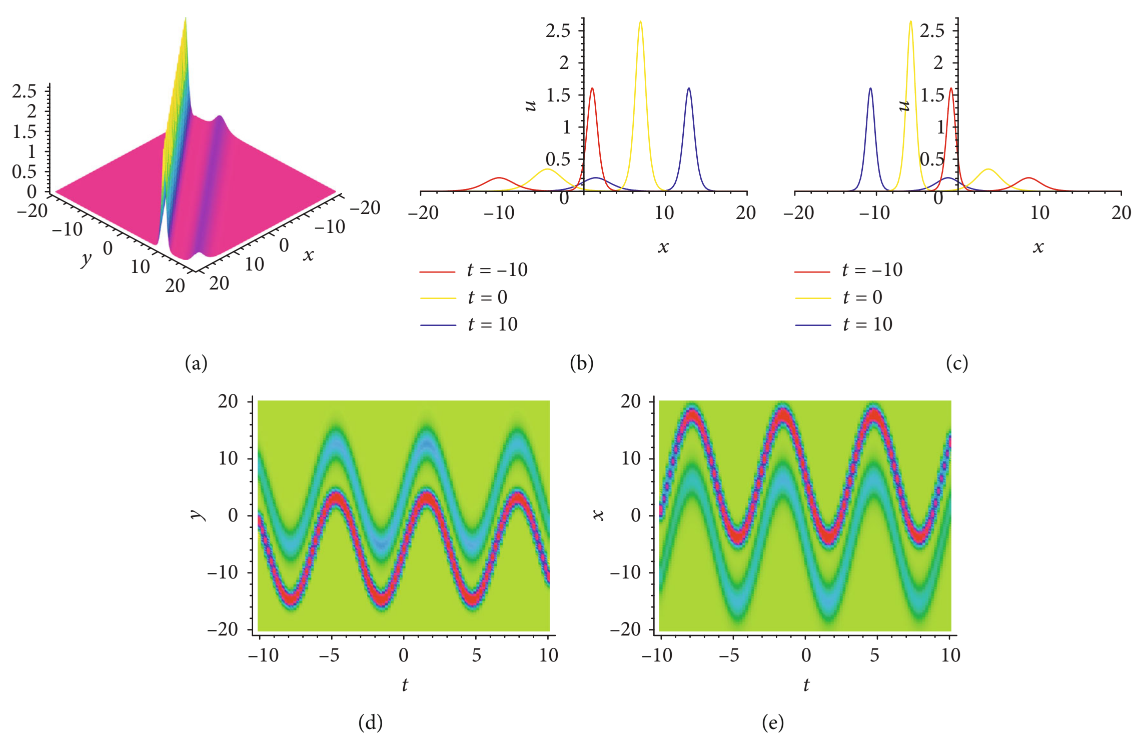


Figure 3: (Color online) one-soliton molecule for Eq. (1) with the parameter selections (10) except for $a_1 = \cos(t)$, $a_9 = 0.01t$, and $\phi_2 = 16$. (a) Three-dimensional plot at $t = 0$. (b) Two-dimensional plot when $y = 0$ at $t = -10, 0, 10$. (c) Two-dimensional plot when $x = 0$ at $t = -10, 0, 10$.

3. Some Novel Hybrid Solutions Consisting of Soliton Molecules, Breathers, and Lumps

In this section, some novel hybrid solutions will be investigated which are soliton molecules interacting with breathers and lump waves. To our knowledge, soliton molecules interacting with breathers and lump waves has not been studied yet. We study the interactions between soliton molecules and other waves via velocity resonance, module resonance, and long-wave limit method.

The long-wave limit method is a powerful technique to get lump solution; based on the N-soliton solution (5), we can obtain interaction solutions consisting of a soliton molecule and a lump wave. For $N = 4$, we are taking a long wave limit on $k_1, k_2, p_1, p_2 (\varepsilon \to 0), k_3, k_4, p_3, p_4$ which satisfies the velocity resonance condition; the parameters are as follows:

$$\begin{aligned} k_1 &= (1+i)\varepsilon, \\ k_2 &= (1-i)\varepsilon, \\ p_1 &= 2\varepsilon, \\ p_2 &= 2\varepsilon, \\ \phi_1 &= \pi i, \\ \phi_2 &= \pi i, \\ k_3 &= -\frac{5}{6}, \\ k_4 &= -\frac{\sqrt{191}}{6}, \\ p_3 &= 1, \\ p_4 &= \frac{\sqrt{191}}{5}, \\ \phi_3 &= 0, \\ \phi_4 &= 20, \\ a_1(t) &= \sin(t), \\ a_9(t) &= 0.01, \\ c_0 &= 1, \\ c_1 &= 1. \end{aligned} \tag{12}$$

Figure 5 displays the interaction between a soliton molecule and a lump wave. The collisions are also elastic. It is necessary to point out that when $a_9 = 0$, the height of the lump wave and soliton molecules do not change before and after the collisions, but when $a_9 = 0.01$, the height of the lump wave and soliton molecules all decrease with the function $\exp(-\int a_9 dt) = \exp(-0.01t)$. At the same time, the velocities of the soliton molecules are synchronously periodically changed with the function $a_1 = \sin(t)$.

Novel hybrid solutions of a soliton molecule and breather wave can be generated by four solitons. Furthermore, k_i, p_i and $\omega_i (i = 1, 2)$ should satisfy the velocity resonance condition (8) or (9), and the other two solitons satisfy the module resonance condition $\eta_i = \overline{\eta_j}$. For instance, taking the following parameters:

$$\begin{aligned} k_1 &= -\frac{4}{5}, \\ k_2 &= \frac{7\sqrt{7}\sqrt{2}}{10}, \\ k_3 &= \frac{2}{5} - \frac{2}{5} i, \\ k_4 &= \frac{2}{5} + \frac{2}{5} i, \\ p_1 &= -\frac{6}{5}, \\ p_2 &= \frac{21\sqrt{7}\sqrt{2}}{20}, \\ p_3 &= \frac{1}{6} + \frac{i}{2}, \\ p_4 &= \frac{1}{6} - \frac{i}{2}, \\ \phi_1 &= \phi_3 = \phi_4 = 0, \\ \phi_2 &= 25, \\ a_1(t) &= \sin(t), \\ a_9(t) &= 0.01, \\ c_0 &= 1, \\ c_1 &= -1. \end{aligned} \tag{13}$$

As shown in Figure 6, four-soliton solution (Eq. (5)) with parameter selections (Eq. (13)) exhibits the interaction between a soliton molecule and a breather wave under partial velocity resonance and the partial module resonance condition. The interactions of soliton molecules and breather solutions are also elastic. When $a_1 = \text{const.}, a_9 = 0$, the amplitudes and velocities of the soliton molecules and the breather waves remain the same before and after the collisions. If a_1 is a function of t and $a_9 \neq 0$, their amplitudes and velocities can be changed during evolution. Under the parameters in Figure 6, their amplitudes decrease with time due to $a_9 = 0.01$ and their velocities are periodically changed due to $a_1 = \sin(t)$.

More generally, we can obtain the general hybrid solutions consisting of m-soliton molecules, n-breather waves, and q-lump waves under the following parameter constraints:

$$\begin{aligned} &\frac{k_1}{k_2} = \frac{p_1}{p_2} = \frac{w_1}{w_2}, \cdots, \frac{k_{2m-1}}{k_{2m}} = \frac{p_{2m-1}}{p_{2m}} = \frac{w_{2m-1}}{w_{2m}}, \eta_{2m+1} \\ &= \overline{\eta_{2m+2}}, \cdots, \eta_{2m+2n-1} = \overline{\eta_{2m+2n}}, k_{2m+2n+1} \\ &= K_{2m+2n+1}\varepsilon, k_{2m+2n+2} = \overline{K_{2m+2n+1}}\varepsilon, \cdots, k_{2m+2n+2q-1} \\ &= K_{2m+2n+2q-1}\varepsilon, k_{2m+2n+2q} = \overline{K_{2m+2n+2q1}}\varepsilon, p_{2m+2n+1} \\ &= P_{2m+2n+1}\varepsilon, p_{2m+2n+2} = \overline{P_{2m+2n+1}}\varepsilon, \cdots, p_{2m+2n+2q-1} \\ &= P_{2m+2n+2q-1}\varepsilon, p_{2m+2n+2q} = \overline{P_{2m+2n+2q1}}\varepsilon, \phi_{2m+2n+1} \\ &= \pi i, \phi_{2m+2n+2} = \pi i, \cdots, \phi_{2m+2n+2q} = \pi i, \varepsilon \to 0. \end{aligned} \tag{14}$$

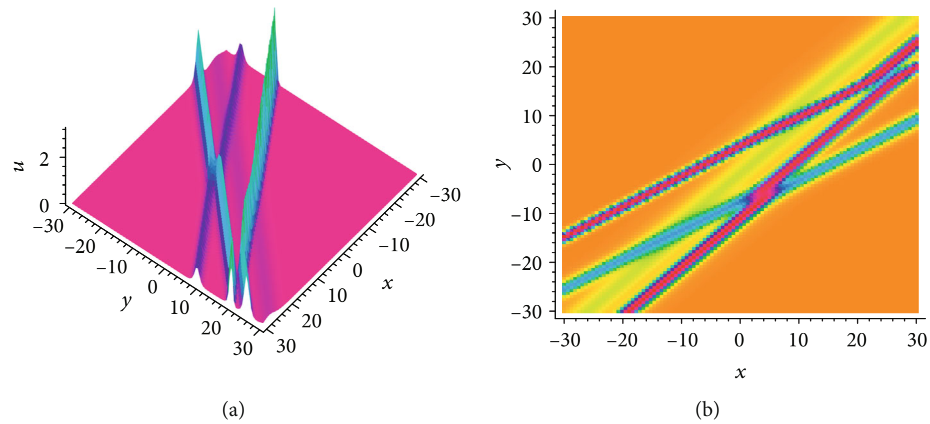


FIGURE 4: (Color online) two-soliton molecules for the solution (5) of Eq. (1) with the parameter selections (11) at $t = 0$. (a) Three-dimensional plot. (b) Density plot.

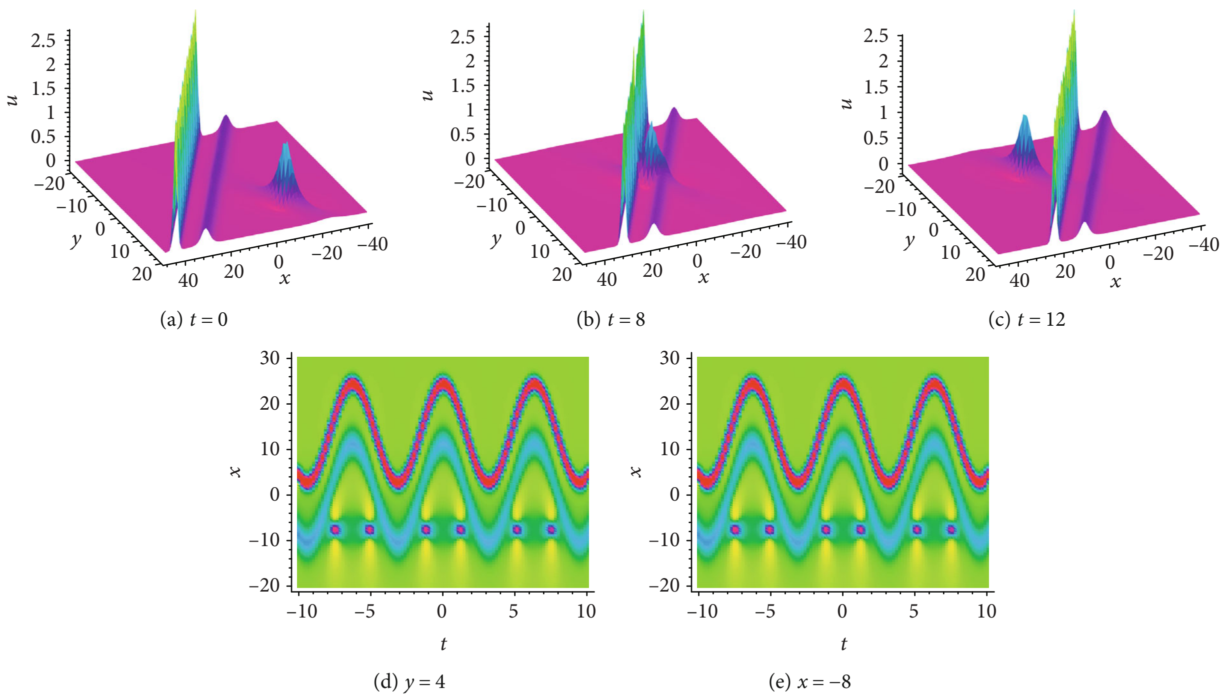


FIGURE 5: (Color online) interaction of a soliton molecule and a lump wave for the $(2+1)$-dimensional variable-coefficient CDGKS equation with parameter (12).

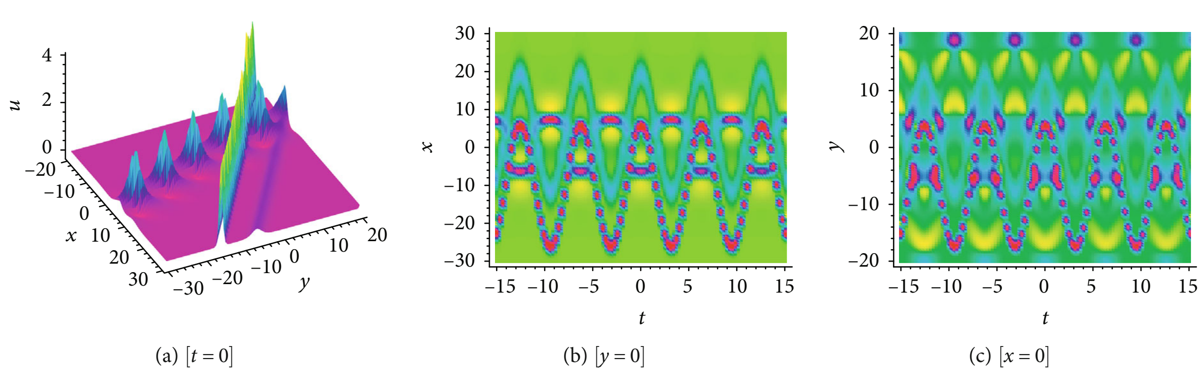
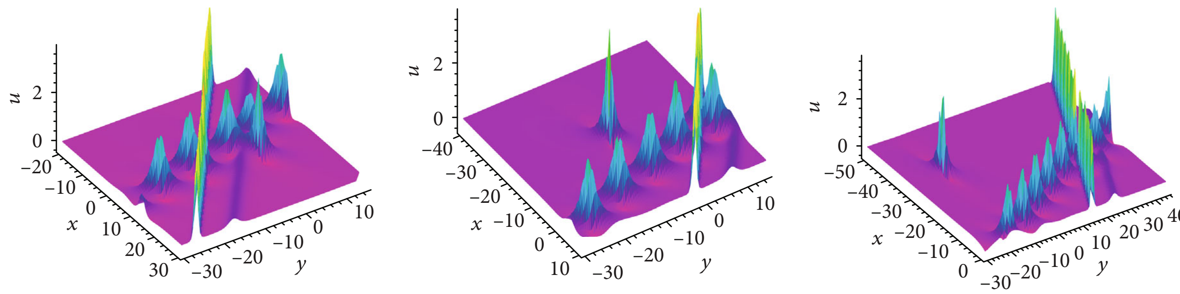


FIGURE 6: (Color online) interaction of a soliton molecule and a breather wave for the $(2+1)$-dimensional variable-coefficient CDGKS equation with parameter (13).



(a) $[t = 0]$ (b) $[t = -1]$ (c) $[t = -2]$

Figure 7: (Color online) elastic interaction property between a soliton molecule, a lump wave, and a breather wave for Eq. (1) described by Eq. (5) with parameters selections Eq. (15).

Then, we can get interactions between soliton molecules, lumps, and breathers by taking $\varepsilon \to 0$. To clearly describe the interaction between them, let us take a simple example of $N = 6$. Next, we set $\{k_1, k_2, p_1, p_2\}$ which satisfies the velocity resonant condition, $\{k_3, k_4, p_3, p_4\}$, which satisfies module resonance condition, then taking a long wave limit on $\{k_5, k_6, p_5, p_6\}(\varepsilon \to 0)$, taking parameters as follows:

$$
\begin{aligned}
c_0 &= 1,\\
c_1 &= -1,\\
k_1 &= -\frac{4}{5},\\
k_2 &= \frac{7\sqrt{7}\sqrt{2}}{10},\\
k_3 &= \frac{2}{5} - \frac{2}{5} i,\\
k_4 &= \frac{2}{5} + \frac{2}{5} i,\\
k_5 &= \left(\frac{1}{2} + i\right)\varepsilon,\\
k_6 &= \left(\frac{1}{2} - i\right)\varepsilon,\\
p_1 &= -\frac{6}{5},\\
p_2 &= \frac{21\sqrt{7}\sqrt{2}}{20},\\
p_3 &= \frac{1}{6} + \frac{i}{2},\\
p_4 &= \frac{1}{6} - \frac{i}{2},\\
p_5 &= -2\varepsilon,\\
p_6 &= -2\varepsilon,\\
\phi_1 &= 0,\\
\phi_2 &= 25,\\
\phi_3 &= 0,\\
\phi_4 &= 0,\\
\phi_5 &= i\pi,\\
\phi_6 &= i\pi,\\
a_1 &= 1,\\
a_9 &= 0.
\end{aligned}
\tag{15}
$$

Figure 7 displays the interaction between a soliton molecule, a lump wave, and a breather wave descried by Eq. (5) with the parameter selections in Eq. (15). The interactions between these waves are also elastic.

4. Conclusion

In this paper, soliton molecules and asymmetric solitons of $(2+1)$-dimensional variable-coefficient Caudrey-Dodd-Gibbon-Kotera-Sawada equation are theoretically obtained by velocity resonance. By introducing a new velocity resonance condition, we obtained the soliton molecules from the general N-soliton expression; see Figures 1, 3, and 4. When taking suitable values of ϕ, the soliton molecules can change to asymmetric soliton; see Figure 2. It is necessary to point out that when $a_1 = \text{const.}$ and $a_9 = 0$, the height of soliton molecules do not change before and after the collisions; but when a_1 is a function of t and $a_9 \neq 0$, the height of soliton molecules are changed with the function $\exp\left(-\int a_9 dt\right)$. At the same time, the velocities of the soliton molecules are synchronously changed with the function $a_1(t)$. Taking a long wave limit on part of parameters and employing resonance condition on others, the new hybrid solutions consisting soliton molecules and lump wave can be obtained; see Figure 5. By employing velocity resonance condition and module resonance condition on wave numbers, we can get a new hybrid solution consisting soliton molecules and breather waves; see Figure 6. By using velocity resonance, module resonance and long-wave limit method to different parts of wave numbers, general hybrid solutions consisting of soliton molecules, lump waves, and breather waves are obtained; see Figure 7. These interaction phenomena may have not been studied. At last, we give the general restrictions to derive these novel interaction solutions containing m-soliton molecules, n-breather waves, and q-lump waves, and their interactions are elastic. The method to construct soliton molecules and some novel types of hybrid solutions would be suitable to investigate in other models in mathematical physics and engineering. Meanwhile, we hope that our results will provide some valuable information in the study of nonlinear science.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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