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Research Article

A New Complex Network Model with Multiweights and Its Synchronization Control

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Based on the weighted complex network model, this paper establishes a multiweight complex network model, which possesses several different weights on the one edge. According to the method of network split, the complex network with multiweights is split into several different complex networks with single weight. Some new static characteristics, such as node weight, node degree, node weight strength, node weight distribution, edge weight distribution, and diversity of weight distribution are defined. Then, by using Lyapunov stability theory, the adaptive feedback synchronization controller is designed, and the complete synchronization of the new complex network model is investigated. Two numerical examples of a triweight network model with the same and diverse structure are given to demonstrate the effectiveness of the control strategies. The synchronization design can achieve good results in the same and diverse structure network models with multiweights, which enrich complex network and control theory, so has certain theoretical and practical significance.

1. Introduction

Complex networks have found widespread use in real life and attracted a large number of scientific researchers who are engaged in natural science and engineering [1–4]. In recent research of complex network, most of them are about static characteristics, chaos control of complex network. The research on static characteristics is mainly concentrating on some classic statistical characteristic and their applications [5–7], and the analysis of network chaos control is mainly involved in chaos synchronization and the method of improving network synchronizability [8–13].

In real-world network, the strength of relationships between nodes is different; it therefore becomes very important to define a new indicator to differentiate the different relationships. The proposed concept of weight successfully solved that problem, and then, weighted complex networks arisen. In the next few years, a lot of research works have fully demonstrate the feasibility and practicality of weighted complex networks [14–19]. For example, Wang et al. [14] designed a novel edge-weight-based compartmental approach and proposed an edge-weight-based removal strategy, which can control the spread of epidemic effectively when the highly weighted edges are preferentially removed. Xue and Bogdan [15] investigated some multifractal estimation algorithms and constructed an effective characterization framework to deal with some challenges. Basu and Maulik [17] proposed a group density-based core analysis approach that overcomes the drawbacks of the node centric approaches. The proposed algorithmic approach focuses on weight density, cohesiveness, and stability of a substructure. Some relevant hotspot issues, such as statistical static feature, evolutionary characteristics, and network dynamics, have been gained the embedded spread.

In recent years, some interesting complex network models continue to surface [20–28]. Wang et al. and Liu et al. [20, 21] studied the progress of spreading dynamics on the multilayer-coupled network, which will enhance understanding and control of real network. For example, they found that an epidemic outbreak on the contact layer can induce an outbreak on the communication layer, and information spreading can effectively raise the epidemic threshold. Gao et al. [22, 23] established a multilink complex dynamical network; the stability analysis and adaptive synchronization of the new network model are also studied. Bian and Yao [24] investigated a multilink complex network model; the synchronization criteria was designed under the idea of network split. Sun et al. [25] further investigated the synchronization of the multilink complex network based on the different time delay. Hu et al. [26] analyzed the stabilities of the multilink complex network using adaptive control theory; then, the pinning synchronization was studied and simulated. An et al. [27, 28] constructed a multiweight complex network model and applied it into a public transit network.

My personal view is that it holds much significance to depict some real networks with multiweight networks. There are more than one weight between two nodes; each weight can describe one type of information, so the new network model contains more properties than the past networks. As you have known, people contact each other by mail, telephone, MSN, and e-mail, so we can build a new complex network model with multiweights to describe the human connection networks. Here is the basic modeling idea: regard each contact as different weight, and there are several different weights between two nodes that are abstracted by a single human. According to the modeling mechanism, it is obvious that the new network model has more complex dynamic characteristic than existing weighted complex network and can embody the multiple attribute of the physical realities using the multiple weights. So it is quite a meaningful work to build this kind of network model and research its rich dynamical behaviors.

Network static geometric quantity and analysis methods that are supplied by graph theory and social networks are the foundation of complex network research works. The researchers abstract the general network static geometric quantity from all kinds of real-life networks and study their general nature, then study more real-life networks. Static characteristics refer to statistical distribution of given network microscopic quantity or the average of macrostatistics [29]. For the weighted complex networks, the static geometric quantity includes degree and distribution feature, node weight, unit weight and distribution feature, weight correlation, the shortest path and distribution feature, weight cluster coefficient and distribution feature, and betweenness and distribution feature. Some scholars discussed the structure and complex of the real-life network characteristics from different areas. For example, Huang et al. [30] modelled a weighted complex network for public transit routes based on passenger flow and collected related data in Beijing to study its structure and complex characteristics. Korunović et al. [31] studied the differences between transient and steady-state characteristics, and the parameters of the SL models are simulated for different seasons.

As a very common and important nonlinear phenomenon, complex network synchronization is one of the most important research subjects. People combined the production of synchronous phenomenon with static geometric quantity and found that complex network topology plays a key role in determining network synchronization. In the meantime, complex network synchronization is a very important phenomenon and has been applied in many fields. Many researchers achieved all kinds of synchronization in the areas of chemistry, biology, engineering, medicine, etc. [32–35]. The synchronization behavior of many complex networks can explain some real-world phenomenon in the natural world and the field of engineering. Based on the research of synchronization in complex networks, on the one hand, we can understand how static geometric quantity affects synchronous ability; on the other hand, when synchronization is advantaged, we can enhance the synchronous ability of networks and reduce its negative factors. Therefore, the phenomenon of network synchronization provides with great theoretical significance and potential applications. At present, the studies of diverse structure network synchronization have attracted many people [36–38], just because the drive systems are different from the response systems for the most part, so the diverse structure synchronization is more challenging, especially the synchronization problems among more different chaotic systems.

Based on above discussions about the various network, we will establish a new multiweight complex network. The big advantage of the new model is that it possesses several weights between two nodes. There must be some new characteristic for the network model and will be more useful in the real world. At the same time, a series of characteristics of the new network must be analyzed and researched in order to use them efficiently and effectively. In this paper, we redefine some static characteristics, such as node weight, node degree, node weight strength, node weight distribution, edge weight distribution, and diversity of weight distribution. We split the triweight network into three single weight networks by using the method of network split. Then, we investigate the complete synchronizations of the same and diverse structure network models by adopting an adaptive control method and simulate it using MATLAB software. The new complex network model with multiweights must have some characteristics different from those of a single-weight complex network, and its topology structure and dynamic characteristics of nodes are more complex. So our motivation is to explore the basic statistical characteristics and synchronization control about the new network model, and then to enrich complex network and control theory, so that we apply them to specific problems, for example, how to design practical networks with better dynamic characters or put them to our advantage in order to understand and interpret the real world better.

We organized the paper as follows. In Section 2, the network model is presented. In Section 3, the new static geometric quantity of the multiweight complex network is defined. In Section 4, the synchronization theory is designed. Conclusion is given in Section 5.

2. The Model of Complex Network with Multiweights

According to the above modeling idea, many important problems in reality can be transformed into multiweight complex networks and then be analyzed from the perspective of complex networks. For example, one can take some bus

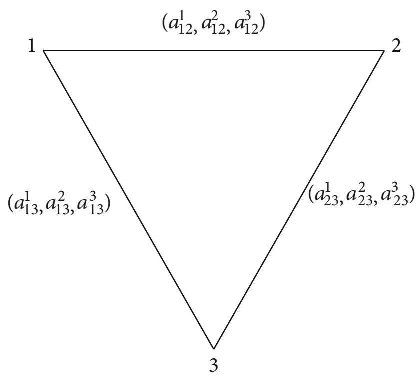

FIGURE 1: The topology map of complex network with triweights.

lines as network nodes, then link one edge from a node to another if there are public stops between the two bus lines. Let us suppose a_{ij}^1, a_{ij}^2, and a_{ij}^3 represent the three different weights on the edge between the ith and the jth network nodes. Then, we can establish a complex public bus line network with triweights, and its topology structure can be shown in Figure 1.

3. The New Static Geometric Quantities of Complex Network with Multiweights

From the model mechanism point of view, the existing static geometric quantities are no longer fit for the new network model, so we must redefine some static geometric quantities, specific as follows.

3.1. Some Static Geometric Quantities of the New Model. Here, we give some notations and definitions: l is the number of weights on each edge. v_{p_j} is the node which is adjacent to the node v_i, e_{ip_j} is the edge connected to the node v_i, and p_j is the label of node v_{p_j}, $j = 1, 2, \cdots$.

3.1.1. Node Weight, Total Node Weight, Mean Node Weight, and Mean Total Node Weight

(i) For the complex network model with multiweights, node weight s_i^r $(1 \le r \le l)$ is defined as the summation of the rth kind of weight on the edge e_{ij}, that is,

$$s_i^r = \sum_{j=N_i} w_{ij}^r, \tag{1}$$

where N_i is the neighboring set of node v_i, w_{ij}^r is the rth kind of weight on the edge e_{ij} $(i \ne j)$. For the complex network with multiweights, the node weight s_i^r can also be defined by the elements of adjacency matrix, that is to say,

$$s_i^r = \sum_{j=1}^{N} a_{ij} w_{ij}^r = \sum_{j=1}^{N} a_{ji} w_{ji}^r, \tag{2}$$

where a_{ij} is the element of adjacency matrix, and $a_{ij} = a_{ji} = 1$ if node v_i and v_j $(i \ne j)$ are connected; otherwise, $a_{ij} = a_{ji} = 0$

(ii) Total node weight S_i^r $(1 \le r \le k)$ is defined as the summation of all the kinds of weights, denoted by

$$S_i^r = \sum_{r=1}^{l} s_i^r \tag{3}$$

(iii) Mean node weight can be defined as

$$\bar{s}_i^l = \frac{1}{n-1} s_i^l \tag{4}$$

(iv) Mean total node weight is denoted by the following formula:

$$\widehat{S}_i^l = \frac{1}{(n-1)^2} \sum_{i=1}^{n} S_i^l \tag{5}$$

From the above definitions, we can see that the meaning of node weight is the same as the traditional weighted networks, and there are multiple weights in the nodes of a complex network model with multiweights. Similarly, the larger the node weight value, the more important the node.

3.1.2. The Node Degree. For the node v_i, the node degree k_i can be defined as the number of weights which is added to each edge. In general, the node degree k_i is invariable.

3.1.3. Node Weight Strength. For the node v_i of the complex network model with multiweights, the rth kind of node weight strength can be defined as the ratio of node weight s_i^r to the node degree k_i, denoted by

$$V_i^r = \frac{s_i^r}{k_i}. \tag{6}$$

The rth kind of node weight reflects the importance of the node to the rth kind of weight.

3.1.4. Node Weight Distribution. The node weight distribution is defined as the ratio of the summation of the rth kind of weight on the edge e_{ip_j} to the summation of all the weights; the specific calculation formula is as follows:

$$S_i^r = \frac{\sum_{p_j \in P} w_{ip_j}^r}{\sum_{r=1}^{k} \sum_{p_j \in P} w_{ip_j}^r}. \tag{7}$$

The definition of node weight distribution reflects the importance of the node v_i.

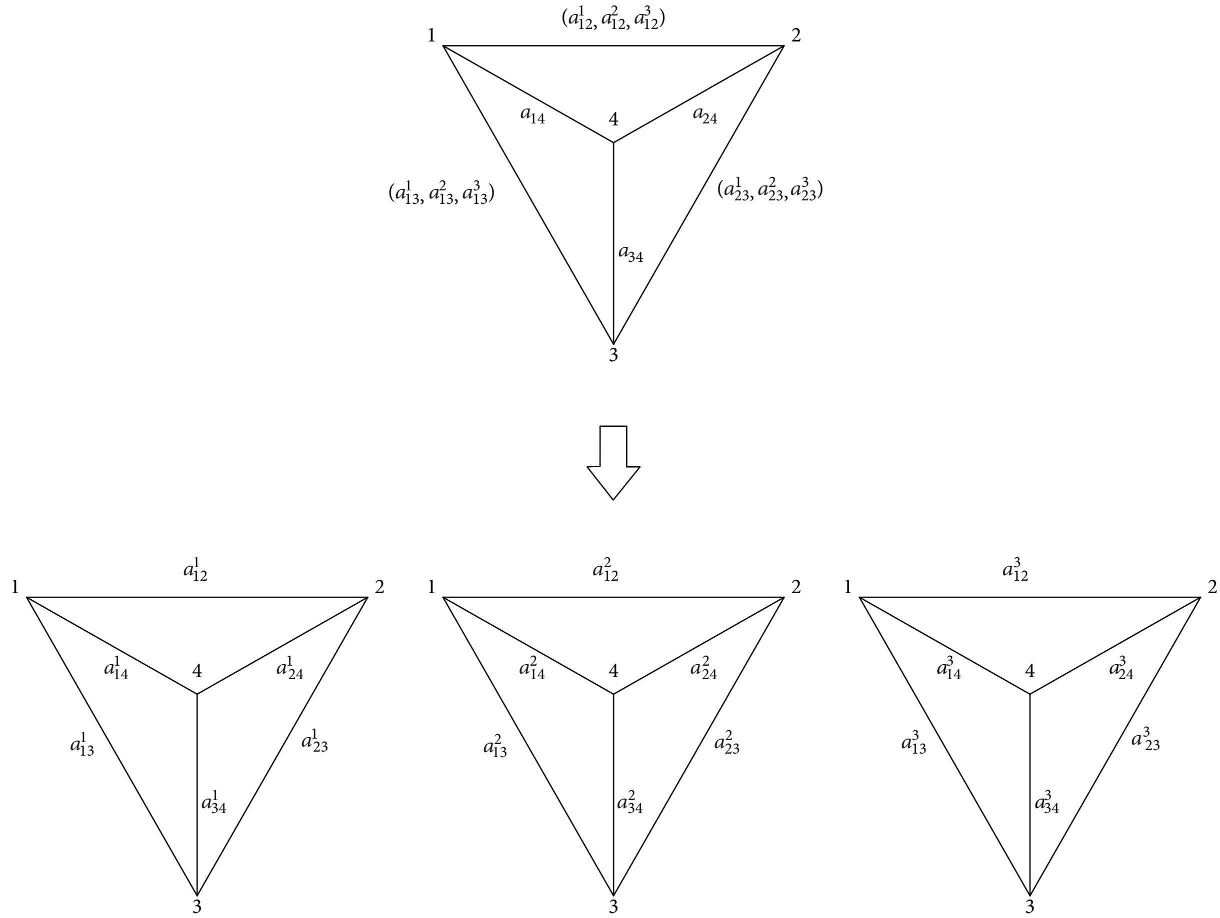

Figure 2: The topology map of complex network with triweights and its split.

3.1.5. Edge Weight Distribution. The edge weight distribution of the rth kind of weight on the edge e_{ij} is defined as the ratio of the rth kind of weight on the edge e_{ij} to the summation of the rth kind of weight, that is to say,

$$P_{ij}^r = \frac{w_{ij}^r}{w^r}, \tag{8}$$

where w^l is the sum of the rth kind of weight, in addition, $\sum_{i=1}^{N}\sum_{j=1}^{N} P_{ij}^l = 1, i \neq j$. The distribution of edge weight reflects the dispersion degree of the rth kind of weight.

3.1.6. The Diversity of Weight Distribution. The rth kind of diversity of weight distribution of the node v_i is defined as the dispersion degree of edge weight distribution, denoted by

$$Y_i^r = \sum_{j \in N_i} \left(\frac{w_{ij}^r}{s_i^r} \right)^2. \tag{9}$$

For the two nodes which have the same node weight and node weight strength, the greater the diversity, the greater the dispersion degree.

3.2. Numerical Example. In this section, we will take public traffic network as an example to analyze statistic characteristic quantities of the new network.

Based on the model method of space R, the public traffic network is established. In particular, the model has four nodes and three weights, and it is modeled based on the following idea: taking bus no.1, 58, 71, and 103 at Lanzhou City as the network nodes, then the corresponding nodes link an edge if there is a common bus stop. When $l = 3$, the three weights are defined as departing frequency [28], accessibility, and passenger flow density [28]. Accessibility can be defined as the reciprocal of the number of the direct public bus between two bus stops. If the number of buses which can be changed two times between two bus stops is m, accessibility can be seen as the double m, and accessibility can be taken as triple n if there are n buses that can be traveled from one stop to another. Based on the above discussion, the triweight traffic network is established by employing a_{ij}^1, a_{ij}^2, and a_{ij}^3 as the three weights (departing frequency, accessibility, and passenger flow density), as shown in the first row of Figure 2.

The key to our analysis in the network model is how to deal with it reasonably; there is an effective way that associate the new model to single-weight complex networks. Here, we design a split method: group the same kind of weights and all the nodes and edges and then establish some single-weight complex networks. So the multiweight network can be split into the combination of some weighted complex network models (the number of network models with single weight is the same to the number of edge weights). Based on the above split method, the topology structure of triweight traffic network and its split are shown in Figure 2; the second row is the three-weighted complex networks spat according to the split idea.

Obviously, this new network model can include more information and analyze the real network more easily. According to the topology map of complex network with triweights, we can design the whole dynamical network equation as

$$\dot{x}_i = f(x_i) + \varepsilon_1 \sum_{j=1}^{4} a_{ij}^1 \mathrm{H}_1 x_j + \varepsilon_2 \sum_{j=1}^{4} a_{ij}^2 \mathrm{H}_2 x_j + \varepsilon_l \sum_{j=1}^{4} a_{ij}^3 \mathrm{H}_3 x_j, \quad 1 \le i \le 4, \tag{10}$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in})^T \in R^n$ is the state vector and the input vector of node i at time t, respectively. $f : R^n \longrightarrow R^n$ is a continuously differentiable vector function. ε_i $(i = 1, 2, 3)$ is the coupling strength of the lth subnetwork, respectively. $H_1, H_2, H_3 \in R^{n\times n}$ are positive diagonal matrices which describe the subnetwork inner couplings. Coupling matrixes $A^l = (a_{ij}^l) \in R^{n\times n}$ $(l = 1, 2, 3)$ is the coupling configuration matrix with zero-sum rows, which represent the coupling strength and the topological structure of the subnetwork, and fulfill the dissipation condition $\sum_{j=1}^{4} a_{ij}^l = 0$. a_{ij}^l is defined as follows: if there exists q connections from node i to node j of the lth subnetwork, then

$$a_{ij}^l = a_{ji}^l = q, \tag{11}$$

otherwise,

$$a_{ij}^l = a_{ji}^l = 0, \quad i \ne j, \tag{12}$$

and define

$$a_{ii}^l = -\sum_{\substack{j=1\\ j\ne i}}^{4} a_{ij}^l = -\sum_{\substack{j=1\\ j\ne i}}^{4} a_{ji}^l, \quad i = 1, 2, 3, 4, \mathrm{l} = 1, 2, 3. \tag{13}$$

For the conventional complex network, we can study its characteristics by distributing the strength of the interaction between nodes and other information on the edges and weights. The complex network with triweights can describe the interaction between two nodes more clearly and exactly, and there must be a lot of different characteristics between the complex network with multiweights and the weighted complex network. Therefore, the complex network with multiweights need to be modeled and explored its particular characteristics.

In this section, we will calculate the statistic characteristic quantities of the complex network with triweights; the three kinds of weights are assigned as follows:

$$\begin{aligned} &a_{12}^1 = 4, a_{13}^1 = 3, a_{14}^1 = 2\,; a_{23}^1 = 4, a_{24}^1 = 7\,; a_{34}^1 = 3, \\ &a_{12}^2 = 2, a_{13}^2 = 3, a_{14}^2 = 6\,; a_{23}^2 = 4, a_{24}^2 = 9\,; a_{34}^2 = 5, \\ &a_{12}^3 = 5, a_{13}^3 = 3, a_{14}^3 = 6\,; a_{23}^3 = 7, a_{24}^3 = 8\,; a_{34}^3 = 3. \end{aligned} \tag{14}$$

3.2.1. Node Weight. For the complex network with triweights as shown in Figure 2, the three kinds of node weights are calculated as follows:

$$\begin{aligned} &(S_1^1, S_2^1, S_3^1, S_4^1) = (9, 15, 10, 12), \\ &(S_1^2, S_2^2, S_3^2, S_4^2) = (11, 15, 12, 20), \\ &(S_1^3, S_2^3, S_3^3, S_4^3) = (14, 20, 13, 17). \end{aligned} \tag{15}$$

As you can see from the data above, the first kind of node weights of nodes 2 and 4 are relatively large, so the two nodes are the more noteworthy. Correspondingly, the bus nos. 58 and 103 have the larger passenger traffic in the public traffic network. Meanwhile, the bus nos.1 and 71 have fewer passengers to change reciprocally because of the relatively small second kind of node weights of nodes 1 and 3. We also can see that node 2 is a rather important node, and there are more passengers in the bus no. 58.

3.2.2. The Node Degree. The degrees of each node are calculated as follows:

$$(v_1, v_2, v_3, v_4) = (2, 2, 2, 2). \tag{16}$$

Obviously, all nodes have equal importance at Lanzhou West Station.

3.2.3. Node Weight Strength. It has been calculated that the three kinds of node weight strength are shown as follows:

$$\begin{aligned} &(V_1^1, V_2^1, V_3^1, V_4^1) = \left(\frac{7}{2}, \frac{15}{2}, 5, 6\right), \\ &(V_1^2, V_2^2, V_3^2, V_4^2) = \left(\frac{11}{2}, \frac{15}{2}, 6, 10\right), \\ &(V_1^3, V_2^3, V_3^3, V_4^3) = \left(7, 10, \frac{13}{2}, \frac{17}{2}\right). \end{aligned} \tag{17}$$

From the calculated result, we can find that the importance of nodes is the same as the conclusions in the example of node degrees.

3.2.4. Node Weight Distribution. The three kinds of node weight distribution are calculated as follows:

$$\left(S_1^1, S_2^1, S_3^1, S_4^1\right) = \left(\frac{9}{34}, \frac{16}{50}, \frac{10}{35}, \frac{12}{49}\right),$$
$$\left(S_1^2, S_2^2, S_3^2, S_4^2\right) = \left(\frac{9}{34}, \frac{15}{50}, \frac{20}{35}, \frac{12}{49}\right), \quad (18)$$
$$\left(S_1^3, S_2^3, S_3^3, S_4^3\right) = \left(\frac{14}{34}, \frac{20}{50}, \frac{13}{35}, \frac{17}{49}\right).$$

Based on the definition of node weight distribution, the nodes 2, 3, and 4 are more important under the three kinds of weights, respectively.

3.2.5. Edge Weight Distribution. The three kinds of edge weight distribution have been calculated as follows:

$$\left(P_{12}^1, P_{13}^1, P_{14}^1, P_{23}^1, P_{24}^1, P_{34}^1\right) = \left(\frac{4}{23}, \frac{3}{23}, \frac{2}{23}, \frac{4}{23}, \frac{7}{23}, \frac{3}{23}\right),$$
$$\left(P_{12}^2, P_{13}^2, P_{14}^2, P_{23}^2, P_{24}^2, P_{34}^2\right) = \left(\frac{2}{29}, \frac{3}{29}, \frac{6}{29}, \frac{4}{29}, \frac{9}{29}, \frac{5}{29}\right),$$
$$\left(P_{12}^3, P_{13}^3, P_{14}^3, P_{23}^3, P_{24}^3, P_{34}^3\right) = \left(\frac{5}{32}, \frac{3}{32}, \frac{6}{32}, \frac{7}{32}, \frac{8}{32}, \frac{3}{32}\right). \quad (19)$$

It can be seen from above data that bus no. 50, no. 109, and no.118 are crowded than other bus lines, bus no. 50 and bus no. 58 have a greater transfer distance, and the number of passengers of transfer from bus no. 50 to bus no. 118 is the largest.

3.2.6. The Diversity of Weight Distribution. The three kinds of edge weight distribution have been calculated as follows:

$$\left(Y_1^1, Y_2^1, Y_3^1, Y_4^1\right) = \left(\frac{29}{49}, \frac{23}{73}, \frac{17}{50}, \frac{31}{72}\right),$$
$$\left(Y_1^2, Y_2^2, Y_3^2, Y_4^2\right) = \left(\frac{49}{121}, \frac{101}{225}, \frac{25}{72}, \frac{71}{200}\right), \quad (20)$$
$$\left(Y_1^3, Y_2^3, Y_3^3, Y_4^3\right) = \left(\frac{5}{14}, \frac{69}{200}, \frac{67}{169}, \frac{109}{289}\right).$$

4. The Synchronization of Triweight Complex Network

In the paper [28], the description and preliminaries of a multiweight network model are described detailedly; now, we will discuss the synchronization by employing the technology.

4.1. The Synchronization Criterion. The controlled dynamical network equation can be given as

$$\dot{x}_i = f(x_i) + \varepsilon_1 \sum_{j=1}^{4} a_{ij}^1 \mathrm{H}_1 x_j + \varepsilon_2 \sum_{j=1}^{4} a_{ij}^2 \mathrm{H}_2 x_j + \varepsilon_l \sum_{j=1}^{4} a_{ij}^3 \mathrm{H}_3 x_j + u_i, \quad 1 \le i \le 4, \quad (21)$$

where $u_i \in R^n$ is the state vector and the input variable of node i at time t, respectively, and fulfills

$$\varepsilon_1 \sum_{j=1}^{3} a_{ij}^1 \mathrm{H}_1 s(t) + \varepsilon_2 \sum_{j=1}^{3} a_{ij}^2 \mathrm{H}_2 s(t) + \varepsilon_3 \sum_{j=1}^{3} a_{ij}^3 \mathrm{H}_3 s(t) + u_i = 0, \quad 1 \le i \le 4, \quad (22)$$

where $s(t)$ is the solution of node equation $\dot{x} = f(x_i(t))$.

To achieve the objective (22), we need the following assumptions and lemmas.

Assumption 1. There exist a positive constant α satisfying

$$\|f(x_i) - f(s)\| \le \alpha \|x_i(t) - s(t)\|, \quad i = 1, 2, \cdots, N. \quad (23)$$

Assumption 2. There exist a positive constant β_l satisfying

$$\left\|H_l\left(x_j(t)\right) - H_l(s(t))\right\| \le \beta_l \left\|x_j(t) - s(t)\right\|, \quad (24)$$
$$l = 1, 2, \cdots, j = 1, 2, \cdots, N.$$

Assumption 3. Assume that all the weights of complex networks with multiweights are nonnegative.

Lemma 4. *For arbitrary* $x, y \in R^n, \mu > 0$, $2x^T y \le \mu x^T x + 1/\mu y^T y$ *is established.*

Theorem 5. *The controller can be selected as*

$$u_i = -p_i e_i, \quad (25)$$

$$\dot{p}_i = k_i e_i^T e_i (1 \le i \le 4). \quad (26)$$

As the positive constant, the role of constant k_i is to realize complete synchronization by adjusting the controller (25).

Proof. The Lyapunov function can be selected as follows:

$$V = \sum_{i=1}^{4} e_i^T e_i + \sum_{i=1}^{4} k_i (p_i - p*)^2 + \sum_{m=1}^{l-1} a_{(m)} \varepsilon_m \beta_m^2 \times \sum_{i=1}^{4} p_i, \quad (27)$$

where

$$a_{(m)} = \max_{1\le i,j\le N} \left| a_{(m)ij} \right| (0 \le m \le l-1), \tag{28}$$

$p*$ is a positive constant.

Taking the derivative of equation (27), we can get

$$\begin{aligned}
\dot{V} &= \sum_{i=1}^{4} e_i^T \dot{e}_i + 2\sum_{i=1}^{4} k_i (p_i - p*) \cdot \dot{p}_i + l \sum_{m=1}^{m-1} a_{(m)} \varepsilon_m \beta_m^2 \times \sum_{i=1}^{4} \dot{p}_i \\
&= 2\sum_{i=1}^{4} e_i^T \dot{e}_i \left[f(x_i) - f(s) + \varepsilon_1 \sum_{j=1}^{4} a_{(1)ij} h_1(e_j) + \varepsilon_2 \sum_{j=1}^{4} a_{(2)ij} h_2(e_j) \right. \\
&\quad \left. + \cdots + \varepsilon_{m-1} \sum_{j=1}^{4} a_{(1)ij} h_1(e_j) \right] + 2\sum_{i=1}^{4} k_i^2 (p_i - p*) e_i^T e_i \\
&\quad + \sum_{m=1}^{l-1} \sum_{i=1}^{4} a_{(m)} \varepsilon_m \beta_l^2 k_i e_i^T e_i.
\end{aligned} \tag{29}$$

Then, one has the following equations based on Assumption 1:

$$\begin{aligned}
e_i^T [f(x_i) - f(s)] &\le \|e_i\| \|f(x_i) - f(s)\| \le \alpha \|e_i\| \|x_i - s\| \\
&= \alpha \|e_i\|^2 = \alpha e_i^T e_i,
\end{aligned} \tag{30}$$

and according to Assumption 2 and Lemma 4, we can get

$$\begin{aligned}
2\sum_{i=1}^{4}\sum_{j=1}^{4} a_{(m)} e_i^T h_m e_j &\le a_{(m)} \sum_{i=1}^{4}\sum_{j=1}^{4} \left[e_i^T e_j + h_m^T e_j^T h_m e_j \right] \\
&\le a_{(m)} \sum_{i=1}^{4}\sum_{j=1}^{4} \left[e_i^T e_j + \beta_l^2 e_j^T e_j \right] \\
&= a_{(m)} \sum_{i=1}^{4} \left[e_i^T e_j + \beta_l^2 e_i^T e_i \right].
\end{aligned} \tag{31}$$

According to Assumption 3, $\dot{V}(t)$ can be rewritten as

$$\begin{aligned}
\dot{V} &\le \sum_{i=1}^{4} \left(2\alpha - 2k_i^2 p*\right) e_i^T e_i + a_{(1)} \varepsilon_1 N \sum_{i=1}^{4} \left[e_i^T e_i + \beta_1^2 e_i^T e_i \right] + \cdots \\
&\quad + a_{(m-1)} \varepsilon_{m-1} \sum_{i=1}^{4} \left[e_i^T e_i + \beta_{m-1}^2 e_i^T e_i \right] + \sum_{l=1}^{m-1} \sum_{i=1}^{4} a_{(l)} \varepsilon_l \beta_l^2 k_i e_i^T e_i \\
&= \sum_{i=1}^{4} \left[2\alpha - 2k_i^2 p* + \sum_{l=1}^{m-1} a_{(l)} \varepsilon_l \beta_l^2 k_i e_i^T e_i \right] + \sum_{i=1}^{4} \sum_{l=1}^{m-1} 4 a_{(l)} \varepsilon_l e_i^T e_i \\
&= \sum_{i=1}^{4} \left[2\alpha - 2k_i^2 p* + \sum_{l=1}^{m-1} a_{(l)} \varepsilon_l \left(\beta_l^2 k_i + 4\right) \right] e_i^T e_i.
\end{aligned} \tag{32}$$

Choose the proper $p*$ and k_i such that $2\alpha - 2k_i^2 p* + \sum_{l=1}^{m-1} a_{(l)} \varepsilon_l (\beta_l^2 k_i + 4)$ is a negative definite matrix and then $\dot{V} < 0$, so we have $\lim_{t\to\infty} \|e_i\| = 0, 1 \le i \le 4$ and the proof is completed.

4.2. The Synchronization of Triweight Network with the Same Structure. Employ Lorenz system as node equation, and let $H_1 = H_2 = H_3 = \mathrm{diag}\,(1, 1, 1)$, so the network node can be described by

$$\begin{pmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{pmatrix} = \begin{pmatrix} 10(x_{i2} - x_{i1}) \\ 28x_{i1} - x_{i2} - x_{i1}x_{i3} \\ x_{i1}x_{i2} - \dfrac{8}{3}x_{i3} \end{pmatrix} + \begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix} + u_i,$$

$$\begin{aligned}
M_{i1} &= x_{11}\left(\varepsilon_1 a_{i1}^1 + \varepsilon_2 a_{i1}^2 + \varepsilon_3 a_{i1}^3\right) + x_{21}\left(\varepsilon_1 a_{i2}^1 + \varepsilon_2 a_{i2}^2 + \varepsilon_3 a_{i2}^3\right) \\
&\quad + x_{31}\left(\varepsilon_1 a_{i3}^1 + \varepsilon_2 a_{i3}^2 + \varepsilon_3 a_{i3}^3\right) + x_{41}\left(\varepsilon_1 a_{i4}^1 + \varepsilon_2 a_{i4}^2 + \varepsilon_3 a_{i4}^3\right), \\
M_{i2} &= x_{12}\left(\varepsilon_1 a_{i1}^1 + \varepsilon_2 a_{i1}^2 + \varepsilon_3 a_{i1}^3\right) + x_{22}\left(\varepsilon_1 a_{i2}^1 + \varepsilon_2 a_{i2}^2 + \varepsilon_3 a_{i2}^3\right) \\
&\quad + x_{32}\left(\varepsilon_1 a_{i3}^1 + \varepsilon_2 a_{i3}^2 + \varepsilon_3 a_{i3}^3\right) + x_{42}\left(\varepsilon_1 a_{i4}^1 + \varepsilon_2 a_{i4}^2 + \varepsilon_3 a_{i4}^3\right), \\
M_{i3} &= x_{13}\left(\varepsilon_1 a_{i1}^1 + \varepsilon_2 a_{i1}^2 + \varepsilon_3 a_{i1}^3\right) + x_{23}\left(\varepsilon_1 a_{i2}^1 + \varepsilon_2 a_{i2}^2 + \varepsilon_3 a_{i2}^3\right) \\
&\quad + x_{33}\left(\varepsilon_1 a_{i3}^1 + \varepsilon_2 a_{i3}^2 + \varepsilon_3 a_{i3}^3\right) + x_{43}\left(\varepsilon_1 a_{i4}^1 + \varepsilon_2 a_{i4}^2 + \varepsilon_3 a_{i4}^3\right),
\end{aligned} \tag{33}$$

$$u_i = -d_i e_i(t), \tag{34}$$

$$\dot{d}_i(t) = k_i e_i^T(t) e_i(t), \quad 1 \le i \le 4. \tag{35}$$

In this example, the values of the parameters for the controller are taken as $\varepsilon_1 = 0.6, \varepsilon_1 = 0.7, \varepsilon_1 = 0.8$, $k_1 = 0.5$, $k_2 = 0.5, k_3 = 0.6, k_4 = 0.3$, according to Theorem 5; the synchronization of the complex network with triweights can be achieved under the adaptive controller (34). Figure 3 show the synchronous errors as the time is evolving.

4.3. The Synchronization of Triweight Network with Diverse Structure. Although there are many papers that have covered the network synchronization, most paper concentrate on the synchronization model with the same nodes, that is, the chaotic systems are the same even if they have different initializations. But in fact, the network models with difference nodes are unavoidable and so pervasive that it has caught many people's heart in recent years. So in this section, we will discuss the synchronization among different nodes, which are abstracted by the following four chaotic systems under the frame of Figure 2.

The Lorenz system:

$$f(x_1) = A_1 x_1 + \begin{pmatrix} 0 \\ -x_{11}x_{13} \\ x_{11}x_{12} \end{pmatrix}, \quad A_1 = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -\dfrac{8}{3} \end{pmatrix}, \tag{36}$$

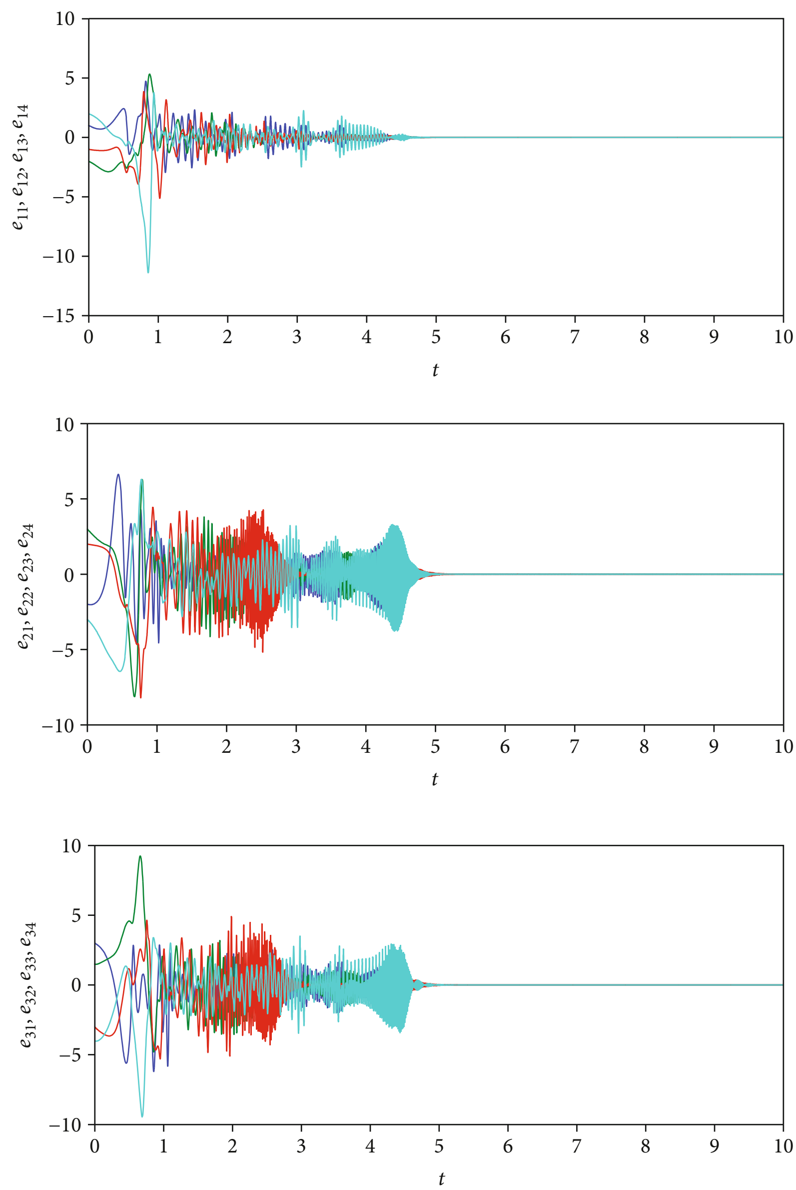

Figure 3: The synchronous errors of the triweight complex network with the same structure.

the Lü system:

$$f(x_2) = A_2 x_2 + \begin{pmatrix} 0 \\ -x_{21}x_{23} \\ x_{21}x_{22} \end{pmatrix}, \quad A_2 = \begin{pmatrix} -36 & 36 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & -3 \end{pmatrix}, \tag{37}$$

the Chen system:

$$f(x_3) = A_3 x_3 + \begin{pmatrix} 0 \\ -x_{31}x_{33} \\ x_{31}x_{32} \end{pmatrix}, \quad A_3 = \begin{pmatrix} -35 & 35 & 0 \\ -7 & 28 & 0 \\ 0 & 0 & -3 \end{pmatrix}, \tag{38}$$

and the Rössler system:

$$f(x_4) = A_4 x_4 + \begin{pmatrix} 0 \\ 0 \\ x_{41}x_{43} + 0.2 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0.2 & 0 \\ 0 & 0 & 5.7 \end{pmatrix}, \tag{39}$$

where $x_i = (x_{i1}, x_{i2}, x_{i3},)^T$, $1 \le i \le 4$. We use the same control to realize diverse structure synchronization under the following the control parameters:

$$\begin{aligned} &\varepsilon_1 = 0.6, \varepsilon_1 = 0.7, \varepsilon_1 = 0.8, \\ &k_1 = 0.5, k_2 = 0.4, k_3 = 0.3, k_4 = 0.3. \end{aligned} \tag{40}$$

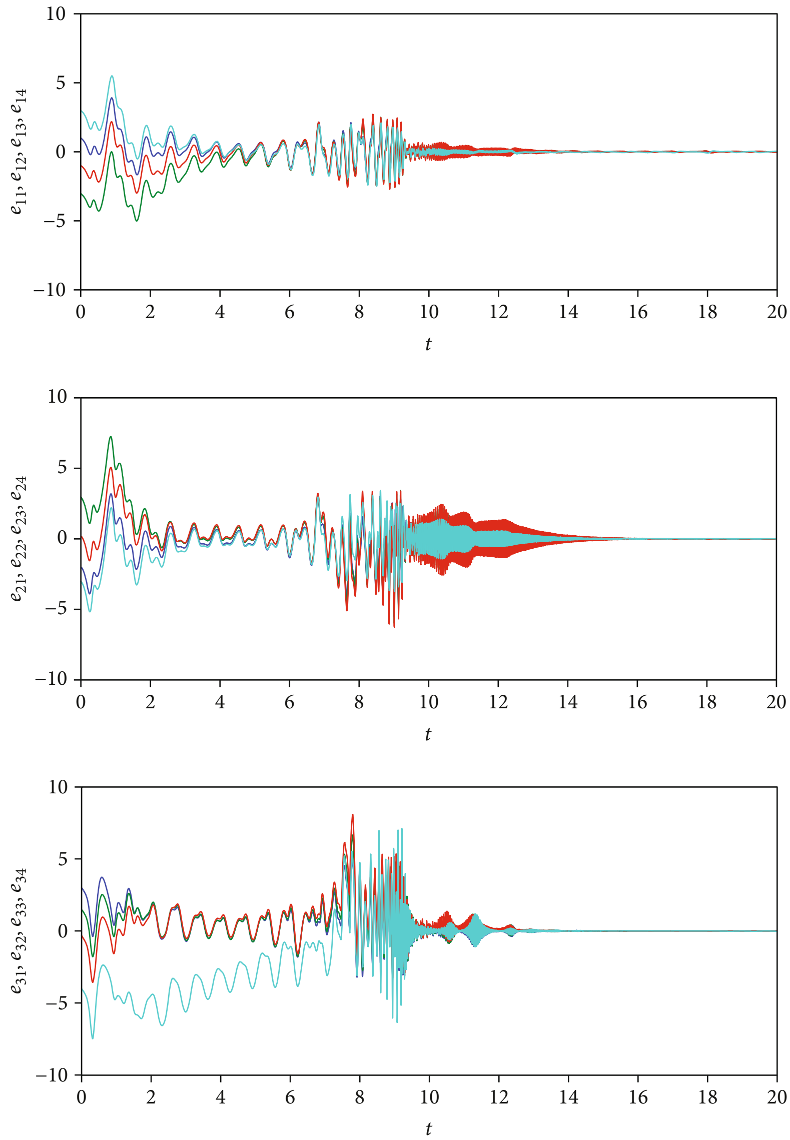

Figure 4: The synchronous errors of the triweight complex network with diverse structure.

According to the controller (26) and control parameters, the four network nodes achieve synchronization although they have different node equation and system initializations; the synchronization sketches are shown in Figure 4.

4.4. Some Deeper Thinking about the Diverse Structure Network Synchronization. Some differences are obvious between the synchronization of the same and diverse structure network models, which is more reasonable and acceptable to model some practical problems, for example, the synchronous ability and the acceptability of messages. There is a great limitation in the same structure network synchronization model due to the difficulty to find some corresponding reality model, while the diverse structure network synchronization has now become more challenging in the field of automation technologies and physical, particularly secure communications. For the studies of multiweight network synchronization, even the problems investigated in the case of different coupled oscillators, we will discuss the following open problems:

(1) When the message is being transmitted, the distribution of multiple weights to each edge might cause the large amount of message, which can lead to overload and congestion, then reduce the transmission rate. The solution is that we can change the topology by readjusting some edges appropriately [37]. There is a basic assumption that the interactions of oscillators are global, so that the synchronous ability relate to the ability of receiving signal. The more receptive the signal, the faster synchronization it becomes. But in the network synchronization model with different nodes, the ability of receiving signal is relatively poorer than the nodes of the same structure network model, just like that the information does not appear to spread easily among people strangers.

In fact, the ability of receiving signal can be readjusted by changing the value β in the formula $L_{ij} = -S_j/P_i^{\beta}$, where L_{ij} is the element of coupling matrix L [29, 39]

(2) In the paper [28], we defined the error as $e_i = x_i - s$; when complete synchronization happens, we have

$$x_1 \longrightarrow x_2 \longrightarrow x_3 \longrightarrow x_4 \longrightarrow s, \quad (41)$$

where $s(t)$ is the soluble vector of node equations when simultaneously happening. That is, all the state variables are tending towards $s(t)$; this pattern is similar to the leader-following consensus problem of the multiagent systems. In fact, we can view $s(t)$ as the leader, and $x_i(t)$ as the follower, so the synchronization problem can be transformed to consensus problem [40]. What makes the problem interesting is that we can introduce some linear or nonlinear systems as the follower systems. This model is scarcely reported, at the same time, has no papers to involve it under the frame of multiweight complex network. Here, we give the leader-following consensus model of the multiagent systems:

$$\dot{x}_i = f(x_i) + \sum_{r=1}^{l} \varepsilon_r \sum_{j=1}^{N} a_{ij}^1 H_1 x_j + B_i u_i + C_i v_i \; (1 \le i \le N), \quad (42)$$

$$e_i = x_i - v_i, \quad (43)$$

where $x_i \in R^n$, ε_i and a_{ij}^l are state vector, coupling strength, and the lth weight, and u_i is control protocol and the ith follower; B_i and C_i can be selected as

$$B_i = (0, 0, 1)^T,$$
$$C_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.8i \end{pmatrix}. \quad (44)$$

v_i is the exogenous signal, denoted by

$$\dot{v}_1 = v_2,$$
$$\dot{v}_2 = -v_1 \quad (45)$$

(3) For the problem of output regulation, we can also apply the multiweight network in consensus problem of the multiagent systems, especially the urgent issue in asynchronously switched multiagent systems [41–43]. Firstly, packets or the changing of communication channel may cause the system transform from one format to another. Therefore, the addition of switching topologies in the problem of output regulation will be of great theoretical and practical significance. Secondly, the state models of agent systems are different and switched with each other, so are the controller modes. Based on formula (42), the asynchronously switched multiagent systems can be described as follows:

$$\dot{x}_i = f(x_i) + \sum_{r=1}^{l} \varepsilon_r \sum_{j=1}^{N} a_{ij}^1 H_1 x_j + B_i^{\varphi_i} u_i + C_i^{\varphi_i} v_i \; (1 \le i \le N), \quad (46)$$

where φ_i is the switching signal of the ith system model; $B_i^{\varphi_i}$ and $C_i^{\varphi_i}$ can be selected as

$$B_i^{\varphi_i} = (0, 0, 1)^T,$$
$$C_i^{\varphi_i} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \varphi_i^l \end{pmatrix}, \quad (47)$$

where l is the number of multiagent systems.

The multiweight complex network can contain more information than the conventional network model, and it is more accordant with reality if we apply it to the consensus problem of multiagent systems; the next work will be the hot issue as discussed above.

5. Conclusion

Real networks are usually the huge and complex systems; we established a multiweight complex network to better characterize them. Some new static geometric quantities are redefined and validated in a living example. Then, we study the complete synchronization in the case of the same and different network nodes; the asymptotically stable network synchronization criteria are deduced and built.

In paper [14], the authors have verified the spread of epidemic when the highly weighted edges are preferentially removed by using an edge-weight-based removal strategy. That is a very interesting finding, and we believe that studying the spreading dynamics on the multiweight complex network will be extremely significant. Meanwhile, to establish the network model, it is more effective to model using a multiweight network. In the coming article, the research will involve spreading dynamics, dynamic characteristics, and the consensus problem of the multiagent systems so that the new network model is more practical.

Data Availability

All data included in this study are available upon request by contact with the corresponding author.

Conflicts of Interest

There is no conflict of interest.

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