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Research Article

An Efficient Alternating Segment Parallel Difference Method for the Time Fractional Telegraph Equation

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The fractional telegraph equation is a kind of important evolution equation, which has an important application in signal analysis such as transmission and propagation of electrical signals. However, it is difficult to obtain the corresponding analytical solution, so it is of great practical value to study the numerical solution. In this paper, the alternating segment pure explicit-implicit (PASE-I) and implicit-explicit (PASI-E) parallel difference schemes are constructed for time fractional telegraph equation. Based on the alternating segment technology, the PASE-I and PASI-E schemes are constructed of the classic explicit scheme and implicit scheme. It can be concluded that the schemes are unconditionally stable and convergent by theoretical analysis. The convergence order of the PASE-I and PASI-E methods is second order in spatial direction and 3-α order in temporal direction. The numerical results are in agreement with the theoretical analysis, which shows that the PASE-I and PASI-E schemes are superior to the classical implicit schemes in both accuracy and efficiency. This implies that the parallel difference schemes are efficient for solving the time fractional telegraph equation.

1. Introduction

Nowadays, the fractional derivative and fractional differential equation have many applications in various fields, such as boundary layer effects in ducts, colored noise, dielectric polarization, electromagnetic waves, fractional kinetics, power-law phenomenon in fluid and complex network, quantitative finance, and viscoelastic mechanics [1–3]. The fractional telegraph equation especially is applied into signal analysis for transmission, propagation of electrical signals, and so on [4]. Because the analytical solution of the fractional telegraph equation is difficult to give explicitly or contains special functions such as the Mittag-Leffler function, which are difficult to calculate, the development of effectively numerical algorithms for solving the fractional telegraph equation is important [5–7].

Because of the historical dependence and global correlation of fractional calculus, the computational and storage requirements of fractional differential equations' numerical simulation are enormous [8–10]. Even with high-performance computers, it is difficult to simulate long-term or large-scale computational domains. With the rapid development of multicore and cluster technology, parallel algorithm has become one of the mainstream technologies in improving the computational efficiency [11]. Zhang et al. constructed a segment implicit scheme by using Saul'yev asymmetric scheme and used the alternative technique to establish a variety of alternating explicit-implicit and implicit parallel methods [12]. The methods have been well applied to numerical solving integer partial differential equation [13–16]. However, efficient parallel numerical methods for integer order differential equations may not be effective for fractional order differential equations. It may even produce completely different numerical analysis processes. How to extend the existing parallel difference method of integer order differential equation to the method of fractional order differential equation is a great challenge to computational mathematics (physics).

In recent years, many scholars have studied the numerical algorithms of the fractional telegraph equation. For example, Ford et al. used the finite difference method to numerically solve the fractional telegraph equation [17]. Saadatmandi and Mohabbati proposed a numerical solution by combining orthogonal Legendre polynomial and Tau method for the fractional telegraph equation [18]. Niu et al. applied the Chebyshev polynomial to get the numerical solution of the fractional telegraph equation [19]. Zhao and Li applied the finite difference method and Galerkin finite element method to solve the time-space fractional telegraph equation numerically [20]. Chen constructed the implicit difference scheme for the Riesz space fractional order telegraph equation [21]. Ren and Liu presented a high-order compact finite difference method for a class of time fractional Black-Scholes equation [22]. The scheme has the second-order temporal accuracy and the fourth-order spatial accuracy. The existing numerical algorithms of fractional differential equations are mostly serial algorithms, which have low computational efficiency.

In recent years, some progress has been made in fast algorithms for fractional partial differential equations [23, 24], most of which are parallel algorithms for algebraic systems based on the view of numerical algebra. For example, Diethelm implemented the second-order Adams-Bashforth-Moulton method of fractional diffusion equations on a parallel computer and discussed the accuracy of parallel method [25]. Gong et al. parallelizes the explicit difference scheme of the space fractional reaction-diffusion equation [26]. Gong et al. parallelizes the implicit scheme of the time fractional diffusion equation [27] and the core of parallelization is to do parallel computation for the product of matrix and vector and the addition of vector and vector. Sweilam et al. applied preconditioned conjugate gradient method was used to solve discrete algebraic equations in parallel, based on Crank-Nicolson difference schemes for time fractional parabolic equations [28]. Wang et al. studied the parallel algorithm of implicit difference schemes for fractional reaction-diffusion equations [29]. The algorithm is based on the principle of minimizing communication, allocating computing tasks reasonably, and not changing the original serial difference schemes as much as possible.

In order to obtain more accurate and stable parallel difference schemes, we are prepared to take the parallel path of traditional difference schemes and hope to find another way to parallelize them beyond the difficulty of numerical algebra [30, 31]. In this paper, a kind of alternating segment pure explicit-implicit (PASE-I) and implicit-explicit (PASI-E) parallel difference schemes is obtained. The numerical experiments and theoretical analysis are consistent, showing that the PASE-I and PASI-E schemes are unconditionally stable and convergent. The numerical examples show that the PASE-I and PASI-E difference schemes have obvious parallel computational properties. It shows that the parallel intrinsic difference schemes proposed in this paper is efficient for solving the time fractional telegraph equations.

2. PASE-I Scheme for Fractional Telegraph Equation

2.1. Time Fractional Telegraph Equation. Consider the following time fractional telegraph equation [3–5]:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \frac{\partial^{\alpha-1} u(x,t)}{\partial t^{\alpha-1}} = K\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad 0 \le x \le \mathrm{L}, t \ge 0, 1 < \alpha \le 2. \tag{1}$$

The initial conditions: $u(x,0) = \varphi_0(x)$, $(\partial u(x,0))/(\partial t) = \varphi_1(x)$.

The boundary conditions: $u(0,t) = 0$, $u(L,t) = 0$.

2.2. Construction of PASE-I Scheme. In order to obtain the PASE-I difference scheme for the fractional telegraph equation, the solution region is first meshed: the space and time steps are taken $h = L/M$ and $\tau = T/N$, where M and N are natural numbers; $x_i = ih (i = 1, 2, \cdots, M)$, $Mh = L$; $t_k = k\tau$ $(k = 1, 2, \cdots, N)$, $N\tau = T$; and the grid nodes are (x_i, t_k).

The time fractional derivative can be approximated by L1 formula [8, 9]:

$$\begin{aligned}\frac{\partial^{\alpha-1} u(x_i,t_{k+1})}{\partial t^{\alpha-1}} &= \frac{1}{\Gamma(2-\alpha)}\sum_{j=0}^{k}\frac{u(x_i,t_{j+1})-u(x_i,t_j)}{\tau}\int_{j\tau}^{(j+1)\tau} \cdot \frac{d\xi}{(t_{k+1}-\xi)^{\alpha-1}} + O(\tau^{3-\alpha}) \\ &\approx \frac{\tau^{2-\alpha}}{(2-\alpha)\Gamma(2-\alpha)}\sum_{j=0}^{k}\frac{u(x_i,t_{k+1-j})-u(x_i,t_{k-j})}{\tau} \cdot \left[(j+1)^{2-\alpha}-j^{2-\alpha}\right] \\ &= \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\sum_{j=0}^{k} c_j \nabla_t u(x_i,t_{k-j}),\end{aligned} \tag{2}$$

$$\begin{aligned}\frac{\partial^{\alpha} u(x_i,t_{k+1})}{\partial t^{\alpha}} &= \frac{1}{\Gamma(2-\alpha)}\sum_{j=0}^{k} \cdot \frac{u(x_i,t_{k+1-j})-2u(x_i,t_{k-j})+u(x_i,t_{k-j-1})}{\tau^2} \cdot \int_{j\tau}^{(j+1)\tau}\frac{d\xi}{(t_{k+1}-\xi)^{\alpha-1}} + O(\tau^{3-\alpha}) \\ &\approx \frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}\sum_{j=0}^{k-1} d_j\delta_t^2 u(x_i,t_{k-j}) + 2d_k(u(x_i,t_1)-u(x_i,t_0)-\tau u_t(x_i)).\end{aligned} \tag{3}$$

Spatial second derivatives are discretized by central difference method:

$$\frac{\partial^2 u(x_i,t_{k+1})}{\partial x^2} = \delta_x^2[\theta u(x_i,t_{k+1}) + (1-\theta)u(x_i,t_k)] + O(h^2). \tag{4}$$

Here, $\delta_x^2 u(x_i,t_k) = (1/h^2)[u(x_{i-1},t_k) - 2u(x_i,t_k) + u(x_{i+1},t_k)]$, $\nabla_t u(x_i,t_k) = (1/\tau)u(x_i,t_{k+1}) - u(x_i,t_k)$, $c_j = (j+1)^{2-\alpha} - j^{2-\alpha}$, $d_j = (j+1)^{2-\alpha} - j^{2-\alpha}$, $j = 0, 1, 2, \cdots, k-1$.

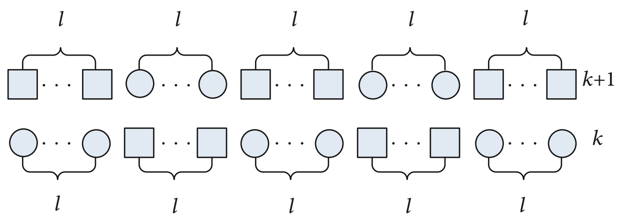

Figure 1: Computational lattice diagram of PASE-I scheme.

By substituting formulas (2), (3), (4) into equation (1), the universal difference schemes are obtained as follows:

$$\frac{\tau^{-\alpha}}{\Gamma(3-\alpha)}\sum_{j=0}^{k-1} c_j\delta_t^2 u(x_i,t_{k-j}) + \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)}\sum_{j=0}^{k} d_j\nabla_t u(x_i,t_{k-j}) = \frac{K}{h^2}\delta_x^2[\theta u(x_i,t_{k+1}) + (1-\theta)u(x_i,t_k)] + f(x_i,t_{k+1}). \quad (5)$$

Let u_i^k denote the value of $u(x,t)$ at the point (x_i,t_k), f_i^k denote the value of $f(x,t)$ at the point(x_i,t_k), and $a=(\tau^{-\alpha}/(\Gamma(3-\alpha)))$,$b=(\tau^{1-\alpha}/(\Gamma(3-\alpha)))$,$r=K/h^2$. The universal difference scheme can be written as follows:

$$a\sum_{j=0}^{k-1} d_j\delta_t^2 u_i^{k-j} + 2d_k[u(x_i,t_1) - u(x_i,t_0) - \tau u_t(x_i)] + b\sum_{j=0}^{k-1} c_j\nabla_t u_i^{k-j} = r\delta_x^2\left(\theta u_i^{k+1} + (1-\theta)u_i^k\right) + f_i^{k+1}, \quad k=0,1,2,\cdots. \quad (6)$$

When $\theta=0$, (6) is the explicit scheme, its advantage is explicit parallel computation, but its disadvantage is conditional stability. When $\theta=1$, (6) is the implicit scheme, its advantage is unconditional stability. The disadvantage is that it needs to calculate tridiagonal equations, which is not easy to parallel computation and has low computational efficiency.

The matrix form of universal difference scheme is as follows:

$$\begin{cases} [(2a+b)I + r\theta G]U^1 = [(2a+b)I - r(1-\theta)G]U^0 + 2a\tau H + b^1 + F^1, \\ AU^{k+1} = BU^k + q_2U^{k-1} + \cdots + q_{k-2}U^2 + q_{k-1}U^1 + q_kU^0 + 2ad_k\tau H + b^{k+1} + F^{k+1}. \end{cases} \quad (7)$$

Here, $q_0 = ad_0 + bc_0$, $q_1 = a(2d_0 - d_1) + b(c_0 - c_1)$, $q_j = -a(d_{j-2} - 2d_{j-1} + d_j) + b(c_{j-1} - c_j)$,$j=2,3,\cdots,k-1$, $q_k = a\ (-d_{k-2} + 2d_{k-1} - d_k) + b(c_{k-1} - c_k)$,$q_{k+1} = a(-d_{k-1} + d_k) + bc_k$, $b^{k+1} = (r(\theta u_1^{k+1} + (1-\theta)u_1^k), 0, \cdots, 0, r(\theta u_{M+1}^{k+1} + (1-\theta)\ u_{M+1}^k))'$,$U^k = [u_1^k, u_2^k, \cdots, u_{M-1}^k]'$,$U^0 = [\varphi_0(x_1), \varphi_0(x_2), \cdots, \varphi_0(x_{M-1})]'$,$H = [\varphi_1(x_1), \varphi_1(x_2), \cdots, \varphi_1(x_{M-1})]'$,$F = [f_1, f_2, \cdots, f_{M-1}]'$,$A = (ad_0 + bc_0)I + r\theta G$, $B = (a(2d_0 - d_1) + b(c_0 - c_1))I - r(1-\theta)G$,$\mathbf{E}$ is the unit matrix of order $(M-1)$,

$$G = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}_{(M-1)\times(M-1)}. \quad (8)$$

By alternatingly segment applying explicit and implicit schemes, the alternating segment explicit-implicit (PASE-I) scheme for fractional telegraph equation is designed as follows: Let $M-1 = ql$, $q, l \in N^+$, $l \geq 3$, q an odd number, and $q \geq 3$. The points to be computed in the same even time layer are divided into q segments, which are computed accordingly by the rule of "explicit segment-implicit segment-explicit segment". Similarly, the next odd layer is also divided into q segment calculation. The calculation rule is changed to "implicit segment-explicit segment-implicit segment". The computational lattice diagram of PASE-I scheme is shown in Figure 1. The blue circle indicates the classical explicit scheme, and the blue square indicates the classical implicit scheme. The PASE-I format will be obtained.

For the PASE-I scheme and $i_0 \geq 0$, consider the calculation of points$(i_0 + i, k+1)$, $i = 1, 2, \cdots, l$ in implicit segments. The implicit segment format is as follows:

$$[(2ad_0 + bc_0)I + rG]V^{k+1} = [(a(2d_0 - d_1) + b(c_0 - c_1))I]V^k + q_2V^{k-1} + \cdots + q_{k-1}V^2 + q_kV^1 + q_{k+1}V^0 + 2ad_k\tau H_1 + b_{im}^{k+1} + F_1{}^{k+1}. \quad (9)$$

The explicit segment format is as follows:

$$[(2ad_0 + bc_0)I]V^{k+1} = [(a(2d_0 - d_1) + b(c_0 - c_1))I + rG]V^k + q_2V^{k-1} + \cdots + q_{k-1}V^2 + q_kV^1 + q_{k+1}V^0 + 2ad_k\tau H_1 + b_{ex}^{k+1} + F_1{}^{k+1}. \quad (10)$$

$$\begin{cases} (q_0 I + rG_1)U^{n+1} = (q_1 I - rG_2)U^n + q_2 U^{n-1} \cdots + q_{n-1}U^2 + q_n U^1 + q_{n+1}U^0 + 2ad_k \tau H + b_1^{n+1} + F^{k+1}, \\ (q_0 I + rG_2)U^{n+2} = (q_1 I - rG_1)U^{n+1} + q_2 U^n \cdots + q_n U^2 + q_{n+1}U^1 + q_{n+2}U^0 + 2ad_{k+1}\tau H + b_2^{n+2} + F^{n+2}. \end{cases} \quad (11)$$

Here, $V^n = (u_{i_0+1}^n, u_{i_0+2}^n, \cdots, u_{i_0+l}^n)'$, $H_1 = [\varphi_1(x_{i_0+1}), \varphi_1(x_{i_0+2}), \cdots, \varphi_1(x_{i_0+l})]'$, $F_1 = [f_{i_0+1}, f_{i_0+2}, \cdots, f_{i_0+l}]'$,

$n = 1, 3, 5 \cdots$, $b_1^{n+1} = (ru_0^n, 0, \cdots, 0, ru_M^n)'$, $b_2^{n+2} = (ru_0^{n+2}, 0, \cdots, 0, ru_M^{n+2})'$, $Q_{l'}$ is a zero matrix of order l', $l' = ((l-1)/l-2)$; G_1, G_2 is martix of order $(M-1)$ and the definition is as follows:

$$G_1 = \begin{pmatrix} Q_{l-1} & & & & & & \\ & G_{l+2} & & & & & \\ & & Q_{l-2} & & & & \\ & & & G_{l+2} & & & \\ & & & & \ddots & & \\ & & & & & Q_{l-2} & \\ & & & & & & G_{l+2} \\ & & & & & & & Q_{l-1} \end{pmatrix}_{(M-1)\times(M-1)},$$

$$G_2 = \begin{pmatrix} \tilde{G}_{l+1} & & & & & & \\ & Q_{l-2} & & & & & \\ & & G_{l+2} & & & & \\ & & & Q_{l-2} & & & \\ & & & & \ddots & & \\ & & & & & G_{l+2} & \\ & & & & & & Q_{l-2} \\ & & & & & & & \bar{G}_{l+1} \end{pmatrix}_{(M-1)\times(M-1)}. \quad (12)$$

Here,

$$G_{l+2} = \begin{pmatrix} 0 & 0 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & 0 & 0 \end{pmatrix}_{(l+2)\times(l+2)}, \quad (13)$$

$$\tilde{G}_{l+1} = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & 0 & 0 \end{pmatrix}_{(l+1)\times(l+1)}, \quad (14)$$

$b_{im}^{k+1} = (ru_{i_0-1}^{k+1}, 0, \cdots, 0, ru_{i_0+l+1}^{k+1})'$,

$b_{ex}^{k+1} = (ru_1^k, 0, \cdots, 0, ru_{i_0+l+1}^k)'$.

The PASE-I scheme is as follows:

$$\bar{G}_{l+1} = \begin{pmatrix} 0 & 0 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}_{(l+1)\times(l+1)}. \quad (15)$$

3. Numerical Analysis of PASE-I and PASI-E Difference Schemes

3.1. Existence and Uniqueness of PASE-I Solution

Lemma 1. *The matrices $(a(2d_0 - d_1) + b(c_0 - c_1))I + G_1$ and $(a(2d_0 - d_1) + b(c_0 - c_1))I + G_2$ defined by PASE-I format are nonsingular matrices.*

Proof. From the definition of G_1 and $a(2d_0 - d_1) + b(c_0 - c_1) > 0$, we can know that $(a(2d_0 - d_1) + b(c_0 - c_1))I + G_1$ is a strictly diagonally dominant matrix, and the principal diagonal element is a positive real number. So $(a(2d_0 - d_1) + b(c_0 - c_1))I + G_1$ is a nonsingular matrix, and its inverse matrix also exists. Thus, there is the following theorem.

Theorem 2. *The PASE-I difference scheme (11) for the time fractional telegraph equation is uniquely solvable.*

3.2. Stability Analysis of PASE-I Scheme

Lemma 3. *If the matrix C is a nonnegative real matrix, for any parameter $\rho \geq 0, 0 \leq \sigma_1 \leq \sigma_2$, there is an estimate*

$$\left\| (\sigma_1 I - \rho C)(\sigma_2 I + \rho C)^{-1} \right\|_2 \leq 1. \quad (16)$$

Proof. $\|(\sigma_1 I - \rho C)(\sigma_2 I + \rho C)^{-1}\|_2^2 = \max_{\varphi \in R^n, \varphi \neq 0} \{((\sigma_1 I - \rho C)(\sigma_2 I + \rho C)^{-1}\varphi, (\sigma_1 I - \rho C)(\sigma_2 I + \rho C)^{-1}\varphi)/(\varphi, \varphi)\}$.

Make a transformation $\psi = (\sigma_2 I + \rho C)^{-1}\varphi$, then

$$\begin{aligned} &\left\| (\sigma_1 I - \rho C)(\sigma_2 I + \rho C)^{-1} \right\|_2^2 \\ &\quad = \max_{\varphi \in R^n, \varphi \neq 0} \frac{((\sigma_1 I - \rho C)\psi, (\sigma_1 I - \rho C)\psi)}{((\sigma_2 I + \rho C)\psi, (\sigma_2 I + \rho C)\psi)} \\ &\quad = \max_{\psi \in R^n, \psi \neq 0} \frac{\sigma_1{}^2(\psi, \psi) - 2\sigma_1\rho(C\psi, \psi) + \rho^2(C\psi, C\psi)}{\sigma_2{}^2(\psi, \psi) + 2\sigma_2\rho(C\psi, \psi) + \rho^2(C\psi, C\psi)}. \end{aligned} \quad (17)$$

From $0 \le \sigma_1 \le \sigma_2$, we have $\|(\sigma_1 I - \rho C)(\sigma_2 I + \rho C)^{-1}\|_2 \le 1$.

Let $\varepsilon_i^n = \tilde{u}_i^n - u_i^n, E^n = (\varepsilon_1^n, \varepsilon_3^n, \cdots, \varepsilon_{M-1}^n), 1 \le n \le N+1$, $\varepsilon_0^n = 0, \varepsilon_M^n = 0$, so we have

$$\begin{cases} (a_0 I + G_1)E^{n+1} = (w_1 I - G_2)E^n + w_2 E^{n-1} \cdots + w_{n-1}E^2 + a_n E^1, \\ (a_0 I + G_2)E^{n+2} = (w_1 I - G_1)E^{n+1} + w_2 E^n \cdots + w_n E^2 + a_{n+1}E^1. \end{cases} \tag{18}$$

Let $\tilde{u}_i^n$ be the approximate solution of the difference scheme, and u_i^n be the exact solution of difference scheme. where $n = 1, 3, \cdots, w_1 = ad_0 + bc_0$, $w_2 = a(2d_0 - d_1) + b(c_0 - c_1)$.

When $n \ge 3$, we have

$$\begin{aligned} E^{n+2} &= (q_0 I + G_2)^{-1}(q_1 I - G_1)(q_0 I + G_1)^{-1}(q_1 I - G_2)E^n \\ &\quad + (q_0 I + G_2)^{-1}(q_1 I - G_1)(q_0 I + G_1)^{-1} \\ &\quad \cdot \left(q_2 E^{n-1} + \cdots + q_{n-1}E^2 + q_n E^1\right) \\ &\quad + (q_0 I + G_2)^{-1}\left(q_2 E^n + \cdots + q_n E^2 + q_{n+1}E^1\right). \end{aligned} \tag{19}$$

Take norms on both sides of formula (19):

$$\begin{aligned} \left\|E^{n+2}\right\| &= \left\|(q_0 I + G_2)^{-1}(q_1 I - G_1)(q_0 I + G_1)^{-1}(q_1 I - G_2)\right\| \\ &\quad \cdot \|E^n\| + \left\|(q_0 I + G_2)^{-1}(q_1 I - G_1)(q_0 I + G_1)^{-1}\right\| \\ &\quad \cdot \left\|\left(q_2 E^{n-1} + \cdots + q_{n-1}E^2 + q_n E^1\right)\right\| + \left\|(q_0 I + G_2)^{-1}\right\| \\ &\quad \cdot \left\|\left(q_2 E^n + \cdots + q_n E^2 + q_{n+1}E^1\right)\right\|. \end{aligned} \tag{20}$$

The growth matrix of PASE-I scheme is $T = (q_0 I + G_2)^{-1}(q_1 I - G_1)(q_0 I + G_1)^{-1}(q_1 I - G_2)$. According to the definition of the matrix, G_l,G_1, and G_2 have the same eigenvalue. Let $\tilde{T} = (q_0 I + G_2)T(q_0 I + G_2)^{-1}$, suppose the eigenvalue of G_1 or G_2 in the matrix is r. From the Lemma 3, we have

$$\begin{aligned} \|T\| = \left\|\tilde{T}\right\| &= \left\|(q_1 I - rG_1)(q_0 I + rG_1)^{-1}\right. \\ &\quad \left.\cdot (q_1 I - rG_2)(q_0 I + rG_2)^{-1}\right\| \\ &= \max\left\{\left|\left(\frac{q_1 - r}{q_0 + r}\right)^2\right|\right\} \le 1. \end{aligned} \tag{21}$$

If $q_1 > r$, let $q_1 = \mu r (\mu > 1)$, because $|(q_1 - r)/(q_0 + r)| \le 1$, we have $\mu \le 2$.

The following is proven by mathematical induction $\|E^n\| \le \|E^1\|$.

When $n = 1$,

$$\left\|E^2\right\| = \left\|(q_0 I + rG_1)^{-1}(q_1 I - rG_2)E^1\right\| \le \left\|E^1\right\|. \tag{22}$$

When $n = 2$, case I $\max\{q_1, r\} = q_1$, and $r < q_1 \le 2r$.

$$\begin{aligned} \left\|E^3\right\| &\le \left\|(q_0 I + rG_2)^{-1}(q_1 I - rG_1)(q_0 I + rG_1)^{-1}(q_1 I - rG_2)\right\| \\ &\quad \cdot \left\|E^1\right\| + \left\|(q_0 I + rG_2)^{-1}\right\|\left\|q_2 E^1\right\| \\ &\le \max\left\{\left|\frac{q_1 - r}{q_0 + r}\right|^2\right\}\left\|E^1\right\| + \max\left\{\left|\frac{q_2}{q_0 + r}\right|\right\}\left\|E^1\right\| \\ &\le \max\left\{\left|\frac{r - q_2}{q_0 + r}\right|\right\}\left\|E^1\right\| \le \left\|E^1\right\|. \end{aligned} \tag{23}$$

When $\max\{q_1, r\} = r$,

$$\begin{aligned} \left\|E^3\right\| &\le \left\|q_0(I + rG_2)^{-1}(q_1 I - rG_1)(q_0 I + rG_1)^{-1}(q_1 I - rG_2)\right\| \\ &\quad \cdot \left\|E^1\right\| + \left\|(q_0 I + rG_2)^{-1}\right\|\left\|q_2 E^1\right\| \\ &\le \max\left\{\left|\frac{q_1 - r}{q_0 + r}\right|\right\}\left\|E^1\right\| + \max\left\{\left|\frac{q_2}{q_0 + r}\right|\right\}\left\|E^1\right\| \\ &\le \max\left\{\frac{r + |q_2|}{r + q_0}\right\}\left\|E^1\right\| \le \left\|E^1\right\|. \end{aligned} \tag{24}$$

Suppose $\|E^n\| \le \|E^1\|$ when $n \le k+1$. When $n = k+2$, we consider the following two situations.

Case 1. $\max\{q_1, r\} \le q_1$, and $r < q_1 \le 2r$.

$$\begin{aligned} \left\|E^{k+2}\right\| &= \left\|\left(q_0 I + r\bar{G}_2\right)^{-1}\left(q_1 I - r\bar{G}_1\right)\right. \\ &\quad \left.\cdot \left(q_0 I + r\bar{G}_1\right)^{-1}\left(q_1 I - r\bar{G}_2\right)\right\| \\ &\quad \cdot \left\|E^k\right\| + \left\|\left(q_0 I + r\bar{G}_2\right)^{-1}\left(q_1 I - r\bar{G}_1\right)\left(q_0 I + r\bar{G}_1\right)^{-1}\right\| \\ &\quad \cdot \left\|\left(q_2 E^{k-1} + \cdots + q_{k-1}E^2 + q_k E^1\right)\right\| \\ &\quad + \left\|\left(q_0 I + r\bar{G}_2\right)^{-1}\right\|\left\|\left(q_2 E^k + \cdots + q_k E^2 + q_{k+1}E^1\right)\right\| \\ &\le \left(\frac{r}{q_0 + r}\right)^2\left\|E^1\right\| + \frac{r|q_0 - q_1|}{(q_0 + r)^2}\left\|E^1\right\| + \frac{|q_0 - q_1|}{q_0 + r}\left\|E^1\right\| \\ &\le \frac{r}{q_0 + r}\left\|E^1\right\| + \frac{|q_0 - q_1|}{q_0 + r}\left\|E^1\right\| \le \left\|E^1\right\|. \end{aligned} \tag{25}$$

Case 2. $\max\{q_1, r\} \le r$.

$$\begin{aligned}\left\|E^{k+2}\right\| &= \left\|\left(q_0 I + \bar{G}_2\right)^{-1}\left(q_1 I - \bar{G}_1\right)\left(q_0 I + \bar{G}_1\right)^{-1}\left(q_1 I - \bar{G}_2\right)\right\| \\ &\quad \cdot \left\|E^k\right\| + \left\|\left(q_0 I + \bar{G}_2\right)^{-1}\left(q_1 I - \bar{G}_1\right)\left(q_0 I + \bar{G}_1\right)^{-1}\right\| \\ &\quad \cdot \left\|\left(q_2 E^{k-1} + \cdots + q_{k-1}E^2 + q_k E^1\right)\right\| + \left\|\left(q_0 I + \bar{G}_2\right)^{-1}\right\| \\ &\quad \cdot \left\|\left(q_2 E^k + \cdots + q_k E^2 + q_{k+1}E^1\right)\right\| \\ &\le \left(\frac{r}{a_0 + r}\right)^2 \left\|E^1\right\| + \frac{r(q_1 - q_0)}{(a_0 + r)^2}\left\|E^1\right\| + \frac{q_1 - q_0}{a_0 + r}\left\|E^1\right\| \\ &= \frac{r}{q_0 + r}\left(\frac{r}{q_0 + r} + \frac{q_1 - q_0}{q_0 + r}\right)\left\|E^1\right\| + \frac{q_1 - q_0}{q_0 + r}\left\|E^1\right\| \\ &\le \frac{r}{q_0 + r}\left\|E^1\right\| + \frac{q_1 - q_0}{q_0 + r}\left\|E^1\right\| \le \left\|E^1\right\|. \end{aligned} \tag{26}$$

To sum up, we have the following theorem.

Theorem 4. *The PASE-I difference scheme (11) for the time fractional telegraph equation is unconditionally stable.*

3.3. Convergence of PASE-I Scheme. Firstly, the accuracy of explicit and implicit schemes is analyzed, respectively, and Taylor expansion will be carried out at (x_i, t_{n+1}). The truncation errors are recorded as $T_1(\tau, h)$ and $T_2(\tau, h)$. It is known that ${}_0^C D_t^\alpha u(x_i, t_{n+1})$ has second-order accuracy. We have

$$\begin{aligned} T_1(\tau, h) &= D_t^\alpha u + D_t^{\alpha-1} u - K\left(u_{xx} + \tau u_{xxt} - \frac{\tau^2}{2}u_{xxtt}\right. \\ &\quad \left. - \frac{h^2}{12}u_{xxxx} + \frac{\tau h^2}{12}u_{xxxxt}\right) + O\left(\tau^{3-\alpha} + h^2\right), \\ T_2(\tau, h) &= D_t^\alpha u + D_t^{\alpha-1} u - K\left(u_{xx} - \tau u_{xxt} - \frac{\tau^2}{2}u_{xxtt}\right. \\ &\quad \left. - \frac{h^2}{12}u_{xxxx} - \frac{\tau h^2}{12}u_{xxxxt}\right) + O\left(\tau^{3-\alpha} + h^2\right). \end{aligned} \tag{27}$$

For the PASE-I scheme, the explicit and implicit schemes are alternately used for each grid point in spatial direction. For $T_1(\tau, h)$ and $T_2(\tau, h)$, the coefficients of u_{xxt} and u_{xxxxt} are equal and symbolically opposite. When the explicit and implicit schemes are alternately used, the errors of the two terms will be offset. Therefore, the accuracy of the PASE-I scheme is second order in spatial direction and $3-\alpha$ order in time direction.

Let $u(x_i, t_n), (i = 1, 2, \cdots, M; n = 1, 2, \cdots, N)$ be the exact solution of equation (1) at grid point (x_i, t_n). Let $e_i^n = u(x_i, t_n) - u_i^n, e^n = (e_1^n, e_3^n, \cdots, e_{M-1}^n)^T$. $e_i^n = u(x_i, t_n) - u_i^n$ is introduced into the PASE-I scheme,

$$\begin{cases} (a_0 I + G_1)e^{n+1} = (q_1 I - G_2)e^n + q_2 e^{n-1} \cdots + q_{n-1}e^2 + q_n e^1 + \tau^\alpha R^{n+1}, \\ (a_0 I + G_2)e^{n+2} = (q_1 I - G_1)e^{n+1} + q_2 e^n \cdots + q_n e^2 + q_{n+1}e^1 + \tau^\alpha R^{n+2}. \end{cases} \tag{28}$$

Obviously, $e^1 = 0$, $R^n = O \le (\tau^{3-\alpha} + h^2)$, there exists a positive constant C, such that $\tau^\alpha R^n = \tau^\alpha C(\tau^{3-\alpha} + h^2)$.

When $n = 1$,

$$\begin{aligned} e^2 &= (I + G_1)^{-1}(I - G_2)e^1 + (I + G_1)^{-1}R^2 \\ &= (I + G_1)^{-1}R. \end{aligned} \tag{29}$$

Combination Lemma 3 has

$$\begin{aligned} \left\|e^2\right\|_\infty &= \left\|(I + G_1)^{-1}R^1\right\|_\infty \le \|R\|_\infty \le C\left(\tau^{3-\alpha} + h^2\right) \\ &= q_1^{-1}C\left(\tau^{3-\alpha} + h^2\right). \end{aligned} \tag{30}$$

When $n = 2$,

$$\begin{aligned} &(a_0 I + G_2)e^3 = (q_1 I - G_1)e^2 + q_2 e^1 + R^3, \\ &\qquad \left\|e^3\right\| = (q_0 I + G_2)^{-1}\left[(q_1 I - G_1)e^2 - q_2 e^1 + R^3\right] \\ &\qquad \le \left\|(q_0 I + G_2)^{-1}\right\|\left[\left\|q_1 I - G_1\right\| - q_2 - q_3\right]q_3^{-1}R. \end{aligned} \tag{31}$$

Case 3. $\max\{w_1, r\} \le w_1$,

$$\begin{aligned} \left\|e^3\right\| &\le \left\|(q_0 I + G_2)^{-1}\right\|\left[\left\|q_1 I - G_1\right\| - q_2 - q_3\right]q_3^{-1}R \\ &\le \frac{1}{q_0 + r}\left[(q_1 + (q_1 - q_0))q_3^{-1}R\right] \le q_3^{-1}R \\ &\le q_3^{-1}C\left(\tau^{3-\alpha} + h^2\right). \end{aligned} \tag{32}$$

Case 4. $\max\{w_1, r\} \le r$,

$$\begin{aligned} \left\|e^3\right\| &= (q_0 I + G_2)^{-1}\left[(q_1 I - G_1)e^2 - q_2 e^1 + R^3\right] \\ &\le \left\|(q_0 I + G_2)^{-1}\right\|\left[\left\|w_1 I - G_1\right\| - q_2 - q_3\right]q_3^{-1}R \\ &\le \frac{1}{q_0 + r}\left[(r - q_1 - (q_0 - q_1))q_3^{-1}R\right] \\ &\le \frac{2q_1 - q_0 + r}{q_0 + r}q_3^{-1}R \le q_3^{-1}R \le q_3^{-1}C\left(\tau^{3-\alpha} + h^2\right). \end{aligned} \tag{33}$$

Suppose the inequality $\|e^n\| \le \|e^1\|$ holds, when $n \le k+1$. When $n = k+2$, reference to the proof process of stability, have

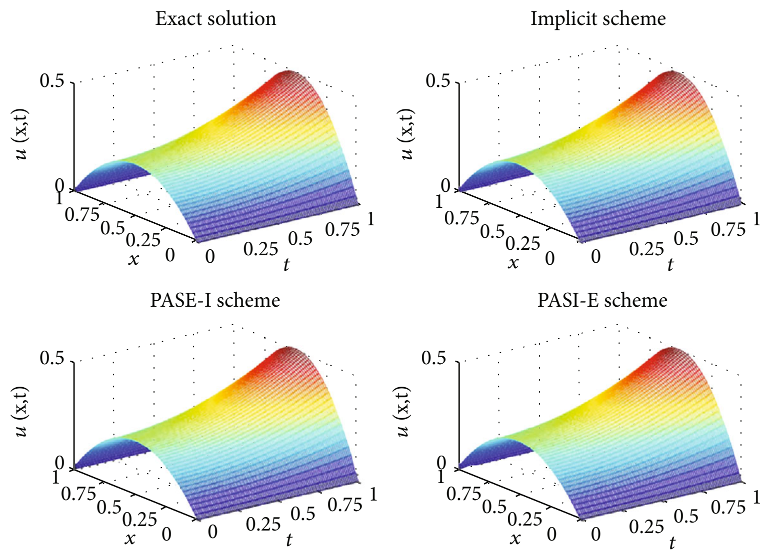

FIGURE 2: Exact and numerical solutions of surfaces ($M = 50$, $N = 100$).

$$\begin{aligned}\left\|e^{k+2}\right\| &= \left\|(q_0I+G_2)^{-1}(q_1I-G_1)(q_0I+G_1)^{-1}(q_1I-G_2)\right\| \\ &\cdot \left\|e^k\right\| + \left\|(q_0I+G_2)^{-1}(q_1I-G_1)(q_0I+G_1)^{-1}\right\| \\ &\cdot \left\|\left(q_2e^{k-1}+\cdots+q_{k-1}e^2+q_ke^1\right)\right\| \\ &+ \left\|(q_0I+G_2)^{-1}\right\|\left\|\left(q_2e^k+\cdots+q_ke^2+q_{k+1}e^1\right)\right\| \\ &+ \left\|(q_0I+G)^{-1}\right\|R^{k+2} \le \left(\frac{r}{q_0+r}\right)^2 R + \frac{r(1-w_1)}{(q_0+r)^2}R \\ &+ \frac{1}{q_0+r}(q_2+\cdots+q_k+q_{k+1})q_{k+1}^{-1}R \\ &\le \left[\frac{r^2}{(q_0+r)^2} + \frac{r|q_0-q_1|}{(q_0+r)^2} + \frac{|q_0-q_1|}{q_0+r}\right]a_{k+1}^{-1}R \\ &\le \left[\frac{r}{q_0+r}\left(\frac{r}{q_0+r}+\frac{|q_0-q_1|}{q_0+r}\right)+\frac{|q_0-q_1|}{q_0+r}\right]q_{k+1}^{-1}R \\ &\le \left[\frac{r}{q_0+r}+\frac{|q_0-q_1|}{q_0+r}\right]q_{k+1}^{-1}R \le q_{k+1}^{-1}R \\ &\le q_{k+1}^{-1}C\left(\tau^{3-\alpha}+h^2\right).\end{aligned} \tag{34}$$

Because $q_k = -a(d_{k-1} - d_k) + bc_k$, then there is a positive finite constant C_1, makes the inequality $\|e^{n+1}\| \le q_{n+1}^{-1} C_1(\tau^{3-\alpha} + h^2)$ hold. We have the following conclusions:

$$\|u(x_i, t_n) - u_i^n\|_\infty \le C_1\left(\tau^{3-\alpha} + h^2\right). \tag{35}$$

In summary, the theorem is obtained.

Theorem 5. *The PASE-I difference scheme (11) for the time fractional telegraph equation is convergent, and* $\|u(x_i, t_n) - u_i^n\|_\infty \le C(\tau^{3-\alpha} + h^2)$, *C is a positive number.*

3.4. PASI-E Parallel Difference Scheme. The PASI-E scheme of the time fractional telegraph equation can be obtained by changing the computational order of explicit segment and implicit segment. In the even time layer, the PASI-E scheme is obtained by using the rule "implicit segment-explicit segment-explicit segment" and the rule "explicit segment-implicit segment-explicit segment" in the odd time layer. The PASI-E scheme for solving the time fractional telegraph equation is as follows:

$$\begin{cases}(q_0I+G_2)V^{n+1} = (q_1I-G_1)V^n + q_2V^{n-1}\cdots+q_{n-1}V^2+q_nV^1+2ad_k\tau H+b_1^{n+1}+F^{k+1}, \\ (q_0I+G_1)V^{n+2} = (q_1I-G_2)V^{n+1} + q_2V^n\cdots+q_nV^2+q_{n+1}V^1+2ad_{k+1}\tau H+b_1^{n+2}+F^{n+2}.\end{cases} \tag{36}$$

Here, $n = 1, 3, 5 \cdots$, the definition of G_1, G_2, H, F^n, b_1^n is defined as before.

Since the difference between PASE-I format (11) and PASI-E (36) lies only in the order of using explicit format and implicit format, the computational complexity of the two formats should be equal in theory.

By the same proof process, the following theorem can be obtained.

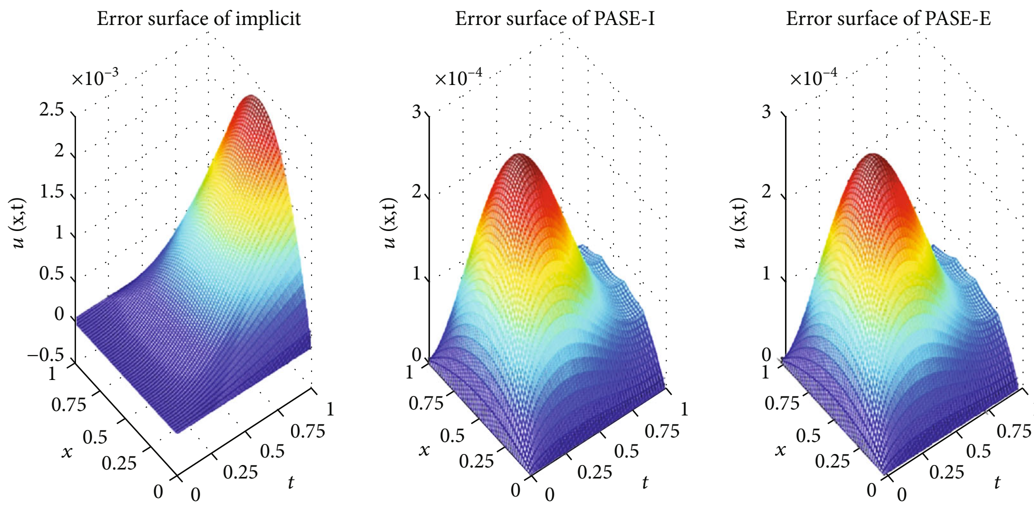

FIGURE 3: The error surface of numerical solutions.

Theorem 6. *The PASI-E difference scheme (36) of the time fractional telegraph equation is unconditionally stable and convergent. Moreover,* $\|u(x_i,t_n)-u_i^n\|_\infty \le C(\tau^{3-\alpha}+h^2)$, *C is a positive number.*

4. Numerical Experiments

The numerical experiment is based on Intel Core i5-2400 CPU@3.10GHz and is carried out under the environment of MATLAB R2014a. We will verify the theoretical analysis by numerical experiments.

Example 1. Consider the following time fractional telegraph equation, take $\alpha = 1.8$ [6, 7].

$$\frac{\partial^{1.8}u(x,t)}{\partial t^{1.8}} + \frac{\partial^{0.8}u(x,t)}{\partial t^{0.8}} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad 0 \le x \le 1, 0 \le t \le 1. \tag{37}$$

Here, $f(x,t) = x(1-x)((2/(\Gamma(1.2)))t^{0.2} + (2/(\Gamma(2.2)))t^{1.2}) + 2(1+t^2)$.

The initial condition: $u(x,0) = x(1-x)$, $u_t(x,0) = 0$, $0 < x < 1$.

The boundary condition: $u(0,t) = 0$, $u(1,t) = 0$.

Then, the equation has exact solutions: $u(x,t) = (1+t^2)x(1-x)$.

When $M = 50$, $N = 100$, the analytical solution surface and numerical solution surface of the three difference schemes of the fractional telegraph equation are shown in Figure 2. From Figure 2, we can see that the PASE-I and PASI-E difference schemes and implicit schemes are smooth and can approximate analytical solutions very well. The error surfaces of the three difference schemes are shown in Figure 3. From Figure 3, we can see that the error of implicit difference schemes is less than 2.5e-3, and the error limits of PASE-I and PASI-E schemes are less than 3e-4. The accuracy of PASE-I or PASI-E scheme is better than the implicit difference scheme.

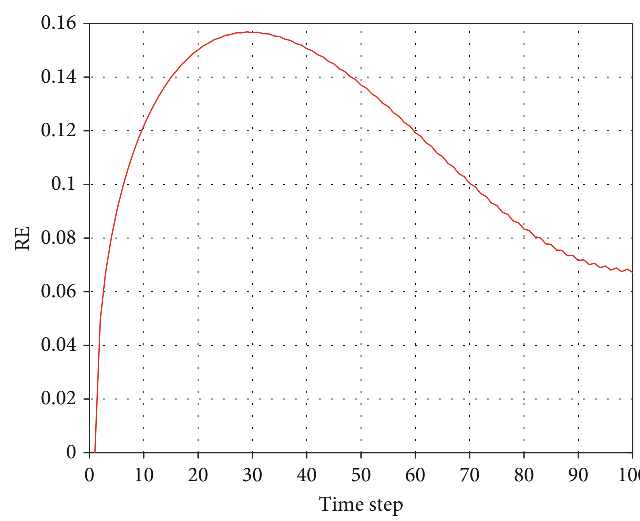

FIGURE 4: The curve of RE changes with time.

Because the accuracy of PASE-I and PASI-E are similar, we take PASE-I as an example to investigate the variation of relative error (RE) of PASE-I in time direction. The relative error is defined as follows. The analytical solution is regarded as the control solution, and the solution of PASE-I scheme is regarded as the perturbation solution. The formula for relative error is

$$\mathrm{RE}(i) = \sum_{j=1}^{M} \frac{\left|u_j^i - \bar{u}_j^i\right|}{u_j^i}. \tag{38}$$

The space step h is 1/50 and the time step is 1/100, respectively. The RE of PASE-I is shown in Figure 4. From Figure 4, we can know that the RE of PASE-I scheme is less than 0.16. The RE decreases with the advance of time step,

Table 1: Time order (*order1*) of difference schemes ($h = 1/200$).

τ	Implicit E_∞	Implicit *order1*	PASE-I E_∞	PASE-I *order1*	PASI-E E_∞	PASI-E *order1*
1/200	2.987928e-5		2.792870e-5		2.794564e-5	
1/400	1.520501e-5	0.974597	1.197192e-5	1.222094	1.197806e-5	1.222228
1/800	7.720232e-6	0.977831	5.127047e-6	1.223454	5.129525e-6	1.223497
1/1600	3.911829e-6	0.980800	2.190633e-6	1.226780	2.191679e-6	1.226788
1/3200	1.978420e-6	0.983494	9.332082e-7	1.231076	9.336585e-7	1.231069

Table 2: Space order (*order2*) of difference schemes.

h	Implicit E_∞	Implicit *order2*	PASE-I E_∞	PASE-I *order2*	PASI-E E_∞	PASI-E *order2*
1/10	1.500088e-4		6.358423e-4		6.304834e-4	
1/20	4.820269e-5	1.637862	1.597008e-4	1.993297	1.596684e-4	1.981378
1/40	1.576648e-5	1.612252	3.994980e-5	1.999111	3.998717e-5	1.997470
1/80	5.148237e-6	1.614710	1.001469e-5	1.996069	1.002560e-5	1.995848
1/160	1.676390e-6	1.618720	2.529035e-6	1.985459	2.531798e-6	1.985453
1/320	5.420003e-7	1.628992	6.398285e-7	1.982830	6.405186e-7	1.982850

which indicates that the PASE-I scheme of time fractional telegraph equation is computationally stable.

Next, time convergence order (*order1*) and space convergence order (*order2*) are defined by the maximum modulus error. The theoretical analysis is validated by the following numerical experiments. With fixed space step h, the *order1* is defined as follows [8, 9]:

$$order1 = \log_2\left(\frac{E_\infty\left(\tau^{3-\alpha}+h^2\right)}{E_\infty\left((\tau/2)^{3-\alpha}+h^2\right)}\right) \approx 3-\alpha. \tag{39}$$

Let $\tau^{3-\alpha} = h^2$, the *order2* is defined as follows:

$$order2 = \log_2\left(\frac{E_\infty\left({\tau_1}^{3-\alpha}+h^2\right)}{E_\infty\left({\tau_2}^{3-\alpha}+(h/2)^2\right)}\right) = \log_2\left(\frac{E_\infty\left(h^2\right)}{E_\infty\left((h/2)^2\right)}\right) \approx 2. \tag{40}$$

From Table 1, the convergence order of PASE-I and PASI-E schemes in time direction is $3-\alpha$ order, and the accuracy of implicit schemes is equal to that of PASE-I and PASI-E schemes. From Table 2, we can see the spatial convergence orders of PASE-I and PASI-E schemes are both of second order, which is consistent with the theoretical analysis.

Example 2. Consider the following time fractional telegraph equation [1, 5].

$$\frac{\partial^{1.8}u(x,t)}{\partial t^{1.8}} + \frac{\partial^{0.8}u(x,t)}{\partial t^{0.8}} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 \le x \le 1, 0 \le t \le 1. \tag{41}$$

The initial conditions: $u(x,0) = x(1-x)$, $u_t(x,0) = 0$, $0 < x < 1$.

The boundary conditions: $u(0,t) = 0$, $u(1,t) = 0$.

When the time division is 500 and the space mesh points is 50, the numerical solution surfaces of the three schemes are shown in Figure 5. From Figure 5, it can be seen that the numerical solution surfaces of the PASE-I and PASI-E schemes are the same and smooth as those of the classical implicit scheme.

We define the speedup as $S_p = T_1/T_p$ (T_1 is the CPU time of implicit, T_p is the CPU time of parallel scheme) and the efficiency as $E_p = S_p/p$ (p is the number of processors in parallel processor) [30, 31]. We use four cores for this numerical experiment. When the number of spatial grid points is 100, 200, 400, 800, 1600, and 3200, respectively, the CPU time required for the three schemes is shown in Table 3. From Table 3, it can be seen that the CPU time of serial difference increases exponentially, and the parallel difference schemes increase relatively slowly, with the increase of the number of spatial grid points. Compared with serial difference schemes, the computational efficiency of the PASE-I and PASI-E parallel schemes are greatly improved with the refinement of the spatial mesh. When the number of spatial grids is small (100), the computing time of parallel difference scheme is almost the same as that of the serial implicit scheme, because the communication between modules consumes a lot of CPU time. With the increase of computational domain, the parallel computing characteristics of PASE-I and PASI-E schemes will become more prominent. When the number of grid points is 1600, the efficiency of parallel difference schemes is optimal in this example. The linear acceleration ratio can be achieved when the number of space grid points is more than 800. Compared with the serial difference scheme, the computation time of PASE-I

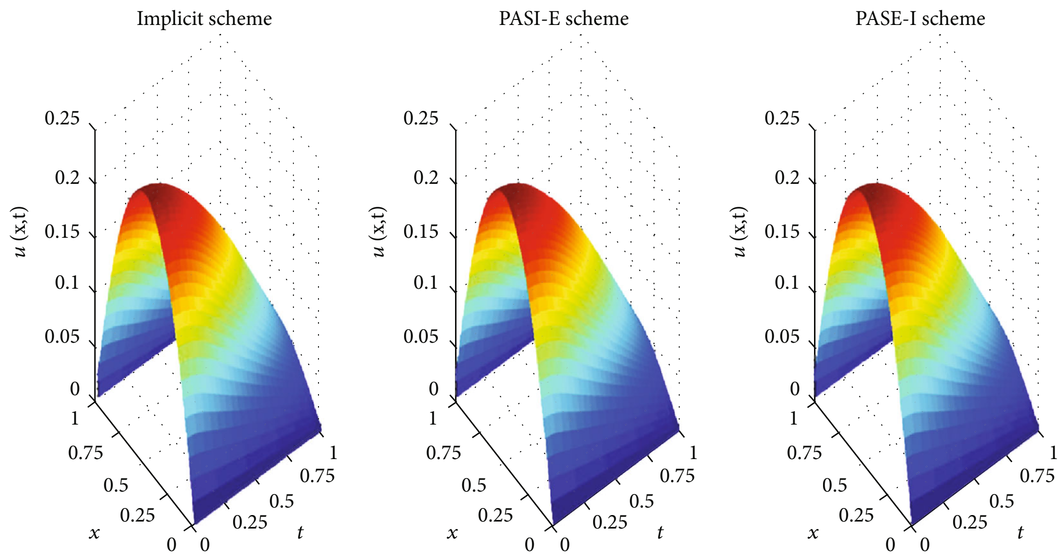

Figure 5: Surfaces of three difference schemes.

Table 3: Comparison of the three difference schemes' CPU time.

Grid numbers	Implicit	PASE-I	PASI-E	S_p	E_p
100	29.26	18.72	18.74	1.56	0.39
200	52.45	23.63	23.72	2.21	0.55
400	95.11	33.69	33.50	2.82	0.70
800	226.02	48.19	48.91	4.68	1.17
1600	433.85	81.76	83.42	5.30	1.32
3200	895.42	170.69	170.18	5.24	1.31

and PASI-E scheme can save about 80% when the number of grid points is great 800.

5. Conclusion

In this paper, a class of alternating segment pure explicit-implicit (PASE-I) and implicit-explicit (PASI-E) parallel difference methods are constructed for the time fractional telegraph equations. From theoretical analysis, it can be concluded that the parallel difference methods are unconditionally convergent and stable, with second-order convergence in spatial direction and $3-\alpha$ order in temporal direction. The numerical experiments show that the parallel difference schemes are more efficient than the serial difference scheme with the increase of space mesh generation. The numerical experiments are in agreement with the theoretical analysis, which show that the numerical algorithm is efficient and feasible for solving the time fractional telegraph equation.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no competing interests.

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