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Research Article

Scattering Theory for 2N Parameter Models of Finitely Many Relativistic δ-Sphere and δ-Sphere plus Coulomb Interactions Supported by Concentric Spheres in Quantum Mechanics

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We study scattering theory for 2N parameter models of finitely many relativistic δ-sphere and δ-sphere plus Coulomb interactions. We provide the mathematical definitions of the Hamiltonians, solve the resolvent equations, and compute the nonrelativistic limits for both models. We obtain new results related to spectral properties and scattering data.

1. Introduction

Over the past three decades, δ-sphere interactions in quantum mechanics have been the subject of several research studies, both from the mathematical point of view and for their applications to the modeling of physical phenomena. The Hamiltonians describing these interactions are defined using the theory of self-adjoint extensions of closed symmetric operators in Hilbert spaces. Initially, the emphasis was placed on studying these interactions in nonrelativistic mechanics [1–10]. The extension of this research work to relativistic quantum mechanics began in 1989 [11–16].

In [17], we studied Dirac Hamiltonians with delta interactions and delta plus Coulomb interactions. We obtained a series of new results for this model including the resolvent equation, the spectral properties, the nonrelativistic limit, and the various quantities related to the scattering theory and the generalization of these results to the case of a delta plus Coulomb interaction.

In this paper, we extend the results obtained in [17] to the case of Dirac Hamiltonian with finitely many delta and delta plus Coulomb interactions with support on N concentric spheres.

The paper is organized as follows: in Section 2, we provide a rigorous mathematical definition of the Dirac Hamiltonian with finitely many delta interactions supported by N concentric spheres and generalize all results obtained in [17] to this case. In Section 3, we expand the results obtained in Section 2 to the case of a Dirac Hamiltonian with finitely many spheres plus Coulomb interactions with support on N concentric spheres.

2. Basic Properties for the 2N Parameter Model of Finitely Many Relativistic δ-Sphere Interactions: Separated Boundary Conditions

2.1. Definition of the Model. In this section, we provide in dimension $n=3$ the rigorous mathematical definition of finitely many relativistic δ-sphere interactions. The formal expression of the hamiltonian describing the 2N parameter model of finitely many δ-sphere interactions with support on concentric spheres of radii $0 < R_1 < \cdots < R_N$ is given by

$$H = H_D + \sum_{n=1}^{N} \Gamma_n \delta(|x| - R_n), \quad x \in \mathbb{R}^3, 0 < R_1 < \cdots < R_N, \quad (1)$$

where we define the Dirac Hamiltonian in Hilbert space $\mathscr{H} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^4$ by

$$H_D = -i\,\underline{\alpha}\nabla + \underline{\beta}\frac{c^2}{2}, \mathscr{D}(H_D) = H^{1,2}(\mathbb{R}^3)\otimes\mathbb{C}^4. \quad (2)$$

The 4×4 matrix Γ_n is defined by

$$\Gamma_n = \begin{pmatrix} \underline{A}_n & 0 \\ 0 & \underline{B}_n \end{pmatrix}, \quad n = 1,\cdots,N, \quad (3)$$

and $\underline{A}_n = A_n\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\underline{B}_n = B_n\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

A_n and B_n, $n = 1,\cdots,N$, are $2N$ parameters corresponding to the relativistic δ-sphere interaction supported by the sphere of radius R_n. $c, H^{m,n}, \underline{\alpha}$, and $\underline{\beta}$ are, respectively, defined by the following: c is the velocity of the light, $H^{m,n}$ is the Sobolev space of indices (m,n), and $\underline{\alpha}$ and $\underline{\beta}$ are 4×4 Dirac matrices given by

$$\underline{\alpha} = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \quad \underline{\beta} = \begin{pmatrix} 1\mathrm{l} & 0 \\ 0 & 1\mathrm{l} \end{pmatrix}. \quad (4)$$

σ are Pauli's spin matrices defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (5)$$

Considering $L^2(\mathbb{R}^3)\otimes\mathbb{C}^4$ the symmetric closed operator, $\bar{H}_D$ is defined by

$$\bar{H}_D = -i\,\underline{\alpha}\nabla + \underline{\beta}\frac{c^2}{2}, \quad \bar{H}_D = \left\{\psi \in H^{1,2}(\mathbb{R}^3)\otimes\mathbb{C}^4 \,\middle|\, \psi(S_{R_n} = 0), \quad 1 < n \le N\right\}, \quad (6)$$

where $S_{R_n} = \{x \in \mathbb{R}^3 : |x| = R_n\}$ is the sphere of radius R_n.

Let us look for self-adjoint extensions of $\bar{H}_D$ which are rotationally and space-reflection symmetric.

The Hilbert space $\mathscr{H}$ can be decomposed as follows:

$$\mathscr{H} = \bigoplus_{j=1/2}^{\infty}\bigoplus_{l=j-1/2}^{j+1/2}\bigoplus_{\mu=-j}^{j}\mathscr{H}_{jl\mu}, \quad (7)$$

where

$$\mathscr{H}_{jl\mu} = \left\{\psi \in \mathscr{H} \,\middle|\, \psi(r,n) = \begin{pmatrix} f(r)\Omega_{jl\mu} \\ g(r)\Omega_{jl'\mu} \end{pmatrix}, \quad f,g \in L^2((0,\infty), r^2\mathrm{d}r)\right\}. \quad (8)$$

The spherical spinors $\Omega_{jl\mu}$ are defined by [18]

$$\Omega_{jl\mu} = \begin{pmatrix} \sqrt{\dfrac{j+\mu}{2l+1}}Y_{l,\mu-1/2}(\theta,\varphi) \\ \sqrt{\dfrac{j-\mu}{2l+1}}Y_{l,\mu+1/2}(\theta,\varphi) \end{pmatrix}, \quad \text{for } l = j - \frac{1}{2},$$
$$\Omega_{jl'\mu} = \begin{pmatrix} -\sqrt{\dfrac{j-\mu+1}{2l+1}}Y_{l,\mu-1/2}(\theta,\varphi) \\ \sqrt{\dfrac{j+\mu+1}{2l+1}}Y_{l,\mu+1/2}(\theta,\varphi) \end{pmatrix}, \quad \text{for } l = j + \frac{1}{2}. \quad (9)$$

We have $l' = j \mp 1/2$ for $l = j \pm 1/2$. The Hilbert space $\mathscr{H}$ then takes the following form:

$$\mathscr{H} = \bigoplus_{j=1/2}^{\infty}\bigoplus_{l=j-1/2}^{j+1/2}\bigoplus_{\mu=-j}^{j}\left[U_{jl}^{-1}\tilde{\mathscr{H}}\right]\otimes\left[\Omega_{jl\mu}(\theta,\varphi)\right], \quad (10)$$

where the isomorphism U_{jl} is defined by

$$U_{lj} : L^2((0,\infty); r^2\mathrm{d}r)\otimes\mathbb{C}^2 \longrightarrow \tilde{\mathscr{H}} = L^2((0,\infty); \mathrm{d}r)\otimes\mathbb{C}^2,$$
$$(U_{jl})\psi(r) = \begin{pmatrix} rf(r) \\ (-1)^{j-l-1/2}rg(r) \end{pmatrix}, \quad (11)$$

and $[\Omega_{jl}(\theta,\varphi)]$ represents the vector space formed by the spherical spinors.

Following decomposition (10), the operator $\bar{H}_D$ reads

$$\bar{H}_D = \bigoplus_{j=1/2}^{\infty}\bigoplus_{l=j-1/2}^{j+1/2}\left[U_{jl}^{-1}h_{jl,\{R\}}U_{jl}\right]\otimes 1\mathrm{l}, \quad (12)$$

where the radial Dirac operator $h_{jl,\{R\}}$ reads

$$h_{jl,\{R\}} = \begin{pmatrix} \frac{c^2}{2} & -c\frac{\mathrm{d}}{\mathrm{d}r} + c\frac{\kappa_{jl}}{r} \\ c\frac{\mathrm{d}}{\mathrm{d}r} + c\frac{\kappa_{jl}}{r} & -\frac{c^2}{2} \end{pmatrix} \equiv \tau, \kappa_{jl} = (-)^{j-l+1/2}\left(j+\frac{1}{2}\right),$$

$$\mathscr{D}\left(h_{jl,\{R\}}\right) = \{\psi \in L^2((0,\infty)) \otimes \mathbb{C}^2 \mid \psi \in AC_{loc}((0,\infty)), \psi(R_n \pm) = 0, \tau\psi \in L^2((0,\infty)) \otimes \mathbb{C}^2\},$$
$$\{R\} = \{R_1, R_2, \cdots, R_N\}, \quad (13)$$

where $AC_{loc}(\Omega)$ denotes the set of locally absolutely continuous functions on Ω and

$$\psi(x\pm) = \lim_{\varepsilon\to 0^+} \psi(x \pm \varepsilon). \quad (14)$$

The adjoint $h^*_{jl,\{R\}}$ of $h_{jl,\{R\}}$ reads

$$h^*_{jl,\{R\}} = \tau,$$
$$\mathscr{D}\left(h^*_{jl,\{R\}}\right) = \{f \in L^2((0,\infty)) \otimes \mathbb{C}^2 \mid f \in AC_{loc}((0,\infty) \setminus \{R\}), \quad \tau f \in L^2((0,\infty)) \otimes \mathbb{C}^2\}. \quad (15)$$

A straightforward computation shows that the equation

$$\left(h^*_{jl,\{R\}} - z\right)\varphi = 0, \quad \varphi \in \mathbb{C} - \left\{\left(-\infty, -\frac{c^2}{2}\right] \bigcup \left[\frac{c^2}{2}, \infty\right)\right\}, \quad (16)$$

has $2N$ linearly independent solutions

$$\varphi^{(1)}_{jl,n}(r) = \begin{cases} \begin{pmatrix} G_{jl}(z,R_n)F_{jl}(z,r) \\ \tilde{G}_{jl}(z,R_n)\tilde{F}_{jl}(z,r) \end{pmatrix}, & r < R_n, \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & r > R_n, n = 1, \cdots, N, \end{cases}$$

$$\varphi^{(2)}_{jl,n}(r) = \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & r < R_n, \\ \begin{pmatrix} F_{jl}(z,R_n)G_{jl}(z,r) \\ \tilde{F}_{jl}(z,R_n)\tilde{G}_{jl}(z,r) \end{pmatrix}, & r > R_n, n = 1, \cdots, N, \end{cases} \quad (17)$$

where

$$F_{jl}(z,r) = \left(\frac{k}{2}\right)^{-\kappa_{jl}-1/2} \Gamma\left(\kappa_{jl} + \frac{3}{2}\right) r^{1/2} J_{\kappa_{jl}+1/2}(kr), \quad (18)$$

$$\tilde{F}_{jl}(z,r) = \frac{1}{c}\left(\frac{1}{2}\right)^{-\kappa_{jl}-1/2} k^{-\kappa_{jl}+1/2} \Gamma\left(\kappa_{jl} + \frac{3}{2}\right) r^{1/2} J_{\kappa_{jl}-1/2}(kr), \quad (19)$$

$$G_{jl}(z,r) = \mathrm{i}\,\frac{\pi}{2}\left(\frac{k}{2}\right)^{\kappa_{jl}+1/2} 1/2\Gamma\left(\kappa_{jl} + \frac{3}{2}\right) r^{1/2} H^{(1)}_{\kappa_{jl}+1/2}(kr), \quad (20)$$

$$\tilde{G}_{jl}(z,r) = \mathrm{i}\,\frac{\pi}{2c}\left(\frac{1}{2}\right)^{\kappa_{jl}+1/2} k^{\kappa_{jl}+3/2} \Gamma\left(\kappa_{jl} + \frac{3}{2}\right)^{-1} r^{1/2} H^{(1)}_{\kappa_{jl}-1/2}(kr). \quad (21)$$

$J_\nu(\bullet)$ is the Bessel function and $H^{(1)}_\nu(\bullet)$ the Hankel function of the first type of order ν.

Therefore, all self-adjoint extensions of $h_{jl,\{R\}}$ are given by a $4N^2$ parameter family of self-adjoint operators.

In this section, we consider the following special $2N$ parameter family of self-adjoint extensions of $h_{jl,\{R\}}$:

$$h_{jl,\{\Gamma\},\{R\}} = \begin{pmatrix} \frac{c^2}{2} & -c\frac{d}{dr} + c\frac{\kappa_{jl}}{r} \\ -c\frac{d}{dr} + c\frac{\kappa_{jl}}{r} & -\frac{c^2}{2} \end{pmatrix}, \quad (22)$$

$$\mathscr{D}\left(h_{jl,\{\Gamma\},\{R\}}\right) = \left\{g \in \mathscr{D}\left(h^*_{jl,\{\Gamma\},\{R\}}\right) \middle| \left(1 - \tau_0\frac{\Gamma_{jl,n}}{2c}\right) g(R_n+) - \left(1 + \tau_0\frac{\Gamma_{jl,n}}{2c}\right) g(R_n-) = 0\right\}, \quad (23)$$

where the 2×2 matrix $\Gamma_{jl,n}$ reads

$$\Gamma_{jl,n} = \begin{pmatrix} A_{jl,n} & 0 \\ 0 & B_{jl,n} \end{pmatrix}, \quad A_{jl,n}, B_{jl,n} \in \mathbb{R}. \quad (24)$$

The Hamiltonian $h_{jl,\{\Gamma\},\{R\}}$ gives the mathematical definition of the formal expression

$$h_{\{\Gamma\},\{R\}} = h_D + \sum_{n=1}^{N} \Gamma_{jl,n}\delta(r - R_n), \quad R_n > 0, n = 1, 2, \cdots, N, \quad (25)$$

where h_D is the radial Dirac operator defined by

$$h_D = \begin{pmatrix} \frac{c^2}{2} & -c\frac{d}{dr} + c\frac{\kappa_{jl}}{r} \\ -c\frac{d}{dr} + c\frac{\kappa_{jl}}{r} & -\frac{c^2}{2} \end{pmatrix}, \quad (26)$$

$$\mathscr{D}(h_D) = H^{1,2}((0,\infty)) \otimes \mathbb{C}^2. \quad (27)$$

Given decomposition (10), a rigorous mathematical definition of formal expression (1) reads

$$H_{\{\Gamma\},\{R\}} = \bigoplus_{j=1/2}^{\infty} \bigoplus_{l=j-1/2}^{j+1/2} \left[U_{jl}^{-1} h_{jl,\{\Gamma\},\{R\}} U_{jl}\right] \otimes 1\!\!1. \quad (28)$$

The case $\Gamma = 0$, i.e., $\Gamma_{jl,n} = 0$ for all j and l in equation (28), yields the Dirac Hamiltonian H_D defined by equation (2).

The case $A_{jl,n} \neq 0$ and $B_{jl,n} = 0$ for all j and l in equation (28) yields the relativistic N-parameter δ-sphere interactions of the first type [17].

The case $A_{jl,n} = 0$ and $B_{jl,n} \neq 0$ for all j and l in equation (28) yields the relativistic N-parameter δ-sphere interactions of the second type [17].

2.2. Resolvent Equation of $h_{jl,\{\Gamma\},\{R\}}$

Theorem 1. *The resolvent of $h_{jl,\{\Gamma\},\{R\}}$ reads*

$$\begin{aligned}
\left(h_{jl,\{\Gamma\},\{R\}} - z\right)^{-1} &= (h_D - z)^{-1} + \sum_{n,m=1}^{N} \Big\{ \vartheta_{nm}^{(1)}(z) \left(\overline{\widehat{M}_{jl,z}^{(2)}(\bullet)},.\right) \widehat{M}_{jl,z}^{(2)}(\bullet) \\
&\quad + \vartheta_{nm}^{(2)}(z) \left(\overline{\tilde{M}_{jl,z}^{(2)}(\bullet)},.\right) \tilde{M}_{jl,z}^{(2)}(\bullet) + \vartheta_{nm}^{(3)}(z) \left(\overline{\widehat{M}_{jl,z}^{(2)}(\bullet)},.\right) \\
&\quad \cdot \tilde{M}_{jl,z}^{(1)}(\bullet) + \vartheta_{nm}^{(4)}(z) \left(\overline{\tilde{M}_{jl,z}^{(2)}(\bullet)},.\right) \widehat{M}_{jl,,z}^{(1)}(\bullet) \Big\}, \\
&\quad z \in \rho\left(h_{jl,\{\Gamma\},\{R\}}\right), \operatorname{Im} k > 0, l \in \left[j - \frac{1}{2}, j + \frac{1}{2}\right], j \in \left[\frac{1}{2}, \infty\right),
\end{aligned} \quad (29)$$

where ρ is the resolvent set and

$$\left[\vartheta^{(1)}(z)\right]_{nm}^{-1} = \begin{cases} -\left[\dfrac{1}{A_{jl}} - \dfrac{B_{jl}}{4c^2} + \dfrac{B_{jl}}{A_{jl}} G_{22}^{(jl)}(z, R_n, R_n) + G_{11}^{(jl)}(z, R_n, R_n)\right], & n = m, \\ -G_{11}^{(jl)}(z, R_m, R_n), & n \neq m, \end{cases} \quad (30)$$

$$\left[\vartheta^{(2)}(z)\right]_{nm}^{-1} = \begin{cases} -\left[\dfrac{1}{B_{jl}} - \dfrac{A_{jl}}{4c^2} + \dfrac{A_{jl}}{B_{jl}} G_{11}^{(jl)}(z, R_n, R_n) + G_{22}^{(jl)}(z, R_n, R_n)\right], & n = m, \\ -G_{22}^{(jl)}(z, R_m, R_n), & n \neq m, \end{cases} \quad (31)$$

$$\left[\vartheta^{(3)}(z)\right]_{nm}^{-1} = \begin{cases} -\left[\dfrac{2c}{B_{jl}A_{jl}} - \dfrac{1}{2c} + \dfrac{2c}{B_{jl}} G_{11}^{(jl)}(z, R_n, R_n) + \dfrac{2c}{A_{jl}} G_{22}^{(jl)}(z, R_n, R_n)\right], & n = m, \\ \dfrac{1}{2c}, & n \neq m, \end{cases} \quad (32)$$

$$\left[\vartheta^{(4)}(z)\right]_{nm}^{-1} = \begin{cases} -\left[-\dfrac{2c}{B_{jl}A_{jl}} + \dfrac{1}{2c} - \dfrac{2c}{B_{jl}} G_{11}^{(jl)}(z, R_n, R_n) - \dfrac{2c}{A_{jl}} G_{22}^{(jl)}(z, R_n, R_n)\right], & n = m, \\ \dfrac{-1}{2c}, & n \neq m, \end{cases} \quad (33)$$

$$\widehat{M}_{jl,z}^{(m)}(r) = \begin{cases} G^{jl}(z, r, R_n) 10, & r < R_n, \\ (-)^m G^{jl}(z, r, R_n) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & r > R_n, m = 1, 2, n = 1, 2, \cdots, N, \end{cases} \quad (34)$$

$$\tilde{M}_{jl,z}^{(m)}(r) = \begin{cases} G^{jl}(z, r, R_n) 01, & r < R_n, \\ (-)^m G^{jl}(z, r, R_n) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & r > R_n, m = 1, 2, n = 1, 2, \cdots, N. \end{cases} \quad (35)$$

$(h_D - z)^{-1}$,Im $k > 0$, *is the radial Dirac resolvent with kernel:*

$$G^{jl}(z,r,R_n) = \begin{pmatrix} G^{jl}_{11}(z,r,R_n) & G^{jl}_{12}(z,r,R_n) \\ G^{jl}_{21}(z,r,R_n) & G^{jl}_{22}(z,r,R_n) \end{pmatrix}, \tag{36}$$

where

$$\begin{aligned} G^{jl}_{11}(z,r,R_n) &= \begin{cases} G_{jl}(z,R_n)F_{jl}(z,r), & r < R_n, \\ F_{jl}(z,R_n)G_{jl}(z,r), & r > R_n, \end{cases} \\ G^{jl}_{12}(z,r,R_n) &= \begin{cases} \tilde{G}_{jl}(z,R_n)F_{jl}(z,r), & r < R_n, \\ \tilde{F}_{jl}(z,R_n)G_{jl}(z,r), & r > R_n, \end{cases} \\ G^{jl}_{21}(z,r,R_n) &= \begin{cases} G_{jl}(z,R_n)\tilde{F}_{jl}(z,r), & r < R_n, \\ F_{jl}(z,R_n)\tilde{G}_{jl}(z,r), & r > R_n, \end{cases} \\ G^{jl}_{22}(z,r,R_n) &= \begin{cases} \tilde{G}_{jl}(z,R_n)\tilde{F}_{jl}(z,r), & r < R_n, \\ \tilde{F}_{jl}(z,R_n)\tilde{G}_{jl}(z,r), & r > R_n. \end{cases} \end{aligned} \tag{37}$$

Proof. Equation (29) follows from Krein's formula [19].

Let $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in L((0,\infty)) \otimes \mathbb{C}^2$ and define the function ξ_{jl} by

$$\xi_{jl}(r) = \left(\left(h_{jl,\{\Gamma\},\{R\}} - z\right)^{-1}\phi\right)(r). \tag{38}$$

Since $\xi_{jl} \in \mathcal{D}(h_{jl,\{\Gamma\},\{R\}})$, it follows that ξ_{jl} satisfies the separated boundary conditions in equation (23).

The implementation of these boundary conditions provides equation (29).

2.3. Spectral Properties of $h_{jl,\{\Gamma\},\{R\}}$

Theorem 2. *For $A_{jl,n}, B_{jl,n} \in \mathbb{R}$, $l \in [j - 1/2, j + 1/2]$, $j \in [1/2, \infty)$, the essential spectrum of $h_{jl,\{\Gamma\},\{R\}}$ is purely absolutely continuous and coincides with $(-\infty, -c^2/2] \cup [c^2/2, \infty)$. Its singularly continuous and residual spectra are empty.*

Proof. Follow step by step the proof of theorem 6.2 and proposition 6.1 in ref. [12].

According to decomposition (10), equations (12) and (29), the resolvent equation of $H_{\{\Gamma\},\{R\}}$ reads

$$\begin{aligned} &\left(H_{\{\Gamma\},\{R\}} - z\right)^{-1} \\ &= (H_D - z)^{-1} + \bigoplus_{j=1/2}^{\infty} \bigoplus_{l=j+1/2}^{j-1/2} \bigoplus_{\mu=-j}^{j} \\ &\times \sum_{n,m=1}^{N} \Big\{ \vartheta^{(1)}_{nm}(z) \Big(|.|^{-1} \overline{\widehat{M}^{(2)}_{jl,z}(.)} \otimes \tilde{\Omega}_{jl\mu}, . \Big) \\ &\times |.|^{-1} \widehat{M}^{(2)}_{jl,z}(.) \otimes \tilde{\Omega}_{jl\mu} + \vartheta^{(2)}_{nm}(z) \Big(|.|^{-1} \overline{\tilde{M}^{(2)}_{jl,z}(.)} \otimes \tilde{\Omega}_{jl\mu}, . \Big) \\ &\times |.|^{-1} \tilde{M}^{(2)}_{jl,z}(.) \otimes \tilde{\Omega}_{jl\mu} + \vartheta^{(3)}_{nm}(z) \Big(|.|^{-1} \overline{\widehat{M}^{(2)}_{jl,z}(.)} \otimes \tilde{\Omega}_{jl\mu}, . \Big) \\ &\times |.|^{-1} \tilde{M}^{(1)}_{jl,z}(.) \otimes \tilde{\Omega}_{jl\mu} + \vartheta^{(4)}_{nm}(z) \Big(|.|^{-1} \overline{\tilde{M}^{(2)}_{jl,z}(.)} \otimes \tilde{\Omega}_{jl\mu}, . \Big) \\ &\times |.|^{-1} \widehat{M}^{(1)}_{jl,z}(.) \otimes \tilde{\Omega}_{jl\mu} \Big\}, \\ &\qquad z \in \rho\left(h_{jl,\{\Gamma\},\{R\}}\right), \operatorname{Im} k > 0, \\ &\qquad l \in \left[j - \frac{1}{2}, j + \frac{1}{2}\right], j \in \left[\frac{1}{2}, \infty\right), \end{aligned} \tag{39}$$

where the notations $\widehat{M}^{(n)}_{jl,z}(.) \otimes \tilde{\Omega}_{jl\mu}$ and $\tilde{M}^{(n)}_{jl,z}(.) \otimes \tilde{\Omega}_{jl\mu}$, $n = 1, 2$, mean

$$\begin{aligned} \widehat{M}^{(n)}_{jl,z}(.) \otimes \tilde{\Omega}_{jl\mu} &= \begin{pmatrix} \widehat{M}^{(n)}_{jl,z,1}(.)\Omega_{jl\mu} \\ \widehat{M}^{(n)}_{jl,z,2}(.)\Omega_{jl'\mu} \end{pmatrix}, \quad n = 1, 2, \\ \tilde{M}^{(n)}_{jl,z}(.) \otimes \tilde{\Omega}_{jl\mu} &= \begin{pmatrix} \tilde{M}^{(n)}_{jl,z,1}(.)\Omega_{jl\mu} \\ \tilde{M}^{(n)}_{jl,z,2}(.)\Omega_{jl'\mu} \end{pmatrix}, \quad n = 1, 2. \end{aligned} \tag{40}$$

2.4. The Nonrelativistic Limit. The nonrelativistic limit of $h_{jl,\{\Gamma\},\{R\}}$ as $c \longrightarrow \infty$ is given by the following.

Theorem 3. *For the spin 1/2 particules, the operator $h_{jl,\{\Gamma\},\{R\}} - c^2/2$ converges in norm resolvent sense to the Schrödinger operator $h_{l,\alpha_l,\{R\}}$ times the projector onto $\tilde{\mathscr{H}}_+ = L^2((0,\infty))$:*

$$\begin{aligned} &n \cdot \lim_{c \to \infty} \left(h_{jl,\{\Gamma\},\{R\}} - \frac{c^2}{2} - z\right)^{-1} \\ &\quad = \left(h_{jl,\{\alpha\},\{R\}} - z\right)^{-1} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad z \in \mathbb{C} \setminus \mathbb{R}, \end{aligned} \tag{41}$$

where the operator $h_{jl,\{\alpha\},\{R\}}$ is defined by

$$\begin{aligned}
h_{jl,\{\alpha\},\{R\}} &= -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2}, \\
\mathscr{D}\left(h_{jl,\{\alpha\},\{R\}}\right) &= \Big\{ f \in L^2((0,\infty)) \big| f, f' \\
&\quad \in AC_{loc}((0,\infty) \setminus \{R_1, \cdots, R_N\}), \\
&\quad g(0+) = 0, \\
f(R_n+) - f(R_n-) &= \frac{\beta_{l,n}}{2}\left[f'(R_n+) + f'(R_n-)\right], \\
&\quad f'(R_n+) - f'(R_n-) \\
&= \frac{\alpha_{l,n}}{2}\left[f(R_n+) + f(R_n-)\right], \\
&\quad -f'' + l(l+1)r^{-2} f \in L^2((0,\infty)) \Big\}, \\
l \geq 1, \alpha_{l,n}\beta_{l,n} - 4 = 0, &\ \alpha_{l,n}, \beta_{l,n} \in \mathbb{R}, 1 \leq n \leq N.
\end{aligned} \tag{42}$$

Proof. One can use the strategy of Gesztesy and Seba [15] where a similar case was discussed for point interactions.

The Hamiltonian $h_{jl,\{\alpha\},\{R\}}$ describes a $2N$ parameter model of finitely many nonrelativistic δ-sphere interactions in quantum mechanics.

2.5. Scattering Theory for the Pair $(h_{jl,\{\alpha\},\{R\}}, h_D)$. Let us define for $k > 0$ the function

$$\begin{aligned}
F_{jl,\{\Gamma\},\{R\}}(r) &= \begin{pmatrix} F_{jl}(r) \\ \tilde{F}_{jl}(r) \end{pmatrix} + \sum_{n,m=1} \Big\{ \vartheta_{nm}^{(1)}(z) F_{jl}(z, R_n) \\
&\quad \cdot \widehat{M}_{jl,z}^{(2)}(r) + \vartheta_{nm}^{(2)}(z) \tilde{F}_{jl}(z, R_n) \tilde{M}_{jl,z}^{(2)}(r) \\
&\quad + \vartheta_{nm}^{(3)}(z) F_{jl}(z, R_n) \tilde{M}_{jl,z}^{(1)}(r) \\
&\quad + \vartheta_{nm}^{(4)}(z) \tilde{F}_{jl}(z, R_n) \widehat{M}_{jl,z}^{(1)}(r) \Big\},
\end{aligned} \tag{43}$$

where functions $F_{jl}(z,r)$, $\tilde{F}_{jl}(z,r)$, $\tilde{M}_{jl,z}^{(2)}(r)$, $\widehat{M}_{jl,z}^{(2)}(r)$, $\tilde{M}_{jl,z}^{(1)}(r)$, $\widehat{M}_{jl,z}^{(1)}(r)$, and $\vartheta_{nm}^{(i)}(z)$, $i = 1, 2, 3, 4$, are, respectively, defined by equations (18), (35), and (33).

The asymptotic behaviour of the function $F_{jl,\{\Gamma\},\{R\}}(r)$ as $r \longrightarrow \infty$ yields [20]

$$\begin{aligned}
F_{jl,\{\Gamma\},\{R\}}(r) &\xrightarrow[r\to\infty]{k>0} \begin{pmatrix} \widehat{A}_{jl}\left(z \sin\left[kr - \kappa_{jl}\dfrac{\pi}{2}\right]\right) \\ \widehat{B}_{jl}(z) \sin\left[kr - (\kappa_{jl} - 1)\dfrac{\pi}{2}\right] \end{pmatrix} \\
&\quad + \sum_{n,m=1}^{N} \vartheta'_{nm}(z) \begin{pmatrix} \widehat{C}_{jl}(z) \exp - i\left[kr - \kappa_{jl}\dfrac{\pi}{2}\right] \\ \widehat{O}_{jl}(z) \exp - i\left[kr - (\kappa_{jl} - 1)\dfrac{\pi}{2}\right] \end{pmatrix} \\
&= \begin{pmatrix} \left[\widehat{A}_{jl}(z) - i\widehat{C}_{jl}(z) \displaystyle\sum_{n,m=1}^{N} \vartheta'_{nm}(z)\right] \sin\left[kr - \kappa_{jl}\dfrac{\pi}{2}\right] \\ \left[\widehat{B}_{jl}(z) - i\widehat{O}_{jl}(z) \displaystyle\sum_{n,m=1}^{N} \vartheta'_{nm}(z)\right] \sin\left[kr - (\kappa_{jl} - 1)\dfrac{\pi}{2}\right] \end{pmatrix} \\
&\quad + \begin{pmatrix} \widehat{C}_{jl}(z) \displaystyle\sum_{n,m=1}^{N} \vartheta'_{nm}(z) \cos\left[kr - \kappa_{jl}\dfrac{\pi}{2}\right] \\ \widehat{O}_{jl}(z) \displaystyle\sum_{n,m=1}^{N} \vartheta'_{nm}(z) \cos\left[kr - (\kappa_{jl} - 1)\dfrac{\pi}{2}\right] \end{pmatrix},
\end{aligned} \tag{44}$$

where

$$\begin{aligned}
\vartheta'_{nm}(z) &= \vartheta_{nm}^{(1)}(z) F_{jl}(z, R_n) F_{jl}(z, R_n) \\
&\quad + \vartheta_{nm}^{(2)}(z) \tilde{F}_{jl}(z, R_n) \tilde{F}_{jl}(z, R_n), \\
\widehat{A}_{jl}(z) &= 2^{-\kappa_{jl}} k^{-\kappa_{jl}-1} \Gamma\left(2\kappa_{jl} + 2\right) \Gamma\left(\kappa_{jl} + 1\right)^{-1}, \\
\widehat{C}_{jl}(z) &= 2^{\kappa_{jl}} k^{\kappa_{jl}} \Gamma\left(2\kappa_{jl} + 2\right)^{-1} \Gamma\left(\kappa_{jl} + 1\right), \\
\widehat{B}_{jl}(z) &= \frac{1}{c} 2^{-\kappa_{jl}} k^{-\kappa_{jl}} \Gamma\left(2\kappa_{jl} + 2\right) \Gamma\left(\kappa_{jl} + 1\right)^{-1}, \\
\widehat{O}_{jl}(z) &= \frac{1}{c} 2^{\kappa_{jl}} k^{\kappa_{jl}+1} \Gamma\left(2\kappa_{jl} + 2\right)^{-1} \Gamma\left(\kappa_{jl} + 1\right).
\end{aligned} \tag{45}$$

A straight computation shows that equation (44) reads

$$\begin{aligned}
&F_{jl,\{\Gamma\},\{R\}}(r) \xrightarrow[r\to\infty]{k>0} \\
&\begin{pmatrix} \left[\tilde{P}_{jl,1}^2(z) + \tilde{P}_{jl,2}^2(z)\right]^{1/2} \sin\left[kr - \kappa_{jl}\dfrac{\pi}{2} + \delta_{jl,\{\Gamma\},\{R\}}\right] + 0(1) \\ \left[\tilde{P}_{jl,3}^2(z) + \tilde{P}_{jl,4}^2(z)\right]^{1/2} \sin\left[kr - (\kappa_{jl} - 1)\dfrac{\pi}{2} + \delta_{jl,\{\Gamma\},\{R\}}\right] \end{pmatrix}.
\end{aligned} \tag{46}$$

Then, the phase shift $\delta_{jl,\{\Gamma\},\{R\}}$ corresponding to $h_{jl,\{\alpha\},\{R\}}$ is defined by

$$\begin{aligned}
\delta_{jl,\{\Gamma\},\{R\}} &= -\arctan \frac{\tilde{P}_{jl,2}^2(z)}{\tilde{P}_{jl,1}^2(z)}, = -\arctan \frac{\tilde{P}_{jl,4}^2(z)}{\tilde{P}_{jl,3}^2(z)}, \\
&= -\arctan \frac{\widehat{C}_{jl}(z) \sum_{n,m=1}^{N} \vartheta'_{nm}(z)}{\widehat{A}_{jl}(z) - i\widehat{C}_{jl}(z) \sum_{n,m=1}^{N} \vartheta'_{nm}(z)}.
\end{aligned} \tag{47}$$

The elements of the on-shell scattering matrix are given by

$$S_{jl,\{\Gamma\},\{R\}} = \exp\left[2i\delta_{jl,\{\Gamma\},\{R\}}(z)\right]. \tag{48}$$

The partial wave scattering amplitude is given by

$$f_{jl,\{\Gamma\},\{R\}} = \frac{\exp\left[2i\delta_{jl,\{\Gamma\},\{R\}}(z)\right] - 1}{2ik}. \tag{49}$$

3. Basic Properties for the $2N$ Parameter Model of Relativistic Many δ-Sphere plus Coulomb Interactions: Separated Boundary Conditions

3.1. Definition of the Hamiltonian. In this section, we give in dimension $n = 3$ the rigorous mathematical definition of relativistic many δ-sphere plus Coulomb interactions. The formal expression of the Hamiltonian describing N finitely δ-sphere plus Coulomb interactions with support on concentric spheres of radii $0 < R_1 < \cdots < R_N$ is given by

$$H_{\gamma,\{\Gamma\}} = H_D + \frac{\gamma}{|x|} + \sum_{n=1}^{N} \Gamma_n \delta(|x| - R_n), \quad x \in \mathbb{R}^3, 0 < R_1 < \cdots < R_N, \tag{50}$$

where the operator H_D is defined by equation (2).

Following decomposition (10), we consider the operator $\bar{H}_{\gamma,\{\Gamma\}}$ that reads

$$\bar{H}_{\gamma,\{\Gamma\}} = \bigoplus_{j=1/2}^{\infty} \bigoplus_{l=j-1/2}^{j+1/2} \left[U_{jl}^{-1} h_{jl,\gamma,\{R\}} U_{jl}\right] \otimes 1\mathrm{l}, \tag{51}$$

where the operator $h_{jl,\gamma,\{R\}}$ is defined in $L^2((0,\infty)) \otimes \mathbb{C}^2$ by

$$\begin{aligned} h_{jl,\gamma,\{R\}} &= \begin{pmatrix} \frac{c^2}{2} + \frac{\gamma}{r} & -c\frac{\mathrm{d}}{\mathrm{d}r} + c\frac{\kappa_{jl}}{r} \\ c\frac{\mathrm{d}}{\mathrm{d}r} + c\frac{\kappa_{jl}}{r} & -\frac{c^2}{2} + \frac{\gamma}{r} \end{pmatrix} \equiv \tau_\gamma, \\ \mathscr{D}\left(h_{jl,\gamma,\{R\}}\right) &= \{\psi \in L^2((0,\infty)) \otimes \mathbb{C}^2 \mid \psi \in AC_{loc}((0,\infty)), \\ &\quad \psi(R_n \pm) = 0, \tau_\gamma \psi \in L^2((0,\infty)) \otimes \mathbb{C}^2\}, \\ &\quad \{R\} = \{R_1, R_2, \cdots, R_N\}. \end{aligned} \tag{52}$$

The adjoint $h^*_{jl,\gamma,\{R\}}$ of $h_{jl,\gamma,\{R\}}$ reads

$$\begin{aligned} h^*_{jl,\gamma,\{R\}} = \tau_\gamma, \mathscr{D}\left(h^*_{jl,\gamma,\{R\}}\right) &= \{f \in L^2((0,\infty)) \otimes \mathbb{C}^2 \mid f \\ &\in AC_{loc}((0,\infty) \setminus \{R\}), \quad \tau_\gamma f \in L^2((0,\infty)) \otimes \mathbb{C}^2\}. \end{aligned} \tag{53}$$

The deficiency index equation

$$\begin{aligned} &\left(h^*_{jl,\gamma,\{R\}} - z\right)\varphi = 0, \\ &f \in \mathbb{C} - \left\{\left(-\infty, -\frac{c^2}{2}\right] \bigcup \left[\frac{c^2}{2}, \infty\right)\right\} \end{aligned} \tag{54}$$

has $2N$ linearly independent solutions

$$\begin{aligned} \varphi^{(1)}_{jl,\gamma} &= \begin{cases} \begin{pmatrix} g_{jl,\gamma,1}(z,R_n) f_{jl,\gamma,1}(z,r) \\ g_{jl,\gamma,2}(z,R_n) f_{jl,\gamma,2}(z,r) \end{pmatrix}, & r < R_n, \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & r > R_n, n = 1, \cdots, N, \end{cases} \\ \varphi^{(2)}_{jl,\gamma} &= \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & r < R_n, \\ \begin{pmatrix} f_{jl,\gamma}(z,R_n) g_{jl,\gamma}(z,r) \\ \tilde{f}_{jl,\gamma}(z,R_n) \tilde{g}_{jl,\gamma}(z,r) \end{pmatrix}, & r > R_n, n = 1, \cdots, N, \end{cases} \end{aligned} \tag{55}$$

where

$$\begin{aligned} f_{jl,\gamma,1}(z,r) &= \left(1 - \frac{\gamma^2}{(\kappa_{jl}c + \zeta)^2}\right)^{-1/2} \left[\cos\left(\arctan\frac{\tilde{\gamma}}{2k\tilde{\zeta}}\right)\right]^{-1/2} \\ &\quad \cdot \left[F_{jl,\gamma}(z,r) - \frac{\gamma}{\kappa_{jl}c + \zeta} \tilde{F}_{jl,\gamma}(z,r)\right], \end{aligned} \tag{56}$$

$$\begin{aligned} f_{jl,\gamma,2}(z,r) &= \left(1 - \frac{\gamma^2}{(\kappa_{jl}c + \zeta)^2}\right)^{-1/2} \left[\cos\left(\arctan\frac{\tilde{\gamma}}{2k\tilde{\zeta}}\right)\right]^{-1/2} \\ &\quad \cdot \left[\tilde{F}_{jl,\gamma}(z,r) - \frac{\gamma}{\kappa_{jl}c + \zeta} F_{jl,\gamma}(z,r)\right], \end{aligned} \tag{57}$$

$$\begin{aligned} g_{jl,\gamma,1}(z,r) &= \left(1 - \frac{\gamma^2}{(\kappa_{jl}c + \zeta)^2}\right)^{-1/2} \left[\cos\left(\arctan\frac{\tilde{\gamma}}{2k\tilde{\zeta}}\right)\right]^{-1/2} \\ &\quad \cdot \left[G_{jl,\gamma}(z,r) - \frac{\gamma}{\kappa_{jl}c + \zeta} \tilde{G}_{jl,\gamma}(z,r)\right], \end{aligned} \tag{58}$$

$$\begin{aligned} g_{jl,\gamma,2}(z,r) &= \left(1 - \frac{\gamma^2}{(\kappa_{jl}c + \zeta)^2}\right)^{-1/2} \left[\cos\left(\arctan\frac{\tilde{\gamma}}{2k\tilde{\zeta}}\right)\right]^{-1/2} \\ &\quad \cdot \left[\tilde{G}_{jl,\gamma}(z,r) - \frac{\gamma}{\kappa_{jl}c + \zeta} G_{jl,\gamma}(z,r)\right], \end{aligned} \tag{59}$$

where

$$F_{jl,\gamma}(z,r)=r^{\tilde{\zeta}+1}e^{-ikr}{}_1F_1\left(\tilde{\zeta}+1-i\frac{\tilde{\gamma}}{2k},2\tilde{\zeta}+2,2ikr\right),$$
$$G_{jl,\gamma}(z,r)=\Gamma\left(2\tilde{\zeta}+2\right)^{-1}\Gamma\left(\tilde{\zeta}+1-\frac{i\tilde{\gamma}}{2k}\right)(2ik)^{2\tilde{\zeta}+1}r^{\tilde{\zeta}+1}e^{-ikr}U\cdot\left(\tilde{\zeta}+1-i\frac{\tilde{\gamma}}{2k},2\tilde{\zeta}+2,2ikr\right),$$
$$\tilde{F}_{jl,\gamma}(z,r)=\frac{\tilde{\zeta}}{c}\left(2\tilde{\zeta}+1\right)\left|\Gamma\left(\tilde{\zeta}+\frac{i\tilde{\gamma}}{2k}\right)\right|\left|\Gamma\left(\tilde{\zeta}+1+\frac{i\tilde{\gamma}}{2k}\right)\right|^{-1}r^{\tilde{\zeta}}e^{-ikr}\times{}_1F_1\left(\tilde{\zeta}-i\frac{\tilde{\gamma}}{2k},2\tilde{\zeta},2ikr\right),$$
$$\tilde{G}_{jl,\gamma}(z,r)=\frac{\tilde{\zeta}^{-1}}{c}\left(2\tilde{\zeta}+1\right)^{-1}\Gamma\left(2\tilde{\zeta}\right)^{-1}\Gamma\left(\tilde{\zeta}+\frac{i\tilde{\gamma}}{2k}\right)\cdot\left|\Gamma\left(\tilde{\zeta}+\frac{i\tilde{\gamma}}{2k}\right)\right|^{-1}\left|\Gamma\left(\tilde{\zeta}+1+\frac{i\tilde{\gamma}}{2k}\right)\right|\times k^2(2ik)^{2\tilde{\zeta}-1}r^{\tilde{\zeta}}e^{-ikr}U\left(\tilde{\zeta}-i\frac{\tilde{\gamma}}{2k},2\tilde{\zeta},2ikr\right). \tag{60}$$

We use the following notations:

$$k=\frac{1}{c}\sqrt{z^2-\frac{c^4}{4}},\quad \zeta=\sqrt{\kappa_{jl}^2c^2-\gamma^2},\quad \tilde{\zeta}=\frac{1}{c}\zeta,\quad \tilde{\gamma}=\frac{2z\gamma}{c^2}. \tag{61}$$

The operator $h_{jl,\gamma,\{R\}}$ has deficiency indices $(2N,2N)$, and therefore, all self-adjoint extensions of $h_{jl,\gamma,\{R\}}$ are given by $4N^2$ parameter family of self-adjoint operators.

Consider the following special $2N$-parameter family of self adjoint extensions of $h_{jl,\gamma,\{R\}}$:

$$h_{jl,\gamma,\{\Gamma\},\{R\}}=\begin{pmatrix}\frac{c^2}{2}+\frac{\gamma}{r} & -c\frac{d}{dr}+c\frac{\kappa_{jl}}{r}\\ -c\frac{d}{dr}+c\frac{\kappa_{jl}}{r} & -\frac{c^2}{2}+\frac{\gamma}{r}\end{pmatrix},$$
$$\mathscr{D}\left(h_{jl,\gamma,\{\Gamma\},\{R\}}\right)=\left\{g\in\mathscr{D}\left(h^*_{jl,\{\Gamma\},\{R\}}\right)\middle|\left(1-\tau_0\frac{\Gamma_{jl,n}}{2c}\right)g(R_n+)-\left(1+\tau_0\frac{\Gamma_{jl,n}}{2c}\right)g(R_n-)=0\right\}, \tag{62}$$

where the 2×2 matrix $\Gamma_{jl,\mathrm{n}}$ is defined by equation (24).

The Hamiltonian $h_{jl,\gamma,\{\Gamma\},\{R\}}$ gives the mathematical definition of the formal expression

$$h_{\gamma,\{\Gamma\},\{R\}}=h_D+\frac{\gamma}{r}+\sum_{n=1}^{N}\Gamma_{jl,n}\delta(r-R_n),\quad R_n>0,\ n=1,2,\cdots,N, \tag{63}$$

where h_D is the radial Dirac operator defined by equation (27).

The rigous mathematical definition of formal expression (50) reads

$$H_{\gamma,\{\Gamma\},\{R\}}=\bigoplus_{j=1/2}^{\infty}\bigoplus_{l=j-1/2}^{j+1/2}\left[U_{jl}^{-1}h_{jl,\gamma,\{\Gamma\},\{R\}}U_{jl}\right]\otimes 1\!\!1. \tag{64}$$

The case $\gamma=0$ and $\Gamma=0$, i.e., $\Gamma_{jl,n}=0$, for all j and l in equation (64), yields the Dirac Hamiltonian H_D defined by equation (2).

The case $\Gamma=0$, for all j and l in equation (64), yields $H_{\gamma,\{R\}}\equiv H_{\gamma,D.}$

The case $A_{jl,n}\neq 0$ and $B_{jl,n}=0$, for all j and l in equation (64), yields the relativistic many δ-sphere plus Coulomb interactions of the first type.

The case $A_{jl,n}=0$ and $B_{jl,n}\neq 0$, for all j and l in equation (64), yields the relativistic many δ-sphere plus Coulomb interactions of the second type.

3.2. Resolvent Equation of $h_{jl,\gamma,\{\Gamma\},\{R\}}$

Theorem 43.1. *The resolvent of $h_{jl,\gamma,\{\Gamma\},\{R\}}$ reads*

$$\left(h_{jl,\gamma,\{\Gamma\},\{R\}}-z\right)^{-1}=\left(h_{\gamma,D}-z\right)^{-1}+\sum_{n,m=1}^{N}\left\{\vartheta^{(1)}_{\gamma,nm}(z)\cdot\left(\overline{\widehat{M}^{(2)}_{jl,\gamma}(.)},.\right)\widehat{M}^{(2)}_{jl,\gamma}(.)+\vartheta^{(2)}_{\gamma,nm}(z)\cdot\left(\overline{\tilde{M}^{(2)}_{jl,\gamma}(.)},.\right)\tilde{M}^{(2)}_{jl,\gamma}(.)+\vartheta^{(3)}_{\gamma,nm}(z)\cdot\left(\overline{\widehat{M}^{(2)}_{jl,\gamma}(.)},.\right)\tilde{M}^{(1)}_{jl,\gamma}(.)+\vartheta^{(4)}_{\gamma,nm}(z)\cdot\left(\overline{\tilde{M}^{(2)}_{jl,\gamma}(.)},.\right)\widehat{M}^{(1)}_{jl,,\gamma}(.)\right\},$$
$$z\in\rho\left(h_{jl,\gamma,\{\Gamma\},\{R\}}\right),\ \mathrm{Im}\,k>0,\quad l\in\left[j-\frac{1}{2},j+\frac{1}{2}\right],\ j\in\left[\frac{1}{2},\infty\right), \tag{65}$$

where ρ is the resolvent set and

$$\left[\vartheta_{\gamma}^{(1)}(z)\right]_{nm}^{-1} = \begin{cases} -\left[\dfrac{1}{A_{jl}} - \dfrac{B_{jl}}{4c^2} + \dfrac{B_{jl}}{A_{jl}} G_{22}^{(jl,\gamma)}(z, R_n, R_n) + G_{11}^{(jl,\gamma)}(z, R_n, R_n)\right], & n = m, \\ -G_{11}^{(jl,\gamma)}(z, R_m, R_n), & n \neq m, \end{cases} \tag{66}$$

$$\left[\vartheta_{\gamma}^{(2)}(z)\right]_{nm}^{-1} = \begin{cases} -\left[\dfrac{1}{B_{jl}} - \dfrac{A_{jl}}{4c^2} + \dfrac{A_{jl}}{B_{jl}} G_{11}^{(jl,\gamma)}(z, R_n, R_n) + G_{22}^{(jl,\gamma)}(z, R_n, R_n)\right], & n = m, \\ -G_{22}^{(jl,\gamma)}(z, R_m, R_n), & n \neq m, \end{cases} \tag{67}$$

$$\left[\vartheta_{\gamma}^{(3)}(z)\right]_{nm}^{-1} = \begin{cases} -\left[\dfrac{2c}{B_{jl}A_{jl}} - \dfrac{1}{2c} + \dfrac{2c}{B_{jl}} G_{11}^{(jl,\gamma)}(z, R_n, R_n) + \dfrac{2c}{A_{jl}} G_{22}^{(jl,\gamma)}(z, R_n, R_n)\right], & n = m, \\ \dfrac{1}{2c}, & n \neq m, \end{cases} \tag{68}$$

$$\left[\vartheta_{\gamma}^{(4)}(z)\right]_{nm}^{-1} = \begin{cases} -\left[-\dfrac{2c}{B_{jl}A_{jl}} + \dfrac{1}{2c} - \dfrac{2c}{B_{jl}} G_{11}^{(jl,\gamma)}(z, R_n, R_n) - \dfrac{2c}{A_{jl}} G_{22}^{(jl,\gamma)}(z, R_n, R_n)\right], & n = m, \\ \dfrac{-1}{2c}, & n \neq m, \end{cases} \tag{69}$$

$$\widehat{M}_{jl,\gamma}^{(m)}(r) = \begin{cases} G^{(jl,\gamma)}(z, r, R_n)\begin{pmatrix} 1 \\ 0 \end{pmatrix}, & r < R_n, \\ (-)^m G^{(jl,\gamma)}(z, r, R_n)\begin{pmatrix} 1 \\ 0 \end{pmatrix}, & r > R_n, m = 1, 2, n = 1, 2, \cdots, N, \end{cases} \tag{70}$$

$$\tilde{M}_{jl,\gamma}^{(m)}(r) = \begin{cases} G^{(jl,\gamma)}(z, r, R_n)\begin{pmatrix} 0 \\ 1 \end{pmatrix}, & r < R_n, \\ (-)^m G^{(jl,\gamma)}(z, r, R_n)\begin{pmatrix} 0 \\ 1 \end{pmatrix}, & r > R_n, m = 1, 2, n = 1, 2, \cdots, N. \end{cases} \tag{71}$$

$(h_{\gamma,D} - z)^{-1}$, Im $k > 0$, *is the radial Dirac resolvent with kernel:*

$$G^{(jl,\gamma)}(z, r, R_n) = \begin{pmatrix} G_{11}^{(jl,\gamma)}(z, r, R_n) & G_{12}^{(jl,\gamma)}(z, r, R_n) \\ G_{21}^{(jl,\gamma)}(z, r, R_n) & G_{22}^{(jl,\gamma)}(z, r, R_n) \end{pmatrix}, \tag{72}$$

where

$$G_{11}^{(jl,\gamma)}(z, r, R_n) = \begin{cases} g_{\gamma,1}(z, R_n) f_{\gamma,1}(z, r), & r < R_n, \\ f_{\gamma,1}(z, R_n) g_{\gamma,1}(z, r), & r > R_n, \end{cases}$$

$$G_{12}^{(jl,\gamma)}(z, r, R_n) = \begin{cases} g_{\gamma,2}(z, R_n) f_{\gamma,1}(z, r), & r < R_n, \\ f_{\gamma,2}(z, R_n) g_{\gamma,1}(z, r), & r > R_n, \end{cases}$$

$$G_{21}^{(jl,\gamma)}(z, r, R_n) = \begin{cases} g_{\gamma,1}(z, R_n) f_{\gamma,2}(z, r), & r < R_n, \\ f_{\gamma,1}(z, R_n) g_{\gamma,2}(z, r), & r > R_n, \end{cases}$$

$$G_{22}^{(jl,\gamma)}(z, r, R_n) = \begin{cases} g_{\gamma,2}(z, R_n) f_{\gamma,2}(z, r), & r < R_n, \\ f_{\gamma,2}(z, R_n) g_{\gamma,2}(z, r), & r > R_n. \end{cases} \tag{73}$$

Proof. Similar to the proof of Theorem 2.

Consider to decomposition (10), equations (51) and (65), the resolvent equation of $H_{\gamma,\{\Gamma\},\{R\}}$ reads

$$\left(H_{\gamma,\{\Gamma\},\{R\}} - z\right)^{-1} = \left(H_{\gamma,D} - z\right)^{-1} + \bigoplus_{j=1/2}^{\infty} \bigoplus_{l=j+1/2}^{j-1/2} \bigoplus_{\mu=-j}^{j} \times \sum_{n,m=1}^{N} \left\{ \vartheta_{\gamma,nm}^{(1)}(z) \left(|.|^{-1} \overline{\widehat{M}_{jl,\gamma}^{(2)}(.)} \otimes \tilde{\Omega}_{jl\mu}, . \right) \right.$$

$$\times|.|^{-1}\widehat{M}^{(2)}_{jl,\gamma}(.)\otimes\tilde{\Omega}_{jl\mu}+\vartheta^{(2)}_{\gamma,nm}(z)\left(|.|^{-1}\overline{\tilde{M}^{(2)}_{jl,\gamma}(.)}\otimes\tilde{\Omega}_{jl\mu},.\right)$$
$$\times|.|^{-1}\tilde{M}^{(2)}_{jl,\gamma}(.)\otimes\tilde{\Omega}_{jl\mu}+\vartheta^{(3)}_{\gamma,nm}(z)\left(|.|^{-1}\overline{\widehat{M}^{(2)}_{jl,\gamma}(.)}\otimes\tilde{\Omega}_{jl\mu},.\right)$$
$$\times|.|^{-1}\tilde{M}^{(1)}_{jl,\gamma}(.)\otimes\tilde{\Omega}_{jl\mu}+\vartheta^{(4)}_{\gamma,nm}(z)\left(|.|^{-1}\overline{\tilde{M}^{(2)}_{jl,\gamma}(.)}\otimes\tilde{\Omega}_{jl\mu},.\right)$$
$$\times|.|^{-1}\widehat{M}^{(1)}_{jl,\gamma}(.)\otimes\tilde{\Omega}_{jl\mu}\Big\},\quad z\in\rho\left(h_{jl,\gamma,\{\Gamma\},\{R\}}\right),$$
$$\mathrm{Im}k>0,\, l\in\left[j-\frac{1}{2},j+\frac{1}{2}\right],\, j\in\left[\frac{1}{2},\infty\right). \tag{74}$$

3.3. The Nonrelativistic Limit. The nonrelativistic limit of $h_{jl,\gamma,\{\Gamma\},\{R\}}$ as $c\longrightarrow\infty$ is given by the following.

Theorem 53.2. *For the spin 1/2 particles, the operator $h_{jl,\gamma,\{\Gamma\},\{R\}}-c^2/2$ converges in norm resolvent sense to the Schrödinger operator $h_{l,\alpha_l,\{R\}}$ times the projector onto $\tilde{\mathscr{H}}_+=L^2((0,\infty))$:*

$$n\cdot\lim_{c\to\infty}\left(h_{jl,\gamma,\{\Gamma\},\{R\}}-\frac{c^2}{2}-z\right)^{-1}=\left(h_{jl,\gamma,\{\alpha\},\{R\}}-z\right)^{-1}\otimes\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad z\in\mathbb{C}\setminus\mathbb{R}, \tag{75}$$

where the operator $h_{jl,\gamma,\{\alpha\},\{R\}}$ is defined by

$$h_{jl,\gamma,\{\alpha\},\{R\}}=-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}+\frac{\gamma}{r},$$
$$\mathscr{D}\left(h_{jl,\{\alpha\},\{R\}}\right)=\Big\{f\in L^2((0,\infty))\,|\,f,f'\in AC_{loc}((0,\infty)\setminus\{R_1,\cdots,R_N\}),\, g(0+)=0,\, f(R_n+)-f(R_n-)=\frac{\beta_{l,n}}{2}\left[f'(R_n+)+f'(R_n-)\right],\, f'(R_n+)-f'(R_n-)=\frac{\alpha_{l,n}}{2}[f(R_n+)+f(R_n-)],$$
$$-f''+l(l+1)r^{-2}f+\gamma r^{-1}f\in L^2((0,\infty))\Big\},$$
$$l\geq 1,\,\alpha_{l,n}\beta_{l,n}-4=0,\,\alpha_{l,n},\beta_{l,n}\in\mathbb{R},\,1\leq n\leq N. \tag{76}$$

Proof. Similar to the proof of Theorem 3.

The Hamiltonian $h_{jl,\gamma,\{\alpha\},\{R\}}$ describes a $2N$ parameter model of finitely many nonrelativistic δ-sphere plus Coulomb interactions in quantum mechanics.

3.4. Scattering Theory for the Pair $(h_{jl,\gamma,\{\alpha\},\{R\}},h_{\gamma,D})$. Let us define for $k>0$ the function

$$F_{jl,\gamma,\{\Gamma\},\{R\}}(r)=\begin{pmatrix}f_{\gamma,1}(z,r)\\f_{\gamma,2}(z,r)\end{pmatrix}+\sum_{n,m=1}\Big\{\vartheta^{(1)}_{\gamma,nm}(z)f_{\gamma,1}(z,R_n)\widehat{M}^{(2)}_{jl,\gamma}(r)+\vartheta^{(2)}_{\gamma,nm}(z)f_{\gamma,2}(z,R_n)\tilde{M}^{(2)}_{jl,\gamma}(r)+\vartheta^{(3)}_{\gamma,nm}(z)f_{\gamma,1}\cdot(z,R_n)\tilde{M}^{(1)}_{jl,\gamma}(r)+\vartheta^{(4)}_{\gamma,nm}(z)f_{\gamma,2}(z,R_n)\widehat{M}^{(1)}_{jl,\gamma}(r)\Big\},$$
$$=\begin{pmatrix}f_{\gamma,1}(z,r)\\f_{\gamma,2}(z,r)\end{pmatrix}+\sum_{n,m=1}\Big\{\vartheta^{(1)}_{\gamma,nm}G^{(jl,\gamma)}(z,r,R_n)\cdot\begin{pmatrix}f_{\gamma,1}(z,R_n)\\0\end{pmatrix}+\vartheta^{(2)}_{\gamma,nm}G^{(jl,\gamma)}(z,r,R_n)\begin{pmatrix}0\\f_{\gamma,1}(z,R_n)\end{pmatrix}-\vartheta^{(3)}_{\gamma,nm}G^{(jl,\gamma)}(z,r,R_n)\begin{pmatrix}0\\f_{\gamma,1}(z,R_n)\end{pmatrix}-\vartheta^{(4)}_{\gamma,nm}G^{(jl,\gamma)}(z,r,R_n)\begin{pmatrix}f_{\gamma,2}(z,R_n)\\0\end{pmatrix}\Big\},$$
$$=\begin{pmatrix}f_{\gamma,1}(z,r)\\f_{\gamma,2}(z,r)\end{pmatrix}+\sum_{n,m=1}\Big\{\vartheta^{(1)}_{\gamma,nm}G^{(jl,\gamma)}(z,r,R_n)\cdot\begin{pmatrix}f_{\gamma,1}(z,R_n)\\0\end{pmatrix}+\vartheta^{(2)}_{\gamma,nm}G^{(jl,\gamma)}(z,R_n)\cdot\begin{pmatrix}0\\f_{\gamma,1}(z,R_n)\end{pmatrix}-\vartheta^{(3)}_{\gamma,nm}G^{(jl,\gamma)}(z,r,R_n)\cdot\left[\begin{pmatrix}0\\f_{\gamma,1}(z,R_n)\end{pmatrix}-\begin{pmatrix}f_{\gamma,2}(z,R_n)\\0\end{pmatrix}\right]\Big\}, \tag{77}$$

where the functions $f_{\gamma,1}(z,r), f_{\gamma,2}(z,r), \tilde{M}^{(2)}_{jl,\gamma}(r), \widehat{M}^{(2)}_{jl,\gamma}(r), \tilde{M}^{(1)}_{jl,\gamma}(r), \widehat{M}^{(1)}_{jl,\gamma}(r)$, and $\vartheta^{(i)}_{\gamma,nm}(z), i=1,2,3,4$, are defined by equations (59), (71), and (69), respectively.

The asymptotic behaviour of the function $F_{jl,\gamma,\{\Gamma\},\{R\}}(r)$ as $r\longrightarrow\infty$ yields [20]:

$$F_{jl,\gamma,\{\Gamma\},\{R\}}(r)\,\frac{k>0}{r\longrightarrow\infty}\longrightarrow\Bigg(d_1(z)\sin\left[kr-\frac{\tilde{\gamma}}{2k}\ln(2kr)-\tilde{\zeta}\frac{\pi}{2}+\delta^0_\zeta(z)\right]-\frac{\gamma}{\kappa_{jl}c+\zeta}d_1(z)\sin\left[kr-\frac{\tilde{\gamma}}{2k}\ln(2kr)-\tilde{\zeta}\frac{\pi}{2}+\delta^0_\zeta(z)\right]\Bigg)$$

$$+\left(d_2(z)\sin\left[kr-\frac{\tilde{\gamma}}{2k}\ln(2kr)-\left(\tilde{\zeta}-1\right)\frac{\pi}{2}+\delta^0_{\zeta-1}(z)\right]\right.$$
$$-\left(\frac{\gamma}{\kappa_{jl}c+\zeta}\right)^{-1}d_2(z)\sin\left[kr-\frac{\tilde{\gamma}}{2k}\ln(2kr)\right.$$
$$\left.\left.-\left(\tilde{\zeta}-1\right)\frac{\pi}{2}+\delta^0_{\zeta-1}(z)\right]\right)+\sum_{n,m=1}\left\{\vartheta^{(1)}_{\gamma,nm}f_{\gamma,1}(z,R_n)f_{\gamma,1}\right.$$
$$\left.kern3pt(z,R_n)+\vartheta^{(2)}_{\gamma,nm}f_{\gamma,2}(z,R_n)f_{\gamma,2}(z,R_n)\right\}$$
$$\times\left(d_3(z)\exp-\mathrm{i}\left[kr-\frac{\tilde{\gamma}}{2k}\ln(2kr)-\tilde{\zeta}\frac{\pi}{2}+\delta^0_{\zeta}(z)\right]\right.$$
$$-\frac{\gamma}{\kappa_{jl}c+\zeta}d_3(z)\exp-\mathrm{i}\left[kr-\frac{\tilde{\gamma}}{2\mathrm{k}}\ln(2kr)\right.$$
$$\left.\left.-\tilde{\zeta}\frac{\pi}{2}+\delta^0_{\zeta}(z)\right]\right)+\left(d_4(z)\sin\left[kr-\frac{\tilde{\gamma}}{2k}\ln(2kr)\right.\right.$$
$$\left.-\left(\tilde{\zeta}-1\right)\frac{\pi}{2}+\delta^0_{\zeta-1}(z)\right]-\left(\frac{\gamma}{\kappa_{jl}c+\zeta}\right)^{-1}d_4(z)\sin\left[kr\right.$$
$$\left.\left.-\frac{\tilde{\gamma}}{2k}\ln(2kr)-\left(\tilde{\zeta}-1\right)\frac{\pi}{2}+\delta^0_{\zeta-1}(z)\right]\right),\qquad(78)$$

where the constants d_i, $i=1,2,3,4$, read

$$d_1(z)=\left(1-\frac{\gamma^2}{\left(\kappa_{jl}c+\zeta\right)^2}\right)^{-1/2}\left[\cos\left(\arctan\frac{\tilde{\gamma}}{2k\tilde{\zeta}}\right)\right]^{-1/2}$$
$$\cdot 2^{-\tilde{\zeta}}k^{-\tilde{\zeta}-1}\Gamma\left(2\tilde{\zeta}+2\right)\times\left|\Gamma\left(\tilde{\zeta}+1+i\frac{\tilde{\gamma}}{2k}\right)\right|^{-1}e^{\pi(\tilde{\gamma}/2k)},$$
$$d_2(z)=-\frac{\gamma}{c\left(\kappa_{jl}c+\zeta\right)}\left(1-\frac{\gamma^2}{\left(\kappa_{jl}c+\zeta\right)^2}\right)^{-1/2}$$
$$\cdot\left[\cos\left(\arctan\frac{\tilde{\gamma}}{2k\tilde{\zeta}}\right)\right]^{-1/2}2^{-\tilde{\zeta}}k^{-\tilde{\zeta}}$$
$$\times\Gamma\left(2\tilde{\zeta}+2\right)\left|\Gamma\left(\tilde{\zeta}+1+i\frac{\tilde{\gamma}}{2k}\right)\right|^{-1}e^{\pi(\tilde{\gamma}/2k)},$$
$$d_3(z)=\left(1-\frac{\gamma^2}{\left(\kappa_{jl}c+\zeta\right)^2}\right)^{-1/2}\left[\cos\left(\arctan\frac{\tilde{\gamma}}{2k\tilde{\zeta}}\right)\right]^{-1/2}$$
$$\cdot 2^{\tilde{\zeta}}k^{\tilde{\zeta}}\Gamma\left(2\tilde{\zeta}+2\right)^{-1}\times\left|\Gamma\left(\tilde{\zeta}+1+i\frac{\tilde{\gamma}}{2k}\right)\right|e^{-\pi(\gamma/2k)},$$
$$d_4(z)=-\frac{\gamma}{c\left(\kappa_{jl}c+\zeta\right)}\left(1-\frac{\gamma^2}{\left(\kappa_{jl}c+\zeta\right)^2}\right)^{-1/2}$$
$$\cdot\left[\cos\left(\arctan\frac{\tilde{\gamma}}{2k\tilde{\zeta}}\right)\right]^{-1/2}2^{\tilde{\zeta}}k^{\tilde{\zeta}+1}$$
$$\times\Gamma\left(2\tilde{\zeta}+2\right)^{-1}\left|\Gamma\left(\tilde{\zeta}+1+i\frac{\tilde{\gamma}}{2k}\right)\right|e^{-\pi(\gamma/2k)}.\qquad(79)$$

Let us introduce the following notations:

$$x_1=k-\frac{\tilde{\gamma}}{k}\ln(2kr)-\tilde{\zeta}\frac{\pi}{2},$$
$$x_2=k--\frac{\tilde{\gamma}}{k}\ln(2kr)-\left(\tilde{\zeta}-1\right)\frac{\pi}{2}.\qquad(80)$$

A straightforward computation shows that

$$F_{jl,\gamma,\{\Gamma\},\{R\}}(r)\xrightarrow[r\longrightarrow\infty]{k>0}$$
$$\longrightarrow\left(d_1(z)\sin\left[x_1+\delta^0_{\zeta}(z)\right]-\frac{\gamma}{\kappa_{jl}c+\zeta}d_1(z)\cos\right.$$
$$\left.\cdot\left[x_2+\delta^0_{\zeta}(z)\right]\right)+\left(d_2(z)\cos\left[x_1+\delta^0_{\zeta-1}(z)\right]\right.$$
$$\left.-\left(\frac{\gamma}{\kappa_{jl}c+\zeta}\right)^{-1}d_2(z)\sin\left[x_2+\delta^0_{\zeta-1}(z)\right]\right)+\sum_{n,m=1}$$
$$\cdot\left\{\vartheta^{(1)}_{\gamma,nm}f_{\gamma,1}(z,R_n)f_{\gamma,1}(z,R_n)+\vartheta^{(2)}_{\gamma,nm}f_{\gamma,2}(z,R_n)f_{\gamma,2}(z,R_n)\right\}$$
$$\times\left(d_3(z)\left[\cos\left(x_1+\delta^0_{\zeta}\right)-i\sin\left(x_1+\delta^0_{\zeta}\right)\right]\right.$$
$$\left.-i\frac{\gamma}{c\kappa_{jl}+\zeta}d_3(z)\left[\cos\left(x_2+\delta^0_{\zeta}\right)-i\sin\left(x_2+\delta^0_{\zeta}\right)\right]\right)$$
$$+\left(-id_4(z)\left[\cos\left(x_1+\delta^0_{\zeta-1}\right)-i\sin\left(x_1+\delta^0_{\zeta-1}\right)\right]\right.$$
$$\left.-\left(\frac{\gamma}{c\kappa_{jl}+\zeta}\right)^{-1}d_4(z)\left[\cos\left(x_2+\delta^0_{\zeta-1}\right)-i\sin\left(x_2+\delta^0_{\zeta-1}\right)\right]\right),\qquad(81)$$

where

$$\delta^0_{\zeta}(z)=\delta^0_{\zeta-1}(z)+\arctan\left(\frac{\tilde{\gamma}}{2k\tilde{\zeta}}\right),$$
$$\delta^0_{\zeta}(z)=\arg\Gamma\left(\tilde{\zeta}+1+i\frac{\tilde{\gamma}}{2k}\right).\qquad(82)$$

Equation (81) reads

$$F_{jl,\gamma,\{\Gamma\},\{R\}}(r)\xrightarrow[r\longrightarrow\infty]{k>0}$$
$$\longrightarrow\begin{pmatrix}V_1(z)\sin\left[x_1+\delta_{\tilde{\zeta}}\right]+V_2(z)\cos\left[x_1+\delta_{\tilde{\zeta}}\right]\\ V_3(z)\sin\left[x_2+\delta_{\tilde{\zeta}-1}\right]+V_4(z)\cos\left[x_2+\delta_{\tilde{\zeta}-1}\right]\end{pmatrix},$$
$$=\begin{pmatrix}\left[V_1^2(z)+V_2^2(z)\right]^{1/2}\sin\left[x_1+\delta^0_{\tilde{\zeta}}+\delta^C_{\gamma,\{\Gamma\},1}\right]\\ \left[V_3^2(z)+V_4^2(z)\right]^{1/2}\sin\left[x_2+\delta^0_{\tilde{\zeta}-1}+\delta^C_{\gamma,\{\Gamma\},2}\right]\end{pmatrix},\qquad(83)$$

where the Coulomb-modified phase shift $\delta^C_{\gamma,\{\Gamma\}}$ is given by

$$\begin{pmatrix} \delta^C_{\gamma,\{\Gamma\},1} \\ \delta^C_{\gamma,\{\Gamma\},2} \end{pmatrix} = \begin{pmatrix} -\arctan \dfrac{V_2(z)}{V_1(z)} \\ -\arctan \dfrac{V_4(z)}{V_3(z)} \end{pmatrix}, \quad (84)$$

where

$$\begin{aligned} V_1(z) &= d_1(z) + \left[d_2(z) - i\sum_{n,m}\vartheta'_{\gamma,nm}(z)d_4(z)\right]\sin \\ &\quad\cdot\left[\arctan\left(\frac{\tilde{\gamma}}{2k\tilde{\zeta}}\right)\right] - \sum_{n,m}\vartheta'_{\gamma,nm}(z)(id_3(z) \\ &\quad + d_4(z))\cos\left[\arctan\left(\frac{\tilde{\gamma}}{2k\tilde{\zeta}}\right)\right], \\ V_2(z) &= \left[d_2(z) - i\sum_{n,m}\vartheta'_{\gamma,nm}(z)d_4(z)\right]\cos\left[\arctan\left(\frac{\tilde{\gamma}}{2k\tilde{\zeta}}\right)\right] \\ &\quad + \sum_{n,m}\vartheta'_{\gamma,nm}(z)(d_3(z)+d_4(z))\sin\left[\arctan\left(\frac{\tilde{\gamma}}{2k\tilde{\zeta}}\right)\right], \\ V_3(z) &= \frac{\gamma}{\kappa_{jl}c+\zeta}\left[-d_1(z) + i\sum_{n,m}\vartheta'_{\gamma,nm}(z)d_3(z)\right]\sin \\ &\quad\cdot\left[\arctan\left(\frac{\tilde{\gamma}}{2k\tilde{\zeta}}\right)\right] - \left(\frac{\gamma}{\kappa_{jl}c+\zeta}\right)^{-1} d_2(z) \\ &\quad + \sum_{n,m}\vartheta'_{\gamma,nm}(z)\left[-\frac{\gamma}{\kappa_{jl}c+\zeta}d_3(z)\cos\left[\arctan\left(\frac{\tilde{\gamma}}{2k\tilde{\zeta}}\right)\right]\right. \\ &\quad \left. + i\left(\frac{\gamma}{\kappa_{jl}c+\zeta}\right)^{-1} d_4(z)\right], \\ V_4(z) &= \frac{\gamma}{\kappa_{jl}c+\zeta}\left[-d_1(z) + i\sum_{n,m}\vartheta'_{\gamma,nm}(z)d_3(z)\right]\cos \\ &\quad\cdot\left[\arctan\left(\frac{\tilde{\gamma}}{2k\tilde{\zeta}}\right)\right] + \sum_{n,m}\vartheta'_{\gamma,nm}(z) \\ &\quad\cdot\left[-\frac{\gamma}{\kappa_{jl}c+\zeta}d_3(z)\sin\left[\arctan\left(\frac{\tilde{\gamma}}{2k\tilde{\zeta}}\right)\right]\right. \\ &\quad \left. - \left(\frac{\gamma}{\kappa_{jl}c+\zeta}\right)^{-1} d_4(z)\right], \\ \vartheta'_{\gamma,nm}(z) &= \vartheta^{(1)}_{\gamma,nm} f_{\gamma,1}(z,R_n) f_{\gamma,1}(z,R_n) \\ &\quad + \vartheta^{(2)}_{\gamma,nm} f_{\gamma,2}(z,R_n) f_{\gamma,2}(z,R_n). \end{aligned} \quad (85)$$

The Coulomb-modified on-shell scattering matrix is given by

$$S_{\gamma,\{\Gamma\}} = \exp\left[2i\delta^C_{\gamma,\{\Gamma\},n}(z)\right], \quad n = 1, 2. \quad (86)$$

The partial wave scattering amplitude is given by

$$f_{\gamma,\{\Gamma\}} = \frac{\exp\left[2i\delta^C_{\gamma,\{\Gamma\},n}(z)\right] - 1}{2ik}, \quad n = 1, 2. \quad (87)$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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