Research Article

Broer-Kaup System with Corrections via Inverse Scattering Transform

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The Broer-Kaup system with corrections is considered. Based on the inverse scattering transform, we extend the perturbation theory to discuss the adiabatic approximate solution and ε -order approximate solution of one soliton to the Broer-Kaup system with corrections.

1. Introduction

The coupled integrable system or the Broer-Kaup (BK) system [1–5]

$$t_t = \frac{1}{2} (v^2 + 2\omega - v_x)_x, \quad (1)$$

$$\omega_t = \left(v\omega + \frac{1}{2}\omega_x \right)_x, \quad (2)$$

is related to one of the Boussinesq systems by variable transformation. The BK system is used to simulate the bidirectional propagation of long waves in shallow water. Here, we assume that v and ω decay rapidly as $|x| \rightarrow \infty$.

Kaup first proposed the perturbation theory based upon the inverse scattering transform [6]. In addition to the perturbation theory based upon the inverse scattering transform [7–15], there are many other methods, such as the direct soliton perturbation theory [16–30] and the normal form method [31–34]. For more detail about the perturbation theory, see [11, 35] and references therein.

In this paper, we consider the BK system with corrections through perturbation theory based upon the inverse scattering transform [15] and investigate the adiabatic approximate solution and ε -order approximate solution of one soliton to the BK system with corrections. In order to carry out the inverse scattering transform to discuss the

BK system with corrections, we keep the first Lax equation but give up the second one. For this reason, the analyticity and asymptotic behaviors of Jost functions are the same as the BK system. In this case, the scattering data are all dependent on time, and the time evolution of scattering data is discussed in detail.

This paper organized as follows. In Section 2, we discuss the spectral analysis and construct the Riemann-Hilbert problem of the BK system. In Section 3, we present the BK system with corrections. In Section 4, we consider the time evolution of the scattering coefficients, the discrete spectrum, and the normalization factors. In Section 5, the conservation laws of the Broer-Kaup system without corrections and its perturbation corrections are given. In Section 6, we obtain the adiabatic approximate solution of one soliton, find the slow variations of spectral parameters, and discuss the ε -order approximate solution of one soliton. In Section 7, we give some short conclusions.

2. Spectral Analysis and Riemann-Hilbert Problem

Equation (1) is the compatibility condition of the following Lax pair:

$$F_x = LF, F_t = MF, \quad (3)$$

with

$$\begin{aligned} L(x, \lambda) &= -i\lambda\sigma_3 + U(x), \quad M(x, \lambda) = \lambda^2\sigma_3 + V(x), \\ U(x) &= \begin{pmatrix} \frac{\nu}{2} & 1 \\ -\omega & -\frac{\nu}{2} \end{pmatrix}, \\ V(x) &= \begin{pmatrix} -\frac{1}{4}(\nu_x - \nu^2) & i\lambda + \frac{\nu}{2} \\ -i\lambda\omega - \frac{1}{2}(\omega_x + \omega\nu) & \frac{1}{4}(\nu_x - \nu^2) \end{pmatrix}. \end{aligned} \quad (4)$$

Now we introduce the Jost functions $\Phi(x, \lambda)$ and $\Psi(x, \lambda)$, which satisfy the following boundary problem:

$$\begin{aligned} \partial_x \Phi(x, \lambda) &= L(x, \lambda)\Phi(x, \lambda), \\ \Phi(x, \lambda) &\longrightarrow Ae^{-i\lambda x\sigma_3}, \quad x \longrightarrow -\infty, \\ \partial_x \Psi(x, \lambda) &= L(x, \lambda)\Psi(x, \lambda), \\ \Psi(x, \lambda) &\longrightarrow Ae^{-i\lambda x\sigma_3}, \quad x \longrightarrow \infty, \end{aligned} \quad (5)$$

where

$$A = \begin{pmatrix} 1 & \frac{1}{2i\lambda} \\ 0 & 1 \end{pmatrix}. \quad (6)$$

For simplicity, let the Jost functions have the following form:

$$\Phi(x, \lambda) = \begin{pmatrix} \phi(x, \lambda) & \tilde{\phi}(x, \lambda) \end{pmatrix}, \quad \Psi(x, \lambda) = \begin{pmatrix} \tilde{\psi}(x, \lambda) & \psi(x, \lambda) \end{pmatrix}. \quad (7)$$

There exists a scattering matrix $T(\lambda)$ which satisfies

$$\Phi(x, \lambda) = \Psi(x, \lambda)T(\lambda), \quad T(\lambda) = \begin{pmatrix} a(\lambda) & -\tilde{b}(\lambda) \\ b(\lambda) & \tilde{a}(\lambda) \end{pmatrix}, \quad (8)$$

then we obtain

$$a(\lambda) = W[\phi(x, \lambda), \psi(x, \lambda)], \quad b(\lambda) = W[\tilde{\psi}(x, \lambda), \phi(x, \lambda)], \quad (9)$$

$$\tilde{a}(\lambda) = W[\tilde{\psi}(x, \lambda), \tilde{\phi}(x, \lambda)], \quad -\tilde{b}(\lambda) = W[\tilde{\phi}(x, \lambda), \psi(x, \lambda)], \quad (10)$$

where $W(\bullet, \bullet)$ denotes the Wronskian. With the help of the Neumann series, we find that $\phi(x, \lambda)$ and $\psi(x, \lambda)$ are analytic in $D^+ = \{\lambda \in \mathbb{C} \mid \text{Im}\lambda > 0\}$ and that $\tilde{\phi}(x, \lambda)$ and $\tilde{\psi}(x, \lambda)$ are analytic in $D^- = \{\lambda \in \mathbb{C} \mid \text{Im}\lambda < 0\}$. The asymptotic behaviors of the Jost functions $\Phi(x, \lambda)$ and $\Psi(x, \lambda)$ take the following form:

$$\begin{aligned} \Phi(x, \lambda)e^{i\lambda x\sigma_3} &= \begin{pmatrix} e^{\eta_-} \left(1 + \frac{1}{2i\lambda} \int_{-\infty}^x \omega(y) dy\right) & \frac{1}{2i\lambda} e^{-\eta_-} \\ \frac{1}{2i\lambda} e^{\eta_-} \omega & e^{-\eta_-} \left(1 - \frac{1}{2i\lambda} \int_{-\infty}^x \omega(y) dy\right) \end{pmatrix} \\ &\quad + O\left(\frac{1}{\lambda^2}\right), \end{aligned} \quad (11)$$

$$\begin{aligned} \Psi(x, \lambda)e^{i\lambda x\sigma_3} &= \begin{pmatrix} e^{\eta_+} \left(1 + \frac{1}{2i\lambda} \int_{\infty}^x \omega(y) dy\right) & \frac{1}{2i\lambda} e^{-\eta_+} \\ \frac{1}{2i\lambda} e^{\eta_+} \omega & e^{\eta_+} \left(1 - \frac{1}{2i\lambda} \int_{\infty}^x \omega(y) dy\right) \end{pmatrix} \\ &\quad + O\left(\frac{1}{\lambda^2}\right), \end{aligned} \quad (12)$$

where

$$\eta_{\pm} = \frac{1}{2} \int_{\pm\infty}^x \nu(y) dy. \quad (13)$$

If ν and ω are confined to be real functions, we obtain the symmetry condition $T^*(-\lambda^*) = T(\lambda)$, which implies

$$\begin{aligned} a^*(-\lambda^*) &= a(\lambda), \\ \tilde{a}^*(-\lambda^*) &= a(\lambda), \end{aligned} \quad (14)$$

where the star denotes the complex conjugate. Since $a(\lambda)$ is analytic in D^+ , if $\lambda_n \in D^+$ is a zero of $a(\lambda)$, then $-\lambda_n^*$ is also a zero. Similarly, $\tilde{a}(\lambda)$ has the simple zeroes $\tilde{\lambda}_n$, $-\tilde{\lambda}_n^* \in D^-$. There are two cases for the eigenvalues: $\lambda_n = -\lambda_n^*$ and $\lambda_n \neq -\lambda_n^*$.

In this paper, we only consider the first case. Assume that $a(\lambda)$ has N simple zeros $\lambda_n = i\nu_n$ ($\nu_n > 0$), $\tilde{a}(\lambda)$ has N simple zeros $\lambda_n = i\tilde{\nu}_n$ ($\tilde{\nu}_n < 0$).

Next, introducing the first sectional matrices

$$M^+(x, t, \lambda) = \begin{pmatrix} \frac{\phi(x, t, \lambda)e^{i\lambda x}}{a(\lambda)} & \frac{\psi(x, t, \lambda)e^{-i\lambda x}}{a(\lambda)} \end{pmatrix}, \quad (15)$$

$$M^-(x, t, \lambda) = \begin{pmatrix} \tilde{\psi}(x, t, \lambda)e^{i\lambda x} & \tilde{\phi}(x, t, \lambda)e^{-i\lambda x} \end{pmatrix}, \quad (16)$$

we get the jump condition

$$M^-(x, t, \lambda) = M^+(x, t, \lambda)(I - H(x, t, \lambda)), \quad \lambda \in \mathbb{R}, \quad (17)$$

where the matrix $H(x, t, \lambda)$ is

$$H(x, t, \lambda) = \begin{pmatrix} 0 & \tilde{b}(\lambda)e^{-2i\lambda x} \\ b(\lambda)e^{2i\lambda x} & 0 \end{pmatrix}. \quad (18)$$

From the previous asymptotic behaviors, we find that $M(x, t, \lambda)$ admits the following normalization condition:

$$M^\pm(x, t, \lambda) = B + O\left(\frac{1}{\lambda}\right), \quad \lambda \longrightarrow \infty, \quad (19)$$

$$B = \begin{pmatrix} e^{\eta_+} & 0 \\ 0 & e^{-\eta_-} \end{pmatrix}.$$

Introducing the second sectional matrices

$$\tilde{M}^+(x, t, \lambda) = \left(\phi(x, t, \lambda) e^{i\lambda x}, \psi(x, t, \lambda) e^{-i\lambda x} \right), \quad (20)$$

$$\tilde{M}^-(x, t, \lambda) = \left(\frac{\tilde{\psi}(x, t, \lambda) e^{i\lambda x}}{\tilde{a}(\lambda)}, \frac{\tilde{\phi}(x, t, \lambda) e^{-i\lambda x}}{\tilde{a}(\lambda)} \right), \quad (21)$$

we get another jump condition

$$\tilde{M}^+(x, t, \lambda) = \tilde{M}^-(x, t, \lambda) (I + \tilde{H}(x, t, \lambda)), \quad \lambda \in \mathbb{R}, \quad (22)$$

where the matrix $\tilde{H}(x, t, \lambda)$ is

$$\tilde{H}(x, t, \lambda) = \begin{pmatrix} 0 & \tilde{b}(\lambda) e^{-2i\lambda x} \\ b(\lambda) e^{2i\lambda x} & 0 \end{pmatrix}. \quad (23)$$

In a similar way, $\tilde{M}^+(x, t, \lambda)$ has the following normalization condition:

$$\tilde{M}^\pm(x, t, \lambda) = \tilde{B} + O\left(\frac{1}{\lambda}\right), \quad \lambda \longrightarrow \infty, \quad (24)$$

$$\tilde{B} = \begin{pmatrix} e^{\eta_-} & 0 \\ 0 & e^{-\eta_+} \end{pmatrix}.$$

From (11), (15), and (20), we get

$$M^\pm(x, t, \lambda) = B + \frac{1}{2i\lambda} B^{(1)} + O\left(\frac{1}{\lambda^2}\right), \quad \lambda \longrightarrow \infty, \quad (25)$$

$$\tilde{M}^\pm(x, t, \lambda) = \tilde{B} + \frac{1}{2i\lambda} \tilde{B}^{(1)} + O\left(\frac{1}{\lambda^2}\right), \quad \lambda \longrightarrow \infty, \quad (26)$$

$$B^{(1)} = \begin{pmatrix} e^{\eta_+} \int_{-\infty}^x \omega(y) dy & e^{-\eta_-} \\ e^{\eta_+} & -e^{-\eta_-} \int_{-\infty}^x \omega(y) dy \end{pmatrix}, \quad (27)$$

$$\tilde{B}^{(1)} = \begin{pmatrix} e^{\eta_-} \int_{-\infty}^x \omega(y) dy & e^{-\eta_+} \\ e^{\eta_-} & -e^{-\eta_+} \int_{-\infty}^x \omega(y) dy \end{pmatrix}. \quad (28)$$

Solving the above Riemann-Hilbert problem, we have

$$M^\pm(x, t, \lambda) = B + \sum_{n=1}^N \frac{\text{Res}[M^\pm, \lambda_n]}{\lambda - \lambda_n} + \mathbf{P}_\pm(M^\pm H), \quad \tilde{M}^\pm(x, t, \lambda)$$

$$= \tilde{B} + \sum_{n=1}^N \frac{\text{Res}[\tilde{M}^\pm, \tilde{\lambda}_n]}{\lambda - \tilde{\lambda}_n} + \mathbf{P}_\pm(\tilde{M}^\pm H), \quad (29)$$

where

$$\text{Res}[M^\pm, \lambda_n] = \left(\frac{\phi(x, t, \lambda_n) e^{i\lambda_n x}}{\dot{a}(\lambda_n)}, \frac{\psi(x, t, \lambda_n) e^{-i\lambda_n x}}{\dot{a}(\lambda_n)} \right), \quad (30)$$

$$\text{Res}[\tilde{M}^\pm, \tilde{\lambda}_n] = \left(\frac{\tilde{\psi}(x, t, \lambda_n) e^{i\lambda_n x}}{\dot{\tilde{a}}(\lambda_n)}, \frac{\tilde{\phi}(x, t, \lambda_n) e^{-i\lambda_n x}}{\dot{\tilde{a}}(\lambda_n)} \right), \quad (31)$$

and the dot denotes the derivative with respect to λ . Here, we have used the Cauchy projectors $\mathbf{P}_\pm$ over the real axis

$$\mathbf{P}_\pm[f](k) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{f(l)}{l - (k \pm i0)} dl. \quad (32)$$

3. BK System with Corrections

Consider the BK system with corrections

$$v_t - \frac{1}{2} (v^2 + 2\omega - v_x)_x = \varepsilon \frac{q(x)}{2}, \quad (33)$$

$$\omega_t - \left(v\omega + \frac{1}{2} \omega_x \right)_x = \varepsilon p(x), \quad (34)$$

where $q(x)$ and $p(x)$ are functionals of v and ω , and ε is a real parameter. When $\varepsilon \longrightarrow 0$, (33) reduces to the BK system without corrections.

To carry out the inverse scattering transform to the BK system with corrections, we keep the first Lax equation

$$\partial_x \phi(x, \lambda) = L(x, \lambda) \phi(x, \lambda). \quad (35)$$

In this way, the analyticity and the asymptotic behavior of Jost functions are the same as those of the BK system without corrections.

Next, we give up the second Lax equation

$$\partial_t \phi(x, \lambda) = M(x, \lambda) \phi(x, \lambda). \quad (36)$$

And introduce the new functions

$$\check{\phi}(x, \lambda) = \{\partial_t - M\} h(\lambda) \phi(x, \lambda), \quad (37)$$

$$\hat{\phi}(x, \lambda) = \{\partial_t - M\} \tilde{h}(\lambda) \tilde{\phi}(x, \lambda). \quad (38)$$

Here, functions $h(\lambda)$ and $\tilde{h}(\lambda)$ satisfy

$$h_t(\lambda) = \lambda^2 h(\lambda), \quad \tilde{h}_t(\lambda) = -\lambda^2 \tilde{h}(\lambda), \quad (39)$$

and $\lambda_t \neq 0$. Computing $\{\partial_t - L\}\phi^\sim(x, \lambda)$ and $\{\partial_t - L\}\hat{\phi}(x, \lambda)$ and using (33), we get

$$\{\partial_t - L\}\phi^\sim(x, \lambda) = G(x, \lambda)h(\lambda)\phi(x, \lambda), \quad (40)$$

$$\{\partial_t - L\}\hat{\phi}(x, \lambda) = G(x, \lambda)\tilde{h}(\lambda)\tilde{\phi}(x, \lambda), \quad (41)$$

where

$$G(x, \lambda) = -i\lambda_t\sigma_3 + \varepsilon Q(x), \quad (42)$$

$$Q(x) = \begin{pmatrix} \frac{q(x)}{2} & 0 \\ -p(x) & -\frac{q(x)}{2} \end{pmatrix}. \quad (43)$$

We note that the BK system with corrections (33) is equivalent to (42). We note that if $v(x)$ and $\omega(x)$ satisfy the BK system with corrections (33), then (42) will determine the associated time evolution of scattering data. Conversely, if the time evolution of scattering data is determined from (42), the solution of the BK system with corrections can be rebuilt. Here and after, we will start from (3.7) to establish the perturbation theory of the BK system with corrections.

4. Evolution of Scattering Data

4.1. Time Dependence of $a(\lambda)$ and $\tilde{a}(\lambda)$. In this section, we discuss the time evolution of the scattering data from (42). For the decay potential, we find, from (37), that

$$\begin{aligned} \phi^\sim(x, \lambda) &\longrightarrow \{\partial_t - \lambda^2\}h(\lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\lambda x}, \quad x \longrightarrow -\infty, \\ \hat{\phi}(x, \lambda) &\longrightarrow \{\partial_t + \lambda^2\}h(\lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\lambda x}, \quad x \longrightarrow -\infty. \end{aligned} \quad (44)$$

We note that, for the BK system with corrections, $a(\lambda)$, $\tilde{a}(\lambda)$, $b(\lambda)$, and $\tilde{b}(\lambda)$ are dependent on t . One may find that functions $\phi^\sim(x, \lambda)$ and $\hat{\phi}(x, \lambda)$ in (37) take the following asymptotic behaviors as $x \longrightarrow \infty$:

$$\begin{aligned} \phi^\sim(x, \lambda) &\longrightarrow h(\lambda)a_t(\lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\lambda x} \\ &+ h(\lambda)\{b_t(\lambda) + \lambda^2 b(\lambda)\} \begin{pmatrix} 1 \\ 2i\lambda \\ 1 \end{pmatrix} e^{i\lambda x}, \end{aligned} \quad (45)$$

$$\begin{aligned} \hat{\phi}(x, \lambda) &\longrightarrow \tilde{h}(\lambda)\{\tilde{b}_t(\lambda) - \lambda^2 \tilde{b}(\lambda)\} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\lambda x} \\ &+ \tilde{h}(\lambda)\tilde{a}_t(\lambda) \begin{pmatrix} 1 \\ 2i\lambda \\ 1 \end{pmatrix} e^{i\lambda x}. \end{aligned} \quad (46)$$

It is remarked that (40) are nonhomogeneous equations about $\phi^\sim(x, \lambda)$ and $\hat{\phi}(x, \lambda)$; thus, they can be expressed as the linear combination of corresponding homogeneous solutions. So, we have the following expressions:

$$\phi^\sim(x, \lambda) = \alpha(x, \lambda)\psi(x, \lambda) + \beta(x, \lambda)\phi(x, \lambda), \quad (47)$$

$$\hat{\phi}(x, \lambda) = \tilde{\alpha}(x, \lambda)\tilde{\psi}(x, \lambda) + \tilde{\beta}(x, \lambda)\tilde{\phi}(x, \lambda), \quad (48)$$

where the coefficients α and β and $\tilde{\alpha}$ and $\tilde{\beta}$ are defined by the following equations:

$$\alpha(x, \lambda) = i \frac{h(\lambda)}{a(\lambda)} \int_{-\infty}^x \phi^T(y, \lambda) \sigma_2 G(y, \lambda) \phi(y, \lambda) dy, \quad (49)$$

$$\beta(x, \lambda) = -i \frac{h(\lambda)}{a(\lambda)} \int_{-\infty}^x \psi^T(y, \lambda) \sigma_2 G(y, \lambda) \phi(y, \lambda) dy, \quad (50)$$

$$\tilde{\alpha}(x, \lambda) = -i \frac{\tilde{h}(\lambda)}{a^-(\lambda) \int_{-\infty}^x \tilde{\phi}^T(y, \lambda) \sigma_2 G(y, \lambda) \tilde{\phi}(y, \lambda) dy}, \quad (51)$$

$$\tilde{\beta}(x, \lambda) = i \frac{\tilde{h}(\lambda)}{a^-(\lambda) \int_{-\infty}^x \tilde{\psi}^T(y, \lambda) \sigma_2 G(y, \lambda) \tilde{\phi}(y, \lambda) dy}. \quad (52)$$

As $x \longrightarrow \infty$, (47) and (48) become

$$\begin{aligned} \phi^\sim(x, \lambda) &\longrightarrow \beta(\infty, \lambda)a(\lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\lambda x} \\ &+ \{\alpha(\infty, \lambda) + \beta(\infty, \lambda)b(\lambda)\} \begin{pmatrix} 1 \\ 2i\lambda \\ 1 \end{pmatrix} e^{i\lambda x}, \end{aligned} \quad (53)$$

$$\begin{aligned} \hat{\phi}(x, \lambda) &\longrightarrow \{\alpha^-(\infty, \lambda) - \beta^-(\infty, \lambda)\tilde{b}(\lambda)\} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\lambda x} \\ &+ \beta^-(\infty, \lambda)\tilde{a}(\lambda) \begin{pmatrix} 1 \\ 2i\lambda \\ 1 \end{pmatrix} e^{i\lambda x}. \end{aligned} \quad (54)$$

Comparing (45) with (53), we find

$$h(\lambda)a_t(\lambda) = \beta(\infty, \lambda)a(\lambda), \quad (55)$$

$$h(\lambda)\{b_t(\lambda) + \lambda^2 b(\lambda)\} = \alpha(\infty, \lambda) + \beta(\infty, \lambda)b(\lambda), \quad (56)$$

$$\tilde{h}(\lambda)\tilde{a}_t(\lambda) = \beta^-(\infty, \lambda)\tilde{a}(\lambda), \quad (57)$$

$$-\tilde{h}(\lambda) \left\{ \tilde{b}_t(\lambda) - \lambda^2 \tilde{b}(\lambda) \right\} = \alpha^-(\infty, \lambda) - \beta^-(\infty, \lambda) \tilde{b}(\lambda). \quad (58)$$

Substituting (49) into (55), we have

$$a_t(\lambda) = -i \int_{-\infty}^{\infty} \psi^T(x, \lambda) \sigma_2 G(x, \lambda) \phi(x, \lambda) dx, \quad (59)$$

$$b_t(\lambda) + \lambda^2 b(\lambda) = -ia^{-1}(\lambda) \int_{-\infty}^{\infty} \left\{ \phi(x, \lambda) - b(\lambda) \psi(x, \lambda) \right\}^T \sigma_2 G(x, \lambda) \phi(x, \lambda) dx, \quad (60)$$

$$\tilde{a}_t(\lambda) = i \int_{-\infty}^{\infty} \tilde{\psi}^T(x, \lambda) \sigma_2 G(x, \lambda) \tilde{\phi}(x, \lambda) dx, \quad (61)$$

$$\tilde{b}_t(\lambda) - \lambda^2 \tilde{b}(\lambda) = ia^{-1}(\lambda) \int_{-\infty}^{\infty} \left\{ \tilde{\phi}(x, \lambda) + \tilde{b}(\lambda) \tilde{\psi}(x, \lambda) \right\}^T \sigma_2 G(x, \lambda) \tilde{\phi}(x, \lambda) dx. \quad (62)$$

One may find that the BK system with corrections reduces to the BK system as $\varepsilon \rightarrow 0$. Under this limitation, $\lambda_t = 0$ and $G(x, \lambda) \rightarrow 0$, then from (59),

$$\begin{aligned} a_t(\lambda) &= 0, \quad b_t(\lambda) + \lambda^2 b(\lambda) = 0, \\ \tilde{a}_t(\lambda) &= 0, \quad \tilde{b}_t(\lambda) - \lambda^2 \tilde{b}(\lambda) = 0. \end{aligned} \quad (63)$$

Let $\rho(\lambda) = b(\lambda)/a(\lambda)$ and $\tilde{\rho}(\lambda) = \tilde{b}(\lambda)/\tilde{a}(\lambda)$, then from (59), we have

$$\begin{aligned} \rho_t(\lambda) + \lambda^2 \rho(\lambda) &= i\varepsilon a^{-2}(\lambda) \int_{-\infty}^{\infty} \phi^T(x, \lambda) \sigma_2 Q(x, \lambda) \phi(x, \lambda) dx, \\ \tilde{\rho}_t(\lambda) - \lambda^2 \tilde{\rho}(\lambda) &= i\varepsilon \tilde{a}^{-2}(\lambda) \int_{-\infty}^{\infty} \tilde{\phi}^T(x, \lambda) \sigma_2 Q(x, \lambda) \tilde{\phi}(x, \lambda) dx, \end{aligned} \quad (64)$$

which reduces to the corresponding results in the BK system as $\varepsilon \rightarrow 0$.

4.2. Time Dependence of λ_n and $\tilde{\lambda}_n$. The energy in the bound state is discrete. For the bound state of the BK system with corrections, the eigenvalues $\{\lambda_n\}$ depend on time. For the bound state, we still have

$$a(\lambda_n) = 0, \quad \tilde{a}(\tilde{\lambda}_n) = 0, \quad (65)$$

$$\phi(x, \lambda_n) = d_n(t) \psi(x, \lambda_n), \quad \tilde{\phi}(x, \tilde{\lambda}_n) = \tilde{d}_n(t) \tilde{\psi}(x, \tilde{\lambda}_n). \quad (66)$$

We note that equations (65) and (66) are valid for any t ; hence,

$$a_t(\lambda_n) = 0, \quad \tilde{a}_t(\tilde{\lambda}_n) = 0. \quad (67)$$

These conditions mean that the corrections are too small to change the bound state solution of the scattering problem. In other words, the soliton solutions of the nonlinear problem can still be found. Let $\lambda \rightarrow \lambda_n$ in (59), using (67) and (42), we have

$$\begin{aligned} 2\lambda_{n,t} \int_{-\infty}^{\infty} \phi_1(x, \lambda_n) \phi_2(x, \lambda_n) dx \\ + \varepsilon \int_{-\infty}^{\infty} \phi^T(x, \lambda_n) \sigma_2 Q(x) \phi(x, \lambda_n) dx &= 0, \\ 2\tilde{\lambda}_{n,t} \int_{-\infty}^{\infty} \tilde{\phi}_1(x, \tilde{\lambda}_n) \tilde{\phi}_2(x, \tilde{\lambda}_n) dx \\ + \varepsilon \int_{-\infty}^{\infty} \tilde{\phi}^T(x, \tilde{\lambda}_n) \sigma_2 Q(x) \tilde{\phi}(x, \tilde{\lambda}_n) dx &= 0, \end{aligned} \quad (68)$$

which can be further reduced to

$$\begin{aligned} \dot{a}(\lambda_n) \lambda_{n,t} &= -\varepsilon d_n^{-1}(t) \int_{-\infty}^{\infty} q(x) \phi_1(x, \lambda_n) \phi_2(x, \lambda_n) dx \\ &\quad + p(x) \phi_1^2(x, \lambda_n) dx, \end{aligned} \quad (69)$$

$$\begin{aligned} \dot{\tilde{a}}(\tilde{\lambda}_n) \tilde{\lambda}_{n,t} &= \varepsilon \tilde{d}_n^{-1}(t) \int_{-\infty}^{\infty} q(x) \tilde{\phi}_1(x, \tilde{\lambda}_n) \tilde{\phi}_2(x, \tilde{\lambda}_n) dx \\ &\quad + p(x) \tilde{\phi}_1^2(x, \tilde{\lambda}_n) dx, \end{aligned} \quad (70)$$

in terms of (9). These equations give the time evolution of parameter λ_n for the BK system with correction terms. When $\varepsilon \rightarrow 0$, (69) become $\lambda_{n,t} = 0$ and $\tilde{\lambda}_{n,t} = 0$; these are the results of the BK system.

4.3. Time Dependence of $d_n(t)$ and $\tilde{d}_n(t)$. To consider the time evolution of the normalization factors, we introduce another set of functions

$$\begin{aligned} \psi^\sim(x, \lambda) &= \{\partial_t - M\} h^{-1}(\lambda) \psi(x, \lambda), \\ \tilde{\psi}^\sim(x, \lambda) &= \{\partial_t - M\} \tilde{h}^{-1}(\lambda) \tilde{\psi}(x, \lambda), \end{aligned} \quad (71)$$

then the two sets of functions satisfy the following equation:

$$\begin{aligned} \phi^\sim(x, \lambda_n) &= h^2(\lambda_n) d_n(t) \psi^\sim(x, \lambda_n) \\ &\quad + h(\lambda_n) \{d_{n,t}(t) + 2\lambda_n^2 d_n(t) \psi(x, \lambda_n)\}, \end{aligned} \quad (72)$$

$$\begin{aligned} \tilde{\phi}^\sim(x, \tilde{\lambda}_n) &= \tilde{h}^2(\tilde{\lambda}_n) \tilde{d}_n(t) \tilde{\psi}^\sim(x, \tilde{\lambda}_n) \\ &\quad + \tilde{h}(\tilde{\lambda}_n) \{\tilde{d}_{n,t}(t) + 2\tilde{\lambda}_n^2 \tilde{d}_n(t) \tilde{\psi}(x, \tilde{\lambda}_n)\}. \end{aligned} \quad (73)$$

In view of the analyticity of $\phi(x, \lambda)$ and $\psi(x, \lambda)$, let $\lambda = \lambda_n$ in (47) and $\tilde{\lambda} = \tilde{\lambda}_n$ in (48); we have

$$\dot{\phi}(x, \lambda_n) = \zeta(x, \lambda_n)\phi(x, \lambda_n), \quad (74)$$

$$\tilde{\phi}(x, \tilde{\lambda}_n) = \tilde{\zeta}(x, \tilde{\lambda}_n)\tilde{\phi}(x, \tilde{\lambda}_n), \quad (75)$$

in terms of (66).

Substituting the above equation into (42), we get

$$\begin{aligned} \zeta_x(x, \lambda_n)\dot{a}(\lambda_n) &= ih(\lambda_n)d_n^{-1}(t)\left\{\dot{\phi}(x, \lambda_n) \right. \\ &\quad \left. - d_n(t)\dot{\psi}(x, \lambda_n)\right\}^T \sigma_2 G(x, \lambda_n)\phi(x, \lambda_n), \end{aligned} \quad (76)$$

$$\begin{aligned} \tilde{\zeta}_x(x, \tilde{\lambda}_n)\dot{\tilde{a}}(\tilde{\lambda}_n) &= -i\tilde{h}(\lambda_n)\tilde{d}_n^{-1}(t)\left\{\dot{\tilde{\phi}}(x, \lambda_n) - \tilde{d}_n(t)\dot{\tilde{\psi}} \right. \\ &\quad \left. \cdot(x, \tilde{\lambda}_n)\right\}^T \sigma_2 G(x, \tilde{\lambda}_n)\tilde{\phi}(x, \tilde{\lambda}_n). \end{aligned} \quad (77)$$

Integrating (76), we find

$$\begin{aligned} \zeta(x, \lambda_n) &= i \frac{h(\lambda_n)}{\dot{a}(\lambda_n)\tilde{d}_n(t)} \int_{-\infty}^x \left\{ \dot{\phi}(y, \lambda_n) \right. \\ &\quad \left. - d_n(t)\dot{\psi}(y, \lambda_n) \right\}^T \sigma_2 G(y, \lambda_n)\phi(y, \lambda_n) dy, \end{aligned} \quad (78)$$

$$\begin{aligned} \tilde{\zeta}(x, \tilde{\lambda}_n) &= -i \frac{\tilde{h}(\tilde{\lambda}_n)}{\dot{\tilde{a}}(\tilde{\lambda}_n)\tilde{d}_n(t)} \int_{-\infty}^x \left\{ \dot{\tilde{\phi}}(y, \lambda_n) - \tilde{d}_n(t)\dot{\tilde{\psi}}(y, \tilde{\lambda}_n) \right\}^T \sigma_2 G(y, \tilde{\lambda}_n)\tilde{\phi}(y, \tilde{\lambda}_n) dy. \end{aligned} \quad (79)$$

It is verified that $\dot{\psi}(x, \lambda_n)$ and $\dot{\psi}(x, \tilde{\lambda}_n)$ admit

$$\{\partial_x - L(x, \lambda_n)\}\dot{\psi}(x, \lambda_n) = G(x, \lambda_n)h^{-1}(\lambda_n)\psi(x, \lambda_n), \quad (80)$$

$$\{\partial_x - L(x, \tilde{\lambda}_n)\}\dot{\tilde{\psi}}(x, \tilde{\lambda}_n) = G(x, \tilde{\lambda}_n)\tilde{h}^{-1}(\tilde{\lambda}_n)\tilde{\psi}(x, \tilde{\lambda}_n), \quad (81)$$

$$\dot{\psi}(x, \lambda_n) = \gamma(x, \lambda_n)\psi(x, \lambda_n), \quad (82)$$

$$\dot{\tilde{\psi}}(x, \tilde{\lambda}_n) = \tilde{\gamma}(x, \tilde{\lambda}_n)\tilde{\psi}(x, \tilde{\lambda}_n), \quad (83)$$

where

$$\begin{aligned} \gamma(x, \lambda_n) &= \frac{-i}{\dot{a}(\lambda_n)h(\lambda_n)} \int_x^\infty \left\{ \dot{\phi}(y, \lambda_n) \right. \\ &\quad \left. - d_n(t)\dot{\psi}(y, \lambda_n) \right\}^T \sigma_2 G(y, \lambda_n)\psi(y, \lambda_n) dy, \end{aligned} \quad (84)$$

$$\begin{aligned} \tilde{\gamma}(x, \tilde{\lambda}_n) &= \frac{i}{\dot{\tilde{a}}(\tilde{\lambda}_n)\tilde{h}(\tilde{\lambda}_n)} \int_x^\infty \left\{ \dot{\tilde{\phi}}(y, \lambda_n) \right. \\ &\quad \left. - \tilde{d}_n(t)\dot{\tilde{\psi}}(y, \tilde{\lambda}_n) \right\}^T \sigma_2 G(y, \tilde{\lambda}_n)\tilde{\psi}(y, \tilde{\lambda}_n) dy. \end{aligned} \quad (85)$$

Substituting (74) and (78) and (82) and (84) into (72), we get the evolution of $d_n(t)$ and $\tilde{d}_n(t)$

$$\begin{aligned} d_{n,t}(t) + 2\lambda_n^2 d_n(t) &= i \frac{\lambda_{n,t}}{\dot{a}(\lambda_n)\tilde{d}_n(t)} \int_{-\infty}^\infty \left\{ \dot{\phi}(x, \lambda_n) - d_n(t)\dot{\psi}(x, \lambda_n) \right\}^T \sigma_1 \phi(x, \lambda_n) dx \\ &\quad + i \frac{\varepsilon}{\dot{a}(\lambda_n)} \int_{-\infty}^\infty \left\{ \dot{\phi}(x, \lambda_n) - d_n(t)\dot{\psi}(x, \lambda_n) \right\}^T \sigma_2 Q(x)\phi(x, \lambda_n) dx, \\ \tilde{d}_{n,t}(t) - 2\tilde{\lambda}_n^2 \tilde{d}_n(t) &= -i \frac{\tilde{\lambda}_{n,t}}{\dot{\tilde{a}}(\tilde{\lambda}_n)\tilde{d}_n(t)} \int_{-\infty}^\infty \left\{ \dot{\tilde{\phi}}(x, \tilde{\lambda}_n) - \tilde{d}_n(t)\dot{\tilde{\psi}}(x, \tilde{\lambda}_n) \right\}^T \sigma_1 \tilde{\phi}(x, \tilde{\lambda}_n) dx \\ &\quad - i \frac{\varepsilon}{\dot{\tilde{a}}(\tilde{\lambda}_n)} \int_{-\infty}^\infty \left\{ \dot{\tilde{\phi}}(x, \tilde{\lambda}_n) - \tilde{d}_n(t)\dot{\tilde{\psi}}(x, \tilde{\lambda}_n) \right\}^T \sigma_2 Q(x)\tilde{\phi}(x, \tilde{\lambda}_n) dx, \end{aligned} \quad (86)$$

which reduce to

$$d_{n,t}(t) + 2\lambda_n^2 d_n(t) = i \frac{\varepsilon}{\dot{a}(\lambda_n)} \int_{-\infty}^{\infty} \left\{ \dot{\phi}(x, \lambda_n) - d_n(t) \dot{\psi}(x, \lambda_n) \right\}^T \sigma_2 Q(x) \phi(x, \lambda_n) dx, \quad (87)$$

$$\tilde{d}_{n,t}(t) - 2\tilde{\lambda}_n^2 \tilde{d}_n(t) = -i \frac{\varepsilon}{\dot{a}(\tilde{\lambda}_n)} \int_{-\infty}^{\infty} \left\{ \dot{\tilde{\phi}}(x, \tilde{\lambda}_n) - \tilde{d}_n(t) \dot{\tilde{\psi}}(x, \tilde{\lambda}_n) \right\}^T \sigma_2 Q(x) \tilde{\phi}(x, \tilde{\lambda}_n) dx. \quad (88)$$

When $\varepsilon \rightarrow 0$, (87) are the results of the BK system without corrections.

5. Perturbation Corrections of the Conservation Laws

The BK system has an infinite number of conservation laws, which can be derived by considering the linear spectral problem in (3), that is,

$$\phi_{1,x} = \left(-i\lambda + \frac{\nu}{2}\right) \phi_1 + \phi_2, \quad (89)$$

$$\phi_{2,x} = -\omega \phi_1 + \left(i\lambda - \frac{\nu}{2}\right) \phi_2. \quad (90)$$

To this end, we eliminate ϕ_2 from system (89) and introduce a new function $\hat{\phi}$ by

$$\phi_1 = e^{-i\lambda x + \eta_- + \phi \wedge}, \quad (91)$$

then we get a Riccati equation

$$\hat{\phi}_{xx} + \hat{\phi}_x^2 - 2i\lambda \hat{\phi}_x + \nu \hat{\phi}_x + \omega = 0. \quad (92)$$

Assume that $\hat{\phi}$ has the following expansion:

$$\hat{\phi} = \sum_{j=0}^{\infty} \frac{\hat{\phi}_j}{(2i\lambda)^j}, \quad (93)$$

then by substituting (93) into (92), we find

$$\begin{aligned} \hat{\phi}_1 &= \int_{-\infty}^x \omega(y) dy, \quad \hat{\phi}_2 = \int_{-\infty}^x \omega_y(y) + \nu(y) \omega(y) dy, \\ \hat{\phi}_3 &= \int_{-\infty}^x \omega_{yy}(y) + \omega^2(y) + \nu^2(y) \omega(y) \\ &\quad + (\nu \omega)_y(y) + \nu(y) \omega_y(y) dy. \end{aligned} \quad (94)$$

For the BK system, $a(\lambda)$ and $b(\lambda)$ are independent of t , then we obtain the following conserved densities:

$$I_1 = \int_{-\infty}^{\infty} \omega(x) dx, \quad I_2 = \int_{-\infty}^{\infty} \nu(x) \omega(x) dx, \quad (95)$$

$$I_3 = \int_{-\infty}^{\infty} \omega^2(x) + \nu^2(x) \omega(x) + \nu(x) \omega_x(x) dx, \quad (96)$$

$$\frac{d}{dt} I_j = 0, \quad I_j = \int_{-\infty}^{\infty} T_j(x) dx. \quad (97)$$

Hence, there exists an associated flux $X_j(x)$ that satisfies

$$\partial_t T_j(x) + \partial_x X_j(x) = 0, \quad (98)$$

and the BK system (1); we find

$$T_1 = \omega, \quad X_1(x) = -\left(\nu \omega + \frac{1}{2} \omega_x\right). \quad (99)$$

For the BK system with corrections, we still have

$$\hat{\phi}(\infty, \lambda) = \sum_{j=0}^{\infty} \frac{I_j}{(2i\lambda)^j}. \quad (100)$$

Since, $a(\lambda)$ depends on t , (97) is not valid. In fact, the evolution of densities I_j is an ε -order term which denotes perturbation correction of the conservation laws. To find the evolution, we rewrite the BK system with corrections

$$\begin{aligned} \nu_t &= S + \varepsilon q(x), \quad \omega_t = R + \varepsilon p(x) \\ S &= \frac{1}{2} (\nu^2 + 2\omega - \nu_x)_x, \quad R = \left(\nu \omega + \frac{1}{2} \omega_x\right)_x. \end{aligned} \quad (101)$$

Making use of the functional derivative, we get

$$\frac{d}{dt} I_j = \int_{-\infty}^{\infty} \frac{\delta I_j}{\delta \nu(x)} \frac{d}{dt} \nu(x) + \frac{\delta I_j}{\delta \omega(x)} \frac{d}{dt} \omega(x) dx, \quad (102)$$

$$\begin{aligned} \frac{d}{dt} I_j &= \int_{-\infty}^{\infty} \frac{\delta I_j}{\delta \nu(x)} S + \frac{\delta I_j}{\delta \omega(x)} R dx \\ &\quad + \varepsilon \int_{-\infty}^{\infty} \frac{\delta I_j}{\delta \nu(x)} \frac{q(x)}{2} + \frac{\delta I_j}{\delta \omega(x)} p(x) dx. \end{aligned} \quad (103)$$

We note that the integrand in the first integral is a divergence term which makes the integral vanish. Hence, (103) reduces to

$$\frac{d}{dt} I_j = \varepsilon \int_{-\infty}^{\infty} \frac{\delta I_j}{\delta v(x)} q(x) + \frac{\delta I_j}{\delta \omega(x)} p(x) dx. \quad (104)$$

6. Adiabatic Approximate Solutions

In this section, we consider the adiabatic approximate solution of the BK system with corrections. Since the linear spectral problem is valid, from (15) to (30) and in the case of reflectionless potentials, we still have

$$\begin{aligned} \tilde{\psi}_1(x, \lambda) &= e^{\eta_+} e^{-i\lambda x} + \sum_{j=1}^N \frac{ic_j}{\lambda - \tilde{\lambda}_j} \psi_1(x, \lambda) e^{i\tilde{\lambda}_j x} e^{-i\lambda x}, \tilde{\psi}_2(x, \lambda) \\ &= \sum_{j=1}^N \frac{ic_j}{\lambda - \tilde{\lambda}_j} \psi_2(x, \lambda) e^{i\tilde{\lambda}_j x} e^{-i\lambda x}, \end{aligned} \quad (105)$$

$$\begin{aligned} \psi_1(x, \lambda) &= \sum_{j=1}^N \frac{i\tilde{c}_j}{\lambda - \tilde{\lambda}_j} \tilde{\psi}_1(x, \tilde{\lambda}_j) e^{i\tilde{\lambda}_j x} e^{i\lambda x}, \psi_2(x, \lambda) \\ &= e^{-\eta_+} e^{i\lambda x} + \sum_{j=1}^N \frac{i\tilde{c}_j}{\lambda - \tilde{\lambda}_j} \tilde{\psi}_2(x, \tilde{\lambda}_j) e^{i\tilde{\lambda}_j x} e^{i\lambda x}, \end{aligned} \quad (106)$$

where

$$c_j = \frac{d_j}{i\dot{a}(\lambda_j)}, \tilde{c}_j = \frac{\tilde{d}_j}{i\dot{a}(\tilde{\lambda}_j)}. \quad (107)$$

The solutions of this system can be expressed in a closed form. For example, for $N = 1$, let $\lambda = \tilde{\lambda}_1$ in (105) and $\lambda = \lambda_1$ in (106), and using Cramer's rule, we find

$$\tilde{\psi}_1(x, \tilde{\lambda}_1) = \frac{1}{\Delta} e^{\eta_+} e^{-i\tilde{\lambda}_1 x}, \quad (108)$$

$$\psi_1(x, \lambda_1) = -\frac{\tilde{f}_1}{\Delta} e^{\eta_+ + i\lambda_1 x}, \quad (109)$$

$$\tilde{\psi}_2(x, \tilde{\lambda}_1) = \frac{f_1}{\Delta} e^{-\eta_+ - i\tilde{\lambda}_1 x}, \quad (110)$$

$$\psi_2(x, \lambda_1) = \frac{1}{\Delta} e^{-\eta_+ + i\lambda_1 x}, \quad (111)$$

where

$$\Delta = 1 + f_1 \tilde{f}_1, \quad f_1 = \frac{ic_1}{\lambda_1 - \tilde{\lambda}_1} e^{2i\lambda_1 x}, \quad \tilde{f}_1 = \frac{i\tilde{c}_1}{\tilde{\lambda}_1 - \lambda_1} e^{-2i\tilde{\lambda}_1 x}. \quad (112)$$

Due to

$$a(\lambda) = e^{\eta} \frac{\lambda - \lambda_1}{\lambda - \tilde{\lambda}_1}, \tilde{a}(\lambda) = e^{\eta} \frac{\lambda - \tilde{\lambda}_1}{\lambda - \lambda_1}, \quad (113)$$

where

$$\eta = \eta_- - \eta_+ = \frac{1}{2} \int_{-\infty}^{\infty} v(x) dx. \quad (114)$$

Hence,

$$f_1 = -d_1(t) e^{-\eta} e^{-2i\lambda_1 x}, \tilde{f}_1 = \tilde{d}_1(t) e^{-\eta} e^{-2i\tilde{\lambda}_1 x}. \quad (115)$$

For the BK system and $N = 1$, $\lambda_1 = i\nu_1$ and $\tilde{\lambda}_1 = i\tilde{\nu}_1$, using $d_{1,t}(t) + 2\lambda_1^2 d_1(t) = 0$ and $\tilde{d}_{1,t}(t) - 2\tilde{\lambda}_1^2 \tilde{d}_1(t) = 0$,

$$f_1 = e^{-z}, \tilde{f}_1 = e^{-\tilde{z}}, \quad (116)$$

where

$$Z = \eta + 2\nu_1 x - 2\nu_1^2 t + Z_{10}, \quad (117)$$

$$\tilde{Z} = -\eta - 2\tilde{\nu}_1 x - 2\tilde{\nu}_1^2 t + \tilde{Z}_{10}. \quad (118)$$

Comparing (25) and (29), we obtain

$$\begin{aligned} e^{\eta_+} \omega &= -2c_1 \psi_2(x, \lambda_1) e^{i\lambda_1 x}, \\ e^{-\eta_+} &= -2\tilde{c}_1 \tilde{\psi}_1(x, \tilde{\lambda}_1) e^{-i\tilde{\lambda}_1 x}, \end{aligned} \quad (119)$$

which gives rise to the solution of the BK system

$$\begin{aligned} \nu &= -(\nu_1 + \tilde{\nu}_1) + \tanh(F), \\ \omega &= (\nu_1 - \tilde{\nu}_1)^2 \operatorname{sech}^2(F), \end{aligned} \quad (120)$$

where

$$F = \frac{1}{2} (Z + \tilde{Z}). \quad (121)$$

For the BK system with corrections, λ_1 depends on t and its time evolution is ε -order. $d_1(t)$ has ε -order's correction besides the previous time evolution. In this case, the Jost functions $\phi(x, \lambda)$, $\tilde{\phi}(x, \lambda)$, $\tilde{\psi}(x, \lambda)$, and $\psi(x, \lambda)$ will be denoted by $\phi^a(x, \lambda)$, $\tilde{\phi}^a(x, \lambda)$, $\tilde{\psi}^a(x, \lambda)$, and $\psi^a(x, \lambda)$ and the expressions $e^{\eta_+} \omega$ and $e^{-\eta_+}$ denoted by $(e^{\eta_+} \omega)^a$ and $(e^{-\eta_+})^a$ which give the adiabatic approximate solutions of the BK system with correction terms. We note that $\phi^a(x, \lambda)$, $\tilde{\phi}^a(x, \lambda)$, $\tilde{\psi}^a(x, \lambda)$, and $\psi^a(x, \lambda)$ satisfy the same equations as (105) and (106). In addition,

$$(e^{\eta_+} \omega)^a = -2c_1 \psi_2^a(x, \lambda_1) e^{i\lambda_1 x}, \quad (122)$$

$$(e^{-\eta_+})^a = -2\tilde{c}_1 \tilde{\psi}_1^a(x, \tilde{\lambda}_1) e^{-i\tilde{\lambda}_1 x}. \quad (123)$$

It is important to note that the form of the adiabatic approximate solutions is similar to one soliton solution of the BK system without correction terms, while the scattering data follow the time evolution discussed in Section 4. Hence, $(e^{\eta_+}\omega)^a$ and $(e^{-\eta_+})^a$ are neither the solution of the BK system nor the solution of the BK system with correction terms; they give a part of the ε -order approximate solutions to the BK system with correction terms. In Section 8, we will discuss the other part of the ε -order approximate solutions.

7. Slow Variations of the Spectral Parameters

To discuss the slow variations of the spectral parameters, we rewrite Z and $\tilde{Z}$ in (117) as

$$Z = \eta + 2\nu_1 x + \delta, \quad \frac{d\delta}{dt} = -2\nu_1^2, \quad (124)$$

$$\tilde{Z} = -\eta - 2\tilde{\nu}_1 x + \tilde{\delta}, \quad \frac{d\tilde{\delta}}{dt} = -2\tilde{\nu}_1^2. \quad (125)$$

In this case,

$$d_1(t) = e^\delta, \quad \tilde{d}_1(t) = e^{\tilde{\delta}}. \quad (126)$$

In addition, for reflectionless case and $N = 1$, the Jost functions take the following form:

$$\psi_1(x, \lambda) = \frac{1}{2} \frac{\tilde{\lambda}_1 - \lambda_1}{\lambda - \tilde{\lambda}_1} e^{\eta_+} e^{i\lambda x} f_1 e^F \operatorname{sech}(F), \quad (127)$$

$$\psi_2(x, \lambda) = \frac{e^{i\lambda x} e^{-\eta_+}}{\lambda - \tilde{\lambda}_1} \left[\lambda - \frac{1}{2} (\lambda_1 + \tilde{\lambda}_1) + \frac{1}{2} (\lambda_1 - \tilde{\lambda}_1) \tanh(F) \right], \quad (128)$$

$$\tilde{\psi}_1(x, \lambda) = \frac{e^{-i\lambda x} e^{\eta_+}}{\lambda - \tilde{\lambda}_1} \left[\lambda - \frac{1}{2} (\lambda_1 + \tilde{\lambda}_1) - \frac{1}{2} (\lambda_1 - \tilde{\lambda}_1) \tanh(F) \right], \quad (129)$$

$$\tilde{\psi}_2(x, \lambda) = \frac{1}{2} \frac{\tilde{\lambda}_1 - \lambda_1}{\lambda - \tilde{\lambda}_1} e^{-\eta_+} e^{-i\lambda x} \tilde{f}_1 e^F \operatorname{sech}(F), \quad (130)$$

$$\phi_1(x, \lambda) = \frac{e^{-i\lambda x} e^{\eta_-}}{\lambda - \tilde{\lambda}_1} \left[\lambda - \frac{1}{2} (\lambda_1 + \tilde{\lambda}_1) - \frac{1}{2} (\lambda_1 - \tilde{\lambda}_1) \tanh(F) \right], \quad (131)$$

$$\phi_2(x, \lambda) = \frac{1}{2} \frac{\tilde{\lambda}_1 - \lambda_1}{\lambda - \tilde{\lambda}_1} e^{\eta_-} e^{-i\lambda x} \tilde{f}_1 e^F \operatorname{sech}(F), \quad (132)$$

$$\tilde{\phi}_1(x, \lambda) = \frac{1}{2} \frac{\tilde{\lambda}_1 - \lambda_1}{\lambda - \tilde{\lambda}_1} e^{-\eta_-} e^{i\lambda x} f_1 e^F \operatorname{sech}(F), \quad (133)$$

$$\tilde{\phi}_2(x, \lambda) = \frac{e^{i\lambda x} e^{\eta_-}}{\lambda - \tilde{\lambda}_1} \left[\lambda - \frac{1}{2} (\lambda_1 + \tilde{\lambda}_1) + \frac{1}{2} (\lambda_1 - \tilde{\lambda}_1) \tanh(F) \right]. \quad (134)$$

These equations can be derived by taking $N = 1$ and substituting (108) into (29). It is remarked that $\psi^a(x, \lambda)$, $\tilde{\psi}^a(x, \lambda)$, $\phi^a(x, \lambda)$, and $\tilde{\phi}^a(x, \lambda)$ have a similar form to those in (127) for the BK system with corrections. In fact, the eigenfunctions in (69) are $\psi^a(x, \lambda)$, $\tilde{\psi}^a(x, \lambda)$, $\phi^a(x, \lambda)$, and $\tilde{\phi}^a(x, \lambda)$. Substituting (127) into (69), we get

$$\lambda_{1,t} = -\frac{1}{4} i\varepsilon \int_{-\infty}^{\infty} \operatorname{sech}^2(q(x) - p(x)e^{2\eta_+} e^{-\tilde{Z}}) dF, \quad (135)$$

$$\tilde{\lambda}_{1,t} = -\frac{1}{4} i\varepsilon \int_{-\infty}^{\infty} \operatorname{sech}^2(q(x) + p(x)e^{2\eta_+} e^Z) dF,$$

or

$$\nu_{1,t} = -\frac{1}{4} \varepsilon \int_{-\infty}^{\infty} \operatorname{sech}^2(q(x) - p(x)e^{2\eta_+} e^{-\tilde{Z}}) dF, \quad (136)$$

$$\tilde{\nu}_{1,t} = -\frac{1}{4} \varepsilon \int_{-\infty}^{\infty} \operatorname{sech}^2(q(x) + p(x)e^{2\eta_+} e^Z) dF.$$

Next, we investigate the variation of parameter in $d_1(t)$ and $\tilde{d}_1(t)$. To this end, combining (127) and (66), we find

$$d_1(t) = -e^{-2i\lambda_1 x} e^{-Z} e^\eta, \quad (137)$$

$$\tilde{d}_1(t) = e^{2i\tilde{\lambda}_1 x} e^{-\tilde{Z}} e^{-\eta}.$$

For the BK system with corrections, (126) is not valid and (124) implies

$$d_{1,t}(t) + 2\lambda_1^2 d_1(t) = d_1(t) (\delta_t - 2\nu_1^2), \quad (138)$$

$$\tilde{d}_{1,t}(t) - 2\tilde{\lambda}_1^2 \tilde{d}_1(t) = \tilde{d}_1(t) (\tilde{\delta}_t + 2\tilde{\nu}_1^2). \quad (139)$$

Using relevant integral equations in (87), we find from (138) that

$$\delta_t = 2\nu_1^2 - \frac{\varepsilon}{2} \int_{-\infty}^{\infty} x \operatorname{sech}^2(F) (q(x) - p(x)e^{2\eta_+} e^{-\tilde{Z}}) \\ + \frac{\operatorname{sech}(F)}{\nu_1 - \tilde{\nu}_1} (q(x) \sin(F) - p(x)e^{2\eta_+} e^Z e^{-F}) dF, \\ \tilde{\delta}_t = -2\tilde{\nu}_1^2 + \frac{\varepsilon}{2} \int_{-\infty}^{\infty} x \operatorname{sech}^2(F) (q(x) + p(x)e^{2\eta_+} e^{-Z}) \\ - \frac{\operatorname{sech}(F)}{\tilde{\nu}_1 - \nu_1} (q(x) \cos(F) - p(x)e^{2\eta_+} e^{-\tilde{Z}} e^F) dF. \quad (140)$$

Now, for one soliton solution, the formulas of spectral parameters are given.

8. Corrections of the Adiabatic Solutions

In this section, we will discuss the other part of the ε -order approximate solutions to the BK system with correction terms. In the case of $N = 1$,

$$\tilde{\psi}_1(x, \lambda) e^{i\lambda x} = e^{\eta_+} + \frac{ic_1}{\lambda - \lambda_1} \psi_1(x, \lambda_1) e^{i\lambda_1 x} + J_1(x, \lambda), \quad (141)$$

$$\tilde{\psi}_2(x, \lambda) e^{i\lambda x} = \frac{ic_1}{\lambda - \lambda_1} \psi_2(x, \lambda_1) e^{i\lambda_1 x} + J_2(x, \lambda), \quad (142)$$

$$\psi_1(x, \lambda) e^{-i\lambda x} = \frac{i\tilde{c}_1}{\lambda - \tilde{\lambda}_1} \tilde{\psi}_1(x, \tilde{\lambda}_1) e^{-i\tilde{\lambda}_1 x} + \tilde{J}_1(x, \lambda), \quad (143)$$

$$\psi_2(x, \lambda) e^{-i\lambda x} = e^{-\eta_+} + \frac{i\tilde{c}_1}{\lambda - \tilde{\lambda}_1} \tilde{\psi}_2(x, \tilde{\lambda}_1) e^{-i\tilde{\lambda}_1 x} + \tilde{J}_2(x, \lambda), \quad (144)$$

where

$$J_j(x, \lambda) = \frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{\rho(k)}{k - \lambda + i0} \psi_j(x, k) e^{ikx} dk, \quad j = 1, 2,$$

$$\tilde{J}_j(x, \lambda) = \frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{\tilde{\rho}(k)}{k - \lambda - i0} \tilde{\psi}_j(x, k) e^{-ikx} dk, \quad j = 1, 2. \quad (145)$$

To solve system (141) to ε -order, we introduce

$$\begin{aligned} \psi(x, \lambda) &= \psi^a(x, \lambda) + \delta\psi(x, \lambda), \\ \tilde{\psi}(x, \lambda) &= \tilde{\psi}^a(x, \lambda) + \delta\tilde{\psi}(x, \lambda), \end{aligned} \quad (146)$$

and note that $\psi^a(x, \lambda)$ and $\tilde{\psi}^a(x, \lambda)$ satisfy

$$\begin{aligned} \tilde{\psi}_1^a(x, \lambda) &= e^{\eta_+} e^{-i\lambda x} + \frac{ic_1}{\lambda - \lambda_1} \psi_1^a(x, \lambda_1) e^{i\lambda_1 x} e^{-i\lambda x}, \\ \tilde{\psi}_2^a(x, \lambda) &= \frac{ic_1}{\lambda - \lambda_1} \psi_2^a(x, \lambda_1) e^{i\lambda_1 x} e^{-i\lambda x}, \\ \psi_1^a(x, \lambda) &= \frac{i\tilde{c}_1}{\lambda - \tilde{\lambda}_1} \tilde{\psi}_1^a(x, \tilde{\lambda}_1) e^{-i\tilde{\lambda}_1 x} e^{i\lambda x}, \\ \psi_2^a(x, \lambda) &= e^{-\eta_+} e^{i\lambda x} + \frac{i\tilde{c}_1}{\lambda - \tilde{\lambda}_1} \tilde{\psi}_2^a(x, \tilde{\lambda}_1) e^{-i\tilde{\lambda}_1 x} e^{i\lambda x}, \end{aligned} \quad (147)$$

which are called the adiabatic approximate equations of system (141), then equations to the ε -order from (141) take the following form:

$$\delta\tilde{\psi}_1(x, \lambda) e^{i\lambda x} = \frac{ic_1}{\lambda - \lambda_1} \delta\psi_1(x, \lambda_1) e^{i\lambda_1 x} + J_1^a(x, \lambda), \quad (148)$$

$$\delta\tilde{\psi}_2(x, \lambda) e^{i\lambda x} = \frac{ic_1}{\lambda - \lambda_1} \delta\psi_2(x, \lambda_1) e^{i\lambda_1 x} + J_2^a(x, \lambda), \quad (149)$$

$$\delta\psi_1(x, \lambda) e^{-i\lambda x} = \frac{i\tilde{c}_1}{\lambda - \tilde{\lambda}_1} \delta\tilde{\psi}_1(x, \tilde{\lambda}_1) e^{-i\tilde{\lambda}_1 x} + \tilde{J}_1^a(x, \lambda), \quad (150)$$

$$\delta\psi_2(x, \lambda) e^{-i\lambda x} = \frac{i\tilde{c}_1}{\lambda - \tilde{\lambda}_1} \delta\tilde{\psi}_2(x, \tilde{\lambda}_1) e^{-i\tilde{\lambda}_1 x} + \tilde{J}_2^a(x, \lambda), \quad (151)$$

$$J_j^a(x, \lambda) = \frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{\rho(k)}{k - \lambda} \psi_j^a(x, k) e^{ikx} dk, \quad j = 1, 2, \quad (152)$$

$$\tilde{J}_j^a(x, \lambda) = \frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{\tilde{\rho}(k)}{k - \lambda} \tilde{\psi}_j^a(x, k) e^{-ikx} dk, \quad j = 1, 2. \quad (153)$$

Using Cramer's rule, we find, from (148), that

$$\delta\psi_2(x, \lambda_1) = \frac{1}{\Delta} e^{i\lambda_1 x} \left(\tilde{J}_2^a(x, \lambda_1) + J_2^a(x, \tilde{\lambda}_1) \frac{i\tilde{c}_1}{\lambda - \tilde{\lambda}_1} e^{-2i\tilde{\lambda}_1 x} \right), \quad (154)$$

$$\delta\tilde{\psi}_1(x, \tilde{\lambda}_1) = \frac{1}{\Delta} e^{-i\tilde{\lambda}_1 x} \left(J_1^a(x, \tilde{\lambda}_1) + \tilde{J}_1^a(x, \lambda_1) \frac{ic_1}{\tilde{\lambda}_1 - \lambda_1} e^{2i\lambda_1 x} \right). \quad (155)$$

Hence, ε -order approximate solutions of the BK system with correction terms can be written as

$$e^{\eta_+} \omega = (e^{\eta_+} \omega)^a + \delta(e^{\eta_+} \omega), \quad (156)$$

$$e^{-\eta_+} = (e^{-\eta_+})^a + \delta(e^{-\eta_+}), \quad (157)$$

In addition, from (122) and (156), we know that

$$\delta(e^{\eta_+} \omega) = -2c_1 \delta\psi_2(x, \lambda_1) e^{i\lambda_1 x} + \frac{1}{\pi} \int_{-\infty}^{\infty} \rho(k) \psi_2^a(x, k) dk, \quad (158)$$

$$\delta(e^{-\eta_+}) = -2\tilde{c}_1 \delta\tilde{\psi}_1(x, \tilde{\lambda}_1) e^{-i\tilde{\lambda}_1 x} - \frac{1}{\pi} \int_{-\infty}^{\infty} \tilde{\rho}(k) \tilde{\psi}_1^a(x, k) dk. \quad (159)$$

Substituting (154) into (158), we get

$$\begin{aligned} \delta(e^{\eta_+} \omega) &= \frac{(e^{\eta_+})^a}{\pi} \int_{-\infty}^{\infty} \tilde{\rho}(k) \tilde{\psi}_2^a(x, k)^2 - \rho(k) \psi_2^a(x, k)^2 dk, \\ \delta(e^{-\eta_+}) &= \frac{(e^{-\eta_+})^a}{\pi} \int_{-\infty}^{\infty} \rho(k) \psi_1^a(x, k)^2 - \tilde{\rho}(k) \tilde{\psi}_1^a(x, k)^2 dk. \end{aligned} \quad (160)$$

9. Conclusions and Discussions

In this work, we discussed the perturbation theory of the Broer-Kaup equation with corrections which is a nearly integrable system. The technique based on the inverse scattering transform is extended to obtain the evolutions of the scattering data, including the discrete spectrums, the scattering coefficients, and the normalized factors. These evolution equations of the scattering data are determined by integral representations of corrections and eigenfunctions. Hence, the adiabatic dynamic of one soliton can be figured out by the time-dependent scattering data, if the corrections are given.

In general, the perturbations by adding corrections can be dissipative terms.

Data Availability

The data will be made available on reasonable request.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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