Research Article

Mechanical Solving a Few Fractional Partial Differential Equations and Discussing the Effects of the Fractional Order

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With the help of Maple, the precise traveling wave solutions of three fractal-order model equations related to water waves, including hyperbolic solutions, trigonometric solutions, and rational solutions, are obtained by using function expansion method. An isolated wave solution is selected from the solution of each nonlinear dispersive wave model equation, and the influence of fractional order change on these isolated wave solutions is discussed. The results show that the fractional derivatives can modulate the waveform, local periodicity, and structure of the isolated solutions of the three model equations. We also point out the construction rules of the auxiliary equations of the extended (G'/G) -expansion method. In the “The Explanation and Discussion” section, a more generalized auxiliary equation is used to further emphasize the rules, which has certain reference value for the construction of the new auxiliary equations. The solutions of fractional-order nonlinear partial differential equations can be enriched by selecting other solvable equations as auxiliary equations.

1. Introduction

Because of many phenomena, integer-order differential equations cannot be well described, which makes fractional nonlinear differential equations have research significance. As an effective mathematical modeling tool, it is widely used in the mathematical modeling of nonlinear phenomena in biology, physics, signal processing, control theory, system recognition, and other scientific fields [1–4]. In order to better understand the mechanism behind the phenomena described by nonlinear fractional partial differential equations, it is necessary to obtain the exact solution, which also provides a reference for the accuracy and stability of the numerical solution. With the rapid development of computer

algebraic system-based nonlinear sciences like Mathematica or Maple, divers' effective methods have been pulled out to acquire precise solutions to nonlinear fractional-order partial differential equations, such as the fractional first integral method [5, 6], the fractional simplest equation method [7, 8], the improved fractional subequation method [9], the Kudryashov method [10], the fractional subequation method [11, 12], the generalised Kudryashov method [13], the fractional exp-function method [14–19], the sech-tanh function expansion method [20, 21], the fractional (G'/G) -expansion method [22–29], the generalized Sinh-Gorden expansion method [30], the fractional functional variable method [31], the rational (G'/G) -expansion method [32], the modified Khater method [33–36], and the fractional modified trial

equation method [37, 38]. Many of these methods are constructed by fractional complex transform [39, 40] and use of the solutions of some solvable differential equations. However, there is no one way to solve all kinds of nonlinear problems, and for the same nonlinear differential equation, different methods will give you different forms of solutions. There are many articles about solving different equations by different methods, but the effect of fractional order on the solution is rarely discussed.

The first model equation we want to solve is the fractional-order Boussinesq equation in space and time, which is suitable for studying the propagation of water in heterogeneous porous media [41].

$$D_t^{2\alpha}u(x, t) + AD_x^{2\beta}[u(x, t)]^2 + BD_x^{2\beta}u(x, t) + ED_x^{4\beta}u(x, t) = 0, \quad 0 < \alpha, \beta \leq 1, t > 0, \quad (1)$$

for the case of $\beta = \alpha$ [42]:

$$D_t^{2\alpha}u(x, t) + AD_x^{2\alpha}[u(x, t)]^2 + BD_x^{2\alpha}u(x, t) + ED_x^{4\alpha}u(x, t) = 0. \quad (2)$$

where $u(x, t)$ represents displacement. A , B , and E are constant coefficients. α and β are fractional derivatives. When $\alpha = \beta = 1$, equation (1) curtails to the Boussinesq equation of the form

$$u_{tt} + A(u^2)_{xx} + Bu_{xx} + Eu_{xxxx} = 0. \quad (3)$$

Equation (3) was first derived by Boussinesq when he studied the propagation of nonviscous shallow water waves [43–45]. Darvishi et al. obtained solitary wave solutions of some equations similar to Boussinesq in literature [46]. Combined with fractional complex transformation, we obtain multiple traveling wave solutions of equation (2) using extended (G'/G) -expansion method and show the effect of fractional order parameters on the waveform of an isolated wave solution of these solutions.

The second model equation we solved was a diffusion model describing shallow water waves (the time fractional-order Boussinesq-Burgers equation) [47].

$$\begin{aligned} D_t^\alpha u(x, t) - \frac{1}{2}v_x(x, t) + 2u_x(x, t) &= 0, \\ D_t^\alpha v(x, t) - \frac{1}{2}u_{xxx}(x, t) + 2(uv)_x(x, t) &= 0, \end{aligned} \quad (4)$$

$$0 < \alpha \leq 1.$$

There are several ways to solve this equation. For example, Javeed et al. solved it by the first integral method [47], and Kumar et al. solved it by the residual power series method [48].

Combined with fractional complex transformation, we obtain multiple traveling wave solutions of equation (4) using (G'/G) -expansion method and show the effect of fractional order parameters on the waveform of an isolated wave solution of these solutions.

Finally, the third model equation that we want to solve can simulate the propagation of surface water waves with a depth far less than the horizontal scale, which is the fractional coupled Boussinesq equations in space and time [49].

$$\begin{aligned} D_t^\alpha u(x, t) + D_x^\beta v(x, t) &= 0, \\ D_t^\alpha v(x, t) + AD_x^\beta(u^2(x, t)) - ED_{xxx}^\beta u(x, t) &= 0, \end{aligned} \quad (5)$$

$$0 < \alpha, \beta \leq 1.$$

There are several ways to solve this system of equations. For example, Yaslan and Girgin solved it by the first integral method [49], Hosseini and Ansari obtained its solution by the modified Kudryashov method [50], and Hoseini et al. solved it by the $\exp(-\phi(\varepsilon))$ -expansion method [51]. Combined with fractional complex transformation, we obtain multiple traveling wave solutions of equation (5) using extended (G'/G) -expansion method and show the effect of fractional order parameters on the waveform of an isolated wave solution of these solutions.

Given a function $f : [0, \infty] \rightarrow R$. Then, the conformable fractional derivative of f of order $0 < \alpha < 1$ is defined as [52]

$$D_x^\alpha(f)(x) = \lim_{\varepsilon \rightarrow 0} \frac{f(x + \varepsilon x^{1-\alpha}) - f(x)}{\varepsilon}. \quad (6)$$

The derivative has the following properties [53].

$$\begin{cases} D_x^\alpha C = 0, \\ D_x^\alpha x^\gamma = \gamma x^{\gamma-\alpha}, & \text{for all } \gamma \in R, \\ (u(x)v(x))^{(\alpha)} = u^{(\alpha)}(x)v(x) + u(x)v^{(\alpha)}(x), \\ (f[u(x)])^{(\alpha)} = x^{1-\alpha}f'_u(u)u'(x). \end{cases} \quad (7)$$

2. The (G'/G) -Expansion Method Combined with Fractional Complex Transformation and Its Extension Method

Consider nonlinear fractional partial differential equations

$$P\left(u, D_t^\alpha u, D_x^\beta u, D_t^\alpha D_t^\alpha u, D_x^\beta D_t^\alpha u, D_x^\beta D_x^\beta u, \dots\right) = 0, \quad 0 < \alpha, \beta \leq 1, \quad (8)$$

where u is the unsolved function of the variables x and t . P is a polynomial function, which consists of u and its fractional derivatives.

The fractional (G'/G) -expansion method and extended fractional (G'/G) -expansion method are used to solve equation (8); the steps are listed as follows:

Step 1. Under the fractional complex transform,

$$u(x, t) = U(\xi), \quad \xi = \frac{x^\beta}{\beta} - \frac{ct^\alpha}{\alpha}, \quad (9)$$

where c is a constant, and it cannot be zero. When $\alpha = \beta = 1$, equation (9) is the usual travelling wave variation.

In the complex fraction transformation, we get

$$\begin{aligned} D_t^\alpha(\cdot) &= -c \frac{d(\cdot)}{d\xi}, \\ D_x^\beta(\cdot) &= \frac{d(\cdot)}{d\xi}, \\ D_{xx}^{2\alpha}(\cdot) &= c^2 \frac{d^2(\cdot)}{d\xi^2}, \\ D_x^\beta D_x^\alpha(\cdot) &= -c \frac{d^2(\cdot)}{d\xi^2}, \\ D_{xx}^{2\beta}(\cdot) &= \frac{d^2(\cdot)}{d\xi^2} \dots \end{aligned} \quad (10)$$

Substituting (9) and (10) into (8), a nonlinear ordinary differential equation is formulated

$$P(U, -cU', U', c^2U'', -cU'', U'', \dots) = 0, \quad (11)$$

where " U'' " = $dU/d\xi$. If the form of equation (11) allows, we can integrate first and set the integral constant to zero, which will help simplify the following calculation.

Step 2. For the fractional (G'/G) -expansion method, we assume that equation (11) has a quasisolution of equation (11) of the following form

$$U(\xi) = \sum_{i=0}^m a_i \left(\frac{G'}{G} \right)^i. \quad (12)$$

For the extended fractional (G'/G) -expansion method, we assume that equation (11) has a quasisolution of the following form

$$U(\xi) = \sum_{i=0}^m a_i \left(\frac{G'}{G} \right)^i + \sum_{i=1}^m b_i \left(\frac{G}{G'} \right)^i, \quad (13)$$

where $a_i (i = 0, 1, \dots, m)$ and $b_i (i = 1, 2, \dots, m)$ are undetermined constants. In combination with the form of equation (12) or (13), the highest derivative term and the nonlinear term in equation (11) are balanced by the homogeneous equilibrium principle, and the value of the positive integer in equation (12) or (13) can be obtained. Let us say that the degree of $U(\xi)$ is $D(U(\xi)) = m$, and then, we can easily derive the degrees of other forms of terms as follows:

$$D\left(\frac{d^q U}{d\xi^q}\right) = m + q, \quad D\left[U^p \left(\frac{d^q U}{d\xi^q}\right)^s\right] = pm + s(m + q). \quad (14)$$

Thus, the value of m in equation (12) or equation (13) can be determined. The $G = G(\xi)$ appearing in equation (12) or (13) is the solution of the second-order differential equation below.

$$G'' + \lambda G' + \mu G = 0, \quad (15)$$

where λ and μ are undetermined constants. In addition, the derivative of (G'/G) is

$$\frac{d}{d\xi} \left(\frac{G'}{G} \right) = \frac{G''G - (G')^2}{G^2} = \frac{G''G}{G^2} - \left(\frac{G'}{G} \right)^2. \quad (16)$$

Equation (16) reveals that we can set the ordinary differential equation (15) to the following form or some other ordinary differential equation can make equation (11) in polynomial form of (G'/G) [54].

$$G''G = \lambda \left(\frac{G'}{G} \right)^2 + \mu GG' + \omega G^2, \quad (17)$$

where λ , μ , and ω are undetermined constants.

Step 3. Substitute equation (12) or (13) into equation (11), use ordinary differential equation (15) concerning (G'/G) to combine the same power terms of (G'/G) , then set the coefficients of all powers of (G'/G) to zero, we get a nonlinear algebraic system of equations concerning the unknowns a_i , b_i , λ , μ , and c .

Step 4. We can use Maple to solve the equations obtained in the third step. By substituting the obtained results into equation (12) or (13) and using the general solutions of equation (15) in different situations, multiple exact solutions of different types of equation (8) can be obtained.

The solutions of equation (15) under different conditions are shown below.

$$\left(\frac{G'}{G}\right) = \begin{cases} \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left(\frac{C_1 \sinh \left(\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \right) \xi \right) + C_2 \cosh \left(\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \right) \xi \right)}{C_1 \cosh \left(\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \right) \xi \right) + C_2 \sinh \left(\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \right) \xi \right)} \right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu > 0, \\ \frac{\sqrt{4\mu - \lambda^2}}{2} \left(\frac{-C_1 \sin \left(\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \right) \xi \right) + C_2 \cos \left(\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \right) \xi \right)}{C_1 \cos \left(\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \right) \xi \right) + C_2 \sin \left(\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \right) \xi \right)} \right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu < 0, \\ \frac{C_2}{C_1 + C_2 \xi} - \frac{\lambda}{2}, & \lambda^2 - 4\mu = 0, \end{cases} \quad (18)$$

where C_1 and C_2 are free constants.

When C_1 and C_2 satisfy different conditions, these results can be further written in simpler forms.

$$\left(\frac{G'}{G}\right) = \begin{cases} \frac{\sqrt{\lambda^2 - 4\mu}}{2} \tanh \left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi + \xi_0 \right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu > 0, \tanh(\xi_0) = \frac{C_2}{C_1}, \left| \frac{C_2}{C_1} \right| < 1, \\ \frac{\sqrt{\lambda^2 - 4\mu}}{2} \coth \left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi + \xi_0 \right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu > 0, \coth(\xi_0) = \frac{C_2}{C_1}, \left| \frac{C_2}{C_1} \right| > 1, \\ \frac{\sqrt{4\mu - \lambda^2}}{2} \cot \left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi + \xi_0 \right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu < 0, \cot(\xi_0) = \frac{C_2}{C_1}, \\ \frac{C_2}{C_1 + C_2 \xi} - \frac{\lambda}{2}, & \lambda^2 - 4\mu = 0. \end{cases} \quad (19)$$

The solutions of equation (17) under different conditions are shown below.

$$\left(\frac{G'}{G}\right) = \begin{cases} \frac{\sqrt{\mu^2 + 4\omega - 4\lambda\omega}}{2(1-\lambda)} \left(\frac{C_1 \sinh \left(\left(\frac{\sqrt{\mu^2 + 4\omega - 4\lambda\omega}}{2} \right) \xi \right) + C_2 \cosh \left(\left(\frac{\sqrt{\mu^2 + 4\omega - 4\lambda\omega}}{2} \right) \xi \right)}{C_1 \cosh \left(\left(\frac{\sqrt{\mu^2 + 4\omega - 4\lambda\omega}}{2} \right) \xi \right) + C_2 \sinh \left(\left(\frac{\sqrt{\mu^2 + 4\omega - 4\lambda\omega}}{2} \right) \xi \right)} \right) + \frac{\mu}{2(1-\lambda)} (\mu^2 - 4(\lambda - 1)\omega > 0, \lambda \neq 1), \\ \frac{\sqrt{4\lambda\omega - 4\omega - \mu^2}}{2(1-\lambda)} \left(\frac{-C_1 \sin \left(\left(\frac{\sqrt{4\lambda\omega - 4\omega - \mu^2}}{2} \right) \xi \right) + C_2 \cos \left(\left(\frac{\sqrt{4\lambda\omega - 4\omega - \mu^2}}{2} \right) \xi \right)}{C_1 \cos \left(\left(\frac{\sqrt{4\lambda\omega - 4\omega - \mu^2}}{2} \right) \xi \right) + C_2 \sin \left(\left(\frac{\sqrt{4\lambda\omega - 4\omega - \mu^2}}{2} \right) \xi \right)} \right) + \frac{\mu}{2(1-\lambda)} (\mu^2 - 4(\lambda - 1)\omega < 0, \lambda \neq 1), \\ \frac{1}{1-\lambda} \left(\frac{C_1}{C_1 \xi + C_2} + \frac{\mu}{2} \right) (\mu^2 - 4(\lambda - 1)\omega = 0, \lambda \neq 1). \end{cases} \quad (20)$$

3. Applications of Fractional (G'/G)-Expansion Method and Its Extended Methods

3.1. *Precise Solutions of the Fractional Boussinesq Equation in Space and Time with Generalised Fractional (G'/G)-Expansion Method.* Equation (2) is written as follows.

$$D_t^{2\alpha} u(x, t) + AD_x^{2\alpha} [u(x, t)]^2 + BD_x^{2\alpha} u(x, t) + ED_x^{4\alpha} u(x, t) = 0. \quad (21)$$

Under the fractional complex transform,

$$u(x, t) = U(\xi), \quad \xi = \frac{x^\alpha}{\alpha} - \frac{ct^\alpha}{\alpha}. \quad (22)$$

Substituting (22) into (21), we convert our problem into a nonlinear ordinary differential equation.

$$c^2 U'' + A(U^2)'' + BU'' + EU''' = 0, \quad (23)$$

where " U' " = $dU/d\xi$. By integrating twice with respect to travelling wave variable factor ξ and setting the constant from the integral to 0, you get the following equation.

$$(c^2 + B)U + AU^2 + EU'' = 0. \quad (24)$$

Applying the homogeneous equilibrium principle to equation (24), we get $2 + m = 2m \implies m = 2$. By taking m to be 2 in equation (13), we get the form of the proposed solution of equation (24) as follows.

$$U(\xi) = a_0 + a_1 \frac{G'}{G} + a_2 \left(\frac{G'}{G}\right)^2 + b_1 \left(\frac{G'}{G}\right)^{-1} + b_2 \left(\frac{G'}{G}\right)^{-2}. \quad (25)$$

By using equation (15), from equation (25), we have

$$\begin{aligned} U^2(\xi) = & b_2^2 \left(\frac{G'}{G}\right)^{-4} + 2b_1 \left(\frac{G'}{G}\right)^{-3} b_2 + (2a_0 b_2 + b_1^2) \left(\frac{G'}{G}\right)^{-2} \\ & + (2a_0 b_1 + 2a_1 b_2) \left(\frac{G'}{G}\right)^{-1} + a_0^2 + 2a_1 b_1 + 2a_2 b_2 \\ & + (2a_0 a_1 + 2a_2 b_1) \left(\frac{G'}{G}\right)^1 + (2a_0 a_2 + a_1^2) \left(\frac{G'}{G}\right)^2 \\ & + 2a_1 a_2 \left(\frac{G'}{G}\right)^3 + a_2^2 \left(\frac{G'}{G}\right)^4, \end{aligned} \quad (26)$$

$$\begin{aligned} U''(\xi) = & 6b_2 \mu^2 \left(\frac{G'}{G}\right)^{-4} + (2b_1 \mu^2 + 10b_2 \lambda \mu) \left(\frac{G'}{G}\right)^{-3} \\ & + (3b_1 \lambda \mu + 4b_2 \lambda^2 + 8b_2 \mu) \left(\frac{G'}{G}\right)^{-2} \\ & + (b_1 \lambda^2 + 2b_1 \mu + 6b_2 \lambda) \left(\frac{G'}{G}\right)^{-1} \\ & + 2a_2 \mu^2 + a_1 \lambda \mu + b_1 \lambda + 2b_2 \\ & + (6a_2 \lambda \mu + a_1 \lambda^2 + 2a_1 \mu) \left(\frac{G'}{G}\right)^1 \\ & + (4a_2 \lambda^2 + 8a_2 \mu + 3a_1 \lambda) \left(\frac{G'}{G}\right)^2 \\ & + (10a_2 \lambda + 2a_1) \left(\frac{G'}{G}\right)^3 + 6a_2 \left(\frac{G'}{G}\right)^4. \end{aligned} \quad (27)$$

Equations (25), (26), and (27) are substituted into equation (24), and then, we can rearrange and combine equation (24) with respect to (G'/G) and set the coefficients of all powers of (G'/G) to be zero. The resulting nonlinear algebraic system with respect to the unknowns a_0, a_1, a_2, b_1, b_2 , and c is as follows.

$$\begin{aligned} \left(\frac{G'}{G}\right)^{-4} : & 6E\mu^2 b_2 + Ab_2^2 = 0, \\ \left(\frac{G'}{G}\right)^{-3} : & 2E\mu^2 b_1 + 10E\mu\lambda b_2 + 2Ab_1 b_2 = 0, \\ \left(\frac{G'}{G}\right)^{-2} : & 3E\mu\lambda b_1 + 4E\lambda^2 b_2 + 2Aa_0 b_2 + Ab_1^2 \\ & + 8E\mu b_2 + c^2 b_2 + Bb_2 = 0, \\ \left(\frac{G'}{G}\right)^{-1} : & 2Aa_1 b_2 + c^2 b_1 + 2Aa_0 b_1 + 6Eb_2 \lambda \\ & + Bb_1 + Eb_1 \lambda^2 + 2Eb_1 \mu = 0, \\ \left(\frac{G'}{G}\right)^0 : & 2Eb_2 + c^2 a_0 + Ba_0 + 2Ea_2 \mu^2 + Ea_1 \lambda \mu \\ & + 2Aa_2 b_2 + 2Aa_1 b_1 + Aa_0^2 + Eb_1 \lambda = 0, \\ \frac{G'}{G} : & 2Ea_1 \mu + 2Aa_2 b_1 + c^2 a_1 + Ba_1 + 6Ea_2 \lambda \mu \\ & + Ea_1 \lambda^2 + 2Aa_0 a_1 = 0, \end{aligned}$$

$$\begin{aligned}
\left(\frac{G'}{G}\right)^2 : & 4E\lambda^2 a_2 + 2Aa_0 a_2 + Aa_1^2 + 8E\mu a_2 + 3E\lambda a_1 \\
& + c^2 a_2 + Ba_2 = 0, \\
\left(\frac{G'}{G}\right)^3 : & 2Aa_1 a_2 + 10E\lambda a_2 + 2Ea_1 = 0, \\
\left(\frac{G'}{G}\right)^4 : & Aa_2^2 + 6Ea_2 = 0.
\end{aligned} \tag{28}$$

The nonlinear algebraic equations were solved by using Maple symbol computing system, and the following solutions were obtained.

Case 1.

$$\left\{ c = \pm \sqrt{-E\lambda^2 + 4E\mu - B}, a_0 = -\frac{6E\mu}{A}, \right. \tag{29}$$

$$\left. a_1 = -\frac{6E\lambda}{A}, a_2 = -\frac{6E}{A}, b_1 = 0, b_2 = 0 \right\}.$$

Case 2.

$$\left\{ c = \pm \sqrt{E\lambda^2 - 4E\mu - B}, a_0 = -\frac{E(\lambda^2 + 2\mu)}{A}, \right. \tag{30}$$

$$\left. a_1 = -\frac{6E\lambda}{A}, a_2 = -\frac{6E}{A}, b_1 = 0, b_2 = 0 \right\}.$$

Substituting the values from (29) or (30) and equation (18) into (25), the exact solutions of equation (21) in different forms can be obtained under different parameter constraints.

Case 1. When $\lambda^2 - 4\mu > 0$, the exact solution of equation (21) in hyperbolic form is as follows.

$$\begin{aligned}
U_{1,2}^1(\xi) = & -\frac{6E}{A} \left[\mu + \lambda \left[-\frac{\lambda}{2} + \eta_1 \left(\frac{C_1 \sinh(\eta_1 \xi) + C_2 \cosh(\eta_1 \xi)}{C_1 \cosh(\eta_1 \xi) + C_2 \sinh(\eta_1 \xi)} \right) \right] \right. \\
& \left. + \left[-\frac{\lambda}{2} + \eta_1 \left(\frac{C_1 \sinh(\eta_1 \xi) + C_2 \cosh(\eta_1 \xi)}{C_1 \cosh(\eta_1 \xi) + C_2 \sinh(\eta_1 \xi)} \right) \right]^2 \right],
\end{aligned} \tag{31}$$

where $\xi = (x^\alpha/\alpha) \mp (\sqrt{-E\lambda^2 + 4E\mu - B}t^\alpha/\alpha)$, and $\eta_1 = 1/2\sqrt{\lambda^2 - 4\mu}$. C_1 and C_2 are constants that can take any number.

If $C_1 \neq 0$, and $C_2 = 0$, then $U_{1,2}^1(\xi)$ become

$$\begin{aligned}
u_{1,2}^1(x, t) = & U_{1,2}^1(\xi) \\
= & -\frac{6E}{A} \left[\mu + \lambda \left[-\frac{\lambda}{2} + \eta_1 \tanh(\eta_1 \xi) \right] \right. \\
& \left. + \left[-\frac{\lambda}{2} + \eta_1 \tanh(\eta_1 \xi) \right]^2 \right].
\end{aligned} \tag{32}$$

Again, using (19), the general solutions for $U_{1,2}^1(\xi)$ in simplified forms are written as

$$\begin{aligned}
\dot{U}_{1,2}^1(\xi) = & -\frac{6E}{A} \left[\mu + \lambda \left[-\frac{\lambda}{2} + \eta_1 \tanh(\eta_1 \xi + \xi_0) \right] \right. \\
& \left. + \left[-\frac{\lambda}{2} + \eta_1 \tanh(\eta_1 \xi + \xi_0) \right]^2 \right],
\end{aligned} \tag{33}$$

when $|C_2/C_1| < 1$, and $\xi_0 = \tanh^{-1}(C_2/C_1)$.

$$\begin{aligned}
\ddot{U}_{1,2}^1(\xi) = & -\frac{6E}{A} \left[\mu + \lambda \left[-\frac{\lambda}{2} + \eta_1 \coth(\eta_1 \xi + \xi_0) \right] \right. \\
& \left. + \left[-\frac{\lambda}{2} + \eta_1 \coth(\eta_1 \xi + \xi_0) \right]^2 \right],
\end{aligned} \tag{34}$$

when $|C_2/C_1| > 1$, and $\xi_0 = \coth^{-1}(C_2/C_1)$.

When $\lambda^2 - 4\mu < 0$, the exact solution of equation (21) in trigonometric form is as follows.

$$\begin{aligned}
U_{3,4}^1(\xi) = & -\frac{6E}{A} \left[\mu + \lambda \left[-\frac{\lambda}{2} + \eta_2 \left(\frac{-C_1 \sin(\eta_2 \xi) + C_2 \cos(\eta_2 \xi)}{C_1 \cos(\eta_2 \xi) + C_2 \sin(\eta_2 \xi)} \right) \right] \right. \\
& \left. + \left[-\frac{\lambda}{2} + \eta_2 \left(\frac{-C_1 \sin(\eta_2 \xi) + C_2 \cos(\eta_2 \xi)}{C_1 \cos(\eta_2 \xi) + C_2 \sin(\eta_2 \xi)} \right) \right]^2 \right],
\end{aligned} \tag{35}$$

where $\xi = (x^\alpha/\alpha) \mp (\sqrt{-E\lambda^2 + 4E\mu - B}t^\alpha/\alpha)$, and $\eta_2 = 1/2\sqrt{4\mu - \lambda^2}$. C_1 and C_2 are constants that can take any number.

In particular, if $C_1 \neq 0$, and $C_2 = 0$, then $U_{3,4}^1(\xi)$ become

$$\begin{aligned}
u_{3,4}^1(x, t) = & U_{3,4}^1(\xi) \\
= & -\frac{6E}{A} \left[\mu - \lambda \left[\frac{\lambda}{2} + \eta_2 \tan(\eta_2 \xi) \right] \right. \\
& \left. + \left[\frac{\lambda}{2} + \eta_2 \tan(\eta_2 \xi) \right]^2 \right].
\end{aligned} \tag{36}$$

When $\lambda^2 - 4\mu = 0$, the exact solution of equation (21) in rational form is as follows.

$$U_{5,6}^1(\xi) = -\frac{6E}{A} \left[\mu + \lambda \left[-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 \xi} \right] + \left[-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 \xi} \right]^2 \right], \tag{37}$$

where $\xi = (x^\alpha/\alpha) \mp (\sqrt{-E\lambda^2 + 4E\mu - B}t^\alpha/\alpha)$. C_1 and C_2 are constants that can take any number.

Case 2. When $\lambda^2 - 4\mu > 0$, the exact solution of equation (21) in hyperbolic form is as follows.

$$U_{1,2}^2(\xi) = -\frac{6E}{A} \left[\frac{(\lambda^2 + 2\mu)}{6} + \lambda \left[-\frac{\lambda}{2} + \eta_1 \left(\frac{C_1 \sinh(\eta_1 \xi) + C_2 \cosh(\eta_1 \xi)}{C_1 \cosh(\eta_1 \xi) + C_2 \sinh(\eta_1 \xi)} \right) \right] + \left[-\frac{\lambda}{2} + \eta_1 \left(\frac{C_1 \sinh(\eta_1 \xi) + C_2 \cosh(\eta_1 \xi)}{C_1 \cosh(\eta_1 \xi) + C_2 \sinh(\eta_1 \xi)} \right) \right]^2 \right], \quad (38)$$

where $\xi = (x^\alpha/\alpha) \mp (\sqrt{E\lambda^2 - 4E\mu - Bt^\alpha}/\alpha)$, and $\eta_1 = 1/2\sqrt{\lambda^2 - 4\mu}$. C_1 and C_2 are constants that can take any number.

If $C_1 \neq 0$, and $C_2 = 0$, then $U_{1,2}^2(\xi)$ become

$$u_{1,2}^2(x, t) = U_{1,2}^2(\xi) = -\frac{6E}{A} \left[\frac{(\lambda^2 + 2\mu)}{6} + \lambda \left[-\frac{\lambda}{2} + \eta_1 \tanh(\eta_1 \xi) \right] + \left[-\frac{\lambda}{2} + \eta_1 \tanh(\eta_1 \xi) \right]^2 \right]. \quad (39)$$

Again, using (19), the general solutions for $U_{1,2}^2(\xi)$ in simplified forms are written as

$$U_{1,2}^2(\xi) = -\frac{6E}{A} \left[\frac{(\lambda^2 + 2\mu)}{6} + \lambda \left[-\frac{\lambda}{2} + \eta_1 \tanh(\eta_1 \xi + \xi_0) \right] + \left[-\frac{\lambda}{2} + \eta_1 \tanh(\eta_1 \xi + \xi_0) \right]^2 \right], \quad (40)$$

when $|C_2/C_1| < 1$, and $\xi_0 = \tanh^{-1}(C_2/C_1)$.

$$\ddot{U}_{1,2}^2(\xi) = -\frac{6E}{A} \left[\frac{(\lambda^2 + 2\mu)}{6} + \lambda \left[-\frac{\lambda}{2} + \eta_1 \coth(\eta_1 \xi + \xi_0) \right] + \left[-\frac{\lambda}{2} + \eta_1 \coth(\eta_1 \xi + \xi_0) \right]^2 \right], \quad (41)$$

when $|C_2/C_1| > 1$, and $\xi_0 = \coth^{-1}(C_2/C_1)$.

When $\lambda^2 - 4\mu < 0$, the exact solution of equation (21) in trigonometric form is as follows.

$$U_{3,4}^2(\xi) = -\frac{6E}{A} \left[\frac{(\lambda^2 + 2\mu)}{6} + \lambda \left[-\frac{\lambda}{2} + \eta_2 \left(\frac{-C_1 \sin(\eta_2 \xi) + C_2 \cos(\eta_2 \xi)}{C_1 \cos(\eta_2 \xi) + C_2 \sin(\eta_2 \xi)} \right) \right] + \left[-\frac{\lambda}{2} + \eta_2 \left(\frac{-C_1 \sin(\eta_2 \xi) + C_2 \cos(\eta_2 \xi)}{C_1 \cos(\eta_2 \xi) + C_2 \sin(\eta_2 \xi)} \right) \right]^2 \right], \quad (42)$$

where $\xi = (x^\alpha/\alpha) \mp (\sqrt{E\lambda^2 - 4E\mu - Bt^\alpha}/\alpha)$, and $\eta_2 = 1/2\sqrt{4\mu - \lambda^2}$. C_1 and C_2 are constants that can take any number.

In particular, if $C_1 \neq 0$, and $C_2 = 0$, then $U_{3,4}^2(\xi)$ become

$$u_{3,4}^2(x, t) = U_{3,4}^2(\xi) = -\frac{6E}{A} \left[\frac{(\lambda^2 + 2\mu)}{6} - \lambda \left[\frac{\lambda}{2} + \eta_2 \tan(\eta_2 \xi) \right] + \left[\frac{\lambda}{2} + \eta_2 \tan(\eta_2 \xi) \right]^2 \right]. \quad (43)$$

When $\lambda^2 - 4\mu = 0$, the exact solution of equation (21) in rational form is as follows.

$$U_{5,6}^2(\xi) = -\frac{6E}{A} \left[\frac{(\lambda^2 + 2\mu)}{6} + \lambda \left[-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 \xi} \right] + \left[-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 \xi} \right]^2 \right], \quad (44)$$

where $\xi = (x^\alpha/\alpha) \mp (\sqrt{E\lambda^2 - 4E\mu - Bt^\alpha}/\alpha)$. C_1 and C_2 are constants that can take any number.

Similarly, if the auxiliary equation (17) and its solution (20) are used in the process of solving, we should also be able to get exact solutions, which we can prove later.

3.2. Precise Solutions of the Fractional Coupled Boussinesq-Burger Equation with Fractional (G'/G) -Expansion Method. Equation (4) is written as follows.

$$D_t^\alpha u(x, t) - \frac{1}{2} v_x(x, t) + 2u_x(x, t) = 0, \quad (45a)$$

$$D_t^\alpha v(x, t) - \frac{1}{2} u_{xxx}(x, t) + 2(uv)_x(x, t) = 0. \quad (45b)$$

Under the fractional complex transformations,

$$u(x, t) = U(\xi), \quad (46)$$

$$v(x, t) = V(\xi),$$

$$\xi = x - ct^\alpha/\alpha.$$

We get the following output.

$$\begin{aligned} D_t^\alpha(\cdot) &= -c \frac{d(\cdot)}{d\xi}, \\ \frac{\partial(\cdot)}{\partial x} &= \frac{d(\cdot)}{d\xi}, \\ \frac{\partial^2(\cdot)}{\partial x^2} &= \frac{d^2(\cdot)}{d\xi^2}, \end{aligned} \quad (47)$$

$$\partial^3(\cdot)/\partial x^3 = d^3(\cdot)/d\xi^3.$$

Substituting (46) and (47) into (45a) and (45b), we convert our problem into nonlinear ordinary differential equations:

$$-cU' - \frac{1}{2}V' + 2UU' = 0, \quad (48a)$$

$$-cV' - \frac{1}{2}U''' + 2(UV)' = 0, \quad (48b)$$

where " $U'' = dU/d\xi$ ". By integrating once with respect to travelling wave variable ξ and setting the constant from the integral to 0, you get the following equation.

$$-cU - \frac{1}{2}V + U^2 = 0, \quad (49a)$$

$$-cV - \frac{1}{2}U'' + 2UV = 0. \quad (49b)$$

From equation (49a), we get

$$V = 2(U^2 - cU). \quad (50)$$

Surrogating equation (50) in equation (49b)

$$-\frac{1}{2}U'' + 4U^3 - 6cU^2 + 2c^2U = 0. \quad (51)$$

Applying the homogeneous equilibrium principle to equation (51), we get $2 + m = 3m \implies m = 1$. By taking m to be 1 in equation (12), we get the form of the proposed solution of equation (51) as follows.

$$U(\xi) = a_0 + a_1 \frac{G'}{G}. \quad (52)$$

By using equations (15) and (52), from equation (51), we have

$$\begin{aligned} & 4a_0^3 - 6ca_0^2 - 1/2a_1\lambda\mu + 2c^2a_0 + (12a_0^2a_1 + 2c^2a_1 - 1/2a_1\lambda^2 \\ & - 12ca_0a_1 - a_1\mu) \frac{G'}{G} + (12a_0a_1^2 - 6ca_1^2 - 3/2a_1\lambda) \left(\frac{G'}{G}\right)^2 \\ & + (4a_1^3 - a_1) \left(\frac{G'}{G}\right)^3 = 0. \end{aligned} \quad (53)$$

The coefficients before all powers of (G'/G) in equation (53) are set as 0, and the resulting nonlinear algebraic system with respect to the unknowns a_0 , a_1 , and c is as follows.

$$\left(\frac{G'}{G}\right)^0 : 4a_0^3 - 6ca_0^2 - \frac{1}{2}a_1\lambda\mu + 2c^2a_0 = 0,$$

$$\frac{G'}{G} : 12a_0^2a_1 + 2c^2a_1 - \frac{1}{2}a_1\lambda^2 - 12ca_0a_1 - a_1\mu = 0,$$

$$\left(\frac{G'}{G}\right)^2 : 12a_0a_1^2 - 6ca_1^2 - \frac{3}{2}a_1\lambda = 0,$$

$$\left(\frac{G'}{G}\right)^3 : 4a_1^3 - a_1 = 0. \quad (54)$$

The symbolic computing system Maple was used to solve the nonlinear algebraic equations, and four sets of solutions were obtained.

Case 1.

$$\left\{ c = -\frac{1}{2}\sqrt{\lambda^2 - 4\mu}, a_0 = -\frac{1}{4}\lambda - \frac{1}{4}\sqrt{\lambda^2 - 4\mu}, \right. \\ \left. a_1 = -\frac{1}{2} \frac{\lambda^2(\lambda + \sqrt{\lambda^2 - 4\mu})}{(\lambda^2 - 4\mu)^{3/2} + \lambda^3 + 4\sqrt{\lambda^2 - 4\mu}\mu} \right\}. \quad (55)$$

Case 2.

$$\left\{ c = -\frac{1}{2}\sqrt{\lambda^2 - 4\mu}, a_0 = \frac{1}{4}\lambda - \frac{1}{4}\sqrt{\lambda^2 - 4\mu}, \right. \\ \left. a_1 = \frac{1}{2} \frac{\lambda^2(-\lambda + \sqrt{\lambda^2 - 4\mu})}{(\lambda^2 - 4\mu)^{3/2} - \lambda^3 + 4\sqrt{\lambda^2 - 4\mu}\mu} \right\}. \quad (56)$$

Case 3.

$$\left\{ c = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}, a_0 = -\frac{1}{4}\lambda + \frac{1}{4}\sqrt{\lambda^2 - 4\mu}, \right. \\ \left. a_1 = -\frac{1}{2} \frac{\lambda^2(-\lambda + \sqrt{\lambda^2 - 4\mu})}{(\lambda^2 - 4\mu)^{3/2} - \lambda^3 + 4\sqrt{\lambda^2 - 4\mu}\mu} \right\}. \quad (57)$$

Case 4.

$$\left\{ c = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}, a_0 = \frac{1}{4}\lambda + \frac{1}{4}\sqrt{\lambda^2 - 4\mu}, \right. \\ \left. a_1 = \frac{1}{2} \frac{\lambda^2(\lambda + \sqrt{\lambda^2 - 4\mu})}{(\lambda^2 - 4\mu)^{3/2} + \lambda^3 + 4\sqrt{\lambda^2 - 4\mu}\mu} \right\}. \quad (58)$$

Substituting the values from (55), (56), (57), or (58) and equation (18) into (52), the exact solutions of equations (45a) and (45b) in different forms can be obtained under different parameter constraints.

Case 1. When $\lambda^2 - 4\mu > 0$, the exact solution of equations (45a) and (45b) in hyperbolic form is as follows.

$$U_7^1(\xi) = -\frac{1}{4}\lambda - \frac{1}{4}\sqrt{\lambda^2 - 4\mu} - \frac{1}{2} \frac{\lambda^2 \left(\lambda + \sqrt{\lambda^2 - 4\mu} \right)}{(\lambda^2 - 4\mu)^{3/2} + \lambda^3 + 4\sqrt{\lambda^2 - 4\mu}\mu} \cdot \left[-\frac{\lambda}{2} + \eta_1 \left(\frac{C_1 \sinh(\eta_1 \xi) + C_2 \cosh(\eta_1 \xi)}{C_1 \cosh(\eta_1 \xi) + C_2 \sinh(\eta_1 \xi)} \right) \right],$$

$$V_7^1 = 2 \left((U_7^1(\xi))^2 - c U_7^1(\xi) \right), \quad (59)$$

where $\xi = x + (1/2)(\sqrt{\lambda^2 - 4\mu}t^\alpha/\alpha)$, $\eta_1 = 1/2\sqrt{\lambda^2 - 4\mu}$, and C_1 and C_2 are constants that can take any number.

If $C_1 \neq 0$, and $C_2 = 0$, then $U_7^1(\xi)$ become

$$u_7^1(x, t) = U_7^1(\xi) = -\frac{1}{4}\lambda - \frac{1}{4}\sqrt{\lambda^2 - 4\mu} - \frac{1}{2} \frac{\lambda^2 \left(\lambda + \sqrt{\lambda^2 - 4\mu} \right)}{(\lambda^2 - 4\mu)^{3/2} + \lambda^3 + 4\sqrt{\lambda^2 - 4\mu}\mu} \cdot \left[-\frac{\lambda}{2} + \eta_1 \tanh(\eta_1 \xi) \right], \quad (60a)$$

$$v_7^1(x, t) = V_7^1 = 2 \left((u_7^1(x, t))^2 - c u_7^1(x, t) \right). \quad (60b)$$

Again, using (19), the general solutions for $U_7^1(\xi)$ in simplified forms are written as

$$\dot{U}_7^1(\xi) = -\frac{1}{4}\lambda - \frac{1}{4}\sqrt{\lambda^2 - 4\mu} - \frac{1}{2} \frac{\lambda^2 \left(\lambda + \sqrt{\lambda^2 - 4\mu} \right)}{(\lambda^2 - 4\mu)^{3/2} + \lambda^3 + 4\sqrt{\lambda^2 - 4\mu}\mu} \cdot \left[-\frac{\lambda}{2} + \eta_1 \tanh(\eta_1 \xi + \xi_0) \right], \quad (61)$$

$$\dot{V}_7^1 = 2 \left((\dot{U}_7^1(\xi))^2 - c \dot{U}_7^1(\xi) \right),$$

when $|C_2/C_1| < 1$, and $\xi_0 = \tanh^{-1}(C_2/C_1)$.

$$\ddot{U}_7^1(\xi) = -\frac{1}{4}\lambda - \frac{1}{4}\sqrt{\lambda^2 - 4\mu} - \frac{1}{2} \frac{\lambda^2 \left(\lambda + \sqrt{\lambda^2 - 4\mu} \right)}{(\lambda^2 - 4\mu)^{3/2} + \lambda^3 + 4\sqrt{\lambda^2 - 4\mu}\mu} \cdot \left[-\frac{\lambda}{2} + \eta_1 \coth(\eta_1 \xi + \xi_0) \right], \quad (62)$$

$$\ddot{V}_7^1 = 2 \left((\ddot{U}_7^1(\xi))^2 - c \ddot{U}_7^1(\xi) \right),$$

when $|C_2/C_1| > 1$, and $\xi_0 = \coth^{-1}(C_2/C_1)$.

When $\lambda^2 - 4\mu < 0$, the exact solution of equation (45a) and (45b) in trigonometric form is as follows.

$$U_8^1(\xi) = -\frac{1}{4}\lambda - \frac{1}{4}\sqrt{\lambda^2 - 4\mu} - \frac{1}{2} \frac{\lambda^2 \left(\lambda + \sqrt{\lambda^2 - 4\mu} \right)}{(\lambda^2 - 4\mu)^{3/2} + \lambda^3 + 4\sqrt{\lambda^2 - 4\mu}\mu} \cdot \left[-\frac{\lambda}{2} + \eta_2 \left(\frac{-C_1 \sin(\eta_2 \xi) + C_2 \cos(\eta_2 \xi)}{C_1 \cos(\eta_2 \xi) + C_2 \sin(\eta_2 \xi)} \right) \right],$$

$$V_8^1 = 2 \left((U_8^1(\xi))^2 - c U_8^1(\xi) \right), \quad (63)$$

where $\xi = x + (1/2)(\sqrt{\lambda^2 - 4\mu}t^\alpha/\alpha)$, $\eta_2 = 1/2\sqrt{4\mu - \lambda^2}$, and C_1 and C_2 are free constants that can take any number.

If $C_1 \neq 0$, and $C_2 = 0$, then $U_{3,4}^1(\xi)$ become

$$u_8^1(x, t) = U_8^1(\xi) = -\frac{1}{4}\lambda - \frac{1}{4}\sqrt{\lambda^2 - 4\mu} - \frac{1}{2} \frac{\lambda^2 \left(\lambda + \sqrt{\lambda^2 - 4\mu} \right)}{(\lambda^2 - 4\mu)^{3/2} + \lambda^3 + 4\sqrt{\lambda^2 - 4\mu}\mu} \cdot \left[-\frac{\lambda}{2} - \eta_2 \tan(\eta_2 \xi) \right], \quad (64)$$

$$v_8^1(x, t) = V_8^1 = 2 \left((u_8^1(x, t))^2 - c u_8^1(x, t) \right).$$

When $\lambda^2 - 4\mu = 0$, the exact solution of equation (45a) and (45b) in rational form is as follows.

$$U_9^1(\xi) = -\frac{1}{4}\lambda - \frac{1}{4}\sqrt{\lambda^2 - 4\mu} - \frac{1}{2} \frac{\lambda^2 \left(\lambda + \sqrt{\lambda^2 - 4\mu} \right)}{(\lambda^2 - 4\mu)^{3/2} + \lambda^3 + 4\sqrt{\lambda^2 - 4\mu}\mu} \cdot \left[-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 \xi} \right], \quad (65)$$

$$V_9^1 = 2 \left((U_9^1(\xi))^2 - c U_9^1(\xi) \right),$$

where $\xi = x + (1/2)(\sqrt{\lambda^2 - 4\mu}t^\alpha/\alpha)$, and C_1 and C_2 are free constants.

For Case 2, Case 3, and Case 4, we can similarly obtain the exact solutions of equations (45a) and (45b). For simplicity, they are unnecessary to repeat. If the auxiliary equation (17) and its solution (20) are used in the process of solving, we should also be able to get exact solutions, which we can prove later.

3.3. *Precise Solutions of the Fractional Coupled Boussinesq Equations in Space and Time with Generalised Fractional (G'/G) -Expansion Method.* Equation (5) is written as follows.

$$D_t^\alpha u(x, t) + D_x^\beta v(x, t) = 0, \quad (66)$$

$$D_t^\alpha v(x, t) + AD_x^\beta(u^2(x, t)) - ED_{xxx}^{\beta\beta} u(x, t) = 0,$$

$$0 < \alpha, \beta \leq 1.$$

Under the fractional complex transformations:

$$\begin{aligned} u(x, t) &= U(\xi), \\ v(x, t) &= V(\xi), \\ \xi &= \frac{x^\beta}{\beta} - \frac{ct^\alpha}{\alpha}, \end{aligned} \quad (67)$$

where c is a nonzero constant. We get the following output.

$$D_t^\alpha(\cdot) = -c \frac{d(\cdot)}{d\xi}, \quad (68)$$

$$\begin{aligned} D_x^\beta(\cdot) &= d(\cdot)/d\xi, \\ D_{xxx}^{\beta\beta}(\cdot) &= d^3(\cdot)/d\xi^3. \end{aligned}$$

Substituting (67) and (68) into (66), we convert our problem into nonlinear ordinary differential equations

$$-cU' + V' = 0, \quad (69a)$$

$$-cV' + A(U^2)' - EU''' = 0, \quad (69b)$$

where " U' " = $dU/d\xi$. By integrating once with respect to travelling wave variable factor ξ and taking the integral constant to be zero, we get

$$-cU + V = 0, \quad (70a)$$

$$-cV + AU^2 - EU'' = 0. \quad (70b)$$

From equation (70a), we get

$$V = cU. \quad (71)$$

Surrogating equation (71) in equation (70b)

$$-c^2U + AU^2 - EU'' = 0. \quad (72)$$

Applying the homogeneous equilibrium principle to equation (72), we get $2 + m = 2m \implies m = 2$. By taking m to be 2 in equation (13), we get the form of the proposed solution of equation (72) as follows.

$$U(\xi) = a_0 + a_1 \frac{G'}{G} + a_2 \left(\frac{G'}{G} \right)^2 + b_1 \frac{G}{G'} + b_2 \left(\frac{G}{G'} \right)^2. \quad (73)$$

By substituting equations (73) and (15) into ordinary differential equation (72), we can rearrange and combine equation (72) with respect to (G'/G) and set the coefficients before all powers of (G'/G) to be 0. The resulting nonlinear algebraic system with respect to the unknowns a_0, a_1, a_2, b_1, b_2 , and c is as follows.

$$\begin{aligned} \left(\frac{G'}{G} \right)^{-4} &: -6E\mu^2 b_2 + Ab_2^2 = 0, \\ \left(\frac{G'}{G} \right)^{-3} &: -2E\mu^2 b_1 - 10E\mu\lambda b_2 + 2Ab_1 b_2 = 0, \\ \left(\frac{G'}{G} \right)^{-2} &: -3E\mu\lambda b_1 - 4E\lambda^2 b_2 + 2Aa_0 b_2 + Ab_1^2 \\ &\quad - 8E\mu b_2 - c^2 b_2 = 0, \\ \left(\frac{G'}{G} \right)^{-1} &: -E\lambda^2 b_1 + 2Aa_0 b_1 + 2Aa_1 b_2 - 2E\mu b_1 \\ &\quad - 6E\lambda b_2 - c^2 b_1 = 0, \\ \left(\frac{G'}{G} \right)^0 &: -2Eb_2 + Aa_0^2 - 2Ea_2\mu^2 + 2Aa_1 b_1 - Eb_1\lambda - Ea_1\lambda\mu \\ &\quad + 2Aa_2 b_2 - c^2 a_0 = 0, \\ \frac{G'}{G} &: -6E\mu\lambda a_2 - E\lambda^2 a_1 + 2Aa_0 a_1 + 2Aa_2 b_1 \\ &\quad - 2E\mu a_1 - c^2 a_1 = 0, \\ \left(\frac{G'}{G} \right)^2 &: -4E\lambda^2 a_2 + 2Aa_0 a_2 + Aa_1^2 - 8E\mu a_2 \\ &\quad - 3E\lambda a_1 - c^2 a_2 = 0, \\ \left(\frac{G'}{G} \right)^3 &: 2Aa_1 a_2 - 10E\lambda a_2 - 2Ea_1 = 0, \\ \left(\frac{G'}{G} \right)^4 &: Aa_2^2 - 6Ea_2 = 0. \end{aligned} \quad (74)$$

The nonlinear algebraic equations were solved by using Maple symbol computing system, and the following solutions were obtained.

Case 1.

$$\left\{ c = \pm \sqrt{E\lambda^2 - 4E\mu}, a_0 = \frac{E(\lambda^2 + 2\mu)}{A}, a_1 = \frac{6E\lambda}{A}, \right. \\ \left. a_2 = \frac{6E}{A}, b_1 = 0, b_2 = 0 \right\}. \quad (75)$$

Case 2.

$$\left\{ c = \pm \sqrt{-E\lambda^2 + 4E\mu}, a_0 = \frac{6E\mu}{A}, a_1 = \frac{6E\lambda}{A}, \right. \\ \left. a_2 = \frac{6E}{A}, b_1 = 0, b_2 = 0 \right\}. \quad (76)$$

Substituting the values from (75) or (76) and equation (18) into (73), the exact solutions of equation (66) in different forms can be obtained under different parameter constraints.

Case 1. When $\lambda^2 - 4\mu > 0$, the exact solution of equation (66) in hyperbolic form is as follows.

$$U_{10,11}^1(\xi) = \frac{6E}{A} \left[\frac{(\lambda^2 + 2\mu)}{6} \right. \\ \left. + \lambda \left[-\frac{\lambda}{2} + \eta_1 \left(\frac{C_1 \sinh(\eta_1 \xi) + C_2 \cosh(\eta_1 \xi)}{C_1 \cosh(\eta_1 \xi) + C_2 \sinh(\eta_1 \xi)} \right) \right] \right. \\ \left. + \left[-\frac{\lambda}{2} + \eta_1 \left(\frac{C_1 \sinh(\eta_1 \xi) + C_2 \cosh(\eta_1 \xi)}{C_1 \cosh(\eta_1 \xi) + C_2 \sinh(\eta_1 \xi)} \right) \right]^2 \right], \quad (77)$$

where $\xi = (x^\beta/\beta) \mp (\sqrt{E\lambda^2 - 4E\mu}t^\alpha/\alpha)$, $\eta_1 = 1/2\sqrt{\lambda^2 - 4\mu}$, and C_1 and C_2 are constants that can take any number.

If $C_1 \neq 0$, and $C_2 = 0$, then $U_{10,11}^1(\xi)$ become

$$u_{10,11}^1(x, t) = U_{10,11}^1(\xi) \\ = \frac{6E}{A} \left[\frac{(\lambda^2 + 2\mu)}{6} + \lambda \left[-\frac{\lambda}{2} + \eta_1 \tanh(\eta_1 \xi) \right] \right. \\ \left. + \left[-\frac{\lambda}{2} + \eta_1 \tanh(\eta_1 \xi) \right]^2 \right]. \quad (78)$$

Again, using (19), the general solutions for $U_{10,11}^1(\xi)$ in simplified forms are written as

$$\dot{U}_{10,11}^1(\xi) = \frac{6E}{A} \left[\frac{(\lambda^2 + 2\mu)}{6} + \lambda \left[-\frac{\lambda}{2} + \eta_1 \tanh(\eta_1 \xi + \xi_0) \right] \right. \\ \left. + \left[-\frac{\lambda}{2} + \eta_1 \tanh(\eta_1 \xi + \xi_0) \right]^2 \right], \quad (79)$$

when $|C_2/C_1| < 1$, and $\xi_0 = \tanh^{-1}(C_2/C_1)$.

$$\ddot{U}_{10,11}^1(\xi) = \frac{6E}{A} \left[\frac{(\lambda^2 + 2\mu)}{6} + \lambda \left[-\frac{\lambda}{2} + \eta_1 \coth(\eta_1 \xi + \xi_0) \right] \right. \\ \left. + \left[-\frac{\lambda}{2} + \eta_1 \coth(\eta_1 \xi + \xi_0) \right]^2 \right], \quad (80)$$

when $|C_2/C_1| > 1$, and $\xi_0 = \coth^{-1}(C_2/C_1)$.

When $\lambda^2 - 4\mu < 0$, the exact solution of equation (66) in trigonometric form is as follows.

$$U_{12,13}^1(\xi) = \frac{6E}{A} \left[\frac{(\lambda^2 + 2\mu)}{6} \right. \\ \left. + \lambda \left[-\frac{\lambda}{2} + \eta_2 \left(\frac{-C_1 \sin(\eta_2 \xi) + C_2 \cos(\eta_2 \xi)}{C_1 \cos(\eta_2 \xi) + C_2 \sin(\eta_2 \xi)} \right) \right] \right. \\ \left. + \left[-\frac{\lambda}{2} + \eta_2 \left(\frac{-C_1 \sin(\eta_2 \xi) + C_2 \cos(\eta_2 \xi)}{C_1 \cos(\eta_2 \xi) + C_2 \sin(\eta_2 \xi)} \right) \right]^2 \right], \quad (81)$$

where $\xi = (x^\beta/\beta) \mp (\sqrt{E\lambda^2 - 4E\mu}t^\alpha/\alpha)$, $\eta_2 = 1/2\sqrt{4\mu - \lambda^2}$, and C_1 and C_2 are constants that can take any number.

If $C_1 \neq 0$, and $C_2 = 0$, then $U_{12,13}^1(\xi)$ become

$$u_{12,13}^1(x, t) = U_{12,13}^1(\xi) \\ = \frac{6E}{A} \left[\frac{(\lambda^2 + 2\mu)}{6} - \lambda \left[\frac{\lambda}{2} + \eta_2 \tan(\eta_2 \xi) \right] \right. \\ \left. + \left[\frac{\lambda}{2} + \eta_2 \tan(\eta_2 \xi) \right]^2 \right]. \quad (82)$$

When $\lambda^2 - 4\mu = 0$, the exact solution of equation (66) in rational form is as follows.

$$U_{14}^1(\xi) = -\frac{6E}{A} \left[\frac{(\lambda^2 + 2\mu)}{6} + \lambda \left[-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 \xi} \right] \right. \\ \left. + \left[-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 \xi} \right]^2 \right], \quad (83)$$

where $\xi = (x^\beta/\beta) \mp (\sqrt{E\lambda^2 - 4E\mu}t^\alpha/\alpha)$, and C_1 and C_2 are constants that can take any number.

Using the obtained result formula (76), we can similarly obtain the exact solution of equation (66). For simplicity, they are unnecessary to repeat. If the auxiliary equation (17) and its solution (20) are used in the process of solving, we should also be able to get exact solutions, which we can prove later.

4. The Explanation and Discussion

By calculating the operation of Maple software, we obtained the exact travelling wave solutions of three fractional-order equations. Literature [34] uses the simplest Riccati equation of a fractional order as an auxiliary function, directly solving space-time fractional Boussinesq equation, and the coefficients and functions in the obtained solution contained fractional order, which was quite different from the solution obtained by using complex transformation. In addition, literature [34] only obtained a set of solutions of algebraic equations composed of quasisolution coefficients, and we obtained four sets of solutions. For equation (4), literature

Solution (60a) red: $\alpha = 0.9$, blue: $\alpha = 0.7$, yellow: $\alpha = 0.3$ Solution (60b) red: $\alpha = 0.9$, blue: $\alpha = 0.7$, yellow: $\alpha = 0.3$ FIGURE 1: 3D plot of solutions (60a) and (60b) for various values of α , and $c = 1/2\sqrt{\lambda^2 - 4\mu}$, $\lambda = \sqrt{2}$, $\mu = 0.4$.

[40] uses the first integral method to obtain its two sets of solutions, both of which are kinked in the image, which is similar to the solution shown in Figure 1. We obtain four sets of solutions of algebraic equations, each of which contains three types of solutions, one kink, one period, and one rational function. For equation (5), reference [42] obtained a set of solutions of algebraic equations by using the exp-function method, which was illustrated as bell-shaped isolated waves. We obtain four sets of solutions of algebraic equations, each containing three types of solutions, one of which is the kink solution shown in Figure 2 or 3.

In general, there are many solutions to nonlinear partial differential equations, only some of which can be obtained in different ways, and our work enriches the solutions to these equations. The three model equations we studied are all water wave-related models, which are nonlinear dispersive wave equations. We are more concerned with the isolated wave solutions resulting from the equilibrium of the nonlinear and dispersion effects, so we select an isolated wave solution from each equation we study to discuss the effect of fractional order on its waveform. The results are shown in Figures 1, 2, 3, 4, and 5. Figure 4 shows the isolated wave solution of the first model equation. Figures 1 and 5 are the isolated wave solutions of the second model equation. Figures 2 and 3 are the isolated wave solutions of the third model equation. Let us look at each of these results in more details.

Solution (32) of equation (2) represents bell-type soliton solutions, which is the result of the equilibrium between the nonlinear term and the dispersion term in equation (2). When $c = +\sqrt{-E\lambda^2 + 4E\mu - B}$, $A = 3$, $B = -1$, $E = 0.5$, $\lambda = \sqrt{2}$, and $\mu = 0.4$, the graphical form of solution (32) changing with α is shown in Figure 4. After setting values for other parameters, Figure 4 explains the perspective view of solution (32), when the values of α are 0.9, 0.8, 0.7, 0.6, 0.5, and 0.4 in turn. You can see in Figure 4 that with the decrease of α , the width of the waveform is increasing, and the waveform surface is gradually transitioning from concave to convex. We might conclude that the fractional order modulates the waveforms of the isolated waves of this equation.

$\beta = 1$; solution (78) red: $\alpha = 0.9$, blue: $\alpha = 0.6$, yellow: $\alpha = 0.3$ FIGURE 2: 3D plot of solution (78) for various values of α , and $\beta = 1$, $c = \sqrt{E\lambda^2 - 4E\mu}$, $\lambda\sqrt{2}$, $\mu = 0.4$, $A = 3$, $E = 0.5$. $\alpha = 1$; solution (78) red: $\beta = 0.9$, blue: $\beta = 0.6$, yellow: $\beta = 0.3$ FIGURE 3: 3D plot of solution (78) for various values of β , and $\alpha = 1$, $c = \sqrt{E\lambda^2 - 4E\mu}$, $\lambda\sqrt{2}$, $\mu = 0.4$, $A = 3$, $E = 0.5$.

FIGURE 4: Snapshots of solution (32) for various values of α , and $c = +\sqrt{-E\lambda^2 + 4E\mu - B}$, $A = 3$, $B = -1$, $E = 0.5$, $\lambda = \sqrt{2}$, $\mu = 0.4$.

When $c = -1/2\sqrt{\lambda^2 - 4\mu}$, $\lambda = \sqrt{2}$, and $\mu = 0.4$, the graphical form of solutions (60a) and (60b) of equation (4) changing with α is shown in Figure 1. Solution (60a) in Figure 1 represents kink soliton solutions. Solution (60b) in Figure 1 represents bell-type soliton solutions. They are the result of the balance between the nonlinear term and the dispersion term in equation (4). The detailed expansion of the graphical form of solution (60b) is shown in Figure 5. After setting

values for other parameters, Figure 5 explains the perspective view of the Solution (60b), when the values of α are 0.1, 0.5, 0.7, and 0.9 in turn. As you can see from Figure 5, the waveform of the solution changes from the form of an isolated wave to the form of a local period, which shows that for some solutions, fractional-order changes can change the structure of the waveform. In other words, the fractional order may modulate the local periodicity of some solutions.

FIGURE 5: Snapshots of soliton solution to periodic solution of solution (60b) for various values of α , and $c = 1/2\sqrt{\lambda^2 - 4\mu}$, $\lambda = \sqrt{2}$, $\mu = 0.4$.

When $\beta = 1$, $c = \sqrt{E\lambda^2 - 4E\mu}$, $\lambda = \sqrt{2}$, $\mu = 0.4$, $A = 3$, and $E = 0.5$, the graphical form of solution (78) of equation (5) changing with α is shown in Figure 2. When $\alpha = 1$, $c = \sqrt{E\lambda^2 - 4E\mu}$, $\lambda = \sqrt{2}$, $\mu = 0.4$, $A = 3$, and $E = 0.5$, the graphical form of solution (78) of equation (5) changing with β is shown in Figure 3. Solution (78) in Figures 2 and 3 represent kink soliton solutions. For an equation with two fractional-order parameters, we fix one fractional-order parameter and then look at the effect of the other fractional-order parameter on the isolated wave solution waveform. For the case that the fractional-order parameter satisfies a certain relation, it needs further study in the future. The effect of fractional order on other solutions can be similarly graphically analysed.

Remark 1. When $G = G(\xi)$ satisfies equation (17), we have

$$\begin{aligned} \left(\frac{G'(\xi)}{G(\xi)} \right)' &= \frac{G''(\xi)G(\xi) - (G'(\xi))^2}{G^2(\xi)} \\ &= (\lambda - 1) \left(\frac{G'}{G} \right)^2 + \mu \left(\frac{G'}{G} \right) + \omega. \end{aligned} \quad (84)$$

In this way, the Riccati equation satisfied by the extended (G'/G) -expansion method can be regarded as more generalised.

5. Conclusion

Combined with fractional complex transformation, the (G'/G) -expansion method and its extended generalised form are used to obtain abundant travelling wave solutions for three fractal-order model equations related to water waves. For the nonlinear dispersive wave model equations, we are more concerned about their soliton solutions, so we choose a soliton solution from the travelling wave solution of each model equation to illustrate and discuss the effect of fractional order parameters on it. The results show that the fractional derivatives can modulate the waveform, local periodicity, and structure of the isolated solutions of the three model equations. Of course, our discussion of fractional derivatives is not enough. For example, in the future, we will further discuss how to modulate the waveform of a soliton solution when multiple fractional parameters are coupled. With the further discussion of the influence of fractional

derivatives on the waveform of the solution of the equation, we may have a better understanding of the formation and properties of the waveform of the solution of the fractional equation. In addition, in this paper, we point out the rule that the auxiliary equation of the extended (G'/G) -expansion method should satisfy, that is, the result of the differential operation of (G'/G) should be in the polynomial form of (G'/G) , which is the basis for the formation of algebraic equations by collecting (G'/G) power term coefficients later. According to the rules satisfied by the auxiliary equations, we can choose other solvable equations as auxiliary equations, which is also helpful to understand the selection of auxiliary equations in other methods.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors conceived the study, participated in the sequence alignment, and read and approved the final manuscript.

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