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Research Article

Residual Symmetries and Bäcklund Transformations of $(2+1)$-Dimensional Strongly Coupled Burgers System

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In this article, we mainly apply the nonlocal residual symmetry analysis to a $(2+1)$-dimensional strongly coupled Burgers system, which is defined by us through taking values in a commutative subalgebra. On the basis of the general theory of Painlevé analysis, we get a residual symmetry of the strongly coupled Burgers system. Then, we introduce a suitable enlarged system to localize the nonlocal residual symmetry. In addition, a Bäcklund transformation is derived by Lie's first theorem. Further, the linear superposition of the multiple residual symmetries is localized to a Lie point symmetry, and an N-th Bäcklund transformation is also obtained.

1. Introduction

Nonlinear partial differential equations have wide applications in the field of physical science, engineering, and other applied disciplines, e.g., nonlinear optics [1–4], fluid flows [5–7], plasma physics [8, 9], excitable media, and so on [10–14]. Burgers equation $p_t = 2pp_x + p_{xx}$ is a very important nonlinear partial differential equation occurring in various areas of applied sciences, such as fluid mechanics [15], nonlinear acoustics [15], gas dynamics, and traffic flow [16]. The equation was first introduced by Harry Bateman in 1915 and later studied by Johannes Martinus Burgers in 1948 [17, 18]. The study of symmetries plays an important role in branches of some natural sciences especially in integrable systems [19, 20]. In [21, 22], the authors proposed a residual symmetry in the process of the residue of the truncated Painlevé expansion for the bosonized super symmetric KdV equation which is a nonlocal symmetry [23–33]. In [34, 35], the authors concerned with the application of the nonlocal residual symmetry analysis to $(2+1)$-dimensional Burgers system, which has the form as follows:

$$p_t = pp_y + arp_x + bp_{yy} + abp_{xx}, \tag{1a}$$

$$p_x = r_y, \tag{1b}$$

where a and b are arbitrary constants.

In [36, 37], a hierarchy called the Frobenius-valued Kakomtsev–Petviashvili hierarchy which takes values in a maximal commutative subalgebra of $gl(m, \mathbb{C})$ was constructed. Then, in [38], the authors considered the Hirota quadratic equation of the commutative version of extended multicomponent Toda hierarchy, which should be useful in Frobenius manifold theory [39, 40]. Recently, we studied Z_n-Painlevé IV equation, Frobenius Painlevé I equation, and Frobenius Painlevé III equation [41]. In this paper, we consider a new $(2+1)$-dimensional strongly coupled Burgers system which is defined by us through taking values in a commutative subalgebra $Z_2 = \mathbb{C}[\Gamma]/(\Gamma^2)$. We replace the p and r of (1) with the commutative matrix

$$\begin{pmatrix} p & q \\ q & p \end{pmatrix}, \quad \begin{pmatrix} r & s \\ s & r \end{pmatrix}. \tag{2}$$

Then, we can get

$$\begin{aligned} p_t &= pp_y + qq_y + arp_x + asq_x + bp_{yy} + abp_{xx}, \\ q_t &= pq_y + qp_y + arq_x + asp_x + bq_{yy} + abq_{xx}, \end{aligned} \tag{3a}$$

$$\begin{aligned} p_x &= r_y, \\ q_x &= s_y, \end{aligned} \tag{3b}$$

which is called $(2+1)$-dimensional strongly coupled Burgers system.

The aim of this paper is to promote the $(2+1)$-dimensional Burgers system to a Frobenius integrable systems which is called $(2+1)$-dimensional strongly coupled Burgers system. A suitable enlarged system is given to localize the nonlocal residual symmetry. It follows that a Bäcklund transformation is derived by solving an initial value problem. It means that one can find various solutions of the $(2+1)$-dimensional strongly coupled Burgers system from a seed solution.

2. Residual Symmetries of $(2+1)$-Dimensional Strongly Coupled Burgers System

We first introduce the truncated Painlevé expansion:

$$\begin{aligned} p &= \sum_{i=0}^{\alpha_0}\left(p_i(\psi+\phi)^{i-\alpha_0}+p_i(\psi-\phi)^{i-\alpha_0}-q_i(\psi-\phi)^{i-\alpha_0}+q_i(\psi+\phi)^{i-\alpha_0}\right),\\ q &= \sum_{i=0}^{\alpha_1}\left(q_i(\psi+\phi)^{i-\alpha_1}+q_i(\psi-\phi)^{i-\alpha_1}-p_i(\psi-\phi)^{i-\alpha_1}+p_i(\psi+\phi)^{i-\alpha_1}\right),\\ r &= \sum_{i=0}^{\alpha_2}\left(r_i(\psi+\phi)^{i-\alpha_2}+r_i(\psi-\phi)^{i-\alpha_2}-s_i(\psi-\phi)^{i-\alpha_2}+s_i(\psi+\phi)^{i-\alpha_2}\right),\\ s &= \sum_{i=0}^{\alpha_3}\left(s_i(\psi+\phi)^{i-\alpha_3}+s_i(\psi-\phi)^{i-\alpha_3}-r_i(\psi-\phi)^{i-\alpha_3}+r_i(\psi+\phi)^{i-\alpha_3}\right), \end{aligned} \tag{4}$$

where p_α, q_α are a set of arbitrary solutions of the equation, and $p_{\alpha-1}, p_{\alpha-2}, \ldots, p_0, q_{\alpha-1}, q_{\alpha-2}, \ldots, q_0$ to be expressed by derivatives of ψ and ϕ. By balancing the dispersion and nonlinear terms according to the leader order analysis to the system (3a and 3b), the truncated Painlevé expansion has the following form:

$$p = \frac{p_0\psi - q_0\phi}{\psi^2-\phi^2} + p_1, \quad q = \frac{q_0\psi - p_0\phi}{\psi^2-\phi^2} + q_1, \tag{5a}$$

$$r = \frac{r_0\psi - s_0\phi}{\psi^2-\phi^2} + r_1, \quad s = \frac{s_0\psi - r_0\phi}{\psi^2-\phi^2} + s_1. \tag{5b}$$

Then, plugging (5a and 5b) into (3a and 3b) and vanishing all the coefficients of each power of $(\psi+\phi)^{-n}+(\psi-\phi)^{-n}$ and $(\psi-\phi)^{-n}-(\psi+\phi)^{-n}$, we obtain

$$\begin{aligned} p_0 &= 2b\psi_y, \quad q_0 = 2b\phi_y,\\ r_0 &= 2b\psi_x, \quad s_0 = 2b\phi_x, \end{aligned} \tag{6}$$

$$\begin{aligned} a\psi_x r_1 + a\phi_x s_1 + ab\psi_{xx} + p_1\psi_y + q_1\phi_y + b\psi_{yy} - \psi_t &= 0,\\ a\phi_x r_1 + a\psi_x s_1 + ab\phi_{xx} + q_1\psi_y + p_1\phi_y + b\phi_{yy} - \phi_t &= 0, \end{aligned} \tag{7}$$

$$\begin{aligned} p_{1t} - p_1p_{1y} - q_1q_{1y} - ar_1p_{1x} - as_1q_{1x} - bp_{1yy} - abp_{1xx} &= 0,\\ q_{1t} - p_1q_{1y} - q_1p_{1y} - as_1p_{1x} - ar_1q_{1x} - bq_{1yy} - abq_{1xx} &= 0, \end{aligned} \tag{8}$$

$$p_{1x} - r_{1y} = 0, \quad q_{1x} - s_{1y} = 0. \tag{9}$$

It is easy to find that (8) and (9) are just the $(2+1)$-dimensional strongly coupled Burgers system (3) with p_1, q_1, r_1, and s_1 as solutions. We then substitute p_0, q_0, r_0, and s_0 of (6) into the linearized form of (8) and (9) with (7) that one can find the p_0, q_0, r_0, and s_0 in (6) are the symmetries of the strongly coupled Burgers system.

According to the theorem of the residual symmetry [22], the strongly coupled Burgers system has a residual symmetry:

$$\sigma^{p_1} = 2b\psi_y, \quad \sigma^{q_1} = 2b\phi_y, \quad \sigma^{r_1} = 2b\psi_x, \quad \sigma^{s_1} = 2b\phi_x, \tag{10}$$

which is nonlocal for ψ and ϕ related to p_1, q_1, r_1, and s_1 by (7). Then, we introduce auxiliary variables f, g, h, and k with the relations $f = \psi_x$, $g = \phi_x$, $h = \psi_y$, and $k = \phi_y$ to obtain a local symmetry in the following enlarged system:

$$\begin{aligned} p_t - pp_y - qq_y - arp_x - asq_x - bp_{yy} - abp_{xx} &= 0,\\ q_t - pq_y - qp_y - asp_x - arq_x - bq_{yy} - abq_{xx} &= 0, \end{aligned} \tag{11a}$$

$$p_x = r_y, \quad q_x = s_y, \tag{11b}$$

$$\begin{aligned} a\psi_x r + a\phi_x s + ab\psi_{xx} + p\psi_y + q\phi_y + b\psi_{yy} - \psi_t &= 0,\\ a\phi_x r + a\psi_x s + ab\phi_{xx} + q\psi_y + p\phi_y + b\phi_{yy} - \phi_t &= 0, \end{aligned} \tag{11c}$$

$$f = \psi_x, \quad g = \phi_x, \tag{11d}$$

$$h = \psi_y, \quad k = \phi_y. \tag{11e}$$

Further, the residual symmetry can be localized into the Lie point symmetry

$$\sigma^p = h, \quad \sigma^q = k, \tag{12a}$$

$$\sigma^r = f, \quad \sigma^s = g, \tag{12b}$$

$$\sigma^f = -\frac{1}{b}(f\psi + g\phi), \quad \sigma^g = -\frac{1}{b}(g\psi + f\phi), \tag{12c}$$

$$\sigma^h = -\frac{1}{b}(h\psi + k\phi), \quad \sigma^k = -\frac{1}{b}(k\psi + h\phi), \tag{12d}$$

$$\sigma^\psi = -\frac{1}{2b}\left(\psi^2+\phi^2\right), \quad \sigma^\phi = -\frac{1}{b}\psi\phi, \tag{12e}$$

which satisfy

$$\begin{aligned} &\sigma^p_t - ab\sigma^r_{yx} - a\sigma^r_y r - a\sigma^s_y s - ar_y\sigma^r - as_y\sigma^s - b\sigma^p_{yy} - p\sigma^p_y\\ &\quad - q\sigma^q_y - p_y\sigma^p - q_y\sigma^q = 0,\\ &\sigma^q_t - ab\sigma^s_{yx} - a\sigma^s_y r - a\sigma^r_y s - ar_y\sigma^s - as_y\sigma^r - b\sigma^q_{yy} - p\sigma^q_y\\ &\quad - q\sigma^p_y - p_y\sigma^q - q_y\sigma^p = 0, \end{aligned} \tag{13a}$$

$$\sigma^p_x = \sigma^r_y, \quad \sigma^q_x = \sigma^s_y, \tag{13b}$$

$$\begin{aligned} &\sigma^\psi_t - b\sigma^\psi_{yy} - a\sigma^\psi_x r - a\sigma^\phi_x s - a\psi_x\sigma^r - a\phi_x\sigma^s - ab\sigma^\psi_{xx} - \sigma^p\psi_y\\ &\quad - \sigma^q\phi_y - p\sigma^\psi_y - q\sigma^\phi_y = 0,\\ &\sigma^\phi_t - b\sigma^\phi_{yy} - a\sigma^\psi_x s - a\sigma^\phi_x r - a\psi_x\sigma^s - a\phi_x\sigma^r - ab\sigma^\phi_{xx}\\ &\quad - \sigma^p\phi_y - \sigma^q\psi_y - p\sigma^\phi_y - q\sigma^\psi_y = 0, \end{aligned} \tag{13c}$$

$$\sigma^\psi_x = \sigma^f, \quad \sigma^\phi_x = \sigma^g, \tag{13d}$$

$$\sigma_y^{\psi} = \sigma^h, \quad \sigma_y^{\phi} = \sigma^k. \tag{13e}$$

Therefore, the enlarged system (14) has the Lie point symmetry vector:

$$\begin{aligned}\vec{V} &= h\partial_p + f\partial_r - \frac{1}{b}(f\psi + g\phi)\partial_f - \frac{1}{b}(h\psi + k\phi)\partial_h - \frac{1}{2b}(\psi^2 + \phi^2)\partial_\psi,\\ \vec{W} &= k\partial_q + g\partial_s - \frac{1}{b}(g\psi + f\phi)\partial_g - \frac{1}{b}(k\psi + h\phi)\partial_k - \frac{1}{b}\psi\phi\partial_\phi.\end{aligned} \tag{14}$$

Next we will give the Bäcklund symmetry theorem, which is obtained by using a finite transformation of the Lie point symmetry (14).

Theorem 1. *If $\{p, q, r, s, f, g, h, k, \psi, \phi\}$ is a solution of the coupled system (11a–11e), then so is $\{\overline{p}, \overline{q}, \overline{r}, \overline{s}, \overline{f}, \overline{g}, \overline{h}, \overline{k}, \overline{\psi}, \overline{\phi}\}$ with*

$$\overline{p} = p + \frac{2\epsilon b(\epsilon h\psi - \epsilon k\phi + 2bh)}{(\epsilon\psi + 2b)^2 - \epsilon^2\phi^2}, \quad \overline{q} = q + \frac{2\epsilon b(\epsilon k\psi - \epsilon h\phi + 2bk)}{(\epsilon\psi + 2b)^2 - \epsilon^2\phi^2}, \tag{15a}$$

$$\overline{r} = r + \frac{2\epsilon b(\epsilon f\psi - \epsilon g\phi + 2bf)}{(\epsilon\psi + 2b)^2 - \epsilon^2\phi^2}, \quad \overline{s} = s + \frac{2\epsilon b(\epsilon g\psi - \epsilon f\phi + 2bg)}{(\epsilon\psi + 2b)^2 - \epsilon^2\phi^2}, \tag{15b}$$

$$\overline{\psi} = \frac{2b(\epsilon\psi^2 - \epsilon\phi^2 + 2b\psi)}{(\epsilon\psi + 2b)^2 - \epsilon^2\phi^2}, \quad \overline{\phi} = \frac{4b^2\phi}{(\epsilon\psi + 2b)^2 - \epsilon^2\phi^2}, \tag{15c}$$

$$\overline{f} = \frac{4b^2(Cf - Dg)}{C^2 - D^2}, \quad \overline{g} = \frac{4b^2(Cg - Df)}{C^2 - D^2}, \tag{15d}$$

$$\overline{h} = \frac{4b^2(Ch - Dk)}{C^2 - D^2}, \quad \overline{k} = \frac{4b^2(Ck - Dh)}{C^2 - D^2}, \tag{15e}$$

where

$$C = \epsilon^2(\psi^2 + \phi^2) + 4b\epsilon\psi + 4b^2, \quad D = 2\epsilon^2\psi\phi + 4b\epsilon\phi, \tag{16}$$

and ϵ is an arbitrary group parameter.

Proof. According to Lie's first theorem on vector (14), it is not difficult to find that the key to prove this theorem is to solve the following initial value problem:

$$\frac{d\overline{p}}{d\epsilon} = \overline{h}, \quad \frac{d\overline{q}}{d\epsilon} = \overline{k}, \tag{17a}$$

$$\frac{d\overline{r}}{d\epsilon} = \overline{f}, \quad \frac{d\overline{s}}{d\epsilon} = \overline{g}, \tag{17b}$$

$$\frac{d\overline{f}}{d\epsilon} = -\frac{1}{b}(\overline{\psi}\,\overline{f} + \overline{\phi}\,\overline{g}), \quad \frac{d\overline{g}}{d\epsilon} = -\frac{1}{b}(\overline{\psi}\,\overline{g} + \overline{\phi}\,\overline{f}), \tag{17c}$$

$$\frac{d\overline{h}}{d\epsilon} = -\frac{1}{b}(\overline{\psi}\,\overline{h} + \overline{\phi}\,\overline{k}), \quad \frac{d\overline{k}}{d\epsilon} = -\frac{1}{b}(\overline{\psi}\,\overline{k} + \overline{\phi}\,\overline{h}), \tag{17d}$$

$$\frac{d\overline{\psi}}{d\epsilon} = -\frac{1}{2b}(\overline{\psi}^2 + \overline{\phi}^2), \quad \frac{d\overline{\phi}}{d\epsilon} = -\frac{1}{b}\overline{\psi}\,\overline{\phi}, \tag{17e}$$

$$\begin{aligned}&\overline{p}(0) = p, \quad \overline{q}(0) = q, \quad \overline{r}(0) = r, \quad \overline{s}(0) = s, \quad \overline{f}(0) = f,\\ &\overline{g}(0) = g, \quad \overline{h}(0) = h, \quad \overline{k}(0) = k, \quad \overline{\psi}(0) = \psi, \quad \overline{\phi}(0) = \phi.\end{aligned} \tag{17f}$$

And (15a–15e) is the solution to the above system. Therefore, we have completed the proof of Theorem 1. ☐

3. Bäcklund Transformations of Strongly Coupled Burgers System Related to Multiple Residual Symmetries

For the $(2+1)$-dimensional strongly coupled Burgers system (3a and 3b), the original residual symmetry

$$\sigma^p = 2b\psi_y, \quad \sigma^q = 2b\phi_y, \quad \sigma^r = 2b\psi_x, \quad \sigma^s = 2b\phi_x, \tag{18}$$

is related to the solution of Equation (11c). Then, according to the linear property of symmetry equations, the multiple residual symmetries are expressed in terms of any linear superposition of symmetry

$$\begin{aligned}&\sigma_n^p = \sum_{i=1}^n c_i\psi_{i,y}, \quad \sigma_n^q = \sum_{i=1}^n c_i\phi_{i,y},\\ &\sigma_n^r = \sum_{i=1}^n c_i\psi_{i,x}, \quad \sigma_n^s = \sum_{i=1}^n c_i\phi_{i,x}, \quad (n = 1, 2, 3, \ldots),\end{aligned} \tag{19}$$

where ψ_i and ϕ_i are different solutions of (11c). And the symmetry (19) should be localized to a Lie point symmetry by introducing more variables. In this way, one can find the finite transformation group of the symmetry (19).

Theorem 2. *If $\{p, q, r, s, f_i, g_i, h_i, k_i, \psi_i, \phi_i\}$ $(i = 1, 2, \ldots, n)$ is a solution of the enlarged system*

$$\begin{aligned}&p_t - pp_y - qq_y - arp_x - asq_x - bp_{yy} - abp_{xx} = 0,\\ &q_t - pq_y - qp_y - asp_x - arq_x - bq_{yy} - abq_{xx} = 0,\end{aligned} \tag{20a}$$

$$p_x = r_y, \quad q_x = s_y, \tag{20b}$$

$$\begin{aligned}&a\psi_{i,x}r + a\phi_{i,x}s + ab\psi_{i,xx} + p\psi_{i,y} + q\phi_{i,y} + b\psi_{i,yy} - \psi_{i,t} = 0,\\ &a\phi_{i,x}r + a\psi_{i,x}s + ab\phi_{i,xx} + q\psi_{i,y} + p\phi_{i,y} + b\phi_{i,yy} - \phi_{i,t} = 0,\end{aligned} \tag{20c}$$

$$f_i = \psi_{i,x}, \quad g_i = \phi_{i,x}, \tag{20d}$$

$$h_i = \psi_{i,y}, \quad k_i = \phi_{i,y}, \tag{20e}$$

$(i = 1, 2, \ldots, n)$,

then the symmetry (19) is localized to the Lie point symmetry

$$\sigma_n^p = \sum_{i=1}^n c_i h_i, \quad \sigma_n^q = \sum_{i=1}^n c_i k_i, \tag{21a}$$

$$\sigma_n^r = \sum_{i=1}^n c_i f_i, \quad \sigma_n^s = \sum_{i=1}^n c_i g_i, \tag{21b}$$

$$\sigma^{\psi_i} = -\frac{c_i}{2b}\left(\psi_i^2+\phi_i^2\right) - \sum_{j\neq i}^{n}\frac{c_j}{2b}\left(\psi_j\psi_i+\phi_j\phi_i\right),$$
$$\sigma^{\phi_i} = -\frac{c_i}{b}\psi_i\phi_i - \sum_{j\neq i}^{n}\frac{c_j}{2b}\left(\psi_j\phi_i+\phi_j\psi_i\right), \quad (21c)$$

$$\sigma^{f_i} = -\frac{c_i}{2b}\left(f_i\psi_i+g_i\phi_i\right) - \sum_{j\neq i}^{n}\frac{c_j}{2b}\left(f_i\psi_j+g_i\phi_j+f_j\psi_i+g_j\phi_i\right),$$
$$\sigma^{g_i} = -\frac{c_i}{2b}\left(f_i\phi_i+g_i\psi_i\right) - \sum_{j\neq i}^{n}\frac{c_j}{2b}\left(f_i\phi_j+g_i\psi_j+f_j\phi_i+g_j\psi_i\right), \quad (21d)$$

$$\sigma^{h_i} = -\frac{c_i}{2b}\left(h_i\psi_i+k_i\phi_i\right) - \sum_{j\neq i}^{n}\frac{c_j}{2b}\left(h_i\psi_j+k_i\phi_j+h_j\psi_i+k_j\phi_i\right),$$
$$\sigma^{k_i} = -\frac{c_i}{2b}\left(h_i\phi_i+k_i\psi_i\right) - \sum_{j\neq i}^{n}\frac{c_j}{2b}\left(h_i\phi_j+k_i\psi_j+h_j\phi_i+k_j\psi_i\right). \quad (21e)$$

Proof. The extended system (20a)–(20e) has the following linearized form:

$$\begin{aligned}&\sigma_t^p - ab\sigma_{yx}^r - a\sigma_y^r r - a\sigma_y^s s - ar_y\sigma^r - as_y\sigma^s\\&\quad - b\sigma_{yy}^p - p\sigma_y^p - q\sigma_y^q - p_y\sigma^p - q_y\sigma^q = 0,\\&\sigma_t^q - ab\sigma_{yx}^s - a\sigma_y^s r - a\sigma_y^r s - ar_y\sigma^s - as_y\sigma^r\\&\quad - b\sigma_{yy}^q - p\sigma_y^q - q\sigma_y^p - p_y\sigma^q - q_y\sigma^p = 0,\end{aligned} \quad (22a)$$

$$\sigma_x^p = \sigma_y^r, \quad \sigma_x^q = \sigma_y^s, \quad (22b)$$

$$\begin{aligned}&\sigma_t^{\psi_i} - b\sigma_{yy}^{\psi_i} - a\sigma_x^{\psi_i} r - a\sigma_x^{\phi_i} s - a\psi_{i,x}\sigma^r - a\phi_{i,x}\sigma^s\\&\quad - ab\sigma_{xx}^{\psi_i} - \sigma^p\psi_{i,y} - \sigma^q\phi_{i,y} - p\sigma_y^{\psi_i} - q\sigma_y^{\phi_i} = 0,\\&\sigma_t^{\phi_i} - b\sigma_{yy}^{\phi_i} - a\sigma_x^{\psi_i} s - a\sigma_x^{\phi_i} r - a\psi_{i,x}\sigma^s - a\phi_{i,x}\sigma^r\\&\quad - ab\sigma_{xx}^{\phi_i} - \sigma^p\phi_{i,y} - \sigma^q\psi_{i,y} - p\sigma_y^{\phi_i} - q\sigma_y^{\psi_i} = 0,\end{aligned} \quad (22c)$$

$$\sigma_x^{\psi_i} = \sigma^{f_i}, \quad \sigma_x^{\phi_i} = \sigma^{g_i}, \quad (22d)$$

$$\sigma_y^{\psi_i} = \sigma^{h_i}, \quad \sigma_y^{\phi_i} = \sigma^{k_i}. \quad (22e)$$

Let us first consider the special case: for any fixed m, $c_m \neq 0$, while $c_j = 0$, $j \neq m$ in (19), from (12a)–(12e), the localized symmetry for $\{p, q, r, s, f_m, g_m, h_m, k_m, \psi_m, \phi_m\}$ can be obtained as follows:

$$\sigma^p = c_m h_m, \quad \sigma^q = c_m k_m, \quad (23a)$$

$$\sigma^r = c_m f_m, \quad \sigma^s = c_m g_m, \quad (23b)$$

$$\sigma^{\psi_m} = -\frac{c_m}{2b}\left(\psi_m^2+\phi_m^2\right), \quad \sigma^{\phi_m} = -\frac{c_m}{b}\psi_m\phi_m, \quad (23c)$$

$$\sigma^{f_m} = -\frac{c_m}{b}\left(f_m\psi_m+g_m\phi_m\right), \quad \sigma^{g_m} = -\frac{c_m}{b}\left(f_m\phi_m+g_m\psi_m\right), \quad (23d)$$

$$\sigma^{h_m} = -\frac{c_m}{b}\left(h_m\psi_m+k_m\phi_m\right), \quad \sigma^{k_m} = -\frac{c_m}{b}\left(f_m\phi_m+k_m\psi_m\right). \quad (23e)$$

In (20c), we let $i = m$ and $i = j$, and then we obtain

$$\begin{aligned}&a\psi_{j,x}r + a\phi_{j,x}s + ab\psi_{j,xx} + p\psi_{j,y} + q\phi_{j,y} + b\psi_{j,yy} - \psi_{j,t} = 0,\\&a\phi_{j,x}r + a\psi_{j,x}s + ab\phi_{j,xx} + q\psi_{j,y} + p\phi_{j,y} + b\phi_{j,yy} - \phi_{j,t} = 0,\end{aligned} \quad (24)$$

$$\begin{aligned}&a\psi_{m,x}r + a\phi_{m,x}s + ab\psi_{m,xx} + p\psi_{m,y} + q\phi_{m,y} + b\psi_{m,yy} - \psi_{m,t} = 0,\\&a\phi_{m,x}r + a\psi_{m,x}s + ab\phi_{m,xx} + q\psi_{m,y} + p\phi_{m,y} + b\phi_{m,yy} - \phi_{m,t} = 0.\end{aligned} \quad (25)$$

Then, we substitute (23a) and (23b) into (22c) with $i = j$ yield

$$\begin{aligned}&\sigma_t^{\psi_j} - b\sigma_{yy}^{\psi_j} - a\sigma_x^{\psi_j} r - a\sigma_x^{\phi_j} s - ac_m\psi_{j,x}f_m - ac_m\phi_{j,x}g_m\\&\quad - ab\sigma_{xx}^{\psi_j} - c_m h_m\psi_{j,y} - c_m k_m\phi_{j,y} - p\sigma_y^{\psi_j} - q\sigma_y^{\phi_j} = 0,\\&\sigma_t^{\phi_j} - b\sigma_{yy}^{\phi_j} - a\sigma_x^{\phi_j} r - a\sigma_x^{\psi_j} s - ac_m\psi_{j,x}g_m - ac_m\phi_{j,x}f_m\\&\quad - ab\sigma_{xx}^{\phi_j} - c_m h_m\phi_{j,y} - c_m k_m\psi_{j,y} - p\sigma_y^{\phi_j} - q\sigma_y^{\psi_j} = 0.\end{aligned} \quad (26)$$

Further, one can obtain a solution of Equation (26):

$$\sigma^{\psi_j} = -\frac{c_m}{2b}\left(\psi_m\psi_j+\phi_m\phi_j\right), \quad \sigma^{\phi_j} = -\frac{c_m}{2b}\left(\psi_m\phi_j+\phi_m\psi_j\right), \quad (27)$$

which can be verified by using (20b), (24), and (25). From (22d) and (22e) with $i = j$, the symmetry for f_j, g_j, h_j, and k_j can be given by:

$$\begin{aligned}\sigma^{f_j} &= -\frac{c_m}{2b}\left(f_j\psi_m+g_j\phi_m+f_m\psi_j+g_m\phi_j\right),\\\sigma^{g_j} &= -\frac{c_m}{2b}\left(f_j\phi_m+g_j\psi_m+f_m\phi_j+g_m\psi_j\right),\end{aligned} \quad (28)$$

$$\begin{aligned}\sigma^{h_j} &= -\frac{c_m}{2b}\left(h_j\psi_m+k_j\phi_m+h_m\psi_j+k_m\phi_j\right),\\\sigma^{k_j} &= -\frac{c_m}{2b}\left(h_j\phi_m+k_j\psi_m+h_m\phi_j+k_m\psi_j\right).\end{aligned} \quad (29)$$

Further, (21a)–(21e) can be obtained by taking a linear combination of the above results for $m = 1, 2, \ldots, n$. Therefore, we have completed the proof of Theorem 2. □

According to Lie's first theorem, the initial value problem of the Lie point symmetry (21a)–(21e) has the following form:

$$\frac{dP(\epsilon)}{d\epsilon} = \sum_{j=1}^{n}c_j\Psi_{j,y}(\epsilon), \quad \frac{dQ(\epsilon)}{d\epsilon} = \sum_{j=1}^{n}c_j\Phi_{j,y}(\epsilon), \quad (30a)$$

$$\frac{dR(\epsilon)}{d\epsilon} = \sum_{j=1}^{n}c_j\Psi_{j,x}(\epsilon), \quad \frac{dS(\epsilon)}{d\epsilon} = \sum_{j=1}^{n}c_j\Phi_{j,x}(\epsilon), \quad (30b)$$

$$\begin{aligned}\frac{d\Psi_i(\epsilon)}{d\epsilon} &= -\frac{c_i}{2b}\left(\Psi_i^2(\epsilon)+\Phi_i^2(\epsilon)\right)\\&\quad - \sum_{j\neq i}^{n}\frac{c_j}{2b}\left(\Psi_j(\epsilon)\Psi_i(\epsilon)+\Phi_j(\epsilon)\Phi_i(\epsilon)\right),\\\frac{d\Phi_i(\epsilon)}{d\epsilon} &= -\frac{c_i}{b}\Psi_i(\epsilon)\Phi_i(\epsilon)\\&\quad - \sum_{j\neq i}^{n}\frac{c_j}{2b}\left(\Psi_j(\epsilon)\Phi_i(\epsilon)+\Phi_j(\epsilon)\Psi_i(\epsilon)\right),\end{aligned} \quad (30c)$$

$$
\begin{aligned}
\frac{dF_i(\epsilon)}{d\epsilon} &= -\frac{c_i}{b}\left(F_i(\epsilon)\Psi_i(\epsilon)+G_i(\epsilon)\Phi_i(\epsilon)\right) \\
&\quad -\sum_{j\neq i}^{n}\frac{c_j}{2b}\left(F_i(\epsilon)\Psi_j(\epsilon)+G_i(\epsilon)\Phi_j(\epsilon)+F_j(\epsilon)\Psi_i(\epsilon)+G_j(\epsilon)\Phi_i(\epsilon)\right), \\
\frac{dG_i(\epsilon)}{d\epsilon} &= -\frac{c_i}{b}\left(F_i(\epsilon)\Phi_i(\epsilon)+G_i(\epsilon)\Psi_i(\epsilon)\right) \\
&\quad -\sum_{j\neq i}^{n}\frac{c_j}{2b}\left(F_i(\epsilon)\Phi_j(\epsilon)+G_i(\epsilon)\Psi_j(\epsilon)+F_j(\epsilon)\Phi_i(\epsilon)+G_j(\epsilon)\Psi_i(\epsilon)\right),
\end{aligned} \tag{30d}
$$

$$
\begin{aligned}
\frac{dH_i(\epsilon)}{d\epsilon} &= -\frac{c_i}{b}\left(H_i(\epsilon)\Psi_i(\epsilon)+K_i(\epsilon)\Phi_i(\epsilon)\right) \\
&\quad -\sum_{j\neq i}^{n}\frac{c_j}{2b}\left(H_i(\epsilon)\Psi_j(\epsilon)+K_i(\epsilon)\Phi_j(\epsilon)+H_j(\epsilon)\Psi_i(\epsilon)+K_j(\epsilon)\Phi_i(\epsilon)\right), \\
\frac{dK_i(\epsilon)}{d\epsilon} &= -\frac{c_i}{b}\left(H_i(\epsilon)\Phi_i(\epsilon)+K_i(\epsilon)\Psi_i(\epsilon)\right) \\
&\quad -\sum_{j\neq i}^{n}\frac{c_j}{2b}\left(H_i(\epsilon)\Phi_j(\epsilon)+K_i(\epsilon)\Psi_j(\epsilon)+H_j(\epsilon)\Phi_i(\epsilon)+K_j(\epsilon)\Psi_i(\epsilon)\right),
\end{aligned} \tag{30e}
$$

$$
\begin{aligned}
&P(0)=p,\quad Q(0)=q,\quad R(0)=r,\quad S(0)=s,\quad \Psi_i(0)=\psi_i, \\
&\Phi_i(0)=\phi_i, F_i(0)=f_i,\quad G_i(0)=g_i, \\
&H_i(0)=h_i,\quad K_i(0)=k_i,\quad i=1,2,\ldots,n.
\end{aligned} \tag{30f}
$$

Then, one can get the following N-th Bäcklund theorem for the extended system (20a)–(20e) by solving (30a)–(30f).

Theorem 3. *If* $\{p,q,r,s,f_i,g_i,h_i,k_i,\psi_i,\phi_i\}$ *is a solution of the coupled system (20a)–(20e), then so is* $\{P(\epsilon),Q(\epsilon),R(\epsilon),S(\epsilon),F_i(\epsilon),G_i(\epsilon)H_i(\epsilon),K_i(\epsilon),\Psi_i(\epsilon),\Phi_i(\epsilon)\}$, $(i=1,2,\ldots,n)$, *where*

$$
\begin{aligned}
P(\epsilon) &= p+b(\ln(A+B)+\ln(A-B))_y, \\
Q(\epsilon) &= q+b(\ln(A+B)-\ln(A-B))_y,
\end{aligned} \tag{31a}
$$

$$
\begin{aligned}
R(\epsilon) &= r+b(\ln(A+B)+\ln(A-B))_x, \\
S(\epsilon) &= s+b(\ln(A+B)-\ln(A-B))_x,
\end{aligned} \tag{31b}
$$

$$
\Psi_i(\epsilon)=\frac{2b}{A^2-B^2}\left(A_iA-B_iB\right),\quad \Phi_i(\epsilon)=\frac{2b}{A^2-B^2}\left(B_iA-A_iB\right), \tag{31c}
$$

$$
F_i(\epsilon)=\Psi_{i,x}(\epsilon),\quad G_i(\epsilon)=\Phi_{i,x}(\epsilon), \tag{31d}
$$

$$
H_i(\epsilon)=\Psi_{i,x}(\epsilon),\quad K_i(\epsilon)=\Phi_{i,x}(\epsilon), \tag{31e}
$$

where

$$
\begin{aligned}
A &= \begin{vmatrix}
-\frac{c_1}{2b}\epsilon\psi_1-1 & c_1\epsilon M_{1,2} & \cdots & c_1\epsilon M_{1,j} & \cdots & c_1\epsilon M_{1,n} \\
c_2\epsilon M_{1,2} & -\frac{c_2}{2b}\epsilon\psi_2-1 & \cdots & c_2\epsilon M_{2,j} & \cdots & c_2\epsilon M_{2,n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c_j\epsilon M_{1,j} & c_j\epsilon M_{2,j} & \cdots & -\frac{c_j}{2b}\epsilon\psi_j-1 & \cdots & c_j\epsilon M_{j,n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c_n\epsilon M_{1,n} & c_n\epsilon M_{2,n} & \cdots & c_n\epsilon M_{j,n} & \cdots & -\frac{c_n}{2b}\epsilon\psi_n-1
\end{vmatrix}, \\
B &= \begin{vmatrix}
-\frac{c_1}{2b}\epsilon\phi_1-1 & c_1\epsilon N_{1,2} & \cdots & c_1\epsilon N_{1,j} & \cdots & c_1\epsilon N_{1,n} \\
c_2\epsilon N_{1,2} & -\frac{c_2}{2b}\epsilon\phi_2-1 & \cdots & c_2\epsilon N_{2,j} & \cdots & c_2\epsilon N_{2,n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c_j\epsilon N_{1,j} & c_j\epsilon N_{2,j} & \cdots & -\frac{c_j}{2b}\epsilon\phi_j-1 & \cdots & c_j\epsilon N_{j,n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c_n\epsilon N_{1,n} & c_n\epsilon N_{2,n} & \cdots & c_n\epsilon M_{j,n} & \cdots & -\frac{c_n}{2b}\epsilon\phi_n-1
\end{vmatrix},
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
A_i &= \begin{vmatrix}
-\frac{c_1}{2b}\epsilon\psi_1-1 & c_1\epsilon M_{1,2} & \cdots & c_1\epsilon M_{1,i-1} & c_1\epsilon M_{1,i} & c_1\epsilon M_{1,i+1} & \cdots & c_1\epsilon M_{1,n} \\
c_2\epsilon M_{1,2} & -\frac{c_2}{2b}\epsilon\psi_2-1 & \cdots & c_2\epsilon M_{2,i-1} & c_1\epsilon M_{2,i} & c_2\epsilon M_{2,i+1} & \cdots & c_2\epsilon M_{2,n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c_{i-1}\epsilon M_{1,i-1} & c_{i-1}\epsilon M_{2,i-1} & \cdots & -\frac{c_{i-1}}{2b}\epsilon\psi_{i-1}-1 & c_{i-1}\epsilon M_{i-1,i} & c_{i-1}\epsilon M_{i-1,i+1} & \cdots & c_{i-1}\epsilon M_{i-1,n} \\
M_{1,i} & M_{2,i} & \cdots & M_{i-1,i} & -\frac{1}{2b}\psi_i & M_{i,i+1} & \cdots & M_{i,n} \\
c_{i+1}\epsilon M_{1,i+1} & c_{i+1}\epsilon M_{2,i+1} & \cdots & c_{i+1}\epsilon M_{i-1,i+1}-1 & c_{i+1}\epsilon M_{i,i+1} & -\frac{c_{i+1}}{2b}\epsilon\psi_{i+1}-1 & \cdots & c_{i+1}\epsilon M_{i+1,n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
c_n\epsilon M_{1,n} & c_n\epsilon M_{2,n} & \cdots & c_n\epsilon M_{i-1,n} & c_n\epsilon M_{i,n} & c_n\epsilon M_{i+1,n} & \cdots & -\frac{c_n}{2b}\epsilon\psi_n-1
\end{vmatrix}, \\
B_i &= \begin{vmatrix}
-\frac{c_1}{2b}\epsilon\phi_1-1 & c_1\epsilon N_{1,2} & \cdots & c_1\epsilon N_{1,i-1} & c_1\epsilon N_{1,i} & c_1\epsilon N_{1,i+1} & \cdots & c_1\epsilon N_{1,n} \\
c_2\epsilon N_{1,2} & -\frac{c_2}{2b}\epsilon\phi_2-1 & \cdots & c_2\epsilon N_{2,i-1} & c_1\epsilon N_{2,i} & c_2\epsilon N_{2,i+1} & \cdots & c_2\epsilon N_{2,n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c_{i-1}\epsilon N_{1,i-1} & c_{i-1}\epsilon N_{2,i-1} & \cdots & -\frac{c_{i-1}}{2b}\epsilon\phi_{i-1}-1 & c_{i-1}\epsilon N_{i-1,i} & c_{i-1}\epsilon N_{i-1,i+1} & \cdots & c_{i-1}\epsilon N_{i-1,n} \\
N_{1,i} & N_{2,i} & \cdots & N_{i-1,i} & -\frac{1}{2b}\phi_i & N_{i,i+1} & \cdots & N_{i,n} \\
c_{i+1}\epsilon N_{1,i+1} & c_{i+1}\epsilon N_{2,i+1} & \cdots & c_{i+1}\epsilon N_{i-1,i+1}-1 & c_{i+1}\epsilon N_{i,i+1} & -\frac{c_{i+1}}{2b}\epsilon\phi_{i+1}-1 & \cdots & c_{i+1}\epsilon N_{i+1,n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
c_n\epsilon N_{1,n} & c_n\epsilon N_{2,n} & \cdots & c_n\epsilon N_{i-1,n} & c_n\epsilon N_{i,n} & c_n\epsilon N_{i+1,n} & \cdots & -\frac{c_n}{2b}\epsilon\phi_n-1
\end{vmatrix},
\end{aligned} \tag{33}
$$

with

$$M_{i,j} = -\frac{1}{4b}\left(\left(\psi_i\psi_j + \phi_i\phi_j + \psi_i\phi_j + \phi_i\psi_j\right)^{1/2} + \left(\psi_i\psi_j + \phi_i\phi_j - \psi_i\phi_j - \phi_i\psi_j\right)^{1/2}\right),$$
$$N_{i,j} = -\frac{1}{4b}\left(\left(\psi_i\psi_j + \phi_i\phi_j + \psi_i\phi_j + \phi_i\psi_j\right)^{1/2} - \left(\psi_i\psi_j + \phi_i\phi_j - \psi_i\phi_j - \phi_i\psi_j\right)^{1/2}\right). \quad (34)$$

From any seed solution of the $(2+1)$-dimensional strongly coupled Burgers system, one can get an infinite number of new solutions because n is an arbitrary positive integer. We consider a special case about the solution of the $(2+1)$-dimensional strongly coupled Burgers system. The $(2+1)$-dimensional strongly coupled Burgers system has a soliton solution:

$$p = a_0 + \frac{2b_0 b\left(e^{4k_0} - 1\right)}{\left(e^{2k_0+2k_1} + 1\right)\left(e^{2k_0-2k_1} + 1\right)} + \frac{2b_1 b\left(e^{2k_0+2k_1} - e^{2k_0-2k_1}\right)}{\left(e^{2k_0+2k_1} + 1\right)\left(e^{2k_0-2k_1} + 1\right)},$$
$$q = a_1 + \frac{2b_1 b\left(e^{4k_0} - 1\right)}{\left(e^{2k_0+2k_1} + 1\right)\left(e^{2k_0-2k_1} + 1\right)} + \frac{2b_0 b\left(e^{2k_0+2k_1} - e^{2k_0-2k_1}\right)}{\left(e^{2k_0+2k_1} + 1\right)\left(e^{2k_0-2k_1} + 1\right)}, \quad (35)$$
$$r = s = 0,$$

where

$$k_0 = (b_0 a_0 + b_1 a_1)t + b_0 y,$$
$$k_0 = (b_0 a_0 + b_1 a_1)t + b_0 y, \quad (36)$$

and a_0, a_1, b_0, and b_1 are arbitrary constants.

Then, the solution of (20c) can be obtained

$$\psi_i = \frac{d_i\left(e^{4k_0} - 1\right)}{\left(e^{2k_0+2k_1} + 1\right)\left(e^{2k_0-2k_1} + 1\right)} + \frac{e_i\left(e^{2k_0+2k_1} - e^{2k_0-2k_1}\right)}{\left(e^{2k_0+2k_1} + 1\right)\left(e^{2k_0-2k_1} + 1\right)},$$
$$\phi_i = \frac{d_i\left(e^{2k_0+2k_1} - e^{2k_0-2k_1}\right)}{\left(e^{2k_0+2k_1} + 1\right)\left(e^{2k_0-2k_1} + 1\right)} + \frac{e_i\left(e^{4k_0} - 1\right)}{\left(e^{2k_0+2k_1} + 1\right)\left(e^{2k_0-2k_1} + 1\right)}, \quad (37)$$

where d_i and e_i are arbitrary constants.

Further, one can obtain the $M_{i,j}$ and $N_{i,j}$ of Theorem 3

$$M_{i,j} = -\frac{C_0}{4b}\frac{e^{4k_0} - 1}{\left(e^{2k_0+2k_1} + 1\right)\left(e^{2k_0-2k_1} + 1\right)} - \frac{D_0}{4b}\frac{e^{2k_0+2k_1} - e^{2k_0-2k_1}}{\left(e^{2k_0+2k_1} + 1\right)\left(e^{2k_0-2k_1} + 1\right)},$$
$$N_{i,j} = -\frac{D_0}{4b}\frac{e^{4k_0} - 1}{\left(e^{2k_0+2k_1} + 1\right)\left(e^{2k_0-2k_1} + 1\right)} - \frac{C_0}{4b}\frac{e^{2k_0+2k_1} - e^{2k_0-2k_1}}{\left(e^{2k_0+2k_1} + 1\right)\left(e^{2k_0-2k_1} + 1\right)}, \quad (38)$$

where

$$C_0 = \left(d_i d_j + e_i e_j + d_i e_j + e_i d_j\right)^{1/2} + \left(d_i d_j + e_i e_j - d_i e_j - e_i d_j\right)^{1/2},$$
$$D_0 = \left(d_i d_j + e_i e_j + d_i e_j + e_i d_j\right)^{1/2} - \left(d_i d_j + e_i e_j - d_i e_j - e_i d_j\right)^{1/2}. \quad (39)$$

We write C_1 and D_1 as

$$C_1 = \frac{e^{4k_0} - 1}{\left(e^{2k_0+2k_1} + 1\right)\left(e^{2k_0-2k_1} + 1\right)}, \quad D_1 = \frac{e^{2k_0+2k_1} - e^{2k_0-2k_1}}{\left(e^{2k_0+2k_1} + 1\right)\left(e^{2k_0-2k_1} + 1\right)}. \quad (40)$$

Finally, consider the case of $n = 2$ in Theorem 3, one can get a new soliton solution

$$p = \frac{C_2((c_0(d_1 + d_2)C_1 + c_0(e_1 + e_2)D_1 + 2b) - D_2((c_0(e_1 + e_2)C_1 + c_0(d_1 + d_2)D_1)}{(c_0(d_1 + d_2 + e_1 + e_2)(C_1 + D_1) + 2b)(c_0(d_1 + d_2 - e_1 - e_2)(C_1 - D_1) + 2b)},$$
$$q = \frac{D_2((c_0(d_1 + d_2)C_1 + c_0(e_1 + e_2)D_1 + 2b) - C_2((c_0(e_1 + e_2)C_1 + c_0(d_1 + d_2)D_1)}{(c_0(d_1 + d_2 + e_1 + e_2)(C_1 + D_1) + 2b)(c_0(d_1 + d_2 - e_1 - e_2)(C_1 - D_1) + 2b)},$$
$$r = s = 0, \quad (41)$$

where

$$C_2 = C_1\left(c_0\left(a_0 d_1 + a_1 e_1 + a_0 d_2 + a_1 e_2\right) + 4b^2 b_0\right)$$
$$+ D_1\left(c_0\left(a_0 e_1 + a_1 d_1 + a_0 e_2 + a_1 d_2\right) + 4b^2 b_1\right)$$
$$+ 2bc_0\left(b_0 d_1 + b_1 e_1 + b_0 d_2 + b_1 e_2\right) + 2ba_0,$$
$$D_2 = D_1\left(c_0\left(a_0 d_1 + a_1 e_1 + a_0 d_2 + a_1 e_2\right) + 4b^2 b_0\right)$$
$$+ C_1\left(c_0\left(a_0 e_1 + a_1 d_1 + a_0 e_2 + a_1 d_2\right) + 4b^2 b_1\right)$$
$$+ 2bc_0\left(b_1 d_1 + b_0 e_1 + b_1 d_2 + b_0 e_2\right) + 2ba_1. \quad (42)$$

4. Conclusion and Discussion

In this paper, we first defined a new $(2+1)$-dimensional strongly coupled Burgers system which takes values in a two component commutative subalgebra Z_2. Then, the residual symmetry of the strongly coupled Burgers system was obtained by using the truncated Painlevé expansion, and the corresponding residual system was just the nonlocal symmetry. To localize the residual symmetry, we introduced a suitable enlarged system. According to Lie's first theorem, the finite Bäcklund transformation was derived. Further, the N-th Bäcklund transformation of the strongly coupled Burgers system was obtained by localizing the linear superposition of multiple residual symmetries, and the N-th Bäcklund transformation was expressed by determinants in a compact form.

Data Availability

This work does not have any experimental data.

Ethical Approval

This work did not involve any active collection of human data.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Yufeng Zhang contribute the idea and did some calculation. Haifeng Wang get computational results, and wrote the paper. All authors gave final approval for publication.

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