Research Article

An Extension of the Mittag-Leffler Function and Its Associated Properties

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Received 27 June 2020; Revised 17 August 2020; Accepted 29 August 2020; Published 22 September 2020

Academic Editor: Soheil Salahshour

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Inspired by certain fascinating ongoing extensions of the special functions such as an extension of the Pochhammer symbol and generalized hypergeometric function, we present a new extension of the generalized Mittag-Leffler (ML) function $\epsilon_{a,b,p,v}^{\kappa}(z_1)$ in terms of the generalized Pochhammer symbol. We then deliberately find certain various properties and integral transformations of the said function $\epsilon_{a,b,p,v}^{\kappa}(z_1)$. Some particular cases and outcomes of the main results are also established.

1. Introduction

The well-known ML (Mittag-Leffler) function with one parameter is defined by

$$\epsilon_a(z_1) = \sum_{l=0}^{\infty} \frac{z_1^l}{\Gamma(al+1)} \quad (a \in \mathbb{C}; \Re(a) > 0, z_1 \in \mathbb{C}). \quad (1)$$

The generalization of (1) with two parameters defined by

$$\epsilon_{a,b}(z_1) = \sum_{l=0}^{\infty} \frac{z_1^l}{\Gamma(al+b)} \quad (a, b \in \mathbb{C}; \Re(a) > 0, \Re(b) > 0) \quad (2)$$

was presented and contemplated by Mittag-Leffler [1–7] and other researchers. In [8], the generalization of (1) was given by

$$\epsilon_{a,b}^{\kappa}(z_1) = \sum_{l=0}^{\infty} \frac{(\kappa)_l}{\Gamma(al+b)} \frac{z_1^l}{l!} \quad (a, b, \kappa \in \mathbb{C}; \Re(a) > 0, \Re(b) > 0). \quad (3)$$

Shukla and Prajapati [9] defined the following generalization of the ML function by

$$\epsilon_{a,b}^{\kappa,q}(z_1) = \sum_{l=0}^{\infty} \frac{(\kappa)_{lq}}{\Gamma(al+b)} \frac{z_1^l}{l!} \quad (a, b, \kappa \in \mathbb{C}; \Re(a) > 0, \Re(b) > 0, q > 0). \quad (4)$$

Rahman et al. [10] defined the following extension of generalized ML function by

$$\epsilon_{a,b,c}^{\kappa,q,c}(z_1) = \sum_{l=0}^{\infty} \frac{B(\kappa+lq; c-\kappa)(c)_{lq}}{B(\kappa, c-\kappa)\Gamma(al+b)} \frac{z_1^l}{l!}, \quad (5)$$

where $a, b, \kappa, c \in \mathbb{C}; \Re(c) > 0, \Re(a) > 0, \Re(b) > 0, \Re(p) > 0$,

$q > 0$, and $B_p(x, y)$ is the extension of beta function (see [11]).

Moreover, the generalization of ML function (3) was presented by [12] as follows:

$$\varepsilon_{a,b,p}^{\kappa}(z_1) = \sum_{l=0}^{\infty} \frac{(\kappa; p)_l}{\Gamma(al+b)} \frac{z_1^l}{l!} \quad (p \geq 0, a, b, \kappa \in \mathbb{C}; \Re(a) > 0, \Re(b) > 0, \Re(p) > 0), \quad (6)$$

where $(\kappa; p)_l$ is the Pochhammer symbol which is defined as

$$(\lambda; \sigma)_{\mu} = \begin{cases} \frac{\Gamma_{\sigma}(\lambda + \mu)}{\Gamma(\lambda)}, & \Re(\sigma), \lambda, \mu \in \mathbb{C}, \\ (\lambda)_{\mu}, & \sigma = 0, \lambda, \mu \in \mathbb{C} \setminus \{0\}. \end{cases} \quad (7)$$

The researchers studied these extensions (6) and (7) and investigated their further extensions and associated properties and applications. (The readers may consult [13–16].) Recently, Srivastava et al. [17] have presented and concentrated in a fairly productive way the following extension of the generalized hypergeometric function:

$${}_sF_t[(\delta_1; \rho, \nu), \dots, (\delta_s); (\zeta_1), \dots, (\zeta_t); z_1] = \sum_{l=0}^{\infty} \frac{(\delta_1; \rho, \nu)_l \cdots (\delta_s)_l}{(\zeta_1)_l \cdots (\zeta_t)_l} \frac{z_1^l}{l!}, \quad (8)$$

where $\delta_j \in \mathbb{C}$ for $j = 1, 2, \dots, s$, $\zeta_k \in \mathbb{C}$ for $k = 1, 2, \dots, t$, and $\zeta_k \neq 0, -1, -2, \dots$, and where $(\mu; \omega, a)_{\eta}$ is the extension of the generalized Pochhammer symbol defined by [23]:

$$(\mu; \omega, a)_{\eta} = \begin{cases} \frac{\Gamma_a(\mu + \eta; \omega)}{\Gamma(\mu)}, & \Re(\omega), \Re(a) > 0, \mu, \eta \in \mathbb{C}, \\ (\mu; \omega)_{\eta}, & a = 0, \mu, \eta \in \mathbb{C} \setminus \{0\}. \end{cases} \quad (9)$$

The integral representation of $(\mu; \omega, a)_{\eta}$ is explained by

$$(\mu; \omega, a)_{\eta} = \sqrt{\frac{2\omega}{\pi}} \frac{1}{\Gamma(\mu)} \int_0^{\infty} s^{\mu+\eta-3/2} e^{-s} K_{a+1/2}\left(\frac{\omega}{s}\right) ds, \quad (10)$$

where $K_a(\cdot)$ is the modified Bessel function of order a . Clearly, when $a = 0$ in (10), at that point, by utilizing the way that $K_{1/2}(t) = \sqrt{\pi/2} t e^{-t}$, it will lead to formula [(31)]:

$$(\mu; \omega, 0)_{\eta} = (\mu; \omega)_{\eta} = \frac{1}{\Gamma(\mu)} \int_0^{\infty} s^{\mu+\eta-1} e^{-s} ds. \quad (11)$$

Specifically, the relating extensions of the confluent hypergeometric function ${}_1F_1$ and the Gauss hypergeometric function ${}_2F_1$ are given by

$${}_2F_1[(\delta_1; \rho, \nu), b; \kappa; z_1] = \sum_{l=0}^{\infty} \frac{(\delta_1; \rho, \nu)_l (b)_l}{(\kappa)_l} \frac{z_1^l}{l!}, \quad (12)$$

$${}_1F_1[(\delta_1; \rho, \nu); \kappa; z_1] = \Phi[(\delta_1; \rho, \nu); \kappa; z_1] = \sum_{l=0}^{\infty} \frac{(\delta_1; \rho, \nu)_l}{(\kappa)_l} \frac{z_1^l}{l!}. \quad (13)$$

The extension of generalized hypergeometric function ${}_rF_s$ of r numerator and s denominator parameters was investigated by [18]. Recently, the researchers defined various extensions of special functions and their associated properties and applications in the diverse field. (The interested readers may consult [19–22].) In [23–25], the authors introduced an extension of fractional derivative operators based on the extended beta functions.

Next, motivated by the above such extensions of special functions, we define an extension of ML function (6) in terms of the generalized Pochhammer symbol (9) and investigate its certain variations.

2. Extension of ML Function

We present an extension of the generalized ML function in (6) regarding the extended Pochhammer symbol in (9) as follows:

$$\varepsilon_{a,b,p,\nu}^{\kappa}(z_1) = \sum_{l=0}^{\infty} \frac{(\kappa; p, \nu)_l}{\Gamma(al+b)} \frac{z_1^l}{l!} \quad (a, b, \kappa \in \mathbb{C}; \Re(a) > 0, \Re(b) > 0, \Re(p) > 0, p \geq 0, \nu \geq 0), \quad (14)$$

given that the series on the right hand side converges.

Clearly, it diminishes to the extended generalized ML function (6) for $\nu = 0$. The special case for $a = 1$ in (14) can be communicated regarding extended confluent hypergeometric function (13) as follows:

$$\begin{aligned} \varepsilon_{1,b,p,\nu}^{\kappa}(z_1) &= \frac{1}{\Gamma(b)_1} {}_1F_1[(\kappa_1; p, \nu); b; z_1] \\ &= \frac{1}{\Gamma(b)} \Phi[(\kappa_1; p, \nu); b; z_1]. \end{aligned} \quad (15)$$

3. Basic Properties of $\varepsilon_{a,b,p,\nu}^{\kappa}(z_1)$

In this section, we present certain basic properties and integral representations of the extended generalized ML function $\varepsilon_{a,b,p,\nu}^{\kappa}(z_1)$ in (14).

Theorem 1. For the function $\varepsilon_{a,b,p,\nu}^{\kappa}(z_1)$ in (14), the following relation holds true:

$$\begin{aligned} \varepsilon_{a,b,p,\nu}^{\kappa}(z_1) &= b \varepsilon_{a,b+1;p,\nu}^{\kappa}(z_1) + a z_1 \frac{d}{dz} \varepsilon_{a,b+1;p,\nu}^{\kappa}(z_1) \\ &\cdot (a, b, \kappa \in \mathbb{C}; \Re(a) > 0, \Re(b) > 0, \Re(p) > 0, p \geq 0, \nu \geq 0). \end{aligned} \quad (16)$$

Specifically, we have

$$E_{a,b;p}^\kappa(z_1) = bE_{a,b+1;p}^\kappa(z_1) + az_1 \frac{d}{dz_1} E_{a,b+1;p}^\kappa(z_1). \quad (17)$$

Proof. From (14), we have

$$\begin{aligned} bE_{a,b+1;p,v}^\kappa(z_1) + az_1 \frac{d}{dz_1} E_{a,b+1;p,v}^\kappa(z_1) \\ &= b \sum_{l=0}^{\infty} \frac{(\kappa, p, v)_l}{\Gamma(al+b+1)} \frac{z_1^l}{l!} + az_1 \frac{d}{dz_1} \sum_{l=0}^{\infty} \frac{(\kappa, p, v)_l}{\Gamma(al+b+1)} \frac{z_1^l}{l!} \\ &= b \sum_{l=0}^{\infty} \frac{(\kappa, p, v)_l}{\Gamma(al+b+1)} \frac{z_1^l}{l!} + az_1 \frac{d}{dz_1} \sum_{l=0}^{\infty} \frac{(\kappa, p, v)_l}{\Gamma(al+b+1)} \frac{z_1^l}{l!} \\ &= b \sum_{l=0}^{\infty} \frac{(\kappa, p, v)_l}{\Gamma(al+b+1)} \frac{z_1^l}{l!} + \sum_{l=0}^{\infty} \frac{az_1(\kappa, p, v)_l}{\Gamma(al+b+1)} \frac{lz_1^{l-1}}{l!} \\ &= b \sum_{l=0}^{\infty} \frac{(\kappa, p, v)_l}{\Gamma(al+b+1)} \frac{z_1^l}{l!} + \sum_{l=0}^{\infty} \frac{(\kappa, p, v)_l(al)}{\Gamma(al+b+1)} \frac{z_1^l}{l!} \\ &= \sum_{l=0}^{\infty} \frac{(\kappa, p, v)_l}{\Gamma(al+b+1)} \frac{z_1^l}{l!} (al+b) \\ &= \sum_{l=0}^{\infty} \frac{(\kappa, p, v)_l (al+b)}{(al+b) \Gamma(al+b)} \frac{z_1^l}{l!}, \text{ using } \Gamma(z_1+1) \\ &= z_1 \Gamma(z_1) = \sum_{l=0}^{\infty} \frac{(\kappa, p, v)_l}{\Gamma(al+b)} \frac{z_1^l}{l!} = E_{a,b;p,v}^\kappa(z_1). \end{aligned} \quad (18)$$

Equation (17) can be obtained from (16) when we put $v = 0$.

Theorem 2. For the function $E_{a,b;p,v}^\kappa(z_1)$ in (14), the following higher order differentiation formulas hold true:

$$\left(\frac{d}{dz_1} \right)^m E_{a,b;p,v}^\kappa(z_1) = (\kappa)_m E_{a,b+m;p,v}^{\kappa+m}(z_1), \quad (19)$$

$$\begin{aligned} \left(\frac{d}{dz_1} \right)^m \left[z_1^{b-1} E_{a,b;p,v}^\kappa(\omega z_1^a) \right] \\ &= z_1^{b-m-1} E_{a,b-m;p,v}^\kappa(\omega z_1^a) \quad (\Re(b-m) > 0, m \in \mathbb{N}), \end{aligned} \quad (20)$$

$$\begin{aligned} \left(\frac{d}{dz_1} \right)^m \left[z_1^{b-1} \Phi((\kappa; p, v); b; \omega z_1) \right] \\ &= \frac{\Gamma(b)}{\Gamma(b-m)} z_1^{b-m-1} \Phi((\kappa; p, v); b-m; \omega z_1), \end{aligned} \quad (21)$$

where $a, b, \kappa \in \mathbb{C}; \Re(a) > 0, \Re(b) > 0, \Re(p) > 0, p \geq 0, v \geq 0$. Specifically, we have

$$\left(\frac{d}{dz_1} \right)^m E_{a,b;p}^\kappa(z_1) = (\kappa)_m E_{a,b+m;p}^{\kappa+m}(z_1), \quad (22)$$

$$\left(\frac{d}{dz_1} \right)^m \left[z_1^{b-1} E_{a,b;p}^\kappa(\omega z_1^a) \right] = z_1^{b-m-1} E_{a,b-m;p}^\kappa(\omega z_1^a), \quad (23)$$

$$\begin{aligned} \left(\frac{d}{dz_1} \right)^m \left[z_1^{b-1} \Phi((\kappa, p); b; \omega z_1) \right] \\ &= \frac{\Gamma(b)}{\Gamma(b-m)} z_1^{b-m-1} \Phi((\kappa, p); b-m; \omega z_1). \end{aligned} \quad (24)$$

Proof. Operating term wise differentiation m times on (14), we get

$$\begin{aligned} \left(\frac{d}{dz_1} \right)^m E_{a,b;p,v}^\kappa(z_1) &= \left(\frac{d}{dz_1} \right)^m \sum_{l=0}^{\infty} \frac{(\kappa; p, v)_l}{\Gamma(al+b)} \frac{z_1^l}{l!} \\ &= \sum_{l=m}^{\infty} \frac{(\kappa; p, v)_l}{\Gamma(al+b)} \frac{l! z_1^{l-m}}{(l-m)!} \\ &= \sum_{l=0}^{\infty} \frac{(\kappa; p, v)_{l+m}}{\Gamma(al+b+am)} \frac{z_1^l}{(l)!}, \text{ replacing } l \text{ by } l+m, \\ &= \sum_{l=0}^{\infty} \frac{(\kappa)_m (\kappa+m; p, v)_l}{\Gamma(al+b+am)} \frac{z_1^l}{l!}, \text{ using } (\lambda; \sigma, v)_{\mu+p} \\ &= (\lambda)_\mu (\lambda+\mu; \sigma, v)_p, \\ &= (\kappa)_m \sum_{l=0}^{\infty} \frac{(\kappa+m; p, v)_l}{\Gamma(al+b+am)} \frac{z_1^l}{l!} \\ &= (\kappa)_m E_{a,b+m;p,v}^{\kappa+m}(z_1). \end{aligned} \quad (25)$$

In a similar manner from (20), we get

$$\begin{aligned} \left(\frac{d}{dz_1} \right)^m \left[z_1^{b-1} E_{a,b;p,v}^\kappa(\omega z_1^a) \right] \\ &= \left(\frac{d}{dz_1} \right)^m z_1^{b-1} \sum_{l=0}^{\infty} \frac{(\kappa; p, v)_l}{\Gamma(al+b)} \frac{(\omega z_1^a)^l}{l!} \\ &= \left(\frac{d}{dz_1} \right)^m \sum_{l=0}^{\infty} \frac{(\kappa; p, v)_l}{\Gamma(al+b)} \frac{z_1^{b-1+al}}{l!} \omega^l \\ &= \sum_{l=0}^{\infty} \frac{(\kappa; p, v)_l}{\Gamma(al+b)} \frac{\omega^l}{l!} \frac{(al+b-1)!}{(al+b-m-1)!} z_1^{al+b-m-1}, \\ &\quad \text{differentiating } m \text{ times,} \\ &= \sum_{l=0}^{\infty} \frac{(\kappa; p, v)_l}{\Gamma(al+b)} \frac{\Gamma(al+b)}{\Gamma(al+b-m)} \omega^l \frac{z_1^{al+b-1-m}}{l!}, \\ &\quad \text{using } (l-1)! = \Gamma(l), \\ &= z_1^{b-m-1} \sum_{l=0}^{\infty} \frac{(\kappa; p, v)_l}{\Gamma(al+b-m)} \frac{(\omega z_1^a)^l}{l!} \\ &= z_1^{b-m-1} E_{a,b-m;p,v}^\kappa(\omega z_1^a). \end{aligned} \quad (26)$$

Moreover, putting $a = 1$ in (20) gives (21). For the special case of (19), (20), and (21), when we put $v = 0$, we get (22), (23), and (24), respectively.

Corollary 3. The following integral representations for ML function $E_{a,b;p,v}^\kappa(z_1)$ (14) hold true:

$$\int_0^{z_1} t^{b-1} \varepsilon_{a,b;p,v}^\kappa(\omega t^a) dt = z_1^b \varepsilon_{a,b+1;p,v}^\kappa(\omega z_1^a),$$

$$\int_0^{z_1} t^{b-1} \Phi((\kappa, p, v); b; \omega t) dt = \frac{1}{\Gamma(b)} z^b \Phi((\kappa; p, v); b+1; \omega z_1), \quad (27)$$

where $a, b, \kappa, \omega \in \mathbb{C}$; $\Re(a) > 0$, $\Re(b) > 0$, $\Re(p) > 0$, $p \geq 0$, $v \geq 0$.

Specifically, we have

$$\int_0^{z_1} t^{b-1} E_{a,b;p}^\kappa(\omega t^a) dt = z_1^b E_{a,b+1;p}^\kappa(\omega z_1^a),$$

$$\int_0^{z_1} t^{b-1} \Phi((\kappa, p); b; \omega t) dt = \frac{1}{\Gamma(b)} z^b \Phi((\kappa; p); b+1; \omega z_1). \quad (28)$$

4. Representation of $\varepsilon_{a,b;p,v}^\kappa(z_1)$ in terms of Generalized Hypergeometric Function

Here, we establish the representation of $\varepsilon_{a,b;p,v}^\kappa(z_1)$ (14) in terms of generalized hypergeometric function as follows.

Theorem 4. The function $\varepsilon_{a,b;p,v}^\kappa(z_1)$ defined in (14) for $a \in \mathbb{N}$ can be represented in the form of generalized hypergeometric function as given by

$$\begin{aligned} \varepsilon_{a,b;p,v}^\kappa(z_1) &= \frac{1}{\Gamma(b)_1} F_q \left[(\kappa, p, v); \frac{b}{q}, \frac{b+1}{q}, \dots, \frac{b+k-1}{q}; \frac{z_1}{q^q} \right] \\ &= \frac{1}{\Gamma(b)_1} F_q \left[(\kappa, p, v); \Delta(q; b); \frac{z_1}{q^q} \right], \end{aligned} \quad (29)$$

where $q \in \mathbb{N}$ and $\Delta(q; b)$ is an array of q parameters $b/q, (b+1)/q, \dots, (b+q-1)/q$.

Proof. Taking $a = q \in \mathbb{N}$ in (14) and utilizing the well-known multiplication formula for the gamma function, we have

$$\begin{aligned} \varepsilon_{q,b;p,v}^\kappa(z_1) &= \sum_{l=0}^{\infty} \frac{(\kappa, p, v)_l}{\Gamma(qn+b)} \frac{z_1^l}{l!} = \sum_{l=0}^{\infty} \frac{(\kappa, p, v)_l}{(\Gamma(b)/\Gamma(b)) \times \Gamma(qn+b)} \frac{z_1^l}{l!} \\ &= \frac{1}{\Gamma(b)} \sum_{l=0}^{\infty} \frac{(\kappa, p, v)_l}{\Gamma(qn+b)\Gamma(b)} \frac{z_1^l}{l!} = \frac{1}{\Gamma(b)} \sum_{l=0}^{\infty} \frac{(\kappa, p, v)_l}{(b)_{nq}} \frac{z_1^l}{l!} \\ &= \frac{1}{\Gamma(b)} \sum_{l=0}^{\infty} \frac{(\kappa, p, v)_l}{\prod_{j=1}^q ((b+j-1)/q)_n} \left(\frac{z_1}{q^q} \right)^l \frac{1}{l!}, \text{ since } (b)_{qn} \\ &= \prod_{j=1}^q \left(\frac{b+j-1}{q} \right)_n, \\ &= \frac{1}{\Gamma(b)} \sum_{l=0}^{\infty} \frac{(\kappa, p, v)_l}{(b/q)_n, ((b+1)/q)_n, \dots, ((b+q-1)/q)_n} \left(\frac{z_1}{q^q} \right)^l \frac{1}{l!} \\ &= \frac{1}{\Gamma(b)_1} F_q \left[(\kappa, p, v); \frac{b}{q}, \frac{b+1}{q}, \dots, \frac{b+q-1}{q}; \frac{z_1}{q^q} \right] \\ &= \frac{1}{\Gamma(b)_1} F_q \left[(\kappa, p, v); \Delta(q; b); \frac{z_1}{q^q} \right]. \end{aligned} \quad (30)$$

5. Integral Transformation of $\varepsilon_{a,b;p,v}^\kappa(z_1)$

Here, we present various integral representations of the function $\varepsilon_{a,b;p,v}^\kappa(z_1)$ in (14) such as the Mellin, the Euler-beta, and the Laplace transformations.

5.1. Mellin Transform. The well-known Millen transform [26] of integrable function $f(z_1)$ with index s is defined by

$$\mathcal{M}\{f(z_1): z_1 \rightarrow s\} = \int_0^\infty z_1^{s-1} f(z_1) dz_1, \quad (31)$$

if the improper integral in (31) exists.

Theorem 5. For the function $\varepsilon_{a,b;p,v}^\kappa(z_1)$ in (14), the following Mellin transform exists:

$$\mathcal{M}\left\{\varepsilon_{a,b;p,v}^\kappa(z_1): v \rightarrow s\right\}, \quad (32)$$

$$\frac{2^{s-1}}{\sqrt{\pi}} \Gamma\left(\frac{s-v}{2}\right) \Gamma\left(\frac{s+v+1}{2}\right) (\kappa)_s E_{a,b}^{\kappa+s}(z_1) (\Re(s-v) > 0, \Re(\kappa+s) > -1). \quad (33)$$

Proof. By (32), we get from (14)

$$\begin{aligned} \mathcal{M}\left\{\varepsilon_{a,b;p,v}^\kappa(z_1): v \rightarrow s\right\} &= \int_0^\infty v^{s-1} \varepsilon_{a,b;p,v}^\kappa(z_1) dv \\ &= \int_0^\infty v^{s-1} \left(\sum_{l=0}^{\infty} \frac{(\kappa; p, v)_l}{\Gamma(al+b)} \frac{z_1^l}{l!} \right) dv \\ &= \int_0^\infty v^{s-1} \left(\sum_{l=0}^{\infty} \frac{\Gamma_v(\kappa+l; p)}{\Gamma(\kappa)\Gamma(al+b)} \frac{z_1^l}{l!} \right) dv. \end{aligned} \quad (34)$$

Interchanging the order of summation and integration, we get

$$\mathcal{M}\left\{\varepsilon_{a,b;p,v}^\kappa(z_1): v \rightarrow s\right\} = \sum_{l=0}^{\infty} \frac{1}{\Gamma(al+b)} \frac{z_1^l}{l!} \frac{1}{\Gamma(\kappa)} \int_0^\infty v^{s-1} \Gamma_v(\kappa+l; p) dv. \quad (35)$$

Now, by utilizing the well-known result ([11], Equation 4.105),

$$\begin{aligned} \int_0^\infty v^{s-1} \Gamma_v(\kappa+l; p) dv &= \frac{2^{s-1}}{\sqrt{\pi}} \Gamma\left(\frac{s-v}{2}\right) \Gamma\left(\frac{s+v+1}{2}\right) \\ &\cdot \Gamma(\kappa+s+l) (\Re(\kappa+s) > -1, \Re(s-v) > 0). \end{aligned} \quad (36)$$

Therefore, we have

$$\begin{aligned}
 \mathcal{M}\left\{\varepsilon_{a,b;p,v}^{\kappa}(z_1): \nu \rightarrow s\right\} &= \frac{(2^{s-1}/\sqrt{\pi})\Gamma((s-\nu)/2)\Gamma((s+\nu+1)/2)}{\Gamma(\kappa)} \sum_{l=0}^{\infty} \frac{\Gamma(\kappa+s+l)}{\Gamma(al+b)} \frac{z_1^l}{l!} \\
 &= \frac{2^{s-1}}{\sqrt{\pi}} \Gamma\left(\frac{s-\nu}{2}\right) \Gamma\left(\frac{s+\nu+1}{2}\right) (\kappa)_s \sum_{l=0}^{\infty} \frac{(\kappa+s)_l}{\Gamma(al+b)} \frac{z_1^l}{l!} \\
 &= \frac{2^{s-1}}{\sqrt{\pi}} \Gamma\left(\frac{s-\nu}{2}\right) \Gamma\left(\frac{s+\nu+1}{2}\right) (\kappa)_s E_{a,b}^{\kappa+s}(z_1). \tag{37}
 \end{aligned}$$

Corollary 6. The below integral transform exists:

$$\begin{aligned}
 \mathcal{M}\left\{\varepsilon_{a,b;p,v}^{\kappa}(z_1): \nu \rightarrow s\right\} &= \frac{(2^{s-1}/\sqrt{\pi})\Gamma((s-\nu)/2)\Gamma((s+\nu+1)/2)}{\Gamma(\kappa)} \\
 &\cdot {}_1\Psi_1[(\kappa+s, 1); (b, a); z], \tag{38}
 \end{aligned}$$

where ${}_p\Psi_q$ is the Wright hypergeometric function (see [27, 28]).

5.2. Euler-Beta Transform. In [26], the well-known Euler-beta transform of the function $f(z_1)$ is defined by

$$\mathcal{B}\{f(z_1); a_1, b_1\} = \int_0^1 z_1^{a_1-1} (1-z_1)^{b_1-1} f(z_1) dz_1. \tag{39}$$

Theorem 7. For the function $\varepsilon_{a,b;p,v}^{\kappa}(z_1)$ in (14), the following Euler-beta transform holds:

$$\begin{aligned}
 \mathcal{B}\left\{\varepsilon_{a,b;p,v}^{\kappa}(xz_1^{\sigma}): a_1, b_1\right\} &= \frac{\Gamma(b_1)\Gamma(a_1)}{\Gamma(b)\Gamma(a_1+b_1)_{\sigma+1}} F_{a+\sigma}\left[(\kappa, p, \nu), \Delta(\sigma; a_1); \Delta(a; b), \Delta(\sigma; a_1+b_1); \frac{x}{a^a}\right] \\
 &\cdot (a, \sigma \in \mathbb{N}; \Re(p) > 0, \Re(a_1) > 0, \Re(b_1) > 0, \Re(b) > 0, \Re(\kappa) > 0, p \geq 0, \nu \geq 0). \tag{40}
 \end{aligned}$$

Proof. By using (40) and (14), we obtain

$$\begin{aligned}
 \mathcal{B}\left\{\varepsilon_{a,b;p,v}^{\kappa}(xz_1^{\sigma}): a_1, b_1\right\} &= \int_0^1 z_1^{a_1-1} (1-z_1)^{b_1-1} \varepsilon_{a,b;p,v}^{\kappa}(xz_1^{\sigma}) dz_1 \\
 &= \int_0^1 z_1^{a_1-1} (1-z_1)^{b_1-1} \left(\sum_{l=0}^{\infty} \frac{(\kappa; p, \nu)_l}{\Gamma(al+b)} \frac{x^l z_1^{\sigma l}}{l!} \right) dz_1 \\
 &= \sum_{l=0}^{\infty} \frac{(\kappa; p, \nu)_l}{\Gamma(al+b)} \frac{x^l}{l!} \left(\int_0^1 z_1^{a_1+\sigma l-1} (1-z_1)^{b_1-1} dz_1 \right) \\
 &= \sum_{l=0}^{\infty} \frac{(\kappa; p, \nu)_l}{\Gamma(al+b)} \frac{x^l}{l!} \frac{\Gamma(\sigma l + a_1)\Gamma(b_1)}{\Gamma(\sigma l + a_1 + b_1)}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{\Gamma(b_1)\Gamma(a_1)}{\Gamma(a_1+b_1)} \sum_{l=0}^{\infty} \frac{(\kappa; p, \nu)_l (a_1)_{\sigma l}}{\Gamma(al+b) \times (a_1+b_1)_{\sigma l}} \frac{x^l}{l!} \\
 &= \frac{\Gamma(b_1)\Gamma(a_1)}{\Gamma(a_1+b_1)\Gamma(b)} \sum_{l=0}^{\infty} \frac{(\kappa; p, \nu)_l (a_1)_{\sigma l}}{(b)_{al} (a_1+b_1)_{\sigma l}} \frac{x^l}{l!} = \frac{\Gamma(b_1)\Gamma(a_1)}{\Gamma(a_1+b_1)\Gamma(b)} \\
 &\cdot \sum_{l=0}^{\infty} \frac{(\kappa; p, \nu)_l \prod_{j=1}^{\sigma} ((a_1+j-1)/\sigma)_l \sigma^{\sigma l}}{\prod_{j=1}^a ((b+j-1)/a)_l a^{al} \prod_{j=1}^{\sigma} ((a_1+b_1+j-1)/\sigma)_l \sigma^{\sigma l}} \frac{x^l}{l!} \\
 &= \frac{\Gamma(b_1)\Gamma(a_1)}{\Gamma(a_1+b_1)\Gamma(b)} \\
 &\cdot \sum_{l=0}^{\infty} \frac{(\kappa; p, \nu)_l \prod_{j=1}^{\sigma} ((a_1+j-1)/\sigma)_l}{\prod_{j=1}^a ((b+j-1)/a)_l a^{al} \prod_{j=1}^{\sigma} ((a_1+b_1+j-1)/\sigma)_l} \frac{x^l}{l!} \\
 &= \frac{\Gamma(b_1)\Gamma(a_1)}{\Gamma(b)\Gamma(a_1+b_1)_{\sigma+1}} \\
 &\cdot F_{a+\sigma}\left[(\kappa, p, \nu), \Delta(\sigma; a_1); \Delta(a; b), \Delta(\sigma; a_1+b_1); \frac{x}{a^a}\right]. \tag{41}
 \end{aligned}$$

Corollary 8. By putting $a_1 = b$ and $\sigma = a \in \mathbb{C}$ in (33) and then using (14), we have

$$\int_0^1 z_1^{b-1} (1-z_1)^{b-1} \varepsilon_{a,b;p,v}^{\kappa}(xz_1^a) dz_1 = \Gamma(b_1) \varepsilon_{a,b+b_1;p,v}^{\kappa}(x). \tag{42}$$

Similarly, we have

$$\int_0^1 z_1^{a_1-1} (1-z_1)^{b-1} \varepsilon_{a,b;p,v}^{\kappa}(x(1-z_1)^a) dz_1 = \Gamma(a_1) \varepsilon_{a,b+a_1;p,v}^{\kappa}(x). \tag{43}$$

In general, we get

$$\begin{aligned}
 &\int_0^1 (x-z_1)^{a_1-1} (z_1-t)^{b-1} \varepsilon_{a,b;p,v}^{\kappa}(w(z_1-t)^a) dz_1 \\
 &= \Gamma(a_1) (x-t)^{b+a_1-1} \varepsilon_{a,b+a_1;p,v}^{\kappa}(w(z-t)^a). \tag{44}
 \end{aligned}$$

5.3. Laplace Transform. The well-known Laplace transform [26] of $f(z_1)$ is defined by

$$\mathbf{L}\{f(z_1)\} = \int_0^{\infty} e^{-sz_1} f(z_1) dz_1. \tag{45}$$

Theorem 9. For the function $\varepsilon_{a,b;p,v}^{\kappa}(z_1)$ in (14), the following Laplace transform holds:

$$\begin{aligned}
 \mathbf{L}\left\{z_1^{a_1-1} \varepsilon_{a,b;p,v}^{\kappa}(xz_1^{\sigma})\right\} &= \frac{\Gamma(a_1)}{\Gamma(b) s^{a_1}} F_a\left[(\kappa, p, \nu), \Delta(\sigma; a_1); \Delta(a; b); \frac{xa^a}{(s\sigma)^{\sigma}}\right] \\
 &\cdot (a, \sigma \in \mathbb{N}; \Re(p) > 0, \Re(a_1) > 0, \Re(s) > 0, \Re(b) > 0, \\
 &\Re(\kappa) > 0, p \geq 0, \nu \geq 0). \tag{46}
 \end{aligned}$$

Proof. By using (45) and from (14), we have

$$\begin{aligned}
 \mathbf{L}\left\{z_1^{a_1-1} \varepsilon_{a,b;p,v}^\kappa(xz_1^\sigma)\right\} &= \int_0^\infty z_1^{a_1-1} e^{-sz_1} \varepsilon_{a,b;p,v}^\kappa(xz_1^\sigma) dz_1 \\
 &= \int_0^\infty z_1^{a_1-1} e^{-sz_1} \left(\sum_{l=0}^\infty \frac{(\kappa;p,v)_l}{\Gamma(al+b)} \frac{x^l z_1^{al}}{l!} \right) dz_1 \\
 &= \sum_{l=0}^\infty \frac{(\kappa;p,v)_l}{\Gamma(al+b)} \frac{x^l}{l!} \int_0^\infty z_1^{a_1+al-1} e^{-sz_1} dz_1 \\
 &= \sum_{l=0}^\infty \frac{(\kappa;p,v)_l}{\Gamma(al+b)} \frac{x^l}{l!} \frac{\Gamma(a_1+\sigma l)}{s^{a_1+\sigma l}} \\
 &= \sum_{l=0}^\infty \frac{(\kappa;p,v)_l}{\Gamma(al+b)} \frac{x^l}{l!} \frac{\Gamma(a_1+\sigma l)}{s^{a_1+\sigma l}} \\
 &= \frac{\Gamma(a_1)}{\Gamma(b)} \sum_{l=0}^\infty \frac{(\kappa;p,v)_l (a_1)_{al}}{(b)_{al}} \frac{(xa^a)^l}{(s^\sigma s^\sigma)^l} \frac{1}{l!} \\
 &= \frac{\Gamma(a_1)}{\Gamma(b)} \frac{1}{s^{a_1}} F_a \left[(\kappa, p, v), \Delta(\sigma; a_1); \Delta(a; b); \frac{xa^a}{(s^\sigma)^\sigma} \right]. \quad (47)
 \end{aligned}$$

Corollary 10. By setting $a_1 = b$ and $\sigma = a$ in (46), we obtain

$$\int_0^\infty z^{b-1} e^{-sz} \varepsilon_{a,b;p,v}^\kappa(xz^a) dz = \frac{1}{s^{b_1}} F_0 \left[(\kappa, p, v); -; \frac{x}{s^a} \right]. \quad (48)$$

5.4. Whittaker Transformation. To determine the Whittaker transforms, we use the following formula:

$$\int_0^\infty t^{v-1} e^{-(1/2)t} \mathbf{W}_{\eta,\xi}(t) dt = \frac{\Gamma((1/2) \pm \xi + v)}{\Gamma(1 - \eta + v)} \left(\Re \left(v \pm \xi > -\frac{1}{2} \right) \right). \quad (49)$$

Theorem 11. For the function $\varepsilon_{a,b;p,v}^\kappa(z_1)$ in (14), the following Whittaker transform holds:

$$\begin{aligned}
 &\int_0^\infty t^{\rho-1} e^{-(1/2)st} \mathbf{W}_{\eta,\xi}(st) \varepsilon_{a,b;p,v}^\kappa(\omega z^\delta) dt \\
 &= \frac{\Gamma((1/2) \pm \xi + \rho)}{\Gamma(1 - \eta + \rho) \Gamma(b) s^\rho} \\
 &\quad \cdot F_{a+\vartheta} \left[(\kappa, p, v), \Delta \left(\vartheta; \frac{1}{2} \pm \xi + \rho \right); \Delta(a, b), \Delta(\vartheta; 1 - \eta + \rho); \frac{\omega a^a}{(s^\vartheta)^\vartheta} \right] \\
 &\quad \cdot (a, \vartheta \in \mathbb{N}; p \geq 0; \Re(p) > 0, \Re(\rho) > 0, \Re(b) > 0, \Re(\kappa) > 0, v \geq 0). \quad (50)
 \end{aligned}$$

Proof. By the definition of Whittaker transform, we have from (14)

$$\begin{aligned}
 &\int_0^\infty t^{\rho-1} e^{-(1/2)st} \mathbf{W}_{\eta,\xi}(st) \varepsilon_{a,b;p,v}^\kappa(\omega t^\vartheta) dt \\
 &= \int_0^\infty t^{\rho-1} e^{-(1/2)st} \mathbf{W}_{\eta,\xi}(st) \left(\sum_{l=0}^\infty \frac{(\kappa;p,v)_l}{\Gamma(al+b)} \right) \frac{\omega^l t^{\vartheta l}}{l!} dt \\
 &= \sum_{l=0}^\infty \frac{(\kappa;p,v)_l}{\Gamma(al+b)} \frac{\omega^l}{l!} \int_0^\infty t^{\rho+\vartheta l-1} e^{-(1/2)st} \mathbf{W}_{\eta,\xi}(st) dt. \quad (51)
 \end{aligned}$$

By putting $v = st$ and then using the definition of Whittaker transforms, we get

$$\begin{aligned}
 \int_0^\infty t^{\rho-1} e^{-(1/2)t} \mathbf{W}_{\eta,\xi}(st) \varepsilon_{a,b;p,v}^\kappa(\omega z^\vartheta) dt &= \sum_{l=0}^\infty \frac{(\kappa;p,v)_l}{\Gamma(al+b)} \frac{\omega^l}{l!} \frac{\Gamma((1/2) \pm \xi + \rho + \vartheta l)}{\Gamma(1 - \eta + \rho + \vartheta l)} (s)^{-\rho-\vartheta l} \\
 &= \frac{1}{s^\rho} \sum_{l=0}^\infty \frac{(\kappa;p,v)_l}{\Gamma(al+b)} \frac{\Gamma((1/2) \pm \xi + \rho + \vartheta l) \times \Gamma((1/2) \pm \xi + \rho) / \Gamma((1/2) \pm \xi + \rho)}{\Gamma(1 - \eta + \rho + \vartheta l) \times \Gamma(1 - \eta + \rho) / \Gamma(1 - \eta + \rho)} \times \frac{1}{l!} \left(\frac{\omega}{s^\vartheta} \right)^l \\
 &= \frac{\Gamma((1/2) \pm \xi + \rho)}{\Gamma(1 - \eta + \rho) \Gamma(b) s^\rho} \sum_{l=0}^\infty \frac{(\kappa;p,v)_l ((1/2) \pm \xi + \rho)_{\vartheta l}}{(b)_{al} (1 - \eta + \rho)_{\vartheta l}} \frac{1}{l!} \left(\frac{\omega}{s^\vartheta} \right)^l \\
 &= \frac{\Gamma((1/2) \pm \xi + \rho)}{\Gamma(1 - \eta + \rho) \Gamma(b) s^\rho} F_{a+\vartheta} \left[(\kappa, p, v), \Delta \left(\vartheta; \frac{1}{2} \pm \xi + \rho \right); \Delta(a, b), \Delta(\vartheta; 1 - \eta + \rho); \frac{\omega a^a}{(s^\vartheta)^\vartheta} \right]. \quad (52)
 \end{aligned}$$

6. Conclusion

In our current investigation, we presented an extension of the generalized ML function $\varepsilon_{a,b;p,v}^\kappa(z_1)$ in (14) by utilizing an extension of Pochhammer symbol $(\mu; \omega, a)_\eta$ defined in (9). Further, we have investigated several basic properties of the newly defined function $\varepsilon_{a,b;p,v}^\kappa(z_1)$. The

special cases of the main result for $v=0$ can be found in the work of [12]. Thus, the results introduced in this present article are new and an extension of the relating outcomes in the existing literature (see, e.g., [29–31]). The newly defined ML function $\varepsilon_{a,b;p,v}^\kappa(z_1)$ presented in this article will be applicable in different fields of applied sciences.

Data Availability

No data was used for this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed equally.

Acknowledgments

The author Thabet Abdeljawad would like to thank Prince Sultan University for funding this work through research group Nonlinear Analysis Methods in Applied Mathematics (NAMAM) group number RG-DES-2017-01-17.

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