

## Research Article

# Existence of Multispike Positive Solutions for a Nonlocal Problem in $\mathbb{R}^3$

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In this paper, we study the following nonlinear Choquard equation  $-\epsilon^2 \Delta u + K(x)u = ((1)/(8\pi\epsilon^2))(\int_{\mathbb{R}^3} ((u^2(y))/(|x-y|))dy)u$ ,  $x \in \mathbb{R}^3$ , where  $\epsilon > 0$  and  $K(x)$  is a positive bounded continuous potential on  $\mathbb{R}^3$ . By applying the reduction method, we proved that for any positive integer  $k$ , the above equation has a positive solution with  $k$  spikes near the local maximum point of  $K(x)$  if  $\epsilon > 0$  is sufficiently small under some suitable conditions on  $K(x)$ .

## 1. Introduction and Main Results

In this paper, we consider the following nonlinear Choquard equations

$$\begin{cases} -\epsilon^2 \Delta u + K(x)u = \varphi u, & x \in \mathbb{R}^3, \\ -\epsilon^2 \Delta \varphi = \frac{|u|^2}{2}, & x \in \mathbb{R}^3, \end{cases} \quad (1)$$

where  $\epsilon > 0$  and  $K(x)$  is a positive bounded continuous potential. The Choquard equation first appeared as early as in 1954, in a work by Pekar describing the quantum mechanics of a polaron at rest [1]. In 1976, Choquard used it to describe an electron trapped in its own hole in a certain approximation to the Hartree-Fock theory of one component plasma in [2]. Penrose [3] also proposed it as a model of self-gravitating matter, in a programme in which quantum state reduction is understood as a gravitational phenomenon. Moreover, the Choquard equation is also known as the Schrödinger-Newton equation in models coupling the Schrödinger equation of quantum physics together with nonrelativistic Newtonian gravity.

Note that the second equation of (1) can be explicitly solved with respect to  $\varphi$  and then (1) reduces to the following single nonlocal equation

$$-\epsilon^2 \Delta u + K(x)u = \frac{1}{8\pi\epsilon^2} \left( \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy \right) u. \quad (2)$$

Equation (2) has attracted considerable attention in recent period and part of the motivation is due to looking for standing waves for the following nonlinear Hartree equations

$$\begin{aligned} i\epsilon \frac{\partial \psi}{\partial t} + \epsilon^2 \Delta \psi + (K(x) + h)\psi \\ = \frac{1}{8\pi\epsilon^2} \left( \int_{\mathbb{R}^3} \frac{\psi^2(y)}{|x-y|} dy \right) \psi, \quad (x, t) \in \mathbb{R}^3 \times \mathbb{R}^+, \end{aligned} \quad (3)$$

with the form  $\psi(x, t) = e^{-iht/\epsilon} u(x)$ , where  $i$  is the imaginary unit,  $h \in \mathbb{R}$  and  $\epsilon$  is the Planck constant. The above Hartree equations also appear in quantum mechanics models (see [4–6]) and in the semiconductor theory (see [7–9]).

Also, the Choquard equation (2) is a special type of the following generalized Choquard equation

$$-\epsilon^2 \Delta u + K(x)u = \frac{1}{8\pi\epsilon^2} \left( \int_{\mathbb{R}^3} \frac{u^p(y)}{|x-y|^{n-\alpha}} dy \right) |u|^{p-2}u, \quad x \in \mathbb{R}^n, \quad (4)$$

where  $\alpha \in (0, n)$  and  $p > 1$ . The symmetry and the regularity of solutions of (4) have been established by Ma and Zhao [10] and by Cingolani et al. [11], respectively, under the suitable assumptions on  $p$  when  $\epsilon = 1$ . Later, in [12] Moroz and Van Schaftingen derived the regularity, positivity, radial symmetry, and sharp asymptotics of ground state solutions of (4) for the optimal range of parameters (see also [13]).

In particular, taking  $n = 3, p = 2$ , and  $\alpha = 2$  in (4), we get (2). In [14], Lions derived the existence of ground state solutions of (2) under some suitable conditions on  $K(x)$  if  $\epsilon > 0$  is small enough. For any positive integer  $k > 0$ , Wei and Winter [15] proved that there exist a solution of (2) concentrating at  $k$  points which are all local minimums or local maximums or non-degenerate critical points of  $K(x)$  under the conditions that  $\inf_{\mathbb{R}^3} K > 0, K \in C^2(\mathbb{R}^3)$  provided  $\epsilon$  is sufficiently small. Recently, Luo, Peng and Wang [16] showed the uniqueness of positive solutions for (2) concentrating at the non-degenerate critical points of  $K(x)$  by using a local Pohozaev type identity for  $\epsilon > 0$  small enough.

But, when  $\epsilon = 1$  and  $K(x) = 1$ , (2) is written as

$$-\Delta u + u = \frac{1}{8\pi} \left( \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy \right) u, \quad x \in \mathbb{R}^3. \quad (5)$$

In [17], Lieb obtained the existence and uniqueness of ground state solutions of (5) by using variational method (see also [18, 19]). Later, Tod and Moroz and Tod [20] and Wei and Winter [15] proved the nondegeneracy of the ground state solutions of (5).

Applying the existence and the nondegeneracy of ground state solutions for (5), inspired by [21, 22], we want to exploit the finite dimensional reduction method to investigate the existence of positive multi-spikes solutions for (2) under the conditions imposed on  $K(x)$  as follows:

(K<sub>1</sub>)  $K$  has a strict local maximum at some point  $y^0 \in \mathbb{R}^3$ , that is, there is  $\delta > 0$  such that  $K(x) < K(y^0)$  for all  $x \in B_\delta(y^0) \setminus \{y^0\}$ .

(K<sub>2</sub>)  $\inf_{\mathbb{R}^3} K \geq b > 0$  and there exist constants  $L, \theta > 0$  with  $\theta < 1$  such that

$$|K(x) - K(y)| \leq L|x - y|^\theta \quad (6)$$

for all  $x, y \in B_{2\delta}(y^0)$ .

We state our main result as follows:

**Theorem 1.** *Assume that (K<sub>1</sub>), (K<sub>2</sub>) hold, then for any positive integer  $k$ , problem (2) has a  $k$ -spike solution for sufficiently small  $\epsilon > 0$ .*

As in [21–23], we mainly use the finite-dimensional reduction to prove our result. Here, our purpose is to verify

that if  $\epsilon$  is small enough, then for any positive integer  $k$ , (2) has a solution with  $k$ -spikes concentrating near  $y^0$  corresponding to any strict local maximum  $y^0$  of  $K(x)$ , namely, a solution with  $k$  maximum points converging to  $y^0$  as  $\epsilon \rightarrow 0$ .

In the end of this part, let us outline the sketch of our proof of Theorem 1. Denoted by  $w(x)$ , the unique radially positive solution of the following problem

$$\begin{cases} -\Delta u + K(y^0)u = \frac{1}{8\pi} \left( \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy \right) u, & \text{in } \mathbb{R}^3, \\ u(x) > 0 \text{ in } \mathbb{R}^3, u(0) = \max_{\mathbb{R}^3} u(x). \end{cases} \quad (7)$$

It follows from [2, 15] that  $w(x)$  is strictly decreasing and satisfies

$$\lim_{|x| \rightarrow +\infty} w(x)|x|e^{|x|} = c_0, \quad \lim_{|x| \rightarrow +\infty} \frac{w(x)}{w'(x)} = -1, \quad (8)$$

for some constant  $c_0 > 0$ . Also,  $w(x)$  is nondegenerate, namely, if  $\psi(x) \in H^1(\mathbb{R}^3)$  solves the lineared equation

$$\begin{aligned} -\Delta \psi + K(y^0)\psi &= \frac{1}{8\pi} \left( \int_{\mathbb{R}^3} \frac{w^2(y)}{|x-y|} dy \right) \psi(x) \\ &+ \frac{1}{4\pi} \left( \int_{\mathbb{R}^3} \frac{w(y)\psi(y)}{|x-y|} dy \right) w(x), \end{aligned} \quad (9)$$

then  $\psi(x)$  is a linear combination of  $(\partial w / \partial x_j), j = 1, 2, 3$ .

We will use the unique solution  $w$  of (7) to establish the solutions of (2). In what follows, without loss of generality, we assume that  $y^0 = 0$  and  $K(0) = 1$ . Let  $B_r(0) = \{x \in \mathbb{R}^3 : |x| < r\}$  and denote its closure by  $\bar{B}_r(0)$ . For any positive integer  $k$  and large  $R$ , we define

$$\begin{aligned} D_k^{\epsilon, \delta} &= \left\{ \mathbf{y} = (y^1, \dots, y^k) \in (\mathbb{R}^3)^k : y^j \in B_{\delta/2}(y^0), \frac{|y^i - y^j|}{\epsilon} \right. \\ &\left. \geq R, i \neq j, i, j = 1, 2, \dots, k \right\}. \end{aligned} \quad (10)$$

Furthermore, since  $\inf_{\mathbb{R}^3} K > 0$ , we can define the following Soblev space

$$H_\epsilon := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} (\epsilon^2 |\nabla u|^2 + K(x)u^2) dx < \infty \right\}, \quad (11)$$

with the corresponding norm  $\|u\|_\epsilon^2 = \langle u, u \rangle_\epsilon$ , where

$$\langle u, v \rangle_\epsilon = \int_{\mathbb{R}^3} (\epsilon^2 \nabla u \nabla v + K(x)uv) dx, \quad (12)$$

and, in this sequel, we denote by  $|\cdot|_p$  the usual norm of  $L^p(\mathbb{R}^3)$  and let  $\|u\|_D := (\int_{\mathbb{R}^3} |\nabla u|^2 dx)^{1/2}$ ,  $\|u\|_{H^1} := (\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx)^{1/2}$  be the norms of  $D^{1,2}(\mathbb{R}^3)$  and  $H^1(\mathbb{R}^3)$ , respectively.

Now fixing  $\mathbf{y} \in D_k^{\epsilon, \delta}$ , set

$$W_{\epsilon, \mathbf{y}} = \sum_{j=1}^k w_{\epsilon, y^j} = \sum_{j=1}^k w\left(\frac{x - y^j}{\epsilon}\right),$$

$$\mathbf{E}_{\epsilon, k} = \left\{ \phi \in H^1(\mathbb{R}^3) : \left\langle \frac{\partial w_{\epsilon, y^j}}{\partial y_s^j}, \phi \right\rangle_{\epsilon} = 0, \|\phi\|_{\epsilon}^2 \leq \delta \epsilon^3 \right\}, \quad (13)$$

for  $j = 1, \dots, k, s = 1, 2, 3$ . We will prove Theorem 1 by verifying the following result.

**Theorem 2.** *Let  $(K_1)$ ,  $(K_2)$  hold. Then, for any positive integer  $k$ , there is  $\epsilon_0 = \epsilon(k)$  such that for  $\epsilon \in (0, \epsilon_0)$ , problem (2) has a solution  $u_{\epsilon}$  of the form*

$$u_{\epsilon}(x) = \sum_{j=1}^k w\left(\frac{x - y_{\epsilon}^j}{\epsilon}\right) + \phi_{\epsilon}, \quad (14)$$

for some points  $y_{\epsilon}^j \in \mathbb{R}^3, j = 1, \dots, k$  and  $\phi_{\epsilon} \in H^1(\mathbb{R}^3)$  satisfying as  $\epsilon \rightarrow 0$ ,

$$y_{\epsilon}^j \rightarrow y^0, \frac{|y_{\epsilon}^i - y_{\epsilon}^j|}{\epsilon} \rightarrow +\infty (i \neq j), \|\phi_{\epsilon}\|_{\epsilon} = o(\epsilon^{3/2}). \quad (15)$$

We want to point out that compared with [15], we introduce a little stronger conditions imposed on  $K(x)$  than that of [15] and the reduction procedure has been modified here to allow for the degenerate of the critical point of  $K(x)$ . Also, the appearance of nonlocal term forces us to face much difficulties in the reduction process which involves some more delicate estimates.

The rest of the paper is organized as follows. In Section 2, we will carry out a reduction procedure. We prove our main result in Section 3. Finally, in Appendix, some technical estimates and an energy expansion for the functional corresponding to problem (2) will be established.

## 2. The Finite-Dimensional Reduction

Observe that the variational functional corresponding to (2) is

$$I_{\epsilon}(u) = \frac{1}{2} \int_{\mathbb{R}^3} (\epsilon^2 |\nabla u|^2 + K(x) u^2) dx - \frac{1}{32\pi\epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x - y|} dy u^2 dx. \quad (16)$$

Then by the direct computation, we have for any  $\phi \in \mathbf{E}_{\epsilon, k}$

$$\begin{aligned} J_{\epsilon}(y^1, \dots, y^k) &:= J_{\epsilon}(\mathbf{y}, \phi) = I_{\epsilon}(W_{\epsilon, \mathbf{y}} + \phi) \\ &= I_{\epsilon}\left(\sum_{j=1}^k w\left(\frac{x - y^j}{\epsilon}\right) + \phi\right) \\ &= \left\{ \frac{1}{2} \int_{\mathbb{R}^3} (\epsilon^2 |\nabla W_{\epsilon, \mathbf{y}}|^2 + K(x) W_{\epsilon, \mathbf{y}}^2) dx \right. \\ &\quad - \frac{1}{32\pi\epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W_{\epsilon, \mathbf{y}}^2(y)}{|x - y|} dy W_{\epsilon, \mathbf{y}}^2 dx \left. \right\} \\ &\quad + \left\{ \int_{\mathbb{R}^3} (K(x) - 1) W_{\epsilon, \mathbf{y}} \phi dx - \frac{1}{8\pi\epsilon^2} \right. \\ &\quad \cdot \left[ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\sum_{i \neq j} w_{\epsilon, y^i} w_{\epsilon, y^j}}{|x - y|} dy \left( \sum_{j=1}^k w_{\epsilon, y^j} \phi \right) dx \right. \\ &\quad \left. \left. + \sum_{i \neq j} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w_{\epsilon, y^i}^2}{|x - y|} dy w_{\epsilon, y^j} \phi dx \right] \right\} \\ &\quad + \left\{ \frac{1}{2} \int_{\mathbb{R}^3} (\epsilon^2 |\nabla \phi|^2 + K(x) \phi^2) dx \right. \\ &\quad - \frac{1}{16\pi\epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W_{\epsilon, \mathbf{y}}^2(y)}{|x - y|} dy \phi^2 dx \\ &\quad - \frac{1}{8\pi\epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W_{\epsilon, \mathbf{y}} \phi}{|x - y|} dy W_{\epsilon, \mathbf{y}} \phi dx \left. \right\} \\ &\quad - \left\{ \frac{1}{32\pi\epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\phi^2(y)}{|x - y|} dy \phi^2 dx \right. \\ &\quad \left. + \frac{1}{8\pi\epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W_{\epsilon, \mathbf{y}} \phi}{|x - y|} dy \phi^2 dx \right\} \\ &=: J_{\epsilon}(\mathbf{y}, 0) + \ell_{\epsilon, \mathbf{y}}(\phi) + \frac{1}{2} \langle L_{\epsilon, \mathbf{y}}(\phi), \phi \rangle - R_{\epsilon, \mathbf{y}}(\phi), \end{aligned} \quad (17)$$

where we use the fact that  $w_{\epsilon, y^j} (j = 1, \dots, k)$  solves

$$-\epsilon^2 \Delta u + u = \frac{1}{8\pi\epsilon^2} \left( \int_{\mathbb{R}^3} \frac{u^2(y)}{|x - y|} dy \right) u. \quad (18)$$

In order to find a critical point for  $J_{\epsilon}(\mathbf{y}, \phi)$ , we need to discuss each terms in the expansion (17). First, we have

**Lemma 3.** *For any  $u \in L^p(\mathbb{R}^3) (2 \leq p \leq 6)$ , there holds*

$$|u|_p \leq C \epsilon^{3((1/p) - (1/2))} \|u\|_{\epsilon}. \quad (19)$$

*Proof.* Taking  $u_\epsilon(x) = u(\epsilon x)$ , then

$$\begin{aligned} \int_{\mathbb{R}^3} |u|^p dx &= \epsilon^3 \int_{\mathbb{R}^3} |u_\epsilon(x)|^p dx \\ &\leq C \epsilon^3 \left[ \int_{\mathbb{R}^3} (|\nabla u_\epsilon|^2 + K(\epsilon x) u_\epsilon^2) dx \right]^{p/2} \\ &= C \epsilon^3 \left[ \epsilon^{-3} \int_{\mathbb{R}^3} (\epsilon^2 |\nabla u|^2 + K(x) u^2) dx \right]^{p/2} \\ &= C \epsilon^{3(1-(p/2))} \|u\|_\epsilon^p, \end{aligned} \quad (20)$$

which implies the conclusion holds.

**Lemma 4.** *There exists a positive constant  $C$  independent of  $\epsilon$  such that*

$$\|R_{\epsilon,y}^i(\phi)\| \leq C \left( \epsilon^{-3} \|\phi\|_\epsilon + \epsilon^{-(3/2)} \right) \|\phi\|_\epsilon^{3-i}, \quad i = 0, 1, 2, \quad (21)$$

where  $R_{\epsilon,y}^i(\phi)$  denotes the  $i$ th derivative of  $R_{\epsilon,y}(\phi)$ .

*Proof.* Note that

$$\begin{aligned} R_{\epsilon,y}(\phi) &= \frac{1}{32\pi\epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\phi^2(y)}{|x-y|} dy \phi^2 dx \\ &\quad + \frac{1}{8\pi\epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W_{\epsilon,y}\phi}{|x-y|} dy \phi^2 dx. \end{aligned} \quad (22)$$

So it is easy to check that

$$\begin{aligned} \langle R'_{\epsilon,y}(\phi), \psi \rangle &= \frac{1}{8\pi\epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\phi^2(y)}{|x-y|} dy \phi \psi dx \\ &\quad + \frac{1}{8\pi\epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W_{\epsilon,y}\psi}{|x-y|} dy \phi^2 dx \\ &\quad + \frac{1}{4\pi\epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W_{\epsilon,y}\phi}{|x-y|} dy \phi \psi dx, \\ \langle R''_{\epsilon,y}(\phi), (\psi, \xi) \rangle &= \frac{1}{4\pi\epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\phi(y)\xi(y)}{|x-y|} dy \phi \psi dx \\ &\quad + \frac{1}{8\pi\epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\phi^2(y)}{|x-y|} dy \psi \xi dx \\ &\quad + \frac{1}{4\pi\epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W_{\epsilon,y}\psi}{|x-y|} dy \phi \xi dx \\ &\quad + \frac{1}{4\pi\epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W_{\epsilon,y}\xi}{|x-y|} dy \phi \psi dx \\ &\quad + \frac{1}{4\pi\epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W_{\epsilon,y}\phi(y)}{|x-y|} dy \psi \xi dx. \end{aligned} \quad (23)$$

First, we estimate  $R_{\epsilon,y}(\phi)$ . Notice that if we denote

$$\varphi_u := \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy, \quad (24)$$

then  $\varphi_u$  satisfies  $-\Delta\varphi_u = (u^2/2)$  in  $\mathbb{R}^3$ . So,

$$\int_{\mathbb{R}^3} |\nabla\varphi_u|^2 dx = \frac{1}{2} \int_{\mathbb{R}^3} \varphi_u u^2 dx \leq c |\varphi_u|_6 |u^2|_{6/5} \leq C \|\varphi_u\|_D \|u\|_{12/5}^2, \quad (25)$$

which implies that

$$\|\varphi_u\|_D \leq C \|u\|_{12/5}^2. \quad (26)$$

As a result, from Lemma 3, we have

$$\begin{aligned} \left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\phi^2(y)}{|x-y|} dy \phi^2 dx \right| \\ \leq C \left| \varphi_\phi \right|_6 |\phi|_{12/5}^2 \leq C |\phi|_{12/5}^4 \leq C \epsilon^{-1} \|\phi\|_\epsilon^4, \end{aligned} \quad (27)$$

$$\begin{aligned} \left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W_{\epsilon,y}\phi}{|x-y|} dy \phi^2 dx \right| \\ \leq \left| \int_{\mathbb{R}^3} \varphi_{W_{\epsilon,y}\phi}^{1/2} \varphi_\phi^{1/2} \phi^2 dx \right| \leq C \left| \varphi_{W_{\epsilon,y}\phi} \right|_6^{1/2} \left| \varphi_\phi \right|_6^{1/2} |\phi|_{12/5}^2 \\ \leq C \left| W_{\epsilon,y} \right|_{12/5} |\phi|_{12/5}^3 \leq C \epsilon^{9((5/12)-(1/2))} \|\phi\|_\epsilon^3 \epsilon^{3-(5/12)} \|w\|_{12/5} \\ \leq C \epsilon^{1/2} \|\phi\|_\epsilon^3. \end{aligned} \quad (28)$$

Combining the definition of  $R_{\epsilon,y}(\phi)$  and (27), (28), we find

$$\|R_{\epsilon,y}(\phi)\| \leq C \left[ \epsilon^{-3} \|\phi\|_\epsilon + \epsilon^{-(3/2)} \right] \|\phi\|_\epsilon^3. \quad (29)$$

Now, we discuss  $R'_{\epsilon,y}(\phi)$ . Similar to (27) and (28), we get

$$\begin{aligned} \left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\phi^2(y)}{|x-y|} dy \phi \psi dx \right| &\leq C \left| \varphi_\phi \right|_6 |\phi|_{12/5} |\psi|_{12/5} \\ &\leq C \epsilon^{-1} \|\phi\|_\epsilon^3 \|\psi\|_\epsilon, \\ \left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W_{\epsilon,y}\psi}{|x-y|} dy \phi^2 dx \right| &\leq C \epsilon^{1/2} \|\phi\|_\epsilon^2 \|\psi\|_\epsilon, \\ \left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W_{\epsilon,y}\phi}{|x-y|} dy \phi \psi dx \right| &\leq C \epsilon^{1/2} \|\phi\|_\epsilon^2 \|\psi\|_\epsilon. \end{aligned} \quad (30)$$

So, by the estimates above, we infer that

$$\left| \langle R'_{\epsilon,y}(\phi), \psi \rangle \right| \leq C \left[ \epsilon^{-3} \|\phi\|_\epsilon^3 + \epsilon^{-(3/2)} \|\phi\|_\epsilon^2 \right] \|\psi\|_\epsilon, \quad (31)$$

and then

$$\|R'_{\epsilon,y}(\phi)\| \leq C \left[ \epsilon^{-3} \|\phi\|_\epsilon + \epsilon^{-(3/2)} \right] \|\phi\|_\epsilon^2. \quad (32)$$

Finally, by the same argument as above, we find

$$\begin{aligned}
 \left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\phi(y)\xi(y)}{|x-y|} dy \phi \psi dx \right| &\leq C \left| \int_{\mathbb{R}^3} \phi_\phi^{1/2} \phi_\xi^{1/2} \phi \psi dx \right| \\
 &\leq C \epsilon^{-1} \|\phi\|_\epsilon^2 \|\psi\|_\epsilon \|\xi\|_\epsilon, \\
 \left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\phi^2(y)}{|x-y|} dy \psi \xi dx \right| &\leq C \epsilon^{-1} \|\phi\|_\epsilon^2 \|\psi\|_\epsilon \|\xi\|_\epsilon, \\
 \left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W_{\epsilon,y} \psi}{|x-y|} dy \phi \xi dx \right| &\leq C \epsilon^{1/2} \|\phi\|_\epsilon \|\psi\|_\epsilon \|\xi\|_\epsilon, \\
 \left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W_{\epsilon,y} \xi}{|x-y|} dy \phi \psi dx \right| &\leq C \epsilon^{1/2} \|\phi\|_\epsilon \|\psi\|_\epsilon \|\xi\|_\epsilon, \\
 \left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W_{\epsilon,y} \phi}{|x-y|} dy \xi \psi dx \right| &\leq C \epsilon^{1/2} \|\phi\|_\epsilon \|\psi\|_\epsilon \|\xi\|_\epsilon.
 \end{aligned} \tag{33}$$

The estimates above imply that

$$\begin{aligned}
 \left| \langle R''_{\epsilon,y}(\phi), (\psi, \xi) \rangle \right| &\leq C \left[ \epsilon^{-3} \|\phi\|_\epsilon^2 + \epsilon^{-(3/2)} \|\phi\|_\epsilon \right] \|\psi\|_\epsilon \|\xi\|_\epsilon, \\
 \|R''_{\epsilon,y}(\phi)\| &\leq C \left[ \epsilon^{-3} \|\phi\|_\epsilon + \epsilon^{-(3/2)} \right] \|\phi\|_\epsilon.
 \end{aligned} \tag{34}$$

This completes our proof.

**Lemma 5.** *There holds*

$$\|\ell_{\epsilon,y}\| \leq C \epsilon^{3/2} \left( \epsilon^\theta + \sum_{j=1}^k (K(y^j) - 1) + \sum_{i \neq j} e^{-(|y^i - y^j|/2\epsilon)} \right), \tag{35}$$

where  $i, j = 1, \dots, k, i \neq j$  and  $C$  is a constant independent of  $\epsilon$ .

*Proof.* Recall that

$$\begin{aligned}
 \ell_{\epsilon,y}(\phi) &= \int_{\mathbb{R}^3} (K(x) - 1) W_{\epsilon,y} \phi dx \\
 &\quad - \frac{1}{8\pi\epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\sum_{i \neq j} w_{\epsilon,y^i} w_{\epsilon,y^j}}{|x-y|} dy \left( \sum_{j=1}^k w_{\epsilon,y^j} \right) \phi dx \\
 &\quad - \frac{1}{8\pi\epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sum_{i \neq j} \frac{w_{\epsilon,y^i}^2}{|x-y|} dy w_{\epsilon,y^i} \phi dx =: \ell_1 + \ell_2 + \ell_3.
 \end{aligned} \tag{36}$$

First, we can write

$$\begin{aligned}
 \ell_1 &= \sum_{j=1}^k \int_{\mathbb{R}^3} (K(x) - K(y^j)) w_{\epsilon,y^j} \phi dx \\
 &\quad + \sum_{j=1}^k \int_{\mathbb{R}^3} (K(y^j) - 1) w_{\epsilon,y^j} \phi dx =: \ell_{1.1} + \ell_{1.2},
 \end{aligned} \tag{37}$$

and we have

$$\begin{aligned}
 \ell_{1.1} &= \sum_{j=1}^k \int_{B_\delta(y^j)} (K(x) - K(y^j)) w_{\epsilon,y^j} \phi dx \\
 &\quad + \sum_{j=1}^k \int_{\mathbb{R}^3 \setminus B_\delta(y^j)} (K(x) - K(y^j)) w_{\epsilon,y^j} \phi dx \\
 &=: \ell_{1.1.1} + \ell_{1.1.2}.
 \end{aligned} \tag{38}$$

Then, by the assumption  $(K_2)$ , Lemma 3 and the decay property of  $w$ , we find

$$\begin{aligned}
 |\ell_{1.1.1}| &\leq \sum_{j=1}^k \int_{B_\delta(y^j)} |K(x) - K(y^j)| w_{\epsilon,y^j} |\phi| dx \\
 &\leq C \epsilon^3 \sum_{j=1}^k \int_{B_{\delta/\epsilon}(0)} |\epsilon x|^\theta w(x) |\phi(\epsilon x + y^j)| dx \\
 &\leq C \epsilon^{3+\theta} \sum_{j=1}^k \left( \int_{B_{\delta/\epsilon}(0)} |x|^{3/2\theta} w^{3/2}(x) dx \right)^{2/3} \\
 &\quad \cdot \left( \int_{B_{\delta/\epsilon}(0)} |\phi(\epsilon x + y^j)|^3 dx \right)^{1/3} \\
 &\leq C \epsilon^{3+\theta} \left( \int_{\mathbb{R}^3} |\phi(x)|^3 e^{-3|x|} dx \right)^{1/3} \leq C \epsilon^{(3/2)+\theta} \|\phi\|_\epsilon,
 \end{aligned} \tag{39}$$

where we used the fact that

$$\begin{aligned}
 \int_{B_{\delta/\epsilon}(0)} |x|^{(3/2)\theta} w^{3/2}(x) dx &\leq \int_{B_R(0)} |x|^{(3/2)\theta} w^{3/2}(x) dx \\
 &\quad + \int_{B_R^c(0)} |x|^{(3/2)\theta} e^{-(3/2)|x|} dx < +\infty.
 \end{aligned} \tag{40}$$

On the other hand, by Hölder inequality, we have

$$\begin{aligned}
 |\ell_{1.1.2}| &\leq C \epsilon^3 \int_{B_{\delta/\epsilon}^c(0)} w(x) |\phi(\epsilon x + y^j)| dx \\
 &\leq C \epsilon^3 \left( \int_{B_{\delta/\epsilon}^c(0)} w^{3/2}(x) dx \right)^{2/3} \left( \int_{B_{\delta/\epsilon}^c(0)} |\phi(\epsilon x + y^j)|^3 dx \right)^{1/3} \\
 &\leq C \epsilon^3 \left( \int_{\delta/\epsilon}^{+\infty} r^{1/2} e^{-(3/2)r} dr \right)^{2/3} \left( \int_{\mathbb{R}^3} e^{-3|x|} |\phi(x)|^3 dx \right)^{1/3} \\
 &\leq C \epsilon^3 e^{-(2\delta/3\epsilon)} \epsilon^{-1} \epsilon^{-(1/2)} \|\phi\|_\epsilon \leq C \epsilon^{(3/2)+\theta} \|\phi\|_\epsilon,
 \end{aligned} \tag{41}$$

which, together with (41), implies that

$$|\ell_{1.1}| \leq C \epsilon^{(3/2)+\theta} \|\phi\|_\epsilon. \tag{42}$$

By the same argument as above, we also deduce that

$$\begin{aligned}
|\ell_{1,2}| &\leq \sum_{j=1}^k (K(y^j) - 1) \int_{\mathbb{R}^3} w_{\epsilon, y^j} |\phi| dx \\
&\leq \sum_{j=1}^k (K(y^j) - 1) \int_{\mathbb{R}^3} w(x) |\phi(\epsilon x + y^j)| \epsilon^3 dx \\
&\leq \epsilon^3 \sum_{j=1}^k (K(y^j) - 1) \left( \int_{\mathbb{R}^3} w(x)^{2/3} dx \right)^{2/3} \\
&\quad \cdot \left( \int_{\mathbb{R}^3} |\phi(\epsilon x + y^j)|^3 dx \right)^{1/3} \\
&\leq C \epsilon^3 \sum_{j=1}^k (K(y^j) - 1) \epsilon^{-1} \epsilon^{-(1/2)} \|\phi\|_\epsilon \\
&= C \epsilon^{3/2} \sum_{j=1}^k (K(y^j) - 1) \|\phi\|_\epsilon.
\end{aligned} \tag{43}$$

Hence,

$$|\ell_1| \leq C \epsilon^{3/2} \left[ \epsilon^\theta + \sum_{j=1}^k (K(y^j) - 1) \right] \|\phi\|_\epsilon. \tag{44}$$

Now, in order to estimate  $\ell_2$ , we recall the Hardy-Littlewood-Sobolev inequality (see [2]): if  $1 < p, q < \infty$ ,  $0 < t < 3$  and  $(1/p) + (1/q) + (t/3) = 2$ ,  $f \in L^p(\mathbb{R}^3)$ ,  $g \in L^q(\mathbb{R}^3)$ , then

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f(x)g(y)}{|x-y|^t} dx dy \leq C(p, q, t) |f|_p |g|_q. \tag{45}$$

Thus, by Hölder inequality and (45), one has

$$\begin{aligned}
|\ell_2| &= \left| \frac{1}{8\pi\epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\sum_{i \neq j} w_{\epsilon, y^i} w_{\epsilon, y^j}}{|x-y|} dy \left( \sum_{j=1}^k w_{\epsilon, y^j} \right) \phi dx \right| \\
&= \left| \frac{\epsilon^3}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\sum_{i \neq j} w(y) w(y - y^i - y^j/\epsilon)}{|x-y|} \right. \\
&\quad \left. \cdot dy \sum_{j=1}^k w(x) \phi(\epsilon x + y^j) dx \right| \\
&\leq C \epsilon^3 \left| \sum_{i \neq j} w(y) w\left(y - \frac{y^i - y^j}{\epsilon}\right) \right|_{6/5} |w(x) \phi(\epsilon x + y^j)|_{6/5} \\
&\leq C \epsilon^3 \sum_{i \neq j} e^{-(y^i - y^j/\epsilon)} |w(x)|_2 |\phi(\epsilon x + y^j)|_3 \\
&\leq C \epsilon^3 \sum_{i \neq j} e^{-(y^i - y^j/2\epsilon)} \epsilon^{-(3/2)} \|\phi\|_\epsilon = C \epsilon^{3/2} \sum_{i \neq j} e^{-(y^i - y^j/2\epsilon)} \|\phi\|_\epsilon.
\end{aligned} \tag{46}$$

Finally, using Lemma A.1 and Hölder inequality, we infer that

$$\begin{aligned}
|\ell_3| &= \left| \frac{1}{8\pi\epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sum_{i \neq j} \frac{w_{\epsilon, y^i}^2}{|x-y|} dy w_{\epsilon, y^i} \phi dx \right| \\
&= \left| \frac{\epsilon^3}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\sum_{i \neq j} w^2(y + (y^i - y^j/\epsilon))}{|x-y|} \right. \\
&\quad \left. \cdot dy w(x) \phi(\epsilon x + y^i) dx \right| \\
&\leq C \epsilon^3 \left\{ \int_{\mathbb{R}^3} \left( \sum_{i \neq j} \left[ e^{-|x - (y^i - y^j/\epsilon)|} \sum_{m=-1}^2 \left| x - \frac{y^i - y^j}{\epsilon} \right|^m \right. \right. \right. \\
&\quad \left. \left. \left. + \left| x - \frac{y^i - y^j}{\epsilon} \right|^{-1} \right]^2 \right) w^2(x) \right\}^{1/2} |\phi(\epsilon x + y^i)|_2 \\
&\leq C \epsilon^3 \sum_{i \neq j} \left\{ \int_{\mathbb{R}^3} \left( \left[ e^{-2|x - (y^i - y^j/\epsilon)|} \sum_{m=-1}^2 \left| x - \frac{y^i - y^j}{\epsilon} \right|^{2m} \right. \right. \right. \\
&\quad \left. \left. \left. + \left| x - \frac{y^i - y^j}{\epsilon} \right|^{-2} \right]^2 \right) w^2(x) \right\}^{1/2} \epsilon^{-(3/2)} \|\phi\|_\epsilon \\
&\leq C \epsilon^{3/2} \sum_{i \neq j} \left[ e^{-(y^i - y^j/2\epsilon)} \left( \frac{y^i - y^j}{\epsilon} \right)^3 + \left( \frac{y^i - y^j}{\epsilon} \right)^{-1} \right] \|\phi\|_\epsilon \\
&\leq C \epsilon^{(3/2)+\theta} \|\phi\|_\epsilon,
\end{aligned} \tag{47}$$

which, together with (44) and (46), concludes this proof.

Now, associated to the quadratic form  $L_{\epsilon, y}(\phi)$ , we define  $L_{\epsilon, y}$  to be a bounded linear map from  $\mathbf{E}_{\epsilon, k}$  to  $\mathbf{E}_{\epsilon, k}$  as

$$\begin{aligned}
\langle L_{\epsilon, y}(v_1), v_2 \rangle &= \int_{\mathbb{R}^3} (\epsilon^2 \nabla v_1 \nabla v_2 + K(x) v_1 v_2) dx \\
&\quad - \frac{1}{8\pi\epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W_{\epsilon, y}^2(y)}{|x-y|} dy v_1 v_2 dx \\
&\quad - \frac{1}{4\pi\epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W_{\epsilon, y} v_1}{|x-y|} dy W_{\epsilon, y} v_2 dx.
\end{aligned} \tag{48}$$

Here, we come to show the invertibility of  $L_{\epsilon, y}$  in  $\mathbf{E}_{\epsilon, k}$ .

**Proposition 6.** *There exist  $\epsilon_0, \delta_0, \rho, R_0 > 0$  such that for  $R \gg R_0, \epsilon \in (0, \epsilon_0), \delta \in (0, \delta_0)$ ,*

$$\|L_{\epsilon, y}(\phi)\| \geq \rho \|\phi\|_\epsilon, \forall \phi \in \mathbf{E}_{\epsilon, k}. \tag{49}$$

*Proof.* We argue by contradiction. Suppose that there exists  $\epsilon_n \rightarrow 0, \mathbf{y}^n = (y^{n,1}, \dots, y^{n,k}) \in D_k^{\epsilon_n, \delta}$ , and  $\phi_n \in \mathbf{E}_{\epsilon_n, k}$  such that

$$\langle L_{\epsilon_n, \mathbf{y}^n}(\phi_n), g \rangle = o_n(1) \|\phi_n\|_{\epsilon_n} \|g\|_{\epsilon_n}, \forall g \in \mathbf{E}_{\epsilon_n, k}. \tag{50}$$

Without loss of generality, we can assume that  $\|\phi_n\|_{\epsilon_n}^2 = \epsilon_n^3$ . Fix  $i \in \{1, \dots, k\}$  and let

$$\bar{\phi}_{n,i} = \phi_n(\epsilon_n x + y^{n,i}). \tag{51}$$

So,

$$\int_{\mathbb{R}^3} (\epsilon_n^2 |\nabla \phi_n|^2 + K(x) \phi_n^2) dx = \epsilon_n^3, \quad (52)$$

$$\int_{\mathbb{R}^3} (|\nabla \bar{\phi}_{n,i}|^2 + K(\epsilon_n x + y^{n,i}) \bar{\phi}_{n,i}^2) dx \leq C,$$

which implies that  $\bar{\phi}_{n,i}$  is bounded in  $H^1(\mathbb{R}^3)$ . Thus, up to a subsequence, there exists  $\phi \in H^1(\mathbb{R}^3)$  such that as  $n \rightarrow +\infty$ ,

$$\begin{cases} \bar{\phi}_{n,i} \rightharpoonup \phi, & \text{in } H^1(\mathbb{R}^3), \\ \bar{\phi}_{n,i} \rightarrow \phi, & \text{a.e. in } \mathbb{R}^3, \\ \bar{\phi}_{n,i} \rightarrow \phi, & \text{in } L^1_{\text{loc}}(\mathbb{R}^3), \quad 2 \leq t \leq 2^*. \end{cases} \quad (53)$$

Next we will prove  $\phi = 0$ . To this end, from (50), we find that  $\bar{\phi}_{n,i}$  satisfies for any  $\bar{g} \in \bar{E}_n$ ,

$$\begin{aligned} & \int_{\mathbb{R}^3} (\nabla \bar{\phi}_{n,i} \nabla \bar{g} + K(\epsilon_n x + y^{n,i}) \bar{\phi}_{n,i} \bar{g}) dx \\ & - \frac{1}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(\sum_{j=1}^k \bar{w}_{\epsilon_n, y^{n,j}})^2}{|x-y|} dy \bar{\phi}_{n,i} \bar{g} dx \\ & - \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\sum_{j=1}^k \bar{w}_{\epsilon_n, y^{n,j}} \bar{\phi}_{n,i}}{|x-y|} dy \sum_{j=1}^k \bar{w}_{\epsilon_n, y^{n,j}} \bar{g} dx \\ & = o_n(1) \|\bar{g}\|_*, \end{aligned} \quad (54)$$

where

$$\begin{aligned} \|\bar{g}\|_*^2 &= \int_{\mathbb{R}^3} (|\nabla \bar{g}|^2 + K(\epsilon_n x + y^{n,i}) \bar{g}^2) dx, \\ \bar{w}_{\epsilon_n, y^{n,j}}(x) &= w\left(x + \frac{y^{n,i} - y^{n,j}}{\epsilon_n}\right), \\ \bar{E}_n &= \left\{ \bar{g} : \bar{g}\left(\frac{x - y^{n,i}}{\epsilon_n}\right) \in \mathbf{E}_{\epsilon_n, k}, \int_{\mathbb{R}^3} \nabla \frac{\partial \bar{w}_{\epsilon_n, y^{n,j}}}{\partial y_s^{n,j}} \nabla \bar{g} dx \right. \\ & \quad \left. + \int_{\mathbb{R}^3} K(\epsilon_n x + y^{n,i}) \frac{\partial \bar{w}_{\epsilon_n, y^{n,j}}}{\partial y_s^{n,j}} \bar{g} dx = 0 \right\}, \end{aligned} \quad (55)$$

for  $j = 1, \dots, k$  and  $s = 1, 2, 3$ .

But, for  $g \in C_0^\infty(\mathbb{R}^3)$ , we can decompose  $g$  as follows

$$g = g_n - \sum_{j=1}^k \sum_{s=1}^3 a_{n,j,s} \frac{\partial \bar{w}_{\epsilon_n, y^{n,j}}}{\partial y_s^{n,j}}, \quad (56)$$

where  $g_n \in \bar{E}_n$  and  $a_{n,j,s} \in \mathbb{R}$ . Then, by the exponential decay of  $(\partial \bar{w}_{\epsilon_n, y^{n,j}} / \partial y_s^{n,j})$ , we have

$$\begin{aligned} & \int_{\mathbb{R}^3} \nabla \frac{\partial \bar{w}_{\epsilon_n, y^{n,j}}}{\partial y_s^{n,i}} \nabla g_n dx \\ & + \int_{\mathbb{R}^3} K(\epsilon_n x + y^{n,i}) \frac{\partial \bar{w}_{\epsilon_n, y^{n,j}}}{\partial y_s^{n,i}} g_n dx = o_n(1), \end{aligned} \quad (57)$$

$$\begin{aligned} & \int_{\mathbb{R}^3} \nabla \frac{\partial \bar{w}_{\epsilon_n, y^{n,j}}}{\partial y_s^{n,i}} \nabla \frac{\partial \bar{w}_{\epsilon_n, y^{n,h}}}{\partial y_s^{n,i}} dx \\ & + \int_{\mathbb{R}^3} K(\epsilon_n x + y^{n,i}) \frac{\partial \bar{w}_{\epsilon_n, y^{n,j}}}{\partial y_s^{n,i}} \frac{\partial \bar{w}_{\epsilon_n, y^{n,h}}}{\partial y_s^{n,i}} dx = o_n(1), \end{aligned}$$

for  $h \neq j$  and  $h, j = 1, \dots, k$ . On the other hand,

$$\int_{\mathbb{R}^3} \left| \nabla \frac{\partial \bar{w}_{\epsilon_n, y^{n,j}}}{\partial y_s^{n,i}} \right|^2 dx + \int_{\mathbb{R}^3} K(\epsilon_n x + y^{n,i}) \left| \frac{\partial \bar{w}_{\epsilon_n, y^{n,j}}}{\partial y_s^{n,i}} \right|^2 dx \geq C. \quad (58)$$

So, up to a subsequence, we can easily check that  $a_{n,j,s} \rightarrow 0$  as  $n \rightarrow \infty$  for  $j \neq i$ , while  $a_{n,i,s} \rightarrow a_{i,s}$  for some  $a_{i,s} \in \mathbb{R}$ . Inserting  $g_n(x - y^{n,i}/\epsilon_n)$  into (54) and letting  $n \rightarrow +\infty$ , we infer that

$$\begin{aligned} & \int_{\mathbb{R}^3} (\nabla \phi \nabla g + \phi g) dx - \frac{1}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w^2(y)}{|x-y|} dy \phi g dx \\ & - \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w(y) \phi(y)}{|x-y|} dy w g dx \\ & + \sum_{s=1}^3 a_{i,s} \left[ \int_{\mathbb{R}^3} \left( \nabla \phi \nabla \frac{\partial w}{\partial x_s} + \phi \frac{\partial w}{\partial x_s} \right) dx \right. \\ & - \frac{1}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w^2(y)}{|x-y|} dy \phi \frac{\partial w}{\partial x_s} dx \\ & \left. - \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w(y) \phi(y)}{|x-y|} dy w \frac{\partial w}{\partial x_s} dx \right] = 0. \end{aligned} \quad (59)$$

Since  $w$  solves

$$-\Delta w + w = \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{w^2(y)}{|x-y|} dy w(x) \quad \text{in } \mathbb{R}^3. \quad (60)$$

We find that

$$\begin{aligned} -\Delta \frac{\partial w}{\partial x_s} + \frac{\partial w}{\partial x_s} &= \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{w^2(y)}{|x-y|} dy \frac{\partial w}{\partial x_s} \\ &+ \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{w(y) (\partial w / \partial x_s)}{|x-y|} dy w(x), \end{aligned} \quad (61)$$

which implies that

$$\begin{aligned} & \int_{\mathbb{R}^3} \left( \nabla \phi \nabla \frac{\partial w}{\partial x_s} + \phi \frac{\partial w}{\partial x_s} \right) dx \\ & = \frac{1}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w^2(y)}{|x-y|} dy \phi \frac{\partial w}{\partial x_s} dx \\ & + \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w(y) \phi(y)}{|x-y|} dy w \frac{\partial w}{\partial x_s} dx. \end{aligned} \quad (62)$$

Combining (59) and (62), we have

$$\begin{aligned} & \int_{\mathbb{R}^3} (\nabla\phi\nabla g + \phi g) dx - \frac{1}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w^2(y)}{|x-y|} dy \phi g dx \\ & - \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w(y)\phi(y)}{|x-y|} dy w g dx = 0. \end{aligned} \quad (63)$$

Considering that  $g \in C_0^\infty(\mathbb{R}^3)$  is arbitrary and  $w$  is non-degenerate, there exist  $a_s, s = 1, 2, 3$  such that

$$\phi = \sum_{s=1}^3 a_s \frac{\partial w}{\partial x_s}. \quad (64)$$

Moreover, being  $\phi_n \in \mathbf{E}_{\epsilon_n, k}$ , we have

$$\int_{\mathbb{R}^3} \left( \nabla\phi\nabla \frac{\partial w}{\partial x_s} + \phi \frac{\partial w}{\partial x_s} \right) dx = 0, \quad (65)$$

which, together with (64), yields  $\phi = 0$ . Finally, by Lemma A.1, we deduce that

$$\begin{aligned} & \frac{1}{8\pi\epsilon_n^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W_{\epsilon_n, \mathbf{y}^n}^2(y)}{|x-y|} dy \phi_n^2 dx \\ & = \frac{\epsilon_n^3}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{[w(y) + \sum_{i \neq j} w(y + (y^{n,j} - y^{n,i}/\epsilon_n))]^2}{|x-y|} \\ & \quad \cdot dy \phi_n^2(\epsilon_n x + y^{n,j}) dx \\ & \leq C\epsilon_n^3 \int_{\mathbb{R}^3} \frac{w^2(y)}{|x-y|} dy \phi_n^2(\epsilon_n x + y^{n,j}) dx \\ & \quad + C\epsilon_n^3 \int_{\mathbb{R}^3} \sum_{i \neq j} \left[ e^{-|x-(y^{n,i}-y^{n,j})/\epsilon_n|} \sum_{m=-1}^2 \left| x - \frac{y^{n,i}-y^{n,j}}{\epsilon_n} \right|^m \right. \\ & \quad \left. + \left| x - \frac{y^{n,i}-y^{n,j}}{\epsilon_n} \right|^{-1} \right] \phi_n^2(\epsilon_n x + y^{n,j}) dx \\ & \leq C\epsilon_n^3 \left[ \int_{\mathbb{R}^3} \left( e^{-|x|} \sum_{m=-1}^2 |x|^m + |x|^{-1} \right) \bar{\phi}_{n,j}^2 dx \right. \\ & \quad \left. + \int_{\mathbb{R}^3} \left( e^{-|x-y^{n,i}/\epsilon_n|} \sum_{m=-1}^2 \left| \frac{x-y^{n,i}}{\epsilon_n} \right|^m + \left| \frac{x-y^{n,i}}{\epsilon_n} \right|^{-1} \right) \phi^2(x) dx \right] \\ & \leq C\epsilon_n^3 \left( \int_{B_R(0)} + \int_{B_R^c(0)} \right) \left( e^{-|x|} \sum_{m=-1}^2 |x|^m + |x|^{-1} \right) \bar{\phi}_{n,j}^2 dx \\ & \quad + C\epsilon_n^3 \left( \int_{\cup_{i=1}^k B_{\epsilon_n R}(y^{n,i})} + \int_{\mathbb{R}^3 \setminus \cup_{i=1}^k B_{\epsilon_n R}(y^{n,i})} \right) \\ & \quad \cdot \left( e^{-|x-y^{n,i}/\epsilon_n|} \sum_{m=-1}^2 \left| \frac{x-y^{n,i}}{\epsilon_n} \right|^m + \left| \frac{x-y^{n,i}}{\epsilon_n} \right|^{-1} \right) \phi^2(x) dx \end{aligned}$$

$$\begin{aligned} & \leq C\epsilon_n^3 \int_{\cup_{i=1}^k B_{\epsilon_n R}(y^{n,i})} \left( e^{-|x-y^{n,i}/\epsilon_n|} \sum_{m=-1}^2 \left| \frac{x-y^{n,i}}{\epsilon_n} \right|^m \right. \\ & \quad \left. + \left| \frac{x-y^{n,i}}{\epsilon_n} \right|^{-1} \right) \phi^2(x) dx + o(\epsilon_n^3) \\ & \quad + C \left( e^{-(1-\theta)R} + \frac{1}{R} \right) \epsilon_n^3 = o(\epsilon_n^3) + o_R(1)\epsilon_n^3, \end{aligned} \quad (66)$$

where  $o_R(1) \rightarrow 0$  as  $R \rightarrow +\infty$ .

Similarly, the Hardy-Littlewood-Sobolev inequality (45) implies that

$$\begin{aligned} & \frac{1}{4\pi\epsilon_n^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W_{\epsilon_n, \mathbf{y}^n}(y)\phi_n(y)}{|x-y|} dy W_{\epsilon_n, \mathbf{y}^n} \phi_n dx \\ & \leq \frac{C}{\epsilon_n^2} \left[ \int_{\cup_{i=1}^k B_{\epsilon_n R}(y^{n,i})} |W_{\epsilon_n, \mathbf{y}^n}|^{6/5} |\phi_n|^{6/5} dx \right. \\ & \quad \left. + \int_{\mathbb{R}^3 \setminus \cup_{i=1}^k B_{\epsilon_n R}(y^{n,i})} |W_{\epsilon_n, \mathbf{y}^n}|^{6/5} |\phi_n|^{6/5} dx \right]^{5/3} \\ & \leq o(\epsilon_n^3) + C e^{-(1-\theta)R} = o(\epsilon_n^3) + o_R(1)\epsilon_n^3. \end{aligned} \quad (67)$$

As a result, by (50),

$$\begin{aligned} o_n(1)\epsilon_n^3 & = o_n(1)\|\phi_n\|_{\epsilon_n}^2 = \langle L_{\epsilon_n, \mathbf{y}^n} \phi_n, \phi_n \rangle \\ & = \|\phi_n\|_{\epsilon_n}^2 - \frac{1}{8\pi\epsilon_n^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W_{\epsilon_n, \mathbf{y}^n}^2(y)}{|x-y|} dy \phi_n^2 dx \\ & \quad - \frac{1}{4\pi\epsilon_n^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W_{\epsilon_n, \mathbf{y}^n} \phi_n}{|x-y|} dy W_{\epsilon_n, \mathbf{y}^n} \phi_n dx \\ & \geq \|\phi_n\|_{\epsilon_n}^2 + o(\epsilon_n^3) + o_R(1)\epsilon_n^3, \end{aligned} \quad (68)$$

which is impossible. So we complete this proof.

**Proposition 7.** *Suppose that  $(K_1)$  and  $(K_2)$  hold. Then, for any given  $k = 1, 2, \dots$ , there exist  $\epsilon_0, \delta_0, R_0 > 0$  such that for  $R \gg R_0, \epsilon \in (0, \epsilon_0), \delta \in (0, \delta_0)$ , there is a  $C^1$  map from  $D_k^{\epsilon, \delta}$  to  $\mathbf{E}_{\epsilon, k}$ ,  $\phi_\epsilon = \phi_\epsilon(\mathbf{y})$  satisfying*

$$\begin{aligned} & \left\langle \frac{\partial J_\epsilon(\mathbf{y}, \phi_\epsilon)}{\partial \phi_\epsilon}, \psi \right\rangle = 0, \forall \psi \in \mathbf{E}_{\epsilon, k}, \\ & \|\phi_\epsilon\|_\epsilon \leq C\epsilon^{3/2} \left( \epsilon^\theta + \sum_{j=1}^k (K(y^j) - 1) + \sum_{i \neq j} e^{-(|y^i - y^j|/2\epsilon)} \right). \end{aligned} \quad (69)$$

*Proof.* We use the contraction mapping theorem to prove the wanted result. It follows from Lemma 5 that  $\ell_{\epsilon, \mathbf{y}}(\phi)$  is a bounded linear map in  $\mathbf{E}_{\epsilon, k}$ . So applying Reisz representation theorem, there exists an  $\ell_{\epsilon, \mathbf{y}}^k \in \mathbf{E}_{\epsilon, k}$  such that

$$\ell_{\epsilon, \mathbf{y}}(\phi) = \langle \ell_{\epsilon, \mathbf{y}}^k, \phi \rangle. \quad (70)$$



Thus, finding a critical point for  $J_\epsilon(\mathbf{y}, \phi)$  is equivalent to solving

$$\ell_{\epsilon, \mathbf{y}}^k + L_{\epsilon, \mathbf{y}}(\phi) - R'_{\epsilon, \mathbf{y}}(\phi) = 0. \quad (71)$$

Since  $L_{\epsilon, \mathbf{y}}$  is invertible in  $\mathbf{E}_{\epsilon, k}$  from Proposition 6, (71) can be rewritten as

$$\phi = L_{\epsilon, \mathbf{y}}^{-1} \left( R'_{\epsilon, \mathbf{y}}(\phi) \right) - L_{\epsilon, \mathbf{y}}^{-1} \left( \ell_{\epsilon, \mathbf{y}}^k \right) := \mathbf{A}(\phi). \quad (72)$$

Define

$$S_\epsilon = \left\{ \phi \in \mathbf{E}_{\epsilon, k} : \|\phi\|_\epsilon \leq C\epsilon^{3/2} \left( e^\theta + \sum_{j=1}^k (K(y^j) - 1) + \sum_{i \neq j} e^{-(1/2)(|y^i - y^j|/\epsilon)} \right) \right\}. \quad (73)$$

We shall verify that  $\mathbf{A}$  is a contraction mapping from  $S_\epsilon$  to itself. First, for  $\forall \phi \in S_\epsilon$ , by Lemmas 4 and 5, we have

$$\begin{aligned} \|\mathbf{A}(\phi)\| &\leq C \left( \|\ell_{\epsilon, \mathbf{y}}^k\|_\epsilon + \|R'_{\epsilon, \mathbf{y}}(\phi)\| \right) \leq C \left( \|\ell_{\epsilon, \mathbf{y}}^k\|_\epsilon + \epsilon^{-(3/2)} \|\phi\|_\epsilon^2 \right) \\ &\leq C\epsilon^{3/2} \left( e^\theta + \sum_{j=1}^k (K(y^j) - 1) + \sum_{i \neq j} e^{-(|y^i - y^j|/2\epsilon)} \right) \\ &\quad + C\epsilon^{3/2} \left( e^\theta + \sum_{j=1}^k (K(y^j) - 1) + \sum_{i \neq j} e^{-(1/2)(|y^i - y^j|/\epsilon)} \right)^2 \\ &\leq \epsilon^{3/2} \left( e^\theta + \sum_{j=1}^k (K(y^j) - 1) + \sum_{i \neq j} e^{-(|y^i - y^j|/2\epsilon)} \right), \end{aligned} \quad (74)$$

which tells that  $\mathbf{A}$  maps  $S_\epsilon$  to  $S_\epsilon$ . On the other hand, for any  $\phi_1, \phi_2 \in S_\epsilon$ , using Lemma 4,

$$\begin{aligned} \|\mathbf{A}(\phi_1) - \mathbf{A}(\phi_2)\| &= \left\| L_{\epsilon, \mathbf{y}}^{-1} \left( R'_{\epsilon, \mathbf{y}}(\phi_1) \right) - L_{\epsilon, \mathbf{y}}^{-1} \left( R'_{\epsilon, \mathbf{y}}(\phi_2) \right) \right\| \\ &\leq C \|R'_{\epsilon, \mathbf{y}}(\phi_1) - R'_{\epsilon, \mathbf{y}}(\phi_2)\| \\ &\leq C \|R''_{\epsilon, \mathbf{y}}(\nu\phi_1 + (1-\nu)\phi_2)\| \|\phi_1 - \phi_2\|_\epsilon \\ &\leq \frac{1}{2} \|\phi_1 - \phi_2\|_\epsilon, \end{aligned} \quad (75)$$

where  $\nu \in (0, 1)$ . Therefore,  $\mathbf{A}$  is a contraction map from  $S_\epsilon$  to  $S_\epsilon$ , and then, applying the contraction mapping theorem, we can find a unique  $\phi_\epsilon$  satisfying (71). So the conclusion follows.

### 3. Proof of the Main Results

In this section, we come to prove our main results. Let  $R \gg R_0$ ,  $\epsilon \in (0, \epsilon_0)$ ,  $\delta \in (0, \delta_0)$  and  $\phi_\epsilon(\mathbf{y})$  be as in Proposition 7.

Define

$$F_\epsilon(\mathbf{y}) = J_\epsilon(\mathbf{y}, \phi_\epsilon(\mathbf{y})), \mathbf{y} \in D_k^{\epsilon, \delta}, \quad (76)$$

and let  $\mathbf{y}_\epsilon = (y_\epsilon^1, \dots, y_\epsilon^k) \in D_k^{\epsilon, \delta}$  satisfies

$$F_\epsilon(\mathbf{y}_\epsilon) = \max \left\{ F_\epsilon(\mathbf{y}) : \mathbf{y} \in D_k^{\epsilon, \delta} \right\}. \quad (77)$$

Next, we can show that  $\mathbf{y}_\epsilon$  is an interior point of  $D_k^{\epsilon, \delta}$  and thus a critical point of  $F_\epsilon$  for small  $\epsilon$ .

**Lemma 8.** *Suppose that  $\mathbf{y}_\epsilon$  satisfies (77). Then, as  $\epsilon \rightarrow 0$ ,  $y_\epsilon^j \rightarrow y^0$ ,  $j = 1, \dots, k$ , and  $|(y_\epsilon^i - y_\epsilon^j)/\epsilon| \rightarrow \infty$ , if  $i \neq j$ .*

*Proof.* It follows from Lemma A.2 and Proposition 7 that

$$\begin{aligned} J_\epsilon(\mathbf{y}_\epsilon, \phi_\epsilon(\mathbf{y}_\epsilon)) &= J_\epsilon(\mathbf{y}_\epsilon, 0) + O \left( \|\ell_{\epsilon, \mathbf{y}_\epsilon}^k\|_\epsilon \|\phi_\epsilon(\mathbf{y}_\epsilon)\|_\epsilon + \|\phi_\epsilon(\mathbf{y}_\epsilon)\|_\epsilon^2 \right) \\ &= J_\epsilon(\mathbf{y}_\epsilon, 0) + O \left( \epsilon^3 \left( \epsilon^{2\theta} + \sum_{j=1}^k (K(y_\epsilon^j) - 1)^2 + \sum_{i \neq j} e^{-|y_\epsilon^i - y_\epsilon^j|/\epsilon} \right) \right) \\ &= \epsilon^3 \left( \frac{k}{2} \|w\|_{H^1}^2 - \frac{1}{2} \left( \int_{\mathbb{R}^3} w^2 dx \right) \sum_{j=1}^k (K(y^0) - K(y_\epsilon^j)) - \frac{k}{32\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w^2(y)}{|x-y|} dy w^2(x) dx \right) \\ &\quad + O \left( \epsilon^3 \left( e^\theta + \sum_{j=1}^k (K(y_\epsilon^j) - 1)^2 + \sum_{i \neq j} e^{-|y_\epsilon^i - y_\epsilon^j|/\epsilon} \right) \right). \end{aligned} \quad (78)$$

Take  $\lambda_\epsilon^i = \epsilon^\sigma \xi_i$ ,  $i = 1, \dots, k$  for some  $\sigma \in ((1/2), 1)$  and some vectors  $\xi_1, \dots, \xi_k$  with  $\xi_i \neq \xi_j$  ( $i \neq j$ ). Then as  $\epsilon \rightarrow 0$ ,

$$\frac{|\lambda_\epsilon^i - \lambda_\epsilon^j|}{\epsilon} = \frac{|\xi_i - \xi_j|}{\epsilon^{1-\sigma}} \rightarrow \infty, \quad (79)$$

which means that  $\lambda_\epsilon = (\lambda_\epsilon^1, \dots, \lambda_\epsilon^k) \in D_k^{\epsilon, \delta}$  if  $\epsilon > 0$  is small enough. Combining (77) and (78), we have

$$\begin{aligned} J_\epsilon(\mathbf{y}_\epsilon, \phi_\epsilon(\mathbf{y}_\epsilon)) &\geq J_\epsilon(\lambda_\epsilon, \phi_\epsilon(\lambda_\epsilon)) \\ &\geq \epsilon^3 \left( \frac{k}{2} \|w\|_{H^1}^2 - \frac{k}{32\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w^2(y)}{|x-y|} dy w^2(x) dx \right) \\ &\quad + O \left( \epsilon^3 \left( e^\theta + \sum_{j=1}^k (K(\lambda_\epsilon^j) - 1)^2 + \sum_{i \neq j} e^{-|\xi_i - \xi_j|/\epsilon^{1-\sigma}} \right) \right) \\ &= \epsilon^3 \left( \frac{k}{2} \|w\|_{H^1}^2 - \frac{k}{32\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w^2(y)}{|x-y|} dy w^2(x) dx \right) \\ &\quad + O(\epsilon^{3+\sigma}), \end{aligned} \quad (80)$$

where  $\bar{\sigma} = \min \{1 - \sigma, \sigma\theta\}$ . Hence,

$$\begin{aligned} & \epsilon^3 \left( \frac{k}{2} \|w\|_{H^1}^2 - \frac{1}{2} \left( \int_{\mathbb{R}^3} w^2 dx \right) \sum_{j=1}^k (K(y^0) - K(y_\epsilon^j)) \right. \\ & \quad \left. - \frac{k}{32\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w^2(y)}{|x-y|} dy w^2(x) dx \right) \\ & \quad + O \left( \epsilon^3 \left( \epsilon^\theta + \sum_{j=1}^k (K(y_\epsilon^j) - 1)^2 + \sum_{i \neq j} e^{-|y_\epsilon^i - y_\epsilon^j|/\epsilon} \right) \right) \quad (81) \\ & \geq \epsilon^3 \left( \frac{k}{2} \|w\|_{H^1}^2 - \frac{k}{32\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w^2(y)}{|x-y|} dy w^2(x) dx \right) \\ & \quad + O(\epsilon^{3+\bar{\sigma}}), \end{aligned}$$

which yields that

$$\begin{aligned} & \frac{1}{2} \left( \int_{\mathbb{R}^3} w^2 dx \right) \sum_{j=1}^k (K(y^0) - K(y_\epsilon^j)) \\ & \leq O \left( \epsilon^3 \left( \epsilon^{\bar{\sigma}} + \sum_{j=1}^k (K(y_\epsilon^j) - K(y^0))^2 + \sum_{i \neq j} e^{-|y_\epsilon^i - y_\epsilon^j|/\epsilon} \right) \right). \quad (82) \end{aligned}$$

So, as  $\epsilon \rightarrow 0$  and for  $i, j = 1, \dots, k, i \neq j$ , we find

$$K(y_\epsilon^j) \rightarrow K(y^0) = 1, y_\epsilon^j \rightarrow y^0, \frac{|y_\epsilon^i - y_\epsilon^j|}{\epsilon} \rightarrow \infty, \quad (83)$$

from which, the conclusion follows.

*Proof of Theorem 2.* By Lemma 8, we can check that (77) can be obtained by some  $\mathbf{y}_\epsilon = (y_\epsilon^1, \dots, y_\epsilon^k) \in D_k^{\epsilon, \delta}$ , which is an interior point of  $D_k^{\epsilon, \delta}$  for small  $\epsilon$  and satisfies

$$y_\epsilon^j \rightarrow y^0, \frac{|y_\epsilon^i - y_\epsilon^j|}{\epsilon} \rightarrow \infty, \frac{\partial J_\epsilon(\mathbf{y}_\epsilon, \phi_\epsilon(\mathbf{y}_\epsilon))}{\partial y_\epsilon^j} = 0 \quad (84)$$

for  $i, j = 1, \dots, k, i \neq j$ . Moreover, from Proposition 7,  $\|\phi_\epsilon(\mathbf{y}_\epsilon)\|_\epsilon = o(\epsilon^{3/2})$  as  $\epsilon \rightarrow 0$ . Finally, it is well-known that if  $\mathbf{y}_\epsilon$  is a critical point of  $F_\epsilon(\mathbf{y}_\epsilon)$ , then  $W_{\epsilon, \mathbf{y}_\epsilon} + \phi_\epsilon(\mathbf{y}_\epsilon)$  is a solution of (2). Thus, we finish this proof.

## Appendix

### Energy Expansion

In this section, we give some basic estimates and the energy expansion for the approximate solutions.

**Lemma A.1.** *There exists a positive constant  $C$  independent of  $\epsilon$  such that*

$$\begin{aligned} & \int_{\mathbb{R}^3} \frac{\sum_{i \neq j} w^2(y + (y^i - y^j/\epsilon))}{|x-y|} dy \\ & \leq C \sum_{i \neq j} \left[ e^{-|x - (y^i - y^j/\epsilon)|} \sum_{m=1}^2 \left| x - \frac{y^i - y^j}{\epsilon} \right|^m + \left| x - \frac{y^i - y^j}{\epsilon} \right|^{-1} \right]. \quad (A.1) \end{aligned}$$

*Proof.* The proof of this Lemma can be obtained arguing as Lemma B.1 of [24] exactly. We omit the details here.

**Lemma A.2.** *There exists a positive constant  $C$  independent of  $\epsilon$  such that*

$$\begin{aligned} I_\epsilon(W_{\epsilon, \mathbf{y}}) &= \epsilon^3 \left( \frac{k}{2} \|w\|_{H^1}^2 - \frac{1}{2} \left( \int_{\mathbb{R}^3} w^2 dx \right) \sum_{j=1}^k (K(y^0) - K(y^j)) \right. \\ & \quad \left. - \frac{k}{32\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w^2(y)}{|x-y|} dy w^2(x) dx \right) \\ & \quad + O \left( \epsilon^{3+\theta} + \epsilon^3 \sum_{i \neq j} e^{-|y^i - y^j|/\epsilon} \right). \quad (A.2) \end{aligned}$$

*Proof.* Recall that

$$\begin{aligned} I_\epsilon(W_{\epsilon, \mathbf{y}}) &= \frac{1}{2} \int_{\mathbb{R}^3} \left( \epsilon^2 |\nabla W_{\epsilon, \mathbf{y}}|^2 + K(x) W_{\epsilon, \mathbf{y}}^2 \right) dx \\ & \quad - \frac{1}{32\pi \epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W_{\epsilon, \mathbf{y}}^2(y)}{|x-y|} dy W_{\epsilon, \mathbf{y}}^2 dx. \quad (A.3) \end{aligned}$$

We have

$$\begin{aligned} I_\epsilon(W_{\epsilon, \mathbf{y}}) &= \frac{1}{2} \int_{\mathbb{R}^3} \left( \epsilon^2 |\nabla W_{\epsilon, \mathbf{y}}|^2 + W_{\epsilon, \mathbf{y}}^2 \right) dx \\ & \quad + \frac{1}{2} \int_{\mathbb{R}^3} (K(x) - 1) W_{\epsilon, \mathbf{y}}^2 dx \\ & \quad - \frac{1}{32\pi \epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W_{\epsilon, \mathbf{y}}^2(y)}{|x-y|} dy W_{\epsilon, \mathbf{y}}^2 dx \\ &= \frac{1}{2} \sum_{j=1}^k \int_{\mathbb{R}^3} \left( \epsilon^2 |\nabla w_{\epsilon, y^j}|^2 + w_{\epsilon, y^j}^2 \right) dx \\ & \quad + \sum_{i < j} \int_{\mathbb{R}^3} \left( \epsilon^2 \nabla w_{\epsilon, y^i} w_{\epsilon, y^j} + w_{\epsilon, y^i} w_{\epsilon, y^j} \right) dx \\ & \quad + \frac{1}{2} \int_{\mathbb{R}^3} (K(x) - 1) W_{\epsilon, \mathbf{y}}^2 dx \\ & \quad - \frac{1}{32\pi \epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W_{\epsilon, \mathbf{y}}^2(y)}{|x-y|} dy W_{\epsilon, \mathbf{y}}^2 dx \end{aligned}$$

$$\begin{aligned}
&= \frac{k\epsilon^3}{2} \|w\|_{H^1}^2 + \frac{1}{8\pi\epsilon^2} \sum_{i<j} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w_{\epsilon,y^i}^2(y)}{|x-y|} dy w_{\epsilon,y^i}^2 dx \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^3} (K(x) - 1) W_{\epsilon,y}^2 dx \\
&\quad - \frac{1}{32\pi\epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W_{\epsilon,y}^2(y)}{|x-y|} dy W_{\epsilon,y}^2 dx.
\end{aligned}$$

(A.4)

Now, we discuss each terms in the right hand of (A.4). First, by the direct computation, one has

$$\begin{aligned}
&\int_{\mathbb{R}^3} (K(x) - 1) W_{\epsilon,y}^2 dx \\
&= \sum_{j=1}^k \int_{\mathbb{R}^3} [K(x) - K(y^j) + K(y^j) - K(y^0)] w_{\epsilon,y^j}^2 dx \\
&\quad + \sum_{i \neq j} \int_{\mathbb{R}^3} [K(x) - K(y^i) + K(y^j) - K(y^0)] w_{\epsilon,y^i} w_{\epsilon,y^j} dx \\
&= \epsilon^3 \sum_{j=1}^k (K(y^j) - K(y^0)) \int_{\mathbb{R}^3} w^2 dx \\
&\quad + \sum_{j=1}^k \int_{\mathbb{R}^3} (K(x) - K(y^j)) w_{\epsilon,y^j}^2 dx \\
&\quad + \sum_{i \neq j} \int_{\mathbb{R}^3} [K(x) - K(y^i) + K(y^j) - K(y^0)] w_{\epsilon,y^i} w_{\epsilon,y^j} dx.
\end{aligned}$$

(A.5)

In view of (A.5), we have

$$\begin{aligned}
&\sum_{j=1}^k \int_{\mathbb{R}^3} (K(x) - K(y^j)) w_{\epsilon,y^j}^2 dx \\
&= \epsilon^3 \sum_{j=1}^k \int_{\mathbb{R}^3} (K(\epsilon x + y^j) - K(y^j)) w^2 dx \\
&= \epsilon^3 \sum_{j=1}^k \left( \int_{B_{\delta/\epsilon}(y^j)} + \int_{B_{\delta/\epsilon}^c(y^j)} \right) (K(\epsilon x + y^j) - K(y^j)) w^2 dx \\
&\leq C\epsilon^3 \left[ \int_{B_{\delta/\epsilon}(y^j)} |\epsilon x|^\theta w^2 dx + \int_{B_{\delta/\epsilon}^c(y^j)} e^{-2|x|} dx \right] \leq C\epsilon^{3+\theta}, \\
&\sum_{i \neq j} \int_{\mathbb{R}^3} (K(x) - K(y^j)) w_{\epsilon,y^i} w_{\epsilon,y^j} dx \\
&= \sum_{i \neq j} \epsilon^3 \int_{\mathbb{R}^3} (K(\epsilon x + y^j) - K(y^j)) w \left( x - \frac{y^i - y^j}{\epsilon} \right) w(x) dx
\end{aligned}$$

$$\begin{aligned}
&\leq C\epsilon^3 \int_{B_{\delta/\epsilon}(y^j)} |\epsilon x|^\theta w \left( x - \frac{y^i - y^j}{\epsilon} \right) w(x) dx \\
&\quad + C\epsilon^3 \int_{B_{\delta/\epsilon}^c(y^j)} w \left( x - \frac{y^i - y^j}{\epsilon} \right) w(x) dx \\
&\leq C\epsilon^3 \left( \epsilon^\theta e^{-|y^i - y^j|/\epsilon} + e^{-|y^i - y^j|/\epsilon} \right) \leq C\epsilon^{3+\theta},
\end{aligned}$$

(A.6)

and similarly,

$$\begin{aligned}
&\sum_{i \neq j} \int_{\mathbb{R}^3} (K(y^i) - K(y^0)) w_{\epsilon,y^i} w_{\epsilon,y^j} dx \\
&\leq C\epsilon^3 \sum_{i \neq j} \int_{\mathbb{R}^3} w \left( x - \frac{y^i - y^j}{\epsilon} \right) w(x) dx \\
&\leq C\epsilon^3 \sum_{i \neq j} e^{-|y^i - y^j|/\epsilon}.
\end{aligned}$$

(A.7)

Thus, from the estimates above and (A.5), we find

$$\begin{aligned}
\int_{\mathbb{R}^3} (K(x) - 1) W_{\epsilon,y}^2 dx &= \epsilon^3 \sum_{j=1}^k (K(y^j) - K(y^0)) \int_{\mathbb{R}^3} w^2 dx \\
&\quad + O(\epsilon^{3+\theta}) + O\left(\epsilon^3 \sum_{i \neq j} e^{-|y^i - y^j|/\epsilon}\right).
\end{aligned}$$

(A.8)

Now, we estimate  $\frac{1}{32\pi\epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (W_{\epsilon,y}^2(y)/|x-y|) dy W_{\epsilon,y}^2 dx$ . We have

$$\begin{aligned}
&\frac{1}{32\pi\epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{W_{\epsilon,y}^2(y)}{|x-y|} dy W_{\epsilon,y}^2 dx \\
&= \frac{1}{32\pi\epsilon^2} \sum_{j=1}^k \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w_{\epsilon,y^j}^2(y)}{|x-y|} dy w_{\epsilon,y^j}^2 dx \\
&\quad + \frac{1}{8\pi\epsilon^2} \sum_{i<j} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w_{\epsilon,y^i}^2(y)}{|x-y|} dy w_{\epsilon,y^i} w_{\epsilon,y^j} dx \\
&\quad + \frac{1}{16\pi\epsilon^2} \sum_{i<j} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w_{\epsilon,y^i}^2(y)}{|x-y|} dy w_{\epsilon,y^i}^2 dx \\
&\quad + \frac{1}{16\pi\epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\sum_{j=1}^k w_{\epsilon,y^j}^2(y)}{|x-y|} dy \sum_{i \neq m \neq j} w_{\epsilon,y^i} w_{\epsilon,y^m} dx \\
&= \frac{\epsilon^3}{32\pi} \sum_{j=1}^k \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w^2(y)}{|x-y|} dy w^2 dx \\
&\quad + \frac{1}{8\pi\epsilon^2} \sum_{i<j} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w_{\epsilon,y^i}^2(y)}{|x-y|} dy w_{\epsilon,y^i} w_{\epsilon,y^j} dx
\end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{16\pi\epsilon^2} \sum_{i < j} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w_{\epsilon,y^i}^2(y)}{|x-y|} dy w_{\epsilon,y^i}^2 dx \\
 & + \frac{1}{16\pi\epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\sum_{j=1}^k w_{\epsilon,y^j}^2(y)}{|x-y|} dy \sum_{i \neq m \neq j} w_{\epsilon,y^i} w_{\epsilon,y^m} dx.
 \end{aligned}
 \tag{A.9}$$

By using the Hardy-Littlewood-Sobolev inequality (45), we have

$$\begin{aligned}
 & \left| \frac{1}{16\pi\epsilon^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\sum_{j=1}^k w_{\epsilon,y^j}^2(y)}{|x-y|} dy \sum_{i \neq m \neq j} w_{\epsilon,y^i} w_{\epsilon,y^m} dx \right| \\
 & \leq \frac{C}{\epsilon^2} \left| \sum_{j=1}^k w_{\epsilon,y^j}^2 \right|_{6/5} \left| \sum_{i \neq m \neq j} w_{\epsilon,y^i} w_{\epsilon,y^m} \right|_{6/5} \\
 & \leq C\epsilon^3 \sum_{i \neq m} e^{-|y^i - y^m|/\epsilon} \leq C\epsilon^{3+\theta}.
 \end{aligned}
 \tag{A.10}$$

Moreover, it follows from Lemma A.1 that

$$\begin{aligned}
 & \left| \frac{1}{16\pi\epsilon^2} \sum_{i < j} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w_{\epsilon,y^i}^2(y)}{|x-y|} dy w_{\epsilon,y^i}^2 dx \right| \\
 & \leq C\epsilon^3 \left| \sum_{i < j} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w(y + (y^i - y^j/\epsilon))}{|x-y|} dy w^2 dx \right| \\
 & \leq C\epsilon^3 \left| \int_{\mathbb{R}^3} \sum_{i < j} \left[ e^{-|x-(y^i - y^j/\epsilon)|} \sum_{m=-1}^2 \left| x - \frac{y^i - y^j}{\epsilon} \right|^m \right. \right. \\
 & \quad \left. \left. + \left| x - \frac{y^i - y^j}{\epsilon} \right|^{-1} \right] w^2(x) dx \right| \\
 & \leq C\epsilon^3 \left| \left( \int_{B_{|y^i - y^j|/2\epsilon}(0)} + \int_{B_{|y^i - y^j|/2\epsilon}^c(0)} \right) \left[ e^{-|x-(y^i - y^j/\epsilon)|} \sum_{m=-1}^2 \right. \right. \\
 & \quad \left. \left. \cdot \left| x - \frac{y^i - y^j}{\epsilon} \right|^m + \left| x - \frac{y^i - y^j}{\epsilon} \right|^{-1} \right] w^2(x) dx \right| \leq C\epsilon^{3+\theta}.
 \end{aligned}
 \tag{A.11}$$

Thus, combining (A.4)-(A.9), we deduce that

$$\begin{aligned}
 I_\epsilon(W_{\epsilon,y}) & = \epsilon^3 \left( \frac{k}{2} \|w\|_{H^1}^2 - \frac{1}{2} \left( \int_{\mathbb{R}^3} w^2 dx \right) \sum_{j=1}^k (K(y^0) \right. \\
 & \quad \left. - K(y^j)) - \frac{k}{32\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w^2(y)}{|x-y|} dy w^2(x) dx \right) \\
 & \quad + O\left( \epsilon^{3+\theta} + \epsilon^3 \sum_{i \neq j} e^{-|y^i - y^j|/\epsilon} \right).
 \end{aligned}
 \tag{A.12}$$

### Data Availability

Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

### Conflicts of Interest

The authors declare that there are no conflicts of interest.

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