Hindawi
Advances in Mathematical Physics
Volume 2020, Article ID 6738247, 9 pages
https://doi.org/10.1155/2020/6738247

Research Article

Response Solutions for a Singularly Perturbed System Involving Reflection of the Argument

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Received 17 November 2019; Revised 3 February 2020; Accepted 13 February 2020; Published 21 March 2020

Academic Editor: Marin Marin

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In this paper, the existence and uniqueness of response solutions, which has the same frequency ω with the nonlinear terms, are investigated for a quasiperiodic singularly perturbed system involving reflection of the argument. Firstly, we prove that all quasiperiodic functions with the frequency ω form a Banach space. Then, we obtain the existence and uniqueness of quasiperiodic solutions by means of the fixed-point methods and the B-property of quasiperiodic functions.

1. Introduction

The following singularly perturbed system

$$\begin{cases} x'(t) = F(t, x(t), y(t), \varepsilon), \\ \varepsilon y'(t) = G(t, x(t), y(t), \varepsilon), \end{cases} \tag{1}$$

occurs in many areas, including biochemical kinetics, genetics, plasma physics, and mechanical and electrical systems involving large damping or resistance [1–4], where x and y are vectors with multiple components and $\varepsilon \geq 0$ is a small parameter. The existence of periodic solutions and almost periodic solutions of (1) had been one of the most attracting topics in the qualitative theory of ordinary differential equations. The early contributions on these topics are due to Anosov [5] and Flatto and Levinson [6]. They investigated system (1) in the case that the degenerate system

$$\begin{cases} x'(t) = F(t, x(t), y(t), 0), \\ 0 = G(t, x(t), y(t), 0), \end{cases} \tag{2}$$

has a periodic solution $\theta(t), \chi(t)$. The authors showed sufficient conditions on F, G which assure that the existence of periodic solutions of (1) and these solutions converge to $\theta(t), \chi(t)$ as $\varepsilon \longrightarrow 0$ uniformly. In 1961, Hale and Seifert [7] generalized the results of Flatto and Levinson to the almost periodic case and gave sufficient conditions for the existence of the almost periodic solutions of (1) using the similar method with [6]. Chang [8] obtained the same result of [7] under generalized hypothesis. But, the above papers [5–8] do not consider the stability properties of the solutions.

Smith [9] considered the existence of almost periodic or periodic solutions for system (1). By the construction of manifolds of initial data, the author investigated the stability properties of these solutions, which approach the given solutions as $t \longrightarrow \infty$ at an exponential rate, α, independent of ε. He also gave the application in a reaction diffusion system with a traveling wave input.

It is natural to ask whether there is a bounded solution of system (1) for sufficiently small ε and how the stability properties of the solutions for the quasiperiodic case are.

For the Silberstein equation

$$x'(t) = x\left(\frac{1}{t}\right), \tag{3}$$

we define $y(t) = x(e^t)$, then Equation (3) is equivalent to $y'(t) = e^{-t}y(-t)$, which is known as the equation involving reflection of argument. This kind of equations has applications in the study of stability of differential-difference equations, see Sharkovskii [10]. One of the earliest contributions

to this kind of equations are due to Wiener and Aftabizadeh [11]. They investigated the boundary value problems for the second-order nonlinear differential equation

$$\begin{cases} y''(t) = f(t, y(t), y(-t)), \\ y(-a) = y_0, y(a) = y_1 \end{cases} \tag{4}$$

by Schauder fixed-point theorem, where $f \in C[[-a, a] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}]$. They also considered the boundary value problems for the following equation

$$\begin{cases} y''(t) = f(t, y(t), y(-t)), \\ y'(-a) - hy(-a) = 0, \quad y'(a) + ky(a) = 0, \end{cases} \tag{5}$$

by changing the equation to a higher order one without reflection of the argument, where $h, k \geq 0, h + k > 0$. Gupta [12, 13] studied more general boundary value problems than Equations (4) and (5) using degree theory arguments. He proved the existence of solutions for the boundary value problems in a simple and straightforward manner. The existence and uniqueness of periodic, almost periodic, pseudo almost periodic, Besicovitch almost periodic, and pseudo almost automorphic solutions of this kind of equations were investigated in [14–19]. Cabada et al. [20–22] studied the first-order equation with two-point boundary conditions and the nth-order differential equations involving reflection, constant coefficients, and initial conditions, adding a new element to the previous studies: the existence of Green's function.

However, as far as we know, the quasiperiodic solutions for the equations involving reflection of the argument have not been considered yet. Our present paper is devoted to discuss the existence and uniqueness of response solutions for the following singularly perturbed system

$$\begin{cases} x'(t) = F(t, x(t), x(-t), y(t), y(-t), \varepsilon), \\ \varepsilon y'(t) = G(t, x(t), x(-t), y(t), y(-t), \varepsilon), \end{cases} \tag{6}$$

where $\varepsilon \geq 0$ is a small real parameter, and the functions F, G are quasiperiodic in t uniformly on $\mathbb{R}^2 \times \mathbb{R}^2$ with frequency $\omega = (\omega_1, \omega_2, \cdots, \omega_d)$. A quasiperiodic solution of (6) with the frequency ω is called response solution.

It is assumed that the degenerate system

$$\begin{cases} x'(t) = F(t, x(t), x(-t), y(t), y(-t), 0), \\ 0 = G(t, x(t), x(-t), y(t), y(-t), 0), \end{cases} \tag{7}$$

has a quasiperiodic "outer" solution which we take to be the trivial solution, that is, we suppose

$$F(t, 0, 0, 0, 0, 0) \equiv G(t, 0, 0, 0, 0, 0) \equiv 0 \tag{8}$$

so that $(x, y) = (0, 0)$ satisfies (7). Expanding (6) about the trivial solution gives

$$\begin{cases} x'(t) = a_1(t, \varepsilon)x(t) + a_2(t, \varepsilon)x(-t) + b_1(t, \varepsilon)y(t) + b_2(t, \varepsilon)y(-t) + f(t, x(t), x(-t), y(t), y(-t), \varepsilon), \\ \varepsilon y'(t) = c_1(t, \varepsilon)x(t) + c_2(t, \varepsilon)x(-t) + d_1(t, \varepsilon)y(t) + d_2(t, \varepsilon)y(-t) + g(t, x(t), x(-t), y(t), y(-t), \varepsilon). \end{cases} \tag{9}$$

One can think of, e.g., $a_1(t, \varepsilon) = \partial F / \partial x(t, 0, 0, 0, 0, \varepsilon)$. In the following discussion, we mainly consider (9).

This paper is organized as follows: in Section 2, we present the Bohr's notion of B-property for quasiperiodic functions and then prove that all ω-frequency continuous quasiperiodic functions form a Banach space under the supremum norm. We prove an existence and uniqueness result for a linear scalar equation with reflection of the argument. In Section 3, the main results on the local existence and uniqueness of response solutions will be stated and proved by means of fixed-point methods in the spirit of Sacker and Sell [23]. We give conclusions of this paper in Section 4.

2. Preliminary

Firstly, we will give some lemmas which are important in proving our main results.

Definition 1. Assume that $\omega_1, \omega_2, \cdots, \omega_d \in \mathbb{R}$ are rationally independent. A continuous function $u(t)$ on $\mathbb{R}$ is said to be quasiperiodic with frequencies $(\omega_1, \omega_2, \cdots, \omega_d)$, if there exists a periodic function $U = U(\theta_1, \theta_2, \cdots, \theta_d)$ (called the lift of u) in $\theta_1, \theta_2, \cdots, \theta_d$ with the same period 2π, such that

$$u(t) = U(\omega_1 t, \omega_2 t, \cdots, \omega_d t), \ \forall t \in \mathbb{R}. \tag{10}$$

Remark 2. This definition for quasiperiodic function can be found in many references, for example [24]. It is not difficult to prove that this definition is equivalent to the definition of quasiperiodic function in [25].

Let $\mathcal{QP}_\omega$ be the set of all quasiperiodic functions with frequency $\omega = (\omega_1, \omega_2, \cdots, \omega_d)$.

Definition 3. (see [4]). A function $H(t)$: $\mathbb{R} \longrightarrow \mathbb{R}$ is said to have a B-property on a set of real numbers $\omega_1, \omega_2, \cdots, \omega_d$, if

(i) H is continuous on $\mathbb{R}$

(ii) for every $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that if a real number τ satisfies the d Diophantine inequalities

$$|\omega_k \tau| \leq \delta (\mathrm{mod}\ 2\pi)\, k = 1, \cdots, d, \tag{11}$$

then, τ is an ε-translation number of H, i.e.,

$$\sup_{t\in\mathbb{R}}|H(t+\tau)-H(t)|\le\varepsilon. \tag{12}$$

Lemma 4. (see [3]). *Suppose that $H(t)$ is a quasiperiodic function with frequencies $(\omega_1,\omega_2,\cdots,\omega_d)$. Then, $H(t)$ has the B-property on $\{\omega_1,\omega_2,\cdots,\omega_d\}$, Conversely, if $H(t)$ has the B-property on a finite rationally independent set $\{\omega_1,\omega_2,\cdots,\omega_d\}$, then $H(t)$ is a quasiperiodic function with frequencies contained in $\{\omega_1,\omega_2,\cdots,\omega_d\}$.*

Proof. The proof of the lemma can be found in [26].

Lemma 5. *$(QP_\omega,\|.\|)$ is a Banach space with the norm $\|f\|=\sup_{t\in\mathbb{R}}|f(t)|$.*

Proof. Suppose that $f_n(t)\in\mathcal{QP}_\omega$, $(n=1,2,\cdots)$ is a Cauchy sequence. By the fact that $\mathcal{QP}_\omega$ is a subspace of $\mathcal{C}_B(\mathbb{R})$, which is a Banach space of bounded continuous function on $\mathbb{R}$ with norm $\|f\|=\sup_{t\in\mathbb{R}}|f(t)|$, there is a $f_0(t)\in\mathcal{C}_B(\mathbb{R})$ such that $\|f_n-f_0\|\longrightarrow 0$ (as $n\longrightarrow\infty$). So for any $\varepsilon>0$ and all $t\in\mathbb{R}$, there exists a $K\in\mathbb{N}$, such that $|f_K(t)-f_0(t)|\le\varepsilon/3$.

Since $f_K(t)\in\mathcal{QP}_\omega$, $f_K(t)$ has the B-property on $\{\omega_1,\omega_2,\cdots,\omega_d\}$ by Lemma 4. So for the ε, there is a $\delta>0$, such that if a real number τ satisfies the inequalities $|\omega_k\tau|\le\delta\ (\mathrm{mod}\ 2\pi)(k=1,\cdots,d)$, then we have $|f_K(t+\tau)-f_K(t)|\le\varepsilon/3$ for all $t\in\mathbb{R}$. Furthermore, we have

$$\begin{aligned}|f_0(t+\tau)-f_0(t)|&\le|f_K(t+\tau)-f_0(t+\tau)|\\&\quad+|f_K(t+\tau)-f_K(t)|\\&\quad+|f_K(t)-f_0(t)|\le\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon,\end{aligned} \tag{13}$$

thus, $f_0(t)$ has the B-property on $\{\omega_1,\omega_2,\cdots,\omega_d\}$. Therefore, $f_0(t)$ is quasiperiodic, i.e., $f_0(t)\in\mathcal{QP}_\omega$.

Corollary 6. *$\mathcal{QP}_\omega^2:=\mathcal{QP}_\omega\times\mathcal{QP}_\omega$ is a Banach space with the norm $\|(f,g)\|:=\|f\|+\|g\|$.*

Lemma 7. *If $H(t)\in\mathcal{QP}_\omega$, then $H(-t)\in\mathcal{QP}_\omega$.*

Proof. Since $H(t)\in\mathcal{QP}_\omega$, $H(t)$ has the B-property on the set $\{\omega_1,\omega_2,\cdots,\omega_d\}$ by Lemma 4. Then for every $\varepsilon>0$, there exists a τ, which satisfies (11), is an ε-translation number of H. For these τ, we have

$$\begin{aligned}\sup_{t\in\mathbb{R}}|H(-(t+\tau))-H(-t)|&\le\sup_{s\in\mathbb{R}}|H(s)-H(s+\tau)|\\&=\sup_{s\in\mathbb{R}}|H(s+\tau)-H(s)|\le\varepsilon.\end{aligned} \tag{14}$$

So $H(-t)$ has the B-property on the set $\{\omega_1,\omega_2,\cdots,\omega_d\}$. Therefore, $H(-t)\in\mathcal{QP}_\omega$ by Lemma 4.

Lemma 8. *There exist $\varepsilon_0>0$ and $L_1>0$ such that for each $h(t)\in\mathcal{QP}_\omega$, $\lambda^2=(\alpha^2-\beta^2)/\varepsilon^2$, $\lambda>0$, the equation*

$$\varepsilon z'(t)=\alpha z(t)+\beta z(-t)+h(t), \tag{15}$$

where $\beta\ne0$, has a unique solution $z(h,\varepsilon)(t)\in\mathcal{QP}_\omega$ for $0<\varepsilon<\varepsilon_0$. Moreover, the operator $\mathbf{L}_\varepsilon^1:h\longrightarrow z(h,\varepsilon)$ is linear and satisfies $\|\mathbf{L}_\varepsilon^1\|\le L_1$. Furthermore, the map $\varepsilon\longrightarrow\mathbf{L}_\varepsilon^1$ is continuous for $0<\varepsilon\le\varepsilon_0$.

Proof. Existence. Similar to the proof of Lemma 2 in [14], we can verify that

$$\begin{aligned}z(h,\varepsilon)(t)=&-\frac{1}{2\lambda}\left[e^{\lambda t}\int_t^\infty e^{-\lambda s}\left(\left(\lambda+\frac{\alpha}{\varepsilon}\right)\frac{h(s)}{\varepsilon}-\frac{\beta h(-s)}{\varepsilon^2}\right)ds\right]\\&+\frac{1}{2\lambda}\left[e^{-\lambda t}\int_{-\infty}^t e^{\lambda s}\left(\left(\lambda-\frac{\alpha}{\varepsilon}\right)\frac{h(s)}{\varepsilon}+\frac{\beta h(-s)}{\varepsilon^2}\right)ds\right],\end{aligned} \tag{16}$$

is a particular solution of Equation (15) for any $h(t)\in\mathcal{QP}_\omega$. Now, we show $z(h,\varepsilon)(t)\in\mathcal{QP}_\omega$.

Since $h(t)\in\mathcal{QP}_\omega$, $h(t),h(-t)$ has the B-property on the set $\{\omega_1,\omega_2,\cdots,\omega_d\}$ by Lemma 4 and Lemma 7. Then for every $\varepsilon>0$, if τ satisfies inequality (11), it will be an ε-translation number of $h(t)$ and $h(-t)$. For this τ, we have

$$\begin{aligned}&|z(h,\varepsilon)(t+\tau)-z(h,\varepsilon)(t)|\\&=\left|-\frac{1}{2\lambda}\left[e^{\lambda t}\int_t^\infty e^{-\lambda s}\left(\left(\lambda+\frac{\alpha}{\varepsilon}\right)\frac{(h(s+\tau)-h(s))}{\varepsilon}\right.\right.\right.\\&\quad\left.\left.\left.-\frac{\beta(h(-s-\tau)-h(-s))}{\varepsilon^2}\right)ds\right]\right.\\&\quad+\frac{1}{2\lambda}\left[e^{-\lambda t}\int_{-\infty}^t e^{\lambda s}\left(\left(\lambda-\frac{\alpha}{\varepsilon}\right)\frac{(h(s+\tau)-h(s))}{\varepsilon}\right.\right.\\&\quad\left.\left.\left.+\frac{\beta(h(-s-\tau)-h(-s))}{\varepsilon^2}\right)ds\right]\right|\\&\le\frac{1}{2\lambda}\left[e^{\lambda t}\int_t^\infty e^{-\lambda s}\left(\left|\lambda+\frac{\alpha}{\varepsilon}\right|\frac{\varepsilon}{\varepsilon}+\left|\frac{\beta\varepsilon}{\varepsilon^2}\right|\right)ds\right]\\&\quad+\frac{1}{2\lambda}\left[e^{-\lambda t}\int_{-\infty}^t e^{\lambda s}\left(\left|\lambda-\frac{\alpha}{\varepsilon}\right|\frac{\varepsilon}{\varepsilon}+\left|\frac{\beta\varepsilon}{\varepsilon^2}\right|\right)ds\right]\\&\le\frac{(|\varepsilon\lambda-\alpha|+|\varepsilon\lambda+\alpha|+2|\beta|)\varepsilon}{2(\alpha^2-\beta^2)}.\end{aligned} \tag{17}$$

So $z(h,\varepsilon)(t)$ has the B-property on the set $\{\omega_1,\omega_2,\cdots,\omega_d\}$ for $0<\varepsilon<\varepsilon_0$. Hence, $z(h,\varepsilon)(t)\in\mathcal{QP}_\omega$.

Uniqueness. If there was another quasiperiodic solution $\tilde{z}(h,\varepsilon)(t)$ for Equation (15), then the difference $u(t)=z(h,\varepsilon)(t)-\tilde{z}(h,\varepsilon)(t)$ should be a solution of the homogeneous equation

$$\varepsilon u'(t)=\alpha u(t)+\beta u(-t). \tag{18}$$

According to the Lemma 2 of [14], we see that $u(t)$ is of the form

$$u(t) = C\left(\frac{\varepsilon\lambda - \alpha}{\beta}e^{-\lambda t} + e^{\lambda t}\right), t \in \mathbb{R}, \tag{19}$$

for some constant C. If $C \neq 0$, then $u(t)$ will be unbounded. This is a contradiction to the boundedness of quasiperiodic function.

So, the operator $\mathbf{L}_\varepsilon^1 : h \longrightarrow z(h, \varepsilon)$ is well defined. From (16), we see the operator $\mathbf{L}_\varepsilon^1$ is linear. On the other hand,

$$|z(h, \varepsilon)(t)| \leq \frac{M_1(|\varepsilon_0\lambda - \alpha| + |\varepsilon_0\lambda + \alpha| + 2|\beta|)}{2(\alpha^2 - \beta^2)}, \tag{20}$$

where $M_1 = \|h\|$. So $\mathbf{L}_\varepsilon^1$ satisfies $\|\mathbf{L}_\varepsilon^1\| \leq L_1$ with $L_1 = (|\varepsilon_0\lambda - \alpha| + |\varepsilon_0\lambda + \alpha| + 2|\beta|)/2(\alpha^2 - \beta^2)$.

To prove the continuity of $\mathbf{L}_\varepsilon^1$ in ε, we write $v(t) = z(h, \varepsilon_1)(t) - z(h, \varepsilon_2)(t)$ for any $0 < \varepsilon_1, \varepsilon_2 < \varepsilon_0$, then $v(t)$ satisfies

$$\begin{aligned}\varepsilon_1 v'(t) = {} & \alpha v(t) + \beta v(-t) + \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2} \\ & \cdot [\alpha z(h, \varepsilon_2)(t) + \beta z(h, \varepsilon_2)(-t) + h(t)].\end{aligned} \tag{21}$$

It follows that

$$\begin{aligned}\|v(t)\| &= \|z(h, \varepsilon_1)(t) - z(h, \varepsilon_2)(t)\| \\ &\leq \frac{|\varepsilon_2 - \varepsilon_1|}{\varepsilon_2}\frac{(|\varepsilon_0\lambda - \alpha| + |\varepsilon_0\lambda + \alpha| + 2|\beta|)}{2(\alpha^2 - \beta^2)} \\ &\quad \cdot [|\alpha|\|z(h, \varepsilon_2)(t)\| + |\beta|\|z(h, \varepsilon_2)(t)\| + \|h(t)\|] \\ &\leq \frac{|\varepsilon_2 - \varepsilon_1|}{\varepsilon_2}\frac{(|\varepsilon_0\lambda - \alpha| + |\varepsilon_0\lambda + \alpha| + 2|\beta|)}{2(\alpha^2 - \beta^2)} \\ &\quad \cdot (|\alpha|L_1 + |\beta|L_1 + 1)M_1\end{aligned} \tag{22}$$

This implies that the map $\varepsilon \longrightarrow \mathbf{L}_\varepsilon^1$ is continuous for $0 < \varepsilon \leq \varepsilon_0$.

Similar to the proof of Lemma 8, one can prove the following Lemma.

Lemma 9. *There exists $L_2 > 0$ such that for each $r(t) \in \mathcal{QP}_\omega$, $\gamma^2 = \alpha^2 - \beta^2$, $\gamma > 0$, the equation*

$$w'(t) = \alpha w(t) + \beta w(-t) + r(t), \tag{23}$$

has a unique solution $w(r)(t) \in \mathcal{QP}_\omega$ for $\beta \neq 0$. The map $r \longrightarrow w(r)$ defines a bounded linear operator satisfying $\|w\| \leq L_2\|r\|$.

For the sake of convenience, we state the following conditions.

(H_1) $a_i(t, \varepsilon), b_i(t, \varepsilon), c_i(t, \varepsilon), d_i(t, \varepsilon) \in \mathcal{QP}_\omega$, $i = 1, 2$ are continuous in ε, uniformly in $t \in \mathbb{R}$. Let M_2 denote a common bound for these functions on $(t, \varepsilon) \in \mathbb{R} \times [0, \varepsilon_0]$.

(H_2) $a_i(t, 0) = a_i^0, b_i(t, 0) = b_i^0, c_i(t, 0) = 0, d_i(t, 0) = d_i^0$, $i = 1, 2$ are constants and $d_2^0 \neq 0$, $a_2^0 \neq 0$. Moreover, $(a_1^0)^2 - (a_2^0)^2 > 0$, $(d_1^0)^2 - (d_2^0)^2 > 0$.

(H_3) The functions f, g are quasiperiodic in t uniformly on (x_1, x_2, y_1, y_2) such that $t \in \mathbb{R}$, $|x_i|, |y_i| \leq \sigma_0 (i = 1, 2)$, $0 \leq \varepsilon \leq \varepsilon_0$, $0 \leq \sigma \leq \sigma_0$. Moreover, there are two nondecreasing functions $\Phi(\varepsilon), \Psi(\varepsilon, \sigma)$, which satisfy

$$\begin{aligned}&\lim_{\varepsilon \to 0}\Phi(\varepsilon) = 0, \\ &\lim_{(\varepsilon,\sigma)\to(0,0)}\Psi(\varepsilon, \sigma) = 0\end{aligned} \tag{24}$$

such that

$$\begin{aligned}&|f(t, 0, 0, 0, 0, \varepsilon)| \leq \Phi(\varepsilon), |g(t, 0, 0, 0, 0, \varepsilon)| \leq \Phi(\varepsilon), t \in \mathbb{R}, \\ &0 \leq \varepsilon \leq \varepsilon_0, |f(t, x_1, x_2, y_1, y_2, \varepsilon) - f(t, \widehat{x}_1, \widehat{x}_2, \widehat{y}_1, \widehat{y}_2, \varepsilon)| \leq \Psi(\varepsilon, \sigma)\sum_{i=1}^{2}[|x_i - \widehat{x}_i| + |y_i - \widehat{y}_i|], |g(t, x_1, x_2, y_1, y_2, \varepsilon) \\ &- g(t, \widehat{x}_1, \widehat{x}_2, \widehat{y}_1, \widehat{y}_2, \varepsilon)| \leq \Psi(\varepsilon, \sigma)\sum_{i=1}^{2}[|x_i - \widehat{x}_i| + |y_i - \widehat{y}_i|]\end{aligned} \tag{25}$$

hold for all $t \in \mathbb{R}$, i f $|x_i|, |\widehat{x}_i|, |y_i|, |\widehat{y}_i| \leq \sigma$, $0 \leq \varepsilon \leq \varepsilon_0$, $0 \leq \sigma \leq \sigma_0$.

3. Main Results

First, we consider the following linear system:

$$\begin{cases} x'(t) = a_1(t, \varepsilon)x(t) + a_2(t, \varepsilon)x(-t) + b_1(t, \varepsilon)y(t) + b_2(t, \varepsilon)y(-t) + \tilde{f}(t), \\ \varepsilon y'(t) = c_1(t, \varepsilon)x(t) + c_2(t, \varepsilon)x(-t) + d_1(t, \varepsilon)y(t) + d_2(t, \varepsilon)y(-t) + \tilde{g}(t), \end{cases} \tag{26}$$

where $\tilde{f}, \tilde{g} \in \mathcal{QP}_\omega$.

Theorem 10. *If (H_1)–(H_2) hold. Then there exist ε_1, $0 < \varepsilon_1 \le \varepsilon_0$, positive functions $A_{i,j}(\varepsilon)$, $1 \le i, j \le 2$ defined for $0 < \varepsilon \le \varepsilon_1$, satisfying*

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} A_{1,1}(\varepsilon) = L_2,\\
&\lim_{\varepsilon\to 0^+} A_{1,2}(\varepsilon) = 2L_1L_2M_2,\\
&\lim_{\varepsilon\to 0^+} A_{2,1}(\varepsilon) = 0,\\
&\lim_{\varepsilon\to 0^+} A_{2,2}(\varepsilon) = L_1,\\
&A_{i,j}(\varepsilon) \le 2L_1L_2M_2 + L_1 + L_2, \quad 1 \le i, j \le 2,
\end{aligned} \tag{27}$$

such that for each $(\tilde f, \tilde g) \in \mathcal{QP}^2_\omega$, $0 < \varepsilon \le \varepsilon_1$, system (26) has a unique solution $(x(\tilde f, \tilde g, \varepsilon)(t), y(\tilde f, \tilde g, \varepsilon)(t)) \in \mathcal{QP}^2_\omega$ which satisfies

$$\begin{aligned}
\|x\| &\le A_{1,1}(\varepsilon)\left\|\tilde f\right\| + A_{1,2}(\varepsilon)\|\tilde g\|,\\
\|y\| &\le A_{2,1}(\varepsilon)\left\|\tilde f\right\| + A_{2,2}(\varepsilon)\|\tilde g\|.
\end{aligned} \tag{28}$$

The map $(\tilde f, \tilde g) \longrightarrow (x(\tilde f, \tilde g, \varepsilon), y(\tilde f, \tilde g, \varepsilon))$ defines a bounded linear operator $\mathbf{K}(\varepsilon)$ satisfying $\|\mathbf{K}(\varepsilon)\| \le 2L_1L_2M_2 + L_1 + L_2$ and $\varepsilon \longrightarrow \mathbf{K}(\varepsilon)$ is continuous for $0 < \varepsilon \le \varepsilon_1$.

Proof. Given $(\tilde f, \tilde g) \in \mathcal{QP}^2_\omega$, $(x_0(t), y_0(t)) \in \mathcal{QP}^2_\omega$. Define $(x(t), y(t))$ as the solution of the system

$$\begin{cases}
x'(t) = a_1^0 x(t) + a_2^0 x(-t) + \left[a_1(t,\varepsilon) - a_1^0\right]x_0(t) + \left[a_2(t,\varepsilon) - a_2^0\right]x_0(-t) + b_1(t,\varepsilon)y(t) + b_2(t,\varepsilon)y(-t) + \tilde f(t),\\
\varepsilon y'(t) = d_1^0 y(t) + d_2^0 y(-t) + \left[d_1(t,\varepsilon) - d_1^0\right]y_0(t) + \left[d_2(t,\varepsilon) - d_2^0\right]y_0(-t) + c_1(t,\varepsilon)x_0(t) + c_2(t,\varepsilon)x_0(-t) + \tilde g(t).
\end{cases} \tag{29}$$

The second equation in (29) has a unique solution $y \in \mathcal{QP}_\omega$ by (H_1), (H_2) and Lemma 8. Then, put this y into the first equation which is solved for a unique $x \in \mathcal{QP}_\omega$ using Lemma 9. Writing $(x, y) = \mathbf{T}_1(x_0, y_0, \tilde f, \tilde g, \varepsilon)$, then solving (26) is equivalent to finding a fixed point of $\mathbf{T}_1(\cdot, \cdot, \tilde f, \tilde g, \varepsilon)$

If $(x_i, y_i) = \mathbf{T}_1(x_0^i, y_0^i, \tilde f, \tilde g, \varepsilon)$, $i = 1, 2$, then we find that $u = x_1 - x_2$, $v = y_1 - y_2$ satisfy

$$\begin{cases}
u'(t) = a_1^0 u(t) + a_2^0 u(-t) + \left(a_1(t,\varepsilon) - a_1^0\right)\left(x_0^1(t) - x_0^2(t)\right) + \left(a_2(t,\varepsilon) - a_2^0\right)\left(x_0^1(-t) - x_0^2(-t)\right) + b_1(t,\varepsilon)v(t) + b_2(t,\varepsilon)v(-t),\\
\varepsilon v'(t) = d_1^0 v(t) + d_2^0 v(-t) + \left(d_1(t,\varepsilon) - d_1^0\right)\left(y_0^1(t) - y_0^2(t)\right) + \left(d_2(t,\varepsilon) - d_2^0\right)\left(y_0^1(-t) - y_0^2(-t)\right) + c_1(t,\varepsilon)\left(x_0^1(t) - x_0^2(t)\right)\\
\qquad + c_2(t,\varepsilon)\left(x_0^1(-t) - x_0^2(-t)\right).
\end{cases} \tag{30}$$

From Lemma 8, Lemma 9 and (H_1), it follows that

$$\begin{aligned}
\|u\| &\le L_2\left[\left(\left\|a_1(t,\varepsilon) - a_1^0\right\| + \left\|a_2(t,\varepsilon) - a_2^0\right\|\right)\right.\\
&\quad \left.\cdot \left\|x_0^1 - x_0^2\right\| + 2M_2\|v\|\right], \|v\|\\
&\le L_1\left[\left(\|c_1(t,\varepsilon)\| + \|c_2(t,\varepsilon)\|\right)\left\|x_0^1 - x_0^2\right\|\right.\\
&\quad \left. + \left(\left\|d_1(t,\varepsilon) - d_1^0\right\| + \left\|d_2(t,\varepsilon) - d_2^0\right\|\right)\left\|y_0^1 - y_0^2\right\|\right].
\end{aligned} \tag{31}$$

And this leads to the estimate

$$\begin{aligned}
\|u\| &\le L_2\left[\left(\left\|a_1(t,\varepsilon) - a_1^0\right\| + \left\|a_2(t,\varepsilon) - a_2^0\right\|\right)\right.\\
&\quad \left.\cdot \left\|x_0^1 - x_0^2\right\| + 2M_2\|v\|\right], \le L_2\left(\left\|a_1(t,\varepsilon) - a_1^0\right\|\right.\\
&\quad \left. + \left\|a_2(t,\varepsilon) - a_2^0\right\|\right)\left\|x_0^1 - x_0^2\right\| + 2L_1L_2M_2\left(\|c_1(t,\varepsilon)\|\right.\\
&\quad \left. + \|c_2(t,\varepsilon)\|\right)\left\|x_0^1 - x_0^2\right\| + 2L_1L_2M_2\left(\left\|d_1(t,\varepsilon) - d_1^0\right\|\right.\\
&\quad \left. + \left\|d_2(t,\varepsilon) - d_2^0\right\|\right)\left\|y_0^1 - y_0^2\right\|.
\end{aligned} \tag{32}$$

From the hypothesis (H_1) and (H_2), it follows that there exists $\varepsilon_1 \le \varepsilon_0$ such that

$$\begin{aligned}
&L_2\left(\left\|a_1(t,\varepsilon) - a_1^0\right\| + \left\|a_2(t,\varepsilon) - a_2^0\right\|\right)\\
&\quad + 2L_1L_2M_2\left(\|c_1(t,\varepsilon)\| + \|c_2(t,\varepsilon)\|\right)\\
&< \frac{1}{2}, 2L_1L_2M_2\left(\left\|d_1(t,\varepsilon) - d_1^0\right\| + \left\|d_2(t,\varepsilon) - d_2^0\right\|\right)\\
&< \frac{1}{2}, L_1\left(\|c_1(t,\varepsilon)\| + \|c_2(t,\varepsilon)\|\right)\\
&< \frac{1}{2}, L_1\left(\left\|d_1(t,\varepsilon) - d_1^0\right\| + \left\|d_2(t,\varepsilon) - d_2^0\right\|\right) < \frac{1}{2}
\end{aligned} \tag{33}$$

for $0 < \varepsilon \le \varepsilon_0$. The contraction mapping principle implies that $\mathbf{T}_1$ has a unique fixed point $(x^*, y^*) \in \mathcal{QP}^2_\omega$. It follows from (29), Lemma 8, and Lemma 9 that

$$\begin{cases} \|x^*\| \le L_2\left[\left(\left\|a_1(t,\varepsilon)-a_1^0\right\|+\left\|a_2(t,\varepsilon)-a_2^0\right\|\right)\|x^*\|+2M_2\|y^*\|+\left\|\tilde{f}\right\|\right], \\ \|y^*\| \le L_1\left[\left(\|c_1(t,\varepsilon)\|+\|c_2(t,\varepsilon)\|\right)\|x^*\|+\left(\left\|d_1(t,\varepsilon)-d_1^0\right\|+\left\|d_2(t,\varepsilon)-d_2^0\right\|\right)\|y^*\|+\|\tilde{g}\|\right], \end{cases} \tag{34}$$

which imply

$$\begin{cases} \|x^*\| \le \left[1-L_2\left(\left\|a_1(t,\varepsilon)-a_1^0\right\|+\left\|a_2(t,\varepsilon)-a_2^0\right\|\right)\right]^{-1}\left[2L_2M_2\|y^*\|+L_2\left\|\tilde{f}\right\|\right], \\ \|y^*\| \le \left[1-L_1\left(\left\|d_1(t,\varepsilon)-d_1^0\right\|+\left\|d_2(t,\varepsilon)-d_2^0\right\|\right)\right]^{-1}[L_1(\|c_1(t,\varepsilon)\|+\|c_2(t,\varepsilon)\|)\|x^*\|+L_1\|\tilde{g}\|]. \end{cases} \tag{35}$$

Putting the second inequality of (35) into the first gives

$$\|x^*\| \le A_{1,1}(\varepsilon)\left\|\tilde{f}\right\|+A_{1,2}(\varepsilon)\|\tilde{g}\|, \tag{36}$$

where

$$\begin{aligned} A_{1,1}(\varepsilon) &= [1-2p(\varepsilon)q(\varepsilon)L_1L_2M_2(\|c_1(t,\varepsilon)\| \\ &\quad +\|c_2(t,\varepsilon)\|)]^{-1}p(\varepsilon)L_2, A_{1,2}(\varepsilon) \\ &= 2[1-2p(\varepsilon)q(\varepsilon)L_1L_2M_2(\|c_1(t,\varepsilon)\| \\ &\quad +\|c_2(t,\varepsilon)\|)]^{-1}p(\varepsilon)q(\varepsilon)L_1L_2M_2, p(\varepsilon) \\ &= \left[1-L_2\left(\left\|a_1(t,\varepsilon)-a_1^0\right\|+\left\|a_2(t,\varepsilon)-a_2^0\right\|\right)\right]^{-1}, q(\varepsilon) \\ &= \left[1-L_1\left(\left\|d_1(t,\varepsilon)-d_1^0\right\|+\left\|d_2(t,\varepsilon)-d_2^0\right\|\right)\right]^{-1}. \end{aligned} \tag{37}$$

Putting (36) into the second inequality of (35) gives

$$\|y^*\| \le A_{2,1}(\varepsilon)\left\|\tilde{f}\right\|+A_{2,2}(\varepsilon)\|\tilde{g}\|, \tag{38}$$

where

$$\begin{aligned} A_{2,1}(\varepsilon) &= A_{1,1}(\varepsilon)q(\varepsilon)L_1(\|c_1(t,\varepsilon)\|+\|c_2(t,\varepsilon)\|), \\ A_{2,2}(\varepsilon) &= A_{1,2}(\varepsilon)q(\varepsilon)L_1(\|c_1(t,\varepsilon)\|+\|c_2(t,\varepsilon)\|)+q(\varepsilon)L_1. \end{aligned} \tag{39}$$

The linear operator $\mathbf{K}(\varepsilon)$: $(\tilde{f},\tilde{g}) \longrightarrow (x^*,y^*)$ is bounded with

$$\begin{aligned} \|x^*\|+\|y^*\| &\le (A_{1,1}(\varepsilon)+A_{2,1}(\varepsilon))\left\|\tilde{f}\right\| \\ &\quad +(A_{1,2}(\varepsilon)+A_{2,2}(\varepsilon))\|\tilde{g}\|, \\ &\le 2(2L_1L_2M_2+L_1+L_2)\left(\left\|\tilde{f}\right\|+\|\tilde{g}\|\right) \end{aligned} \tag{40}$$

provided that ε_1 is so small that $A_{i,j}(\varepsilon) \le 2L_1L_2M_2+L_1+L_2$ for $0<\varepsilon\le\varepsilon_1$. Thus, $\|\mathbf{K}(\varepsilon)\| \le 2(2L_1L_2M_2+L_1+L_2)$.

Now, we consider the continuity of the map $\varepsilon \longrightarrow \mathbf{K}(\varepsilon)$. If we write $x(t,\varepsilon)=x(\tilde{f},\tilde{g},\varepsilon)(t)$ for $0<\varepsilon\le\varepsilon_1$, then

$$\begin{aligned} u(t) &= x(t,\varepsilon_1)-x(t,\varepsilon_2), \\ v(t) &= y(t,\varepsilon_1)-y(t,\varepsilon_2) \end{aligned} \tag{41}$$

satisfy

$$\begin{cases} u'(t) = a_1(t,\varepsilon_1)u(t)+a_2(t,\varepsilon_1)u(-t)+b_1(t,\varepsilon_1)v(t)+b_2(t,\varepsilon_1)v(-t)+[a_1(t,\varepsilon_1)-a_1(t,\varepsilon_2)]x(t,\varepsilon_2) \\ \qquad +[a_2(t,\varepsilon_1)-a_2(t,\varepsilon_2)]x(-t,\varepsilon_2)+[b_1(t,\varepsilon_1)-b_1(t,\varepsilon_2)]y(t,\varepsilon_2)+[b_2(t,\varepsilon_1)-b_2(t,\varepsilon_2)]y(-t,\varepsilon_2), \\ \varepsilon_1 v'(t) = c_1(t,\varepsilon_1)u(t)+c_2(t,\varepsilon_1)u(-t)+d_1(t,\varepsilon_1)v(t)+d_2(t,\varepsilon_1)v(-t)+[c_1(t,\varepsilon_1)-c_1(t,\varepsilon_2)]x(t,\varepsilon_2) \\ \qquad +[c_2(t,\varepsilon_1)-c_2(t,\varepsilon_2)]x(-t,\varepsilon_2)+[d_1(t,\varepsilon_1)-d_1(t,\varepsilon_2)]y(t,\varepsilon_2)+[d_2(t,\varepsilon_1)-d_2(t,\varepsilon_2)]y(-t,\varepsilon_2) \\ \qquad +\dfrac{\varepsilon_2-\varepsilon_1}{\varepsilon_2}[c_1(t,\varepsilon_2)x(t,\varepsilon_2)+c_2(t,\varepsilon_2)x(-t,\varepsilon_2)+d_1(t,\varepsilon_2)y(t,\varepsilon_2)+d_2(t,\varepsilon_2)y(-t,\varepsilon_2)+\tilde{g}(t)]. \end{cases} \tag{42}$$

In view of (36), (38), and (40), it follows that the map $\varepsilon \longrightarrow \mathbf{K}(\varepsilon)$ is continuous for $0 < \varepsilon \le \varepsilon_1$.

We now consider the nonlinear system (9)

$$\begin{cases} x'(t) = a_1(t,\varepsilon)x(t) + a_2(t,\varepsilon)x(-t) + b_1(t,\varepsilon)y(t) + b_2(t,\varepsilon)y(-t) + f(t,x(t),x(-t),y(t),y(-t),\varepsilon), \\ \varepsilon y'(t) = c_1(t,\varepsilon)x(t) + c_2(t,\varepsilon)x(-t) + d_1(t,\varepsilon)y(t) + d_2(t,\varepsilon)y(-t) + g(t,x(t),x(-t),y(t),y(-t),\varepsilon). \end{cases} \tag{43}$$

Theorem 11. *Suppose that (H_1), (H_2) and (H_3) hold. Then there exist $0 < \varepsilon_2 \le \varepsilon_1$, $0 < \sigma_1 \le \sigma_0$ such that for each ε satisfying $0 < \varepsilon \le \varepsilon_2$ and $(f, g) \in \mathcal{QP}_\omega^2$, system (9) has a unique solution $(x(t,\varepsilon), y(t,\varepsilon)) \in \mathcal{QP}_\omega^2$, which satisfies*

$$\begin{aligned} &\|x\| \le \sigma_1, |y|| \le \sigma_1, \\ &\|x\| + \|y\| = O(\Phi(\epsilon)), \epsilon \longrightarrow 0, \end{aligned} \tag{44}$$

and is continuous in ε uniformly for $t \in \mathbb{R}$.

Proof. From (H_3), we can choose σ_1 and ε_2 such that

$$\begin{aligned} &2(2L_1L_2M_2 + L_1 + L_2)(4\sigma_1\Psi(\varepsilon_2,\sigma) + \Phi(\varepsilon_2)) \\ &< \sigma_1, 4(2L_1L_2M_2 + L_1 + L_2)\Psi(\varepsilon_2,\sigma) \le \frac{1}{2}. \end{aligned} \tag{45}$$

For any $(x_0, y_0) \in \mathcal{QP}_\omega^2$ with $\|x_0\| \le \sigma_1$, $\|y_0\| \le \sigma_1$, $0 < \varepsilon \le \varepsilon_2$, consider the system

$$\begin{cases} x'(t) = a_1(t,\varepsilon)x(t) + a_2(t,\varepsilon)x(-t) + b_1(t,\varepsilon)y(t) + b_2(t,\varepsilon)y(-t) + f(t,x_0(t),x_0(-t),y_0(t),y_0(-t),\varepsilon), \\ \varepsilon y'(t) = c_1(t,\varepsilon)x(t) + c_2(t,\varepsilon)x(-t) + d_1(t,\varepsilon)y(t) + d_2(t,\varepsilon)y(-t) + g(t,x_0(t),x_0(-t),y_0(t),y_0(-t),\varepsilon). \end{cases} \tag{46}$$

By Theorem 10, system (46) has a unique solution $(x, y) \in \mathcal{QP}_\omega^2$ and the estimate

$$\begin{aligned} &|f(t,x_0(t),x_0(-t),y_0(t),y_0(-t),\varepsilon)| \\ &\quad \le \Psi(\varepsilon,\sigma)2[\|x_0\| + \|y_0\|] + \Phi(\varepsilon) \\ &\quad \le 4\sigma_1\Psi(\varepsilon_2,\sigma_1) + \Phi(\varepsilon_2). \end{aligned} \tag{47}$$

Writing $(x, y) = \mathbf{L}_\varepsilon^2(f(\cdot, x_0, y_0, \varepsilon), g(\cdot, x_0, y_0, \varepsilon)) \equiv \mathbf{T}_2(x_0, y_0, \varepsilon)$, then the existence of a solution of (9) is equivalent to the existence of a fixed point of the mapping $\mathbf{T}_2$. We may estimate (x, y) by Theorem 10

$$\begin{aligned} \|x\| &\le [A_{1,1}(\varepsilon) + A_{1,2}(\varepsilon))](4\sigma_1\Psi(\varepsilon_2,\sigma_1) + \Phi(\varepsilon_2)), \\ &\le 2(2L_1L_2M_2 + L_1 + L_2)(4\sigma_1\Psi(\varepsilon_2,\sigma_1) + \Phi(\varepsilon_2)), \end{aligned} \tag{48}$$

and similarly for $\|y\|$. If $(x_i, y_i) = \mathbf{T}_2(x_0^i, y_0^i, \varepsilon)$, $i = 1, 2$, then we find that

$$\begin{aligned} \|x_1 - x_2\| &\le 2(2L_1L_2M_2 + L_1 + L_2)\Psi(\varepsilon,\sigma) \\ &\quad \cdot \left[\left\|x_0^1 - x_0^2\right\| + \left\|y_0^1 - y_0^2\right\|\right] \end{aligned} \tag{49}$$

and similarly for $\|y\|$. We conclude

$$\begin{aligned} \|x_1 - x_2\| + \|y_1 - y_2\| &\le 4(2L_1L_2M_2 + L_1 + L_2)\Psi(\varepsilon_2,\sigma) \\ &\quad \cdot \left[\left\|x_0^1 - x_0^2\right\| + \left\|y_0^1 - y_0^2\right\|\right]. \end{aligned} \tag{50}$$

Hence, the mapping $\mathbf{T}_2(\cdot, \cdot, \varepsilon)$ maps the closed set $\Omega = \{(x_0, y_0) \in \mathcal{QP}_\omega^2 : \|x_0\| \le \sigma_1, \|y_0\| \le \sigma_1\}$ into itself for each ε with $0 < \varepsilon \le \varepsilon_2$ and is a uniform contraction in view of (45), (48), and (50).

It follows that $\mathbf{L}_\varepsilon^2$ is continuous since f, g are continuous in (x, y, ε) uniformly for t. For fixed $(x_0, y_0) \in \mathcal{QP}_\omega^2$, the map $\varepsilon \longrightarrow \mathbf{T}_2(x_0, y_0, \varepsilon)$ is continuous on $(0, \varepsilon_2]$. It follows from the uniform contraction principle that $\mathbf{T}_2$ has a unique fixed point $(x^*, y^*) \in \mathcal{QP}_\omega^2$ which is a continuous function of ε with $0 < \varepsilon \le \varepsilon_2$.

Finally, we obtain the estimation of (x^*, y^*) from the defining system as

$$\begin{aligned} \|x^*\| + \|y^*\| &\le 2(2L_1L_2M_2 + L_1 + L_2) \\ &\quad \cdot [2(\|x^*\| + \|y^*\|)\Psi(\varepsilon_2,\sigma_1) + \Phi(\varepsilon)] \\ &\le \frac{1}{2}(\|x^*\| + \|y^*\|) \\ &\quad + 2(2L_1L_2M_2 + L_1 + L_2)\Phi(\varepsilon). \end{aligned} \tag{51}$$

So

$$\|x^*\| + \|y^*\| \le 4(2L_1L_2M_2 + L_1 + L_2)\Phi(\varepsilon). \tag{52}$$

In order to explain the practical application of the system proposed in this paper, we consider the following singularly perturbed equations, which is closely related to a class of equations widely applied in the field of engineering technology and wave theory of physics.

Example 1. (Practical example). Consider a class of singularly perturbed equations, which can be described as follows:

$$\varepsilon U''(s) - U'(s) - U'\left(\frac{1}{s}\right)\frac{1}{s^2} - h\left(s, U(s), U\left(\frac{1}{s}\right), \varepsilon\right) = 0. \tag{53}$$

Introducing the variables

$$U(s) = U_0(s) + u(s), \tag{54}$$

where $U_0(s)$ is the bounded solution of the system,

$$-U'(s) - U'\left(\frac{1}{s}\right)\left(\frac{1}{s^2}\right) - h\left(s, U(s), U\left(\frac{1}{s}\right), 0\right) = 0, \tag{55}$$

and $U_0''(s)$ exists. Then $u(s)$ satisfies

$$\begin{aligned}\varepsilon u''(s) - u'(s) - u'\left(\frac{1}{s}\right)\frac{1}{s^2} - A(s,\varepsilon)u(s) - B(s,\varepsilon)u\left(\frac{1}{s}\right)\\ = R\left(s, u(s), u\left(\frac{1}{s}\right), \varepsilon\right),\end{aligned} \tag{56}$$

where

$$\begin{aligned}A(s,\varepsilon) &= h'_2\left(s, U_0(s), U_0\left(\frac{1}{s}\right), \varepsilon\right), B(s,\varepsilon)\\ &= h'_3\left(s, U_0(s), U_0\left(\frac{1}{s}\right), \varepsilon\right)u\left(\frac{1}{s}\right),\\ R\left(s, u(s), u\left(\frac{1}{s}\right), \varepsilon\right) &= -A(s,\varepsilon)u(s)\\ &- B(s,\varepsilon)u\left(\frac{1}{s}\right) - h\left(s, U_0(s), U_0\left(\frac{1}{s}\right), 0\right)\\ &+ h\left(s, U_0(s) + u(s), U_0\left(\left(\frac{1}{s}\right)\frac{1}{s}\right)\right.\\ &\left.+ u\left(\frac{1}{s}\right), \varepsilon\right) - \varepsilon U''_0(s).\end{aligned} \tag{57}$$

Subsequently, setting $X(s) = \varepsilon u'(s) - u(s) + ku(1/s)$, $Y(s) = \varepsilon u'(s)$, then Equation (56) is equivalent to the system

$$\begin{cases} X'(s) = A(s,\varepsilon)(X(s) - Y(s)) + B(s,\varepsilon)\left(X\left(\frac{1}{s}\right) - Y\left(\frac{1}{s}\right)\right) + R\left(s, X(s) - Y(s), X\left(\frac{1}{s}\right) - Y\left(\frac{1}{s}\right), \varepsilon\right),\\ \varepsilon Y'(s) = Y(s) + Y\left(\frac{1}{s}\right)\frac{1}{s^2} + \varepsilon A(s,\varepsilon)\left(X(s) - Y(s)\right) + \varepsilon B(s,\varepsilon)\left(X\left(\frac{1}{s}\right) - Y\left(\frac{1}{s}\right)\right) + \varepsilon R\left(s, X(s) - Y(s), X\left(\frac{1}{s}\right) - Y\left(\frac{1}{s}\right), \varepsilon\right). \end{cases} \tag{58}$$

Finally, the substitutions $s = e^t, X(s) = X(e^t) \triangleq x(t), Y(s) = y(e^t) \triangleq y(t)$, transform (58) into

$$\begin{cases} x'(s) = e^t\left[A\left(e^t,\varepsilon\right)(x(t) - y(t)) + B\left(e^t,\varepsilon\right)(x(-t) - y(-t)) + R\left(e^t, x(t) - y(t), x(-t) - y(-t), \varepsilon\right)\right],\\ \varepsilon y'(s) = e^t y(t) + e^{-t}y(-t) + \varepsilon e^t\left[A\left(e^t,\varepsilon\right)(x(t) - y(t)) + B\left(e^t,\varepsilon\right)(x(-t) - y(-t)) + R\left(e^t, x(t) - y(t), x(-t) - y(-t), \varepsilon\right)\right], \end{cases} \tag{59}$$

which is a form of (9).

4. Conclusions

In this paper, we consider the existence of a response solution for a singularly perturbed system involving reflection of the argument. Firstly, we prove that all ω-frequency continuous quasiperiodic functions form a Banach space under the supremum norm using the key lemma, that is, Lemma 4. Then, we obtain an existence and uniqueness result for a linear scalar equation with reflection of the argument. Expanding (6) about the trivial solution gives system (9), we firstly prove the existence and uniqueness of response solutions for a linear system (26). Then, we obtain the existence of response solutions for system (9) by means of fixed-point methods.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All four authors contributed equally to this work. They all read and approved the final version of the manuscript.

Acknowledgments

The third author is supported by the NSFC grant no. 11801295 and NSF of Shandong Province grant ZR2018BA008. The fourth author is supported by the NSF of Shandong Province grant ZR2019MEE030.

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