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Research Article

Characterization of Graphs with an Eigenvalue of Large Multiplicity

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Let G be a simple and undirected graph. The eigenvalues of the adjacency matrix of G are called the eigenvalues of G. In this paper, we characterize all the n-vertex graphs with some eigenvalue of multiplicity $n-2$ and $n-3$, respectively. Moreover, as an application of the main result, we present a family of nonregular graphs with four distinct eigenvalues.

1. Introduction

All graphs here considered are simple, undirected, and connected. Let G be a graph with vertex set $V(G)=\{v_1, v_2, \cdots, v_n\}$. The set of all the neighbors of a vertex v_i is denoted by $N_G(v_i)$. Two adjacent vertices v_i and v_j are denoted by $v_i \sim v_j$. The adjacency matrix $A(G)=[a_{ij}]$ of G is a real symmetric matrix, and $a_{ij}=1$ if there is an edge joining the vertices v_i and v_j; otherwise, $a_{ij}=0$. The eigenvalues of $A(G)$ are called the eigenvalues of G, denoted by $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$. The rank of the adjacency matrix $A(G)$ of G is called the rank of G, written as $r(G)$. The rank of a matrix M is also written as $r(M)$. An independent set of G is a subset of $V(G)$ such that there is no edge between any two vertices. The number of vertices in a maximum independent set of G is called the independent number of G, denoted by $\mu(G)$. The distance between two vertices v_i and v_j, denoted by $d(v_i, v_j)$, is the length of a shortest path between v_i and v_j. Denote by $\mathrm{diam}(G)$ the diameter of G, then $\mathrm{diam}(G)=\max\{d(v_i, v_j): v_i, v_j \in V(G)\}$. Let $m(\rho_i)$ be the multiplicity of an eigenvalue ρ_i of a graph G.

The multiplicity of an eigenvalue of a graph has attracted much attention. Rowlinson gives an extensive investigation in this topic [1–8]. Let G be a graph of order n with an eigenvalue ρ. In [1], the author proved that if $m(\rho)=k$ and $\rho \neq 0,-1$, then $k \leq 1/2(t-1)(t+4)$ with $t=n-k$. This upper bound was extended to $k \leq (1/2)t(t-1)$ with $t=n-k>2$ (or equivalently, $k \leq n+1/2-\sqrt{2n+1/4}$) in [2]. The graphs satisfying $k=(1/2)t(t-1)$ were discussed in [5]. In [3, 4, 6–8], the authors studied the multiplicity of an eigenvalue of a graph in some special graph classes. Moreover, Fonseca [9] proved many relations between the multiplicities of an eigenvalue whenever a path is removed from the graph. Bu et al. [10] gave two upper bounds on eigenvalue multiplicity of unicyclic graphs and trees. Wong et al. [11] improved an upper bound on the multiplicity of a positive eigenvalue of a tree in [3].

Notice that the upper bounds in [1, 2] are established for the multiplicity of the eigenvalue not equal to 0 or -1. In other words, the multiplicities of the eigenvalues 0 and -1 cannot be bounded easily. Then, it is interesting to study the multiplicities of the eigenvalues 0 and -1 of graphs. Here, we are interested in searching the graphs with the eigenvalue -1 or 0 of large multiplicity. It is well known that the multiplicity of the eigenvalue 0 is called the nullity of a graph, which has been studied intensively. Hence, attention may be paid to the graphs with the eigenvalue -1 of large multiplicity. More generally, in this paper, we investigate the graphs with some eigenvalue of large multiplicity because they are related to the graphs with few distinct eigenvalues, which have been investigated intensively (see [12–18], for example).

Denote the set of all n-vertex connected graphs with some eigenvalue of multiplicity k by $\mathscr{G}(n,k)$. The following are the main conclusions of this paper.

Theorem 1. *Let G be a graph of order $n>3$, then $G\in\mathscr{G}(n,n-2)$ if and only if G is the complete bipartite graph $K_{s,t}$ with $s+t=n$.*

Theorem 2. *Let G be a graph of order $n>6$, then $G\in\mathscr{G}(n,n-3)$ if and only if G is the complete tripartite graph $K_{a,b,c}$ with $a+b+c=n$ or the graph Γ (see Figure 1) with $s,t,p\geq 1$ and $s+t+p=n$.*

2. Proofs

Before showing the proofs of Theorems 1 and 2, we first present some known results as lemmas.

Lemma 3 (interlacing theorem, [19]). *For a real symmetric matrix A of order n, let M be a principal submatrix of A with order $s(\leq n)$. Then,*

$$\lambda_{i+n-s}(A)\leq\lambda_i(M)\leq\lambda_i(A),\quad 1\leq i\leq s, \tag{1}$$

where λ_i is the ith largest eigenvalue.

Let H be a symmetric real matrix, whose block form is

$$H=\begin{pmatrix} H_{11} & \cdots & H_{1t}\\ \vdots & \ddots & \vdots\\ H_{t1} & \cdots & H_{tt}\end{pmatrix}, \tag{2}$$

where the transpose of H_{ij} is H_{ji}. Let q_{ij} be the average row sum of H_{ij}, then $Q=[q_{ij}]$ is the quotient matrix of H. If the row sum of H_{ij} is constant, then we say H has an equitable partition.

Lemma 4 (see [19]). *Let H be a symmetric real matrix having an equitable partition and Q be the quotient matrix of H. Then, each eigenvalue of Q is an eigenvalue of H.*

Lemma 5 (see [20, 21]). *Let G be a graph, then $r(G)=2$ if and only if G is a complete bipartite graph, and $r(G)=3$ if and only if G is a complete tripartite graph.*

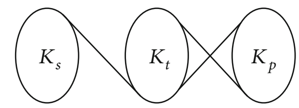

Figure 1: The graph Γ.

Lemma 6 (see [2]). *Let G be a graph of order n and ρ be an eigenvalue of multiplicity k. If $\rho\notin\{0,-1\}$, then*

$$k\leq n+\frac{1}{2}-\sqrt{2n+\frac{1}{4}}, \tag{3}$$

or equivalently, $k\leq(1/2)t(t-1)$ with $t=n-k>2$.

Lemma 7. *Let G be a graph with n vertices and $K=\{v_1,\cdots,v_q\}$ induces a clique in G such that $N_G(v_i)-K=N_G(v_j)-K$ $(1\leq i,j\leq q)$, then -1 is an eigenvalue of G with multiplicity at least $q-1$.*

Proof. Since $K=\{v_1,\cdots,v_q\}$ induces a clique of G and $N_G(v_i)-K=N_G(v_j)-K$ $(1\leq i,j\leq q)$, then the first q rows of the matrix $A(G)+I$ are identical, where I is the identity matrix. Thus $A(G)+I$ contains 0 as an eigenvalue of multiplicity at least $q-1$, which indicates that -1 is an eigenvalue of $A(G)$ with multiplicity at least $q-1$.

In the following, we present the proofs of Theorems 1 and 2.

2.1. Proof of Theorem 1. Let G be a graph of order $n\geq 4$. If G is the complete bipartite graph $K_{s,t}$ with $s+t=n$, then it is easy to know that all the eigenvalues of $K_{s,t}$ are $\{\sqrt{st},-\sqrt{st},0\}$ with multiplicities $\{1,1,n-2\}$, respectively. Thus, $G=K_{s,t}\in\mathscr{G}(n,n-2)$.

Now suppose that $G\in\mathscr{G}(n,n-2)$. We will show that G must be a complete bipartite graph. Let θ be the eigenvalue of G with multiplicity $n-2$. First, assume that $\theta=0$, then the rank $r(G)$ of G is 2, and thus, G is a complete bipartite graph from Lemma 5. Next, assume that $\theta\neq 0$ (this case cannot happen from the following proof). Then, $r(A(G)-\theta I)=2$ with I as the identity matrix, which indicates that the independent number $\mu(G)\leq 2$ (otherwise, $r(A(G)-\theta I)>2$ clearly, a contradiction). Moreover, we claim that G is a cograph, i.e., G contains no path P_4 as an induced subgraph. Otherwise, assume that G contains P_4 as an induced subgraph, and then, $A(P_4)$ (resp., $A(P_4)-\theta I$) is a principal submatrix of $A(G)$ (resp., $A(G)-\theta I$). Thus, one can obtain that

$$r(A(G)-\theta I)\geq r(A(P_4)-\theta I)\geq 3 \tag{4}$$

a contradiction. As a result, $\mathrm{diam}(G)\leq 2$. If $\mathrm{diam}(G)=1$, G is the complete graph K_n whose eigenvalues are $n-1$ and -1 with multiplicity 1 and $n-1$, respectively. Obviously, $K_n\notin\mathscr{G}(n,n-2)$. Suppose that $\mathrm{diam}(G)=2$ and H is an arbitrary connected subgraph with order 4 of G in the following. For the eigenvalues ρ_i of G and λ_i of H, it follows from Lemma 3 that

$$\begin{cases} \rho_1 \geq \lambda_1 \geq \rho_{n-3}, \\ \rho_2 \geq \lambda_2 \geq \rho_{n-2}, \\ \rho_3 \geq \lambda_3 \geq \rho_{n-1}, \\ \rho_4 \geq \lambda_4 \geq \rho_n. \end{cases} \quad (5)$$

Since $G \in \mathcal{G}(n, n-2)$, we obtain that H also contains $\theta(\neq 0)$ as an eigenvalue of multiplicity at least 2. Recalling that $\mu(G) \leq 2$, $\operatorname{diam}(G) = 2$, and G is a cograph, then H must be isomorphic to one of the graphs $\{H_1, H_2, H_3\}$ (see Figure 2). However, by direct calculation, $H_i (1 \leq i \leq 3)$ contains no nonzero eigenvalue of multiplicity at least 2 from Table 1, a contradiction.

Consequently, the proof is completed.

2.2. Proof of Theorem 2. Let G be a graph of order $n > 6$. We first show the sufficiency part. If G is the complete tripartite graph $K_{a,b,c}$ with $a+b+c=n$, then from Lemma 5, it is clear that $K_{a,b,c} \in \mathcal{G}(n, n-3)$ with eigenvalue 0 of multiplicity $n-3$. Suppose that G is the graph Γ with $s, t, p \geq 1$ and $s+t+p=n$ in Figure 1. From Lemma 7, Γ contains -1 as an eigenvalue of multiplicity at least $s+t+p-3 = n-3$. According to the partition $V(\Gamma) = \{V(K_s), V(K_t), V(K_p)\}$, the quotient matrix Q of $A(\Gamma)$ is

$$Q = \begin{pmatrix} s-1 & p & 0 \\ s & p-1 & t \\ 0 & p & t-1 \end{pmatrix}. \quad (6)$$

By calculation, the determinant of the matrix $Q+I$ is

$$\det (Q+I) = -pst \neq 0, \quad (7)$$

which implies that -1 is not an eigenvalue of the quotient matrix Q. Applying Lemma 4, we obtain that -1 is an eigenvalue of Γ with multiplicity $n-3$; that is, $\Gamma \in \mathcal{G}(n, n-3)$.

We now prove the necessity part. Suppose that $G \in \mathcal{G}(n, n-3)$ and θ is the eigenvalue of G with multiplicity $n-3$. First, if $\theta = 0$, then $r(G) = 3$ and G is a complete tripartite graph $K_{a,b,c}$ with $a+b+c=n$ from Lemma 5. Next, suppose that $\theta \neq 0$, then $r(A(G) - \theta I) = 3$. We claim that the independent number $\mu(G) = 2$. Assume on the contrary that $\mu(G) \neq 2$. If $\mu(G) = 1$, G is the complete graph K_n and $K_n \notin \mathcal{G}(n, n-3)$ from the proof of Theorem 1. Suppose that $\mu(G) \geq 4$ with $\{v_1, v_2, v_3, v_4\}$ an independent set of G, and let M be the principal submatrix of $A(G)$ indexed by $\{v_1, v_2, v_3, v_4\}$. Then,

$$r(A(G) - \theta I) \geq r(M - \theta I) = 4, \quad (8)$$

contradicting with $r(A(G) - \theta I) = 3$.

Now suppose that $\mu(G) = 3$ with $S = \{v_1, v_2, v_3\}$ a maximum independent set of G, which yields that each vertex out of S must be adjacent to at least one of S. To complete the proof, the following claims are necessary.

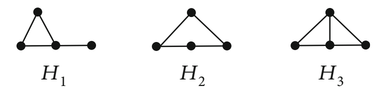

Figure 2: The graphs H_1, H_2, and H_3.

Table 1: The eigenvalues of graphs $H_i (1 \leq i \leq 3)$.

H_1	-1.4812	-1	0.3111	2.1701
H_2	-2	0	0	2
H_3	-1.5616	-1	0	2.5616

Claim 1. The eigenvalue $\theta = -1$.

Recalling that $\theta \neq 0$, further, if $\theta \neq -1$, then from Lemma 6 and $n > 6$,

$$m(\theta) \leq n + \frac{1}{2} - \sqrt{2n + \frac{1}{4}} < n - 3, \quad (9)$$

contradicting with $m(\theta) = n-3$. Hence, $\theta = -1$.

Claim 2. There exists no vertex adjacent to exactly two of $\{v_1, v_2, v_3\}$.

Without loss of generality, suppose for a contradiction that there exists a vertex u such that $u \sim v_1$, $u \sim v_2$ and $u \nsim v_3$. Let M be the principal submatrix of $A(G)$ indexed by $\{v_1, v_2, v_3, u\}$, then $M - \theta I$ is a principal submatrix of $A(G) - \theta I$ and

$$M - \theta I = \begin{pmatrix} -\theta & 0 & 0 & 1 \\ 0 & -\theta & 0 & 1 \\ 0 & 0 & -\theta & 0 \\ 1 & 1 & 0 & -\theta \end{pmatrix}. \quad (10)$$

Denote by R_{v_i} the row of $A(G) - \theta I$ indexed by the vertex v_i. Since $r(A(G) - \theta I) = 3$, it is clear that $\{R_{v_1}, R_{v_2}, R_{v_3}\}$ are linearly independent, which yields that any other rows of $A(G) - \theta I$ can be written as a linear combination of $\{R_{v_1}, R_{v_2}, R_{v_3}\}$. Let

$$R_u = aR_{v_1} + bR_{v_2} + cR_{v_3}. \quad (11)$$

Applying (11) to the first, second, and fourth columns of $M - \theta I$, we get

$$\begin{cases} -a\theta = 1, \\ -b\theta = 1, \\ a + b = -\theta, \end{cases} \quad (12)$$

which yields that $\theta^2 = 2$, contradicting with Claim 1.

Claim 3. There exists no vertex adjacent to each of $\{v_1, v_2, v_3\}$.

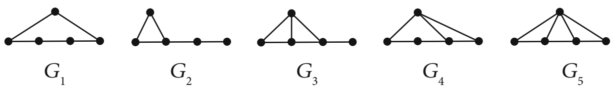

Figure 3: The graphs $G_i (1 \le i \le 5)$.

Suppose for a contradiction that there exists a vertex u such that $u \sim v_i (i = 1, 2, 3)$. Analogous with the proof of Claim 2, let M be the principal submatrix of $A(G)$ indexed by $\{v_1, v_2, v_3, u\}$, then

$$M - \theta I = \begin{pmatrix} -\theta & 0 & 0 & 1 \\ 0 & -\theta & 0 & 1 \\ 0 & 0 & -\theta & 1 \\ 1 & 1 & 1 & -\theta \end{pmatrix}. \tag{13}$$

As $r(A(G) - \theta I) = 3$, then clearly $\{R_{v_1}, R_{v_2}, R_{v_3}\}$ are linearly independent, which span the row space of $A(G) - \theta I$. Let

$$R_u = aR_{v_1} + bR_{v_2} + cR_{v_3}. \tag{14}$$

Applying (14) to the columns of $M - \theta I$, we get

$$\begin{cases} -a\theta = 1, \\ -b\theta = 1, \\ -c\theta = 1, \\ a + b + c = -\theta, \end{cases} \tag{15}$$

which implies that $\theta^2 = 3$, contradicting with Claim 1. Combining the above claims, we see that if $\mu(G) = 3$, then G is not connected, a contradiction. As a result, $\mu(G) \ne 3$. Recalling the discussions before, it can be proved that $\mu(G) = 2$.

In what follows, we prove that G contains no induced path P_4, i.e., G is a cograph. If G contains P_4 as an induced subgraph, then by considering an induced subgraph of order 5 of G, we see that G must contain some G_i $(1 \le i \le 4)$ (see Figure 3) as an induced subgraph (noting that $\mu(G) = 2$). Applying Lemma 3 and Claim 1, we obtain that G_i $(1 \le i \le 5)$ contains $\theta = -1$ as an eigenvalue of multiplicity at least 2. However, by direct calculation, it follows that the multiplicity of -1 as an eigenvalue of G_i $(1 \le i \le 5)$ is not more than one (see Table 2), a contradiction. Therefore, G is a cograph and the diameter $\mathrm{diam}(G) = 2$.

Now we are in a position to complete the proof. Note that $\mu(G) = 2$ and $\mathrm{diam}(G) = 2$ from the above process. Let $P_3 = v_1 v_2 v_3$ be a diameter of G, then $\{v_1, v_3\}$ is a maximum independent set of G and each vertex out of $\{v_1, v_3\}$ is adjacent to at least one of $\{v_1, v_3\}$. Let

Table 2: The eigenvalues of graphs $G_i (1 \le i \le 4)$.

G_1	-1.618	-1.618	0.618	0.618	2
G_2	-1	-0.5392	-1.6751	1	2.2143
G_3	-1	-0.5892	-1.7757	0.7237	2.6412
G_4	-1.1701	-2	0	0.6889	2.4812
G_5	-1.618	-1.4728	-0.4626	0.618	2.9354

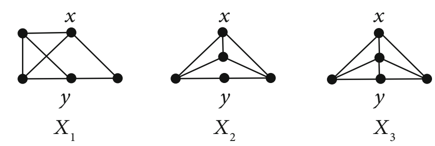

Figure 4: The graphs $X_i (1 \le i \le 3)$.

$$\begin{cases} S_{v_1} = \{v_i \in V(G) \setminus V(P_3) \colon v_i \sim v_1, v_i \nsim v_3\}, \\ S_{v_2} = \{v_i \in V(G) \setminus V(P_3) \colon v_i \sim v_1, v_i \sim v_3\}, \\ S_{v_3} = \{v_i \in V(G) \setminus V(P_3) \colon v_i \nsim v_1, v_i \sim v_3\}, \end{cases} \tag{16}$$

then any vertex out of $V(P_3)$ belongs precisely to one of S_{v_i} $(1 \le i \le 3)$. The following claims are needed for us.

Claim 4. Each vertex of S_{v_1} (resp., S_{v_3}) is adjacent to each one of S_{v_2}.

Suppose $u \in S_{v_1}$ and $w \in S_{v_2}$ such that $u \nsim w$. Then, the vertices $\{u, v_1, w, v_3\}$ induce a path P_4, a contradiction. The proof for the case of S_{v_3} is parallel, omitted.

Claim 5. All the vertices of S_{v_1} (resp., S_{v_3}) induce a clique of G.

We only prove the case of S_{v_1}. If $x, y \in S_{v_1}$ and $x \nsim y$, then $\{x, y, v_3\}$ induce an independent set of G, contradicting with $\mu(G) = 2$.

Claim 6. All the vertices of S_{v_2} induce a clique of G.

Assume that $x, y \in S_{v_2}$ and $x \nsim y$. Considering an induced subgraph H of order 5 of G, we obtain that H is isomorphic to one of $\{X_1, X_2, X_3\}$(see Figure 4). It follows from Lemma 3 that H contains θ as an eigenvalue of multiplicity at least 2. But $X_i (1 \le i \le 3)$ contains no eigenvalue of multiplicity 2 from Table 3, a contradiction.

From Claims 4–6 and the facts $\mu(G) = 2$ and $\mathrm{diam}(G) = 2$, we derive that G is isomorphic to the graph Γ in Figure 1, as required. The proof is completed.

van Dam [14] and Huang and Huang [18] investigated the regular graphs with four distinct eigenvalues. Here, as an application of Theorem 2, we obtain a family of nonregular graphs with four distinct eigenvalues.

Table 3: The eigenvalues of graphs $X_i (1 \le i \le 3)$.

X_1	-2.1774	-1	0	0.3216	2.8558
X_2	-2.1774	-1	0	0.3216	2.8558
X_3	-2	-1.2361	0	0	3.2361

Corollary 8. *The graph Γ with $s, t, p \ge 1$ and $s + t + p = n$ (see Figure 1) contains four distinct eigenvalues, which is not a regular graph.*

Proof. From the proof of Theorem 2, we see that -1 is an eigenvalue of Γ with multiplicity $n - 3$ and the remaining three eigenvalues of Γ are those of the quotient matrix Q of $A(\Gamma)$. Since det $(Q + I) = -pst < 0$, then $Q + I$ contains two positive eigenvalues and one negative eigenvalue. By the Perron-Frobenius theorem, the largest eigenvalue of Γ is simple; then, the largest eigenvalue of Q (resp., $Q + I$) is simple. Thus, $Q + I$ contains three distinct eigenvalues, that is, Q contains three distinct eigenvalues. Recalling that -1 is not an eigenvalue of Q, then Γ contains four distinct eigenvalues. Moreover, it is clear that Γ is not a regular graph.

Data Availability

In this study, we use the theoretical model method to carry out our research. Our conclusions are obtained primarily by using theoretical deduction and numerical study. Of these, numerical study data are derived from the author's assumptions, also illustrated in Tables 1–3. We thereby declare that no further external data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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