

## Research Article

# On the Ground State to Hamiltonian Elliptic System with Choquard's Nonlinear Term

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Received 14 January 2020; Accepted 23 March 2020; Published 5 May 2020

Academic Editor: Wen-Xiu Ma

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In the present paper, we consider the following Hamiltonian elliptic system with Choquard's nonlinear term

$$\begin{cases} -\Delta u + V(x)u = \int_{\Omega} (G(v(y)) / |x - y|^{\beta}) dy g(v) \text{ in } \Omega, \\ -\Delta v + V(x)v = \int_{\Omega} (F(u(y)) / |x - y|^{\alpha}) dy f(u) \text{ in } \Omega, \text{ where } \Omega \subset \mathbb{R}^N \text{ is a bounded domain with a smooth boundary, } 0 < \alpha < N, 0 < \beta < N, \text{ and } F \text{ is the primitive of } f, \text{ similarly for } G. \text{ By establishing a strongly indefinite variational setting, we prove that the above problem has a ground state solution.} \end{cases}$$

## 1. Introduction and Main Results

In this paper, we deal with the existence of ground state solutions for the following Hamiltonian elliptic system with Choquard's nonlinear term:

$$\begin{cases} -\Delta u + V(x)u = \int_{\Omega} \frac{G(v(y))}{|x - y|^{\beta}} dy g(v) \text{ in } \Omega, \\ -\Delta v + V(x)v = \int_{\Omega} \frac{F(u(y))}{|x - y|^{\alpha}} dy f(u) \text{ in } \Omega, \\ u = 0, v = 0 \text{ on } \partial\Omega, \end{cases} \quad (1)$$

where  $\Omega \subset \mathbb{R}^N$  is a bounded domain with a smooth boundary,  $N \geq 3$ ,  $0 < \alpha < N$ ,  $0 < \beta < N$ , and  $F$  is the primitive of  $f$ , similarly for  $G$ . For a single equation in whole space,  $\mathbb{R}^N$  is closely related to the Choquard-Pekar equation:

$$-\Delta u + V(x)u = \int_{\mathbb{R}^N} \frac{F(u(y))}{|x - y|^{\mu}} dy f(u), \text{ in } \mathbb{R}^N, \quad 0 < \mu < N. \quad (2)$$

When  $N = 3$ ,  $\mu = 1$ ,  $V(x) \equiv 1$ , and  $f(u) = u$ , equation (2) has appeared in several contexts of quantum physics. In 1954, Pekar used equation (2) to describe a polaron at rest in quantum theory. In 1976, to model an electron trapped in its own hole, P. Choquard considered (2) as a certain approximation to Hartree-Fock's theory of one component plasma (see [1, 2]). In some particular cases, (2) is also known as the Schrödinger-Newton equation which was introduced by Penrose in [3] to describe the self-gravitational collapse of a quantum mechanical wave function.

For  $N = 3$ ,  $\mu = 1$ ,  $V(x) \equiv 1$ , and  $f(u) = u$ , the existence of ground states of (2) was obtained in [4] by variational methods. Later, Moroz and Van Schaftingen [2] investigated the regularity, radial symmetry, and asymptotic behavior at infinity of positive solutions for a generalized Choquard equation. Other results involving existence, multiplicity, and concentration of Choquard's problems can be found in [5–12] and the references therein.

We also observed that Bhattarai [13] considered the coupled Choquard-type fractional Schrödinger systems:

$$\begin{cases} (-\Delta)^\alpha u + \omega_1 u = \lambda_1 \int_{\mathbb{R}^N} \frac{|u(y)|^{p_1}}{|x-y|^\beta} dy |u|^{p_1-2} u + \int_{\mathbb{R}^N} \frac{|v(y)|^q}{|x-y|^\beta} dy |u|^{q-2} u, \text{ in } \mathbb{R}^N, \\ (-\Delta)^\alpha v + \omega_2 v = \lambda_2 \int_{\mathbb{R}^N} \frac{|v(y)|^{p_2}}{|x-y|^\beta} dy |v|^{p_2-2} v + \int_{\mathbb{R}^N} \frac{|u(y)|^q}{|x-y|^\beta} dy |v|^{q-2} v, \text{ in } \mathbb{R}^N, \end{cases} \quad (3)$$

where  $\omega_1$  and  $\omega_2$  appear as the Lagrange multipliers,  $N \geq 2$ ,  $0 < \alpha < 1$ ,  $0 < \beta < N$ ,  $2 \leq p_1, p_2$ , and  $q < (N + 2\alpha + \beta)/N$ . Bhattarai looked for minimizers of the  $L^2$ -constrained minimization problem via concentration compactness techniques. Recently, Giacomoni et al. [14] were also concerned with the coupled Hardy-Littlewood-Sobolev critical nonlinearity fractional Schrödinger system in a smooth bounded domain  $\Omega \subset \mathbb{R}^N$ :

$$\begin{cases} (-\Delta)^\theta u = \lambda |u|^{q-2} u + \left( \int_{\Omega} \frac{|v(y)|^{2_\mu^*}}{|x-y|^\mu} dy \right) |u|^{2_\mu^*-2} u \text{ in } \Omega, \\ (-\Delta)^\theta v = \delta |v|^{q-2} v + \left( \int_{\Omega} \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu} dy \right) |v|^{2_\mu^*-2} v \text{ in } \Omega, \end{cases} \quad (4)$$

where  $2_\mu^* = (2N - \mu)/(N - 2\theta)$ ,  $0 < q < 2$ . By minimizing over a suitable subset of Nehari’s manifold, they proved the existence of at least two nontrivial solutions for a suitable range of  $\lambda$  and  $\delta$ . For problems involving Hardy-Littlewood-Sobolev’s critical exponent problems, please see [15–17] and the references therein. Among them, a strongly indefinite Choquard equation was studied in [17].

Motivated by the papers mentioned above, in particular, the papers [13, 14, 17], the purpose of the present paper is to investigate the ground state solution of problem (1). To the best of our knowledge, there is no work concerning the existence of ground state solutions to the Choquard-type Hamiltonian elliptic system. For the Hamiltonian elliptic system, we refer the readers to the papers [18–22] and the references therein.

Throughout this paper, we will always assume  $N \geq 3$  and we suppose that  $V, g$ , and  $f$  satisfy the following assumptions.

(V)  $V(x) \in L^\infty(\Omega, \mathbb{R})$ , 0 lies in a gap of  $\sigma(-\Delta + V(x))$ , where  $\sigma(-\Delta + V(x))$  is the spectrum of the operator  $-\Delta + V(x)$ .

(H<sub>1</sub>)  $f, g \in C(\mathbb{R}, \mathbb{R})$ .

(H<sub>2</sub>)  $f(u) = o(u)$  as  $u \rightarrow 0$ ,  $g(v) = o(v)$  as  $v \rightarrow 0$ .

(H<sub>3</sub>)  $\limsup_{u \rightarrow \infty} (f(u)/u^p) < \infty$ ,  $\limsup_{v \rightarrow \infty} (g(v)/v^q) < \infty$ , where  $p, q > 1$  satisfy

$$\frac{N - (\alpha/2)}{(p + 1)N} + \frac{N - (\beta/2)}{(q + 1)N} > \frac{N - 2}{N}. \quad (5)$$

(H<sub>4</sub>) There exists  $\theta > 2$ , such that  $u, v \in \mathbb{R}$ :

$$\begin{aligned} 0 &\leq \theta F(u) \leq f(u)u, \\ 0 &\leq \theta G(v) \leq g(v)v. \end{aligned} \quad (6)$$

(H<sub>5</sub>)  $F(u) > 0$  for  $u \neq 0$  and  $G(v) > 0$  for  $v \neq 0$ .

Before stating our main result, we review the definition of  $(s, t)$  ground state solution about (1). We call  $(u, v) \neq (0, 0)$  as a  $(s, t)$  ground state solution of (1) in work space  $E$  (see Section 2); if  $(\tilde{u}, \tilde{v})$  is another  $(s, t)$  weak solution (see Definition 2), then the corresponding energy functional  $I(u, v) \leq I(\tilde{u}, \tilde{v})$ , where  $I$  will be defined in equation (27) or equation (32).

**Theorem 1.** *Suppose that  $H_1, H_2, H_3, H_4$  and  $H_5$  are satisfied. Then (1) has a  $(s, t)$  ground state solution.*

To prove Theorem 1, here we use a minimizing argument based on the ideas developed in [23]. We are concerned with system (1) in whole space  $\mathbb{R}^N$  involving the Hamiltonian elliptic system with Choquard’s nonlinear term; it is a nonlocal problem, which brings about two obstacles. One is to check the linking structure, and the other one is to prove the boundedness of the corresponding (PS) sequences. To avoid these obstacles, we shall deal with our problem in bounded domain. Indeed, the difficulty is still there for the problem on  $\mathbb{R}^N$ . It is worth pointing out that the monotonicity condition like [23] is not required on the nonlinear terms  $f$  and  $g$ ; it prevents us from using the standard way (see, e.g., [23, 24]) to check that the minimizer is a critical point. Via the basic leitmotiv from Proposition 3.2 of [25], we shall use the deformation lemma to prove it (see Lemma 13). We end this section by giving our arrangements of this paper. In Section 2, we establish the variational settings about (1). In Section 3, we provide some lemmas and then prove Theorem 1.

## 2. Variational Settings

For the system

$$\begin{cases} -\Delta u + V(x)u = \int_{\Omega} \frac{G(v(y))}{|x-y|^\beta} dy g(v), \text{ in } \Omega, \\ -\Delta v + V(x)v = \int_{\Omega} \frac{F(u(y))}{|x-y|^\alpha} dy f(u), \text{ in } \Omega, \end{cases} \quad (7)$$

if  $F(u) = |u|^{\theta+1}$ , by Hardy-Littlewood-Sobolev's inequality (see Theorem 4.3 in [26]), then the term

$$\int_{\Omega} \int_{\Omega} \frac{F(u(y))F(u(x))}{|x-y|^{\alpha}} dy dx, \quad (8)$$

is well defined if  $F(u) \in L^r(\Omega)$  for  $t > 1$  such that  $(2/r) + (\alpha/N) = 2$ . In view of Sobolev's embedding theorem, it requires that  $1 \leq (\theta + 1)r \leq 2N/(N - 2)$ , which leads us to assume that

$$1 - \frac{\alpha}{2N} \leq \theta - 1 \leq \frac{2N - \alpha}{N - 2}. \quad (9)$$

Since we restrict  $p \geq 1, q \geq 1$  and

$$\frac{N - (\alpha/2)}{(p+1)N} + \frac{N - (\beta/2)}{(q+1)N} > \frac{N - 2}{N}. \quad (10)$$

So it holds that  $1 - (\alpha/2N) \leq p + 1$  and  $1 - (\beta/2N) \leq q + 1$  but  $p$  or  $q$  could be supercritical in the sense that

$$\begin{aligned} p + 1 &> \frac{2N - \alpha}{N - 2} \\ \text{or } q + 1 &> \frac{2N - \beta}{N - 2}. \end{aligned} \quad (11)$$

We remark that  $p \geq 1$  is used in (38) and similarly for  $q \geq 1$ . Since  $p$  or  $q$  could be supercritical, we need the fractional Sobolev spaces (see, e.g., [19, 22, 27]). According to  $H_p$ , we can choose  $s, t > 0$  with  $s + t = 2$ , such that

$$\begin{aligned} p + 1 &< \frac{N}{N - 2s} \left(2 - \frac{\alpha}{N}\right), \\ q + 1 &< \frac{N}{N - 2t} \left(2 - \frac{\beta}{N}\right). \end{aligned} \quad (12)$$

Denote  $S = -\Delta + V(x)$ . Since the effective domain  $\mathcal{D}(S) = \mathcal{D}(S^*)$  and  $S$  is symmetric, so  $S$  is self-adjointed on  $L^2(\Omega)$ . Since the self-adjoint operator is closed, according to the polar decomposition theorem (Theorem VIII.32 in [28] and jointly with Theorem 3.2 and 3.3, Ch IV in [29]), it holds that there is a positive self-adjoint operator  $|S|$  (in fact  $|S| = \sqrt{S^*S}$ ), with  $\mathcal{D}(|S|) = \mathcal{D}(S)$  and a partial isometry  $U$  such that  $S = U|S|$ ,  $|S|$  and  $U$  are uniquely determined and

$$\begin{aligned} SU &= US, \\ |S|U &= U|S|. \end{aligned} \quad (13)$$

It is well known that  $U$  and  $|S|$  are both self-adjoint operators on  $L^2(\Omega)$ . In view of Theorem 3.35 in Chapter V of [30] and Corollary 5.5.6 in [31], there is a unique square root operator  $Q$  such that  $Q^2 = |S|$ , furthermore,  $QU = UQ$ . Denote  $|S|^{1/2} := Q$ . Let

$$\begin{aligned} A^s &:= |S|^{s/2}, \\ A^t &:= |S|^{t/2}. \end{aligned} \quad (14)$$

According to [19, 27], we consider a basis of  $L^2(\Omega)$  constituted by eigenfunctions  $\{\phi_n\}$  of

$$-\Delta u + V(x)u = \lambda u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad (15)$$

with associated eigenvalues  $\lambda_n$ . If for  $u \in L^2(\Omega)$ , we write  $u = \sum_{n=1}^{\infty} a_n \phi_n$ , then the effective domain of  $A^s$  and  $A^t$  are

$$\begin{aligned} E^s &:= \mathcal{D}(A^s) = \left\{ u \in L^2(\Omega) : \sum_{n=1}^{\infty} \lambda_n^s |a_n|^2 < \infty \right\}, \\ E^t &:= \mathcal{D}(A^t) = \left\{ u \in L^2(\Omega) : \sum_{n=1}^{\infty} \lambda_n^t |a_n|^2 < \infty \right\}. \end{aligned} \quad (16)$$

They are two Hilbert spaces endowed with the following inner product, respectively,

$$\begin{aligned} \langle u, v \rangle_s &= \langle A^s u, A^s v \rangle_{L^2}, \\ \langle \varphi, \psi \rangle_t &= \langle A^t \varphi, A^t \psi \rangle_{L^2}, \end{aligned} \quad (17)$$

where

$$\begin{aligned} A^s &: E^s \longrightarrow L^2(\Omega), \\ u &\mapsto \sum_{n=1}^{\infty} \lambda_n^{s/2} a_n \phi_n, \end{aligned} \quad (18)$$

is an isometric isomorphism. So,  $A^s$  has the inverse  $(A^s)^{-1}$ , and we denote  $A^{-s} = (A^s)^{-1}$ , similarly for  $A^t$ . Set  $E = E^s \times E^t$ , then  $E$  is a Hilbert space with the inner product and norm

$$\begin{aligned} \langle z, \eta \rangle &= \langle u, \varphi \rangle_s + \langle v, \psi \rangle_t, \\ \|z\|^2 &= \|u\|_s^2 + \|v\|_t^2, \end{aligned} \quad (19)$$

for  $z = (u, v), \eta = (\varphi, \psi) \in E$ . We recall the embedding theorem (see [32]); for  $r > 0$ , the embedding  $E^r \longrightarrow L^k(\mathbb{R}^N)$  is continuous for  $1 \leq k \leq (2N)/(N - 2r)$  and is compact for  $1 \leq k < (2N)/(N - 2r)$ . We also consider the bounded self-adjoint operator  $L : E \longrightarrow E$  defined as follows, for  $z = (u, v), \eta = (\varphi, \psi)$ :

$$\langle Lz, \eta \rangle = \langle A^s Uu, A^t \psi \rangle_{L^2} + \langle A^t v, A^s U\varphi \rangle_{L^2}. \quad (20)$$

A natural question will be asked whether the operator  $L : E \longrightarrow E$  is well defined. Here, we only check if  $\langle A^s Uu, A^t \psi \rangle_{L^2}$  is well defined. In fact, noting that  $A^s$  and  $U$  are self-adjoint operators, we infer that

$$\langle A^s Uu, A^t \psi \rangle_{L^2} = \langle Uu, A^2 \psi \rangle_{L^2} = \langle u, U|S|\psi \rangle_{L^2} = \langle u, S\psi \rangle_{L^2}. \quad (21)$$

By the complex interpolation theory (see Chapter 1.15 of [32]), we get

$$E^s = [L^2(\Omega), H^2(\Omega)]_{1-(s/2)}. \quad (22)$$

So  $\langle A^s U u, A^t \psi \rangle_{L^2} < \infty$ .

Next, we have  $L$  that has only two eigenvalues  $-1$  and  $1$ , whose corresponding eigenspaces are

$$\begin{aligned} E^+ &= \{ (u, A^{-t} A^s U u) : u \in E^s \}, \\ E^- &= \{ (u, -A^{-t} A^s U u) : u \in E^s \}. \end{aligned} \quad (23)$$

Clearly,  $E = E^+ \oplus E^-$ . Indeed,  $0$  lies in a gap of  $\sigma((? + V(x)))$ , and using Theorem 3.3 in Chapter IV of [29], we get

$$L^2(\Omega) = L^+ \oplus L^-, \quad (24)$$

where  $L^\pm = u \in L^2(\Omega) : U u = \pm u$ .

Similar to Proposition 1.1 in [19], for  $z = (u, v)$ , we can easily get

$$Lz = (A^{-s} A^t U v, A^{-t} A^s U u). \quad (25)$$

We consider the eigenvalue problem  $Lz = \lambda z$  in  $E$ , which yields that

$$\lambda^2 v = A^{-t} A^s U A^{-s} A^t U v = U U v = U(v^+ + v^-) = v. \quad (26)$$

Here, we have used the fact that  $A^s U = U A^s$  and  $A^t U = U A^t$  (see [22]).

We consider the corresponding energy functional for (7):

$$\begin{aligned} I(u, v) &= \int_{\Omega} A^s U u A^t v dx - \frac{1}{2} \int_{\Omega} \int_{\Omega} \frac{F(u(y)) F(u(x))}{|x-y|^\alpha} dy dx \\ &\quad - \frac{1}{2} \int_{\Omega} \int_{\Omega} \frac{G(v(y)) G(v(x))}{|x-y|^\beta} dy dx, \quad (u, v) \in E. \end{aligned} \quad (27)$$

Similarly ([27], p. 61), under our assumptions, using the fact that  $A^s U$  and  $A^t$  are linear, jointly with Lebesgue's dominated convergence theorem, the Gateaux derivative of  $I$  in the direction  $\eta = (\varphi, \psi)$  at  $z = (u, v)$  is defined as

$$\begin{aligned} DI(z, \eta) &= \lim_{l \rightarrow 0} \frac{I(z + l\eta) - I(z)}{l} = \langle A^s U u, A^t \psi \rangle_{L^2} + \langle A^s U \varphi, A^t v \rangle_{L^2} \\ &\quad - \lim_{l \rightarrow 0} \frac{1}{2l} \int_{\Omega} \int_{\Omega} \frac{F[u(y) + l\varphi(y)] F[u(x) + l\varphi(x)] - F[u(y)] F[u(x)]}{|x-y|^\alpha} dy dx \\ &\quad - \lim_{l \rightarrow 0} \frac{1}{2l} \int_{\Omega} \int_{\Omega} \frac{G[v(y) + l\psi(y)] G[v(x) + l\psi(x)] - G[v(y)] G[v(x)]}{|x-y|^\beta} dy dx \\ &= \langle A^s U u, A^t \psi \rangle_{L^2} + \langle A^s U \varphi, A^t v \rangle_{L^2} - \int_{\Omega} \int_{\Omega} \frac{F(u(y)) f(u(x)) \varphi(x)}{|x-y|^\alpha} dy dx \\ &\quad - \int_{\Omega} \int_{\Omega} \frac{G(v(y)) g(v(x)) \psi(x)}{|x-y|^\beta} dy dx. \end{aligned} \quad (28)$$

Clearly,  $DI(z, \eta)$  is linear bounded about  $\eta$  and continuous about  $z$ . Thus,  $I \in C^1(E, \mathbb{R})$ . And its Fréchet derivative is given by

$$\begin{aligned} \langle I'(z), \eta \rangle &= \langle A^s U u, A^t \psi \rangle_{L^2} + \langle A^s U \varphi, A^t v \rangle_{L^2} \\ &\quad - \int_{\Omega} \int_{\Omega} \frac{F(u(y)) f(u(x)) \varphi(x)}{|x-y|^\alpha} dy dx \\ &\quad - \int_{\Omega} \int_{\Omega} \frac{G(v(y)) g(v(x)) \psi(x)}{|x-y|^\beta} dy dx \end{aligned} \quad (29)$$

*Definition 2.* We say that  $z = (u, v) \in E \setminus \{0\}$  is a  $(s, t)$  weak solution of (7), if for each  $\eta = (\varphi, \psi) \in E$ , there holds

$$\begin{aligned} \langle A^s U u, A^s \psi \rangle_{L^2} + A^s U \varphi, A^t v_{L^2} \\ = \int_{\Omega} \int_{\Omega} \frac{F(u(y)) f(u(x)) \varphi(x)}{|x-y|^\alpha} dy dx \\ - \int_{\Omega} \int_{\Omega} \frac{G(v(y)) g(v(x)) \psi(x)}{|x-y|^\beta} dy dx. \end{aligned} \quad (30)$$

It is easy to check that the critical point  $(u, v)$  of  $I$  is a  $(s, t)$  weak solution of (7). Moreover, for  $z = (u, v) \in E$ ,  $z = z^+ + z^-$ , in view of Lemma 2.1 in [19], it holds that

$$z^\pm = \left( \frac{u \pm A^{-s}A^tUv}{2}, \frac{v \pm A^{-t}A^sUu}{2} \right). \quad (31)$$

Therefore,

$$I(z) = \frac{1}{2} \left( \|z^+\|^2 - \|z^-\|^2 \right) - \frac{1}{2} \iint_{\Omega} \frac{F(u(y))F(u(x))}{|x-y|^\alpha} dydx - \frac{1}{2} \iint_{\Omega} \frac{G(v(y))G(v(x))}{|x-y|^\beta} dydx. \quad (32)$$

*Remark 3.* If  $\inf_{x \in \Omega} V(x) > 0$ , then  $U = I$ .

### 3. Existence of Ground States

Following [23] (in page 3804), we introduce the generalized Nehari manifold

$$\mathcal{M} = \left\{ z \in E \setminus E^- : \langle I'(z), z \rangle = 0 \text{ and } \langle I'(z), \eta \rangle = 0 \text{ for all } \eta \in E^- \right\}. \quad (33)$$

We need to prove that  $\mathcal{M} \neq \emptyset$ .

For  $z \in E \setminus E^-$ , define  $\gamma \in C^1(\mathbb{R}^+ \times F, \mathbb{R})$  given by

$$\gamma_z(t, \eta) = I(tz^+ + \eta). \quad (34)$$

It is easy to check the following lemma (see, e.g., Lemma 3.1 in [25]).

**Lemma 4.**  $(t, \eta) \in \mathbb{R}^+ \times E^-$  is a critical point of  $\gamma_z$  if and only if  $tz^+ + \eta \in \mathcal{M}$ .

For  $e \in E \setminus E^-$ , set  $\widehat{E}(e) := E^- \oplus \mathbb{R}^+e$ , where  $\mathbb{R}^+ = [0, +\infty)$ .

**Lemma 5.** For each  $z \in E \setminus E^-$ , there exists  $t_z z^+ + \eta_z \in \widehat{E}(z)$ , such that

$$I(t_z z^+ + \eta_z) = \max_{t \geq 0, \eta \in F} I(tz^+ + \eta). \quad (35)$$

Moreover,  $t_z z^+ + \eta_z \in \mathcal{M}$ .

*Proof.* By Lemma 4, it suffices to prove that the maximum exists. This follows from the next two lemmas. Indeed, using Lemma 7 and Lemma 8, combining with  $I(0) = 0$ , it is easy to check that the maximum point exists.

*Remark 6.* Even though  $f$  and  $g$  have satisfied the condition  $(S_5)$  in [23], we cannot have the uniqueness of  $t_z z^+ + \eta_z$ . So, we cannot get the result similar to Proposition 2.3 in [23].

**Lemma 7.** There are  $r > 0$  and  $\rho > 0$  such that  $\inf_{z \in S_\rho^+} I(z) \geq r$ , where  $S_\rho^+ = \{z \in E^+ : \|z\| = \rho\}$ .

*Proof.* For a given  $\varepsilon > 0$ , by  $H_2, H_3$ , and  $H_4$ , there is a  $C(\varepsilon) > 0$  such that

$$|F(u)| \leq \varepsilon |u|^2 + C(\varepsilon) |u|^{p+1}. \quad (36)$$

By Hardy-Littlewood-Sobolev's inequality, one has

$$\begin{aligned} & \iint_{\Omega} \frac{F(u(y))F(u(x))}{|x-y|^\alpha} dydx \\ & \leq C \|F(u)\|_{L^r(\Omega)}^2 \leq C \left( \int_{\Omega} (\varepsilon |u|^2 + C(\varepsilon) |u|^{p+1})^r dx \right)^{2/r} \\ & \leq C_1 \varepsilon \|z\|^4 + C_2 C(\varepsilon) \|z\|^{2(p+1)}, \end{aligned} \quad (37)$$

where  $(2/r) + (\alpha/N) = 2$ . Here we use the fact that

$$2 < 2r \leq r(p+1) < \frac{2N}{N-2s}. \quad (38)$$

Similarly, we have

$$\iint_{\Omega} \frac{G(v(y))G(v(x))}{|x-y|^\alpha} dydx \leq C_1 \varepsilon \|z\|^4 + C_2 C(\varepsilon) \|z\|^{2(p+1)}. \quad (39)$$

Thus, for  $z = (u, A^{-t}A^sUu) \in E^+$ , it holds that

$$I(z) \geq \frac{1}{2} \|z\|^2 - \varepsilon \|z\|^4 - C_1(\varepsilon) \|z\|^{p+1} - C_2(\varepsilon) \|z\|^{q+1}. \quad (40)$$

Hence, we can choose some  $r, \rho > 0$  such that  $I(z) \geq r$  for all  $\|z\| = \rho$ .

**Lemma 8.** For every  $e \in E^+ \setminus \{0\}$ , there is a  $R(e) > 0$  such that  $I < 0$  on  $\widehat{E}(e) \setminus B_{R(e)}(0)$ .

*Proof.* If not, there exists  $\{w_n\} \subset \widehat{E}(e)$  such that  $\|w_n\| \rightarrow \infty$  and  $I(w_n) \geq 0$ .

Without loss of generality, we may assume that  $e \in E^+$ ,  $\|e\| = 1$ . By doing  $(w_{n,1}, w_{n,2}) = w_n = t_n e + w_n^-$  where  $t_n \geq 0$ . Set

$$\bar{w}_n = \frac{w_n}{\|w_n\|} = \frac{t_n}{\|w_n\|} e + \frac{w_n^-}{\|w_n\|} := s_n^2 e + \eta_n^-. \quad (41)$$

Obviously,  $1 = \|\bar{w}_n\|^2 = s_n^2 + \|\eta_n^-\|^2$ . Jointly with

$$\begin{aligned} 0 & \leq \frac{I(w_n)}{\|w_n\|^2} = \frac{1}{2} (s_n^2 - \|\eta_n^-\|^2) \\ & \quad - \frac{1}{2} \iint_{\Omega} \frac{F(w_{n,1}(y))F(w_{n,1}(x))}{|x-y|^\alpha \|w_n\|^2} dydx \\ & \quad - \frac{1}{2} \iint_{\Omega} \frac{G(w_{n,2}(y))G(w_{n,2}(x))}{|x-y|^\beta \|w_n\|^2} dydx \\ & \leq \frac{1}{2} (s_n^2 - \|\eta_n^-\|^2), \end{aligned} \quad (42)$$

one has  $(1/\sqrt{2}) \leq s_n \leq 1$ . So, for a subsequence,  $s_n \rightarrow s > 0$ ,  $\bar{w}_n \rightarrow \bar{w}$ ,  $\bar{w}_n(x) \rightarrow \bar{w}(x)$ , a.e., in  $\Omega$ . Hence,  $\bar{w} = se + \bar{w}^-$  ( $x \neq 0$ ). It means that for

$$\mathcal{A} := \{x \in \Omega : \bar{w}(x) \neq 0\}, \quad (43)$$

where  $|\mathcal{A}| > 0$ . Thus, for  $x \in \mathcal{A}$ ,  $|w_n(x)| \rightarrow \infty$ .

It follows from  $H_2$ ,  $H_3$ , and  $H_5$  that there exists  $C_1, C_2 > 0$  such that

$$\begin{aligned} F(u) &\geq C_1|u|^\theta - C_2, \\ G(v) &\geq C_1|v|^\theta - C_2, \end{aligned} \quad (44)$$

and

$$\begin{aligned} \lim_{u \rightarrow \infty} \frac{F(u)}{u^2} &= \infty, \\ \lim_{v \rightarrow \infty} \frac{G(v)}{v^2} &= \infty. \end{aligned} \quad (45)$$

Thus,  $w_n = (w_{n,1}, w_{n,2})$ , it follows from Fatou's lemma that

$$\begin{aligned} &\int_{\Omega} \int_{\Omega} \frac{F(w_{n,1}(y))F(w_{n,1}(x))}{|x-y|^\alpha \|w_n\|^2} dy dx \\ &+ \int_{\Omega} \int_{\Omega} \frac{G(w_{n,2}(y))F(w_{n,2}(x))}{|x-y|^\beta \|w_n\|^2} dy dx \\ &\geq C \left( \int_{\mathcal{A}} C_1 |w_n|^\theta dx - C_2 |\mathcal{A}| \right) \int_{\mathcal{A}} \frac{F(w_{n,1}) + G(w_{n,2})}{|w_n|^2} \frac{|w_n|^2}{\|w_n\|^2} dx \\ &\geq C_3 \int_{\mathcal{A}} \frac{F(w_{n,1}) + G(w_{n,2})}{|w_n|^2} \frac{|w_n|^2}{\|w_n\|^2} dx \rightarrow \infty, \end{aligned} \quad (46)$$

which would conflict with (38). Here, we use the fact that  $\Omega$  is bounded.

**Lemma 9.**  $\mathcal{M}$  contains all nontrivial critical points.

*Proof.* If  $z = (u, v) \in E$ ,  $z \neq 0$ , and  $I'(z) = 0$ , by  $H_5$  and  $H_6$ , one has

$$\begin{aligned} I(z) &= I(z) - \frac{1}{2} \langle I'(z), z \rangle \\ &= \frac{1}{2} \int_{\Omega} \int_{\Omega} \frac{F(u(y))[f(u(x))u(x) - F(u(x))]}{|x-y|^\alpha} dy dx \\ &+ \frac{1}{2} \int_{\Omega} \int_{\Omega} \frac{G(v(y))[g(v(x))v(x) - G(v(x))]}{|x-y|^\beta} dy dx \\ &\geq C_1 \int_{\Omega} F(u) dx \int_{\Omega} [f(u)u - F(u)] dx \\ &+ C_2 \int_{\Omega} G(v) dx \int_{\Omega} [g(v)v - G(v)] dx > 0. \end{aligned} \quad (47)$$

For  $z \in E^- \setminus \{0\}$ , it follows that

$$\begin{aligned} I(z) &= -\frac{1}{2} \|z^-\|^2 - \int_{\Omega} \int_{\Omega} \frac{F(u(y))F(u(x))}{|x-y|^\alpha} dy dx \\ &- \int_{\Omega} \int_{\Omega} \frac{G(u(y))G(u(x))}{|x-y|^\beta} dy dx \leq 0. \end{aligned} \quad (48)$$

Define

$$m := \inf_{z \in \mathcal{M}} I(z). \quad (49)$$

We shall prove that  $m$  is achieved on  $\mathcal{M}$  and the minimizer is a critical point of  $I$ .

**Lemma 10.**

(1) There exists  $\rho > 0$  such that

$$m = \inf_{z \in \mathcal{M}} I(z) \geq \inf_{z \in S_\rho} I(z) > 0, \quad (50)$$

where  $S_\rho := \{z \in E^+ : \|z\| = \rho\}$ .

(2)  $\|z^+\| \geq \max\{\|z^-\|, \sqrt{2m}\}$  for every  $z \in \mathcal{M}$ .

*Proof.* First, let us prove assertion (1). For every  $z \in \mathcal{M}$ , with Lemma 7 in hand, we can choose some  $\delta > 0$ , small  $\rho > 0$  such that

$$I(z) \geq I\left(\rho \frac{z^+}{\|z^+\|}\right) \geq \delta. \quad (51)$$

Second,  $I(z) \leq 1/2(\|z^+\|^2 - \|z^-\|^2)$  for every  $z \in \mathcal{M}$ , which concludes the proof of assertion (2).

**Lemma 11.**  $I$  is coercive on  $\mathcal{M}$ .

*Proof.* If not, there exists a sequence  $\{z_n\} \subset \mathcal{M}$  such that  $\|z_n\| \rightarrow \infty$  and  $I(z_n) \leq d$  for some  $d \in [c, \infty)$ . Let  $w_n = (w_{n,1}, w_{n,2}) := (z_n / \|z_n\|)$ . Up to a subsequence,  $w_n \rightarrow w$  and  $w_n(x) \rightarrow w(x)$ , a.e., in  $\mathbb{R}^N$ .

If  $w = 0$ , one has  $w_n^+ \rightarrow L^p(\Omega) \times L^q(\Omega)$  for  $1 \leq p < (2N/(N-2s))$ ,  $1 \leq q < (2N/(N-2t))$ . So it holds that

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\Omega} \int_{\Omega} \frac{F(w_{n,1}^+(y))F(w_{n,1}^+(x))}{|x-y|^\alpha} dy dx &= 0, \\ \lim_{n \rightarrow \infty} \int_{\Omega} \int_{\Omega} \frac{G(w_{n,2}^+(y))G(w_{n,2}^+(x))}{|x-y|^\beta} dy dx &= 0. \end{aligned} \quad (52)$$

Jointly with Lemma 5, we obtain

$$d \geq I(z_n) \geq I(sw_n^+) \geq \frac{s^2}{4} + o_n(1). \quad (53)$$

It is absurd if  $s > \sqrt{4d}$ . So  $w \neq 0$ .

Similar to the proof of Lemma 7, it holds that

$$\int_{\Omega} \int_{\Omega} \left[ \frac{F(z_{n,1}(y))F(z_{n,1}(x))}{|x-y|^{\alpha} z_n^2} + \frac{G(z_{n,2}(y))F(z_{n,2}(x))}{|x-y|^{\beta} z_n^2} \right] w_n^2 dy dx \xrightarrow{n \rightarrow \infty} \infty. \quad (54)$$

Therefore,

$$0 \leq \frac{I(z_n)}{\|z_n\|^2} = \frac{1}{2} \left( \|w_n^+\|^2 - \|w_n^-\|^2 \right) - \int_{\Omega} \int_{\Omega} \left[ \frac{F(z_{n,1}(y))F(z_{n,1}(x))}{|x-y|^{\alpha} z_n^2} + \frac{G(z_{n,2}(y))F(z_{n,2}(x))}{|x-y|^{\beta} z_n^2} \right] w_n^2 dy dx \xrightarrow{n \rightarrow \infty} -\infty, \quad (55)$$

which is a clear contradiction.

**Lemma 12.**  $c = \inf_{z \in \mathcal{M}} I(z)$  is achieved.

*Proof.* The proof is similar to Lemma 3.6 in [33], but for readers' convenience, we review the details of the proof.

Let  $\{z_n\} \subset \mathcal{M}$  be a minimizing sequence. In the light of Lemma 10,  $\{z_n\}$  is bounded in  $E$ . Extracting a subsequence if necessary, we have  $z_n \rightharpoonup z$  in  $E$ . By the continuity of the projections,  $z_n^+ \rightharpoonup z^+$ .

Now let us prove that  $z^+ \neq 0$ . If not, by Lemma 5 and the result (2) of Lemma 9, we get

$$m + o_n(1) = I(z_n) \geq I(tz_n^+) = \frac{1}{2} t^2 \|z_n^+\|^2 + o_n(1). \quad (56)$$

It is impossible if  $t$  is large.

By Lemma 5, there exists  $t_z > 0$ ,  $\eta_z \in E^-$  such that

$$t_z z^+ + \eta_z = ((t_z z^+ + \eta_z)_1, (t_z z^+ + \eta_z)_2) \in \mathcal{M}. \quad (57)$$

With Lebesgue's dominated theorem, it follows that

$$\begin{aligned} m &\leq I(t_z z^+ + \eta_z) = \frac{1}{2} t_z^2 \|z^+\|^2 - \frac{1}{2} \|\eta_z^-\|^2 \\ &\quad - \int_{\Omega} \int_{\Omega} \frac{F((t_z z^+ + \eta_z)_1(y))F((t_z z^+ + \eta_z)_1(x))}{|x-y|^{\alpha}} dy dx \\ &\quad - \int_{\Omega} \int_{\Omega} \frac{G((t_z z^+ + \eta_z)_2(y))G((t_z z^+ + \eta_z)_2(x))}{|x-y|^{\beta}} dy dx \\ &= \liminf_{n \rightarrow \infty} I(t_z z_n^+ + \eta_z) \leq \liminf_{n \rightarrow \infty} I(z_n) = c. \end{aligned} \quad (58)$$

**Lemma 13.** If  $z_0 \in \mathcal{M}$  is such that

$$m = \inf_{z \in \mathcal{M}} I(z) = I(z_0), \quad (59)$$

then  $I'(z_0) = 0$ .

*Proof.* Assume by contradiction that  $I'(z_0) \neq 0$ , then  $\delta > 0$  and  $\theta > 0$  exist such that

$$\|I'(z_0)\|_{E^*} \geq \theta, \quad \text{for } z \in B_{2\delta}(z_0) \cap \mathcal{B}, \quad (60)$$

where  $\mathcal{B} := \{z \in \widehat{E}(z_0) : I(z) < m - \varepsilon\}$  (if necessary, we can choose another  $z_0$  and  $\varepsilon$  small enough such that  $B_{2\delta}(z_0) \cap \mathcal{B} \neq \emptyset$ ).

Define

$$T : \widehat{E}(z_0) \longrightarrow \mathbb{R}^+ \times E^-, \quad (61)$$

given by  $T(tz_0^+ + \eta) = (t, \eta)$ .

Denote  $B_{\delta/4} := B_{\delta/4}(z_0) \cap \mathcal{B}$ ,  $\tilde{T} := T|_{B_{\delta/4}}$ , and  $D := \tilde{T}(B_{\delta/4})$ . Define

$$\pi := \tilde{T}^{-1} : D \longrightarrow B_{\delta/4}, \quad (62)$$

given by  $\pi(t, \eta) = tz_0^+ + \eta$ .

It is clear that

$$\max_{(t,\eta) \in \partial D} I(\pi(t, \eta)) \leq m - \varepsilon < m. \quad (63)$$

For

$$0 < \varepsilon < \min \left\{ \frac{\varepsilon}{2}, \frac{\theta\delta}{8} \right\}, \quad (64)$$

$$S := B_{2\delta}(z_0) \cap \mathcal{B},$$

it yields a deformation  $\Gamma$  (see Lemma 2.3 of [34] or A.4 of [35]) such that

- (a)  $\Gamma(1, z) = z$  if  $z \in I^{-1}([m - 2\varepsilon, m + 2\varepsilon]) \cap S_{2\delta}$ ,
- (b)  $\Gamma(1, I_{m+\varepsilon} \cap S) \subset I_{m-\varepsilon}$ , where  $S_{2\delta} := \{z : \text{dist}(z, S) \leq 2\delta\}$ .

Let us define a  $h : D \longrightarrow E$  and for every  $w \in E^-$ ,  $\Psi_0^w, \Psi_1^w : D \longrightarrow \mathbb{R}^2$  by

$$\begin{aligned} h(t, \eta) &= \Gamma(1, \pi(t, \eta)), \\ \Psi_0^w(t, \eta) &= \left( \left\langle I'(\pi(t, \eta)), \pi(t, \eta) \right\rangle, \left\langle I'(\pi(t, \eta)), w \right\rangle \right), \\ \Psi_1^w(t, \eta) &= \left( \left\langle I'(h(t, \eta)), h(t, \eta) \right\rangle, \left\langle I'(h(t, \eta)), w \right\rangle \right). \end{aligned} \quad (65)$$

Lemma 5 and the degree theory now yield that

$$\deg(\Psi_0^w, D, 0) = 1. \quad (66)$$



Since  $(t, \eta) \in \tilde{T}(\partial B_{\delta/4})$ , combining with (53), it therefore holds that

$$I((\pi(t, \eta))) \leq m - \varepsilon < m - 2\varepsilon. \quad (67)$$

It follows from (a) that

$$\Psi_0^w = \Psi_1^w \text{ on } \partial D. \quad (68)$$

Consequently, we obtain

$$\deg(\Psi_1^w, D, 0) = 1. \quad (69)$$

Then, we know that there exists  $(t_0, \eta_0) \in D$  such that  $h(t_0, \eta_0) \in \mathcal{M}$ . This implies that

$$m \leq I(h(t_0, \eta_0)) = I(\Gamma(1, \pi(t_0, \eta_0))). \quad (70)$$

Noting that  $I(\pi(t_0, \eta_0)) < m + \varepsilon$  and also  $\pi(D) = B_{\delta/4} \subset S$ , in view of (b), one has  $m \leq I(\Gamma(1, \pi(t_0, \eta_0))) \leq m - \varepsilon$ , which is a clear contradiction.

*Proof of Theorem 14.* The existence of  $(s, t)$  ground state solution of (1) follows directly from Lemma 12 to Lemma 13.

## Data Availability

Data are available upon request. All data will be such that no personal identifying information will be included.

## Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

## Authors' Contributions

The authors contributed to each part of this paper equally. The authors read and approved the final manuscript.

## Acknowledgments

This work is supported by the National Natural Sciences Foundation of People's Republic of China under Grants 11901514, 11861072, 11961078, and 11561072.

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