Hindawi
Advances in Mathematical Physics
Volume 2020, Article ID 7640717, 4 pages
https://doi.org/10.1155/2020/7640717

Research Article

New Generalized Soliton Solutions for a $(3+1)$-Dimensional Equation

Yiren Chen

College of Mathematics and Statistics, Shenzhen University, Shenzhen 518060, China

Correspondence should be addressed to Yiren Chen; adavidcen@hotmail.com

Received 15 August 2019; Revised 27 October 2019; Accepted 5 November 2019; Published 2 January 2020

Academic Editor: Stephen C. Anco

In this paper, we investigate the nonlinear wave solutions for a $(3+1)$-dimensional equation which can be reduced to the potential KdV equation. We present generalized N-soliton solutions in which some arbitrarily differentiable functions are involved by using a simplified Hirota's method. Our work extends some previous results.

1. Introduction

The research of the exact solutions for nonlinear equations plays a significant role in the investigation of nonlinear science. For nonlinear science, owing to containing more information and describing many natural phenomena, such as plasmas, gas dynamics, fluid mechanics, and elastic media, many nonlinear equations have been investigated [1–10]. Compared with the lower-dimensional nonlinear equations, the higher-dimensional nonlinear equations are more difficult to study because of the complex construction and multiple variables. Clarkson and Mansfield [3] introduced the following equation:

$$u_{xxxt} + \beta(u_x u_{xt} + u_t u_{xx}) - (u_{xx} + u_{xt}) = 0, \quad (1)$$

which they called the SWWI equation, and another equation

$$u_{xxxt} + \beta(2u_x u_{xt} + u_t u_{xx}) - (u_{xx} + u_{xt}) = 0, \quad (2)$$

which they called the SWWII equation.

For Equation (1), Clarkson and Mansfield [3] obtained an interesting solution

$$u(x,t) = \frac{3}{\beta}\tanh\frac{x+f(t)}{2} + \frac{3}{\beta}\tanh\frac{x-f(t)}{2} + \frac{t}{\beta}, \quad (3)$$

which contains an arbitrarily differentiable function $f(t)$. For Equation (2), it seems that there is no solution which is similar to (3). Several other equations and their multiple soliton solutions were investigated by Tian and Zhang [11, 12], Yue and Chen [13], and Wazwaz [14, 15]. Furthermore, the other new and interesting solutions of some important nonlinear equations were investigated, such as the NlS(m,n) equation [16], the GP(m,n) equation [17], and new nonlocal models [18].

Based on Equations (1) and (2), a $(3+1)$-dimensional equation

$$u_{yzt} + u_{xxxyz} - 6u_x u_{xyz} - 6u_{xy} u_{xz} = 0 \quad (4)$$

was introduced by Wazwaz [19] as a higher-dimensional shallow water wave equation. It is easy to see that Equation (4) can be reduced to the potential KdV equation for $z = y = x$.

In [19], Wazwaz investigated multiple soliton solutions and multiple singular soliton solutions of Equation (4). For Equation (4), Wazwaz gave the solutions combined by $e^{k_i x + r_i y + s_i z - k_i^3 t}$ $(i = 1, 2, 3)$, where k_i, r_i, and s_i are arbitrary constants. In [20], we succeed in constructing the following generalized soliton solutions:

(1) The generalized 1-soliton solution

$$u = \frac{-2k_1\varphi_1 e^{\theta_1}}{p(y,z) + \varphi_1 e^{\theta_1}} + \gamma(t,y,z) \tag{5}$$

(2) The generalized 2-soliton solution

$$u = \frac{-2\left(k_1\varphi_1 e^{\theta_1} + k_2\varphi_2 e^{\theta_2} + a_0(k_1+k_2)\varphi_1\varphi_2 e^{\theta_1+\theta_2}\right)}{p(y,z) + \varphi_1 e^{\theta_1} + \varphi_2 e^{\theta_2} + a_0\varphi_1\varphi_2 e^{\theta_1+\theta_2}} + \gamma(t,y,z), \tag{6}$$

where k_1 and k_2 are arbitrary constants; $g(t,y)$, $h(t,z)$, $e(y,z)$, $p(y,z)$, $\varphi_1 = \varphi_1(y,z)$, and $\varphi_2 = \varphi_2(y,z)$ are arbitrarily differentiable functions; $\gamma(t,y,z) = g(t,y) + h(t,z) + e(y,z)$; and $a_0 = (k_1-k_2)^2/((k_1-k_2)^2[p(y,z)])$

However, we failed to gain the generalized N-soliton solutions ($N \geq 3$) with the same form.

In this paper, using a simplified Hirota's method [21] which is different from classical Hirota's method [22–25], we obtain generalized N-soliton solutions and their dispersion relations. For Equation (4), these solutions are combined by $p(y,z)$ and $\varphi_i(y,z)e^{k_i x - k_i^3 t}$, where $\varphi_i(y,z)$ $(i = 1, 2, \cdots, n)$ are arbitrarily differentiable functions and k_i $(i = 1, 2, \cdots, n)$ are arbitrary constants. This construction of N-soliton solution is novel and has not been seen before.

The structure of this paper is organized as follows: In Section 2, our main results are presented. A short conclusion is given in Section 3.

2. The New Generalized N-Soliton Solutions

In this section, we state the main results for Equation (4) by means of simplified Hirota's method [21]. Depending on the different constructions of Equation (4), we use more complex test functions.

Firstly, we consider the linear part of Equation (4)

$$u_{yzt} + u_{xxxyz} = 0. \tag{7}$$

Substituting $u = \varphi_i e^{k_i x - c_i t}$ ($i = 1, 2, \cdots, n$ and c_i should be determined) into Equation (7), we obtain $c_i = k_i^3$.

For $N = 1$, based on the different constructions of Equation (4) and $c_i = k_i^3$, we can let

$$F_1 = p(y,z) + \varphi_1 e^{\theta_1} \tag{8}$$

and assume that Equation (4) has solution of the following form:

$$u = \frac{R(\partial F_1/\partial x)}{F_1} + \gamma(t,y,z), \tag{9}$$

where R is to be determined. Substituting (9) into Equation (4), we acquire $R = -2$ and the following generalized 1-soliton solution:

$$u = \frac{-2k_1\varphi_1 e^{\theta_1}}{p(y,z) + \varphi_1 e^{\theta_1}} + \gamma(t,y,z), \tag{10}$$

which is equal to Equation (5).

For $N = 2$, let

$$F_2 = p(y,z) + \varphi_1 e^{\theta_1} + \varphi_2 e^{\theta_2} + \varphi_3 e^{\theta_3} + \frac{a}{p(y,z)}\varphi_1\varphi_2 e^{\theta_1+\theta_2}, \tag{11}$$

where a is to be determined. Because of $R = -2$, we can assume that Equation (4) has solution of the following form:

$$u = \frac{-2(\partial F_2/\partial x)}{F_2} + \gamma(t,y,z). \tag{12}$$

Substituting (12) into Equation (4), then we obtain $a = (k_1-k_2)^2/(k_1-k_2)^2$, $a_0 = a/p(y,z)$, and the following generalized 2-soliton solution:

$$u = \frac{-2\left(k_1\varphi_1 e^{\theta_1} + k_2\varphi_2 e^{\theta_2} + a_0(k_1+k_2)\varphi_1\varphi_2 e^{\theta_1+\theta_2}\right)}{p(y,z) + \varphi_1 e^{\theta_1} + \varphi_2 e^{\theta_2} + a_0\varphi_1\varphi_2 e^{\theta_1+\theta_2}} + \gamma(t,y,z), \tag{13}$$

which is equal to Equation (6).

For $N = 3$, based on the result of the generalized 1-soliton and 2-soliton solutions above and $a = ((k_1-k_2)/(k_1+k_2))^2$, we should let

$$\begin{aligned}
&\theta_3 = k_3 x - k_3^3 t,\\
&a_{ij} = \left(\frac{k_i - k_j}{k_i + k_j}\right)^2, \quad (1 \leq i < j \leq 3),\\
&F_3 = p(y,z) + \varphi_1 e^{\theta_1} + \varphi_2 e^{\theta_2} + \varphi_3 e^{\theta_3}\\
&\quad + \frac{a_{12}}{p(y,z)}\varphi_1\varphi_2 e^{\theta_1+\theta_2} + \frac{a_{13}}{p(y,z)}\varphi_1\varphi_3 e^{\theta_1+\theta_3}\\
&\quad + \frac{a_{23}}{p(y,z)}\varphi_2\varphi_3 e^{\theta_2+\theta_3} + d_{123}\varphi_1\varphi_2\varphi_3 e^{\theta_1+\theta_2+\theta_3}
\end{aligned} \tag{14}$$

and suppose that Equation (4) has solution of the following form:

$$u = \frac{-2(\partial F_3/\partial x)}{F_3} + \gamma(t,y,z), \tag{15}$$

where d_{123} is to be determined. Substituting (15) into Equation (4), we have

$$d_{123} = \frac{a_{12}a_{13}a_{23}}{p(y,z)^2}. \tag{16}$$

This implies that the function in (15) $(N=3)$ is a generalized 3-soliton solution of Equation (4). For $N>3$, we can obtain the N-soliton solution with the parallel manner.

In conclusion, for $N \geq 1$, arbitrarily given constants k_i and arbitrarily given differentiable functions $g(t,y)$, $h(t,z)$, $e(y,z)$, $p(y,z)$, and $\varphi_i=\varphi_i(y,z)(i=1,2,\cdots,N)$, let

$$
\begin{aligned}
&\gamma=(t,y,z)=g(t,y)+h(t,z)+e(y,z),\\
&\theta_i=k_i x-k_i^3 t \quad (i=1,2,\cdots,N),\\
&a_{ij}=\left(\frac{k_i-k_j}{k_i+k_j}\right)^2 \quad (1\leq i<j\leq N),\\
&f=p(y,z)+\sum_{i=1}^{N} f_i,\\
&f_1=\sum_{i=1}^{N}\varphi_1 \exp\theta_i,\\
&f_k=p(y,z)^{1-k}\sum_{i_1<i_2<\cdots<i_k}^{(N)}\prod_{j=1}^{k}\varphi_{i_j}\prod_{i_m<i_n}^{(k)}a_{i_m i_n}\exp\\
&\quad\cdot\left[\sum_{j=1}^{k}\theta_{i_j}\right], \quad (1<k<N),\\
&\cdots,\\
&f_N=p(y,z)^{1-N}\prod_{i=1}^{N}\varphi_i\prod_{i<j}^{(N)}a_{ij}\exp\left[\sum_{i=1}^{N}\theta_i\right],
\end{aligned} \tag{17}
$$

where the notation $\sum_{i_1<i_2<\cdots<i_k}^{(N)}$ means the summation over all possible combinations of $0<i_1<i_2<\cdots<i_k\leq N, (1<k<N)$. Then Equation (4) has the following generalized N-soliton solutions:

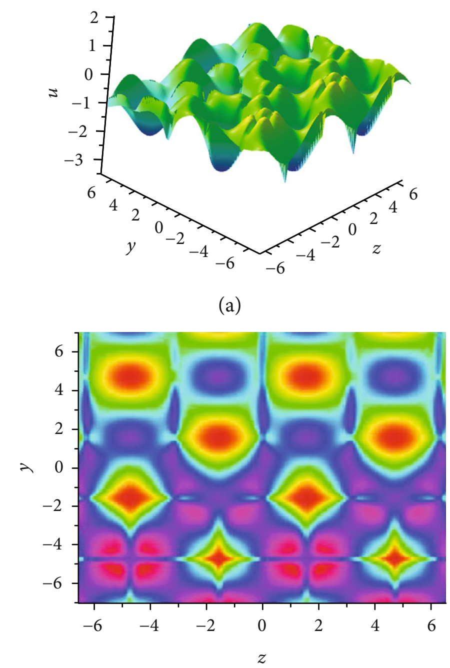
$$
u=\frac{-2(\partial f/\partial x)}{f}+\gamma(t,y,z). \tag{18}
$$

Remark 1. The N-soliton solutions $(N>2)$ contain arbitrarily differentiable function $p(y,z)$, and this phenomenon has not been found in previous works. Because the N-soliton solutions we gained include a lot of arbitrarily differentiable functions, they have some interesting dynamic properties. We take 1- and 2-solitons to display some in Figures 1 and 2.

Remark 2. Since these solutions contain arbitrary functions, we add the word "generalized" to each type of solutions.

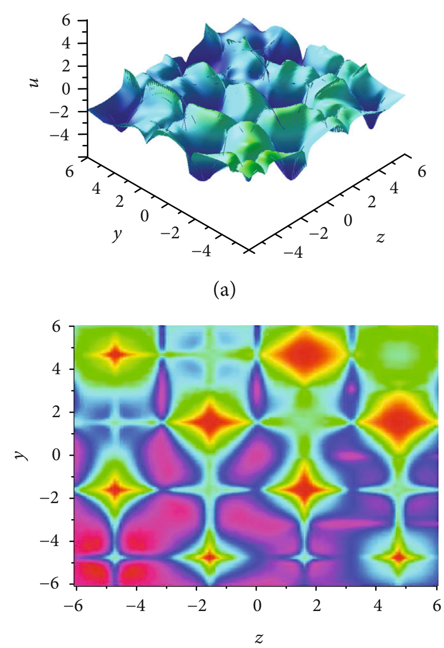
3. Conclusion

In this paper, we succeed in presenting generalized N-soliton solutions by virtue of the special construction of Equation (4). We have obtained many new expressions of the solutions which are listed in Section 2. It is interesting that these expressions contain some arbitrary functions. Moreover, we can see that these generalized N-soliton solutions



(b)

Figure 1: The graph of generalized 1-soliton solution for Equation (4), where $x=t=0$, $k_1=1$, $p(y,z)=\cos^2(y)\cos^2(z)$, $\varphi_1(y,z)=\exp(y)\sin^2(z)$, and $\gamma(t,y,z)=\sin(y)\sin(z)$. (a) Perspective view of the u in the (y,z,u) space and (b) overhead view of the u.



(b)

Figure 2: The graph of generalized 2-soliton solution for Eq.(4), where $x=t=0$, $k_1=2$, $k_2=1$, $p(y,z)=\cos^2(y)\cos^2(z)$, $\varphi_1(y,z)=\exp(y)\sin^2(z)$, $\varphi_2=(y,z)=\exp(z)\sin^2(y)$, and $\gamma(t,y,z)=\sin(y)\sin(z)$. (a) Perspective view of the u in the (y,z,u) space and (b) overhead view of the u.

possess the same phase shift, besides the same dispersion relations. Furthermore, we are certain that this phenomenon will be found in other equation with similar construction and it will be our next goal to utilize our results to other physical equations.

Data Availability

The data used to support this work are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work is supported by the NSF of Shenzhen University (Contact No. 2019082).

References

[1] M. Jimbo and T. Miwa, "Solitons and infinite-dimensional Lie algebras," *Publications of the Research Institute for Mathematical Sciences*, vol. 19, no. 3, pp. 943–1001, 1983.

[2] M. Boiti, J. J. . P. Leon, M. Manna, and F. Pempinelli, "On the spectral transform of a Korteweg-de Vries equation in two spatial dimensions," *Inverse Problems*, vol. 2, no. 3, pp. 271–279, 1986.

[3] P. A. Clarkson and E. L. Mansfield, "On a shallow water wave equation," *Nonlinearity*, vol. 7, no. 3, pp. 975–1000, 1994.

[4] B. F. Feng, L. Ling, and Z. Zhu, "Reply to comment on defocusing complex short-pulse equation and its multi-dark-soliton solution," *Physical Review E*, vol. 96, no. 2, article 026202, 2017.

[5] M. J. Ablowitz, B. F. Feng, X. D. Luo, and Z. H. Musslimani, "Inverse scattering transform for the nonlocal reverse space-time nonlinear Schrödinger equation," *Theoretical and Mathematical Physics*, vol. 196, no. 3, pp. 1241–1267, 2018.

[6] B. Xia, Z. Qiao, and J. Li, "An integrable system with peakon, complex peakon, weak kink, and kink-peakon interactional solutions," *Communications in Nonlinear Science and Numerical Simulation*, vol. 63, pp. 292–306, 2018.

[7] Z. S. Wen and L. J. Shi, "Dynamical behaviors of traveling wave solutions to a Fujimoto-Watanabe equation," *Chinese Physics B*, vol. 27, no. 9, article 090201, 2018.

[8] Y. Shi, X. Li, and B. Zhang, "Traveling wave solutions of two nonlinear wave equations by (G'/G)-expansion method," *Advances in Mathematical Physics*, vol. 2018, Article ID 8583418, 8 pages, 2018.

[9] Z. Wen, "Abundant dynamical behaviors of bounded traveling wave solutions to generalized θ-equation," *Computational Mathematics and Mathematical Physics*, vol. 59, no. 6, pp. 926–935, 2019.

[10] J. Rao, J. He, D. Mihalache, and Y. Cheng, "Dynamics and interaction scenarios of localized wave structures in the Kadomtsev-Petviashvili-based system," *Applied Mathematics Letters*, vol. 94, pp. 166–173, 2019.

[11] S. Tian and H. Zhang, "Riemann theta functions periodic wave solutions and rational characteristics for the nonlinear equations," *Journal of Mathematical Analysis and Applications*, vol. 371, no. 2, pp. 585–608, 2010.

[12] S. F. Tian and H. Q. Zhang, "Riemann theta functions periodic wave solutions and rational characteristics for the (1+1)-dimensional and (2+1)-dimensional Ito equation," *Chaos, Solitons & Fractals*, vol. 47, pp. 27–41, 2013.

[13] Y. Yue and Y. Chen, "Dynamics of localized waves in a (3+1)-dimensional nonlinear evolution equation," *Modern Physics Letters B*, vol. 33, no. 9, article 1950101, 2019.

[14] A. M. Wazwaz, "Multiple-soliton solutions for a (3 + 1)-dimensional generalized KP equation," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 2, pp. 491–495, 2012.

[15] A. M. Wazwaz, "Kadomtsev-Petviashvili hierarchy: N-soliton solutions and distinct dispersion relations," *Applied Mathematics Letters*, vol. 52, pp. 74–79, 2016.

[16] Z. Yan, "Envelope compactons and solitary patterns," *Physics Letters A*, vol. 355, no. 3, pp. 212–215, 2006.

[17] Z. Yan, "Localized analytical solutions and parameters analysis in the nonlinear dispersive Gross-Pitaevskii mean-field GP (m,n) model with space-modulated nonlinearity and potential," *Studies in Applied Mathematics*, vol. 132, no. 3, pp. 266–284, 2014.

[18] Z. Yan, "Integrable $\mathscr{PT}$-symmetric local and nonlocal vector nonlinear Schrodinger equations: a unified two-parameter model," *Applied Mathematics Letters*, vol. 47, pp. 61–68, 2015.

[19] A. M. Wazwaz, "Multiple-soliton solutions and multiple-singular soliton solutions for two higher-dimensional shallow water wave equations," *Applied Mathematics and Computation*, vol. 211, no. 2, pp. 495–501, 2009.

[20] Y. Chen and R. Liu, "Some new nonlinear wave solutions for two (3+1)-dimensional equations," *Applied Mathematics and Computation*, vol. 260, pp. 397–411, 2015.

[21] A. M. Wazwaz, "A two-mode modified KdV equation with multiple soliton solutions," *Applied Mathematics Letters*, vol. 70, pp. 1–6, 2017.

[22] R. Hirota, "Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons," *Physical Review Letters*, vol. 27, no. 18, pp. 1192–1194, 1971.

[23] X. Lü, H. W. Zhu, X. H. Meng, Z. C. Yang, and B. Tian, "Soliton solutions and a Bäcklund transformation for a generalized nonlinear Schrödinger equation with variable coefficients from optical fiber communications," *Journal of Mathematical Analysis and Applications*, vol. 336, no. 2, pp. 1305–1315, 2007.

[24] J. Rao, Y. Zhang, A. S. Fokas, and J. He, "Rogue waves of the nonlocal Davey-Stewartson I equation," *Nonlinearity*, vol. 31, no. 9, pp. 4090–4107, 2018.

[25] J. Rao, K. Porsezian, J. He, and T. Kanna, "Dynamics of lumps and dark–dark solitons in the multi-component long-wave–short-wave resonance interaction system," *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 474, no. 2209, article 20170627, 2018.