Hindawi
Advances in Mathematical Physics
Volume 2020, Article ID 8406509, 12 pages
https://doi.org/10.1155/2020/8406509

Research Article

Existence of Mild Solutions for a Class of Impulsive Hilfer Fractional Coupled Systems

Karim Guida, Khalid Hilal, and Lahcen Ibnelazyz

Sultan Moulay Slimane University, BP 523 Beni Mellal, Morocco

Correspondence should be addressed to Karim Guida; guida.karim@gmail.com

Received 17 June 2020; Accepted 3 September 2020; Published 29 September 2020

Academic Editor: Ming Mei

The aim of this paper is to give existence results for a class of coupled systems of fractional integrodifferential equations with Hilfer fractional derivative in Banach spaces. We first give some definitions, namely the Hilfer fractional derivative and the Hausdorff's measure of noncompactness and the Sadovskii's fixed point theorem.

1. Introduction

Fractional differential equations have been a good tool in many research areas in the last decade, such as engineering, mathematics, physics, and many other sciences [1, 2]. For some basic results on this theory, we refer the readers to the papers [3, 4] and the references therein.

There are many different definitions of fractional derivatives, each one with its importance and application, which helped justify the importance of fractional calculus. We mention here a few of the most notable definitions of fractional derivatives: Hadamard, Caputo–Hadamard, Hilfer, ψ-Hilfer, Caputo–Riesz, Grünwald–Letnikov, for more details we refer the readers to [5–9].

Recently, a lot of attention has been devoted to the existence of fractional differential problems with Hilfer fractional derivative, see [10–12]. The Hilfer fractional derivative, which is a generalization of the Riemann-Liouville fractional derivative, was introduced by nonother than Hilfer [1, 13].

The first results on the existence of general value problems involving Hilfer fractional derivative were investigated in [14] and after that in [12]. Following these results, Gu and Trujilo [15] gave the existence of solutions for fractional differential equations with Hilfer fractional derivative using the notion of measure of noncompactness. These equations are widely employed in the biomedical field.

On the other hand, the concept of noninstantaneous impulses was first introduced in [16] by Hernandez; these conditions appeared in the mathematical description of problems that experience abrupt changes during their evolution in time. In the established works, fractional differential equations (FDEs) involving Caputo's fractional derivative are commonly considered with impulsive conditions for obtaining mild solutions [17–20]. However, in [21], Sousa obtained for the first time the mild solutions for Hilfer fractional differential equations with noninstantaneous impulses. To the best of our knowledge, there are few papers dealing with coupled systems, and on top of that, even fewer existence results for neutral Hilfer fractional differential equations. That is why, to make a little contribution to the already existing results, we consider in this paper a class of coupled systems of Hilfer fractional differential equations with not instantaneous impulses in a Banach space as follows:

$$\begin{cases} {}^{H}D_{0+}^{\beta_1,\gamma_1}[x_1(t)+f_1(t,x_1(t),x_2(t))] = A_1x_1(t) + g_1\left(t, x_1(t), x_2(t), \int_0^t u_1(t,s)\varphi_1(t,s,x_1(s),x_2(s))ds\right), t\in(s_i,t_{i+1}], i=0,1,2,\cdots,m, \\ {}^{H}D_{0+}^{\beta_2,\gamma_2}[x_2(t)+f_2(t,x_1(t),x_2(t))] = A_2x_2(t) + g_2\left(t, x_1(t), x_2(t), \int_0^t u_2(t,s)\varphi_2(t,s,x_1(s),x_2(s))ds\right), t\in(s_i,t_{i+1}], i=0,1,2,\cdots,m, \\ x_1(t) = m_i(t,x_1(t)), t\in(t_i,s_i], i=1,2,\cdots,m, \\ x_2(t) = n_i(t,x_2(t)), t\in(t_i,s_i], i=1,2,\cdots,m, \\ I_{0+}^{1-\alpha_1}(x_1(0)+f_1(0,x_1(0),x_2(0))) = x_{01}, \\ I_{0+}^{1-\alpha_2}(x_2(0)+f_2(0,x_1(0),x_2(0))) = x_{02}, \end{cases} \quad (1)$$

where for $j=1,2$, $I_{0+}^{1-\alpha_j}$ are the RL fractional integrals, ${}^{H}D^{\beta_j,\gamma_j}$ are the Hilfer fractional derivatives of order (β_j,γ_j) with $0\le\beta_j<1, 0\le\gamma_j\le1$, and $0\le\alpha_j=\beta_j+\gamma_j-\beta_j\gamma_j\le1$, the linear operators $A_j : D(A_j)\subset X-\rightarrow X$ are the infinitesimal generators of strongly continuous semigroups $\{T_j(t)\}_{t\ge0}$ in a Banach space $X, 0=t_0=s_0<t_1<s_1<t_2<\cdots<t_i<s_i<t_{i+1}=T$ is a partition of $[0,T], T>0, i=1,2,\cdots,m$.

The functions $f_j : [0,T]\times X\times X-\rightarrow X$ and $g_j : [0,T]\times X\times X\times X-\rightarrow X$ are satisfying some assumptions that will be given later, and the functions m_i: $(t_i,s_i]\times X-\rightarrow X$ and $n_i : (t_i,s_i]\times X-\rightarrow X$ characterize the impulsive conditions and $x_{0j}\in X$. The conditions on $u_j : [0,T]\times[0,T]\rightarrow\mathbb{R}$ and $\varphi_j : [0,T]\times[0,T]\times X\times X-\rightarrow X$ are given in a later part.

This paper is organized as follows: we first give some preliminaries and notions that will be used throughout the work; after that, we will establish the existence results by means of the fixed point theory; last but not least, we will give an example that illustrates the results.

2. Preliminaries and Notations

Let $C(J,X)$ be the complete normed linear space of all continuous functions $x(t)$ defined on the interval $J=[0,T]$ with $\|x(t)\|=\sup_{t\in j}\|x(t)\|$. We define the Banach space $C_{1-\alpha}(J,X)$ introduced in [14] by $C_{1-\alpha}(J,X)=\{x : J\rightarrow X \text{ such that } t^{1-\alpha} x(t)\in C(J,X)\}$ with norm $\|x\|_{C_{1-\alpha}}=\sup_{0\le t\le T}|t^{1-\alpha}x(t)|$.

We also define the Banach space

$$PC_{1-\alpha}(J,X)=\left\{x(t)\text{: } (t-t_i)^{1-\alpha}x(t)\in C((t_i,t_{i+1}],\mathbb{R}) \text{ and } \lim_{t\to t_i}(t-t_i)^{1-\alpha}x(t)\text{exists}\right\}, \quad (2)$$

for $i=1,2,\cdots,m$ with the norm $\|x\|_{PC_{1-\alpha}}=\max\{\sup_{t\in J}\|t^{1-\alpha} x(t^+)\|, \sup_{t\in J}\|t^{1-\alpha}x(t^-)\|\}$.

The space $\mathbb{X}=PC_{1-\alpha_1}\times PC_{1-\alpha_2}$ equipped with the norm $\|(x_1,x_2)\|=\max\{\|x_1\|_{PC_{1-\alpha_1}}, \|x_2\|_{PC_{1-\alpha_2}}\}$ is also a Banach space.

By $L(X)$, we denote the family of bounded linear operators defined on X, and for $j=1,2$, $\{R_{\beta_j,\gamma_j}(t)\}_{t\ge0}$ are the $(\beta_j-\gamma_j)$-resolvent operators generated by A_j.

Definition 1 (see [1]). The Hilfer fractional derivative of order $n-1\le\beta<n, n\in\mathbb{N}; 0\le\gamma\le1$, with lower limit a is defined as follows:

$$D_{a^+}^{\beta,\gamma}f(t)=I_{a^+}^{\beta(n-\gamma)}\frac{d}{dt}I_{a^+}^{(1-\beta)(n-\gamma)}f(t)=I_{a^+}^{\beta(n-\gamma)}D_{a^+}^{\gamma+\beta n-\gamma\beta}f(t), \quad (3)$$

where $I_{a^+}^{\beta(n-\gamma)}$ is the RL integral, and $D_{a^+}^{\gamma+\beta n-\gamma\beta}$ is the RL derivative.

Lemma 2. (see [14]). *Let $0<\beta<1$, $0\le\gamma\le1$, and $\alpha=\beta+\gamma-\beta\gamma$. If $f\in C_{1-\alpha}[a,b]$ is such that $D_{a^+}^{\alpha}f\in C_{1-\alpha}[a,b]$ D, then*

$$I_{a^+}^{\alpha}D_{a^+}^{\alpha}f=I_{a^+}^{\beta}D_{a^+}^{\beta,\gamma}f \text{ and } D_{a^+}^{\alpha}I_{a^+}^{\alpha}f=D_{a^+}^{\gamma(1-\beta)}f. \quad (4)$$

Lemma 3. (see [3]). *Let $0<\beta<1$ and $0\le\alpha\le1$. If $f\in C_{1-\alpha}[a,b]$ and $I_{a^+}^{1-\beta}f\in C_{\alpha}^{1}[a,b]$, then*

$$I_{a^+}^{\beta}D_{a^+}^{\beta}f(x)=f(x)-\frac{I_{a^+}^{1-\beta}f(a)}{\Gamma(\beta)}(x-a)^{\beta-1}, \forall x\in(a,b]. \quad (5)$$

Definition 4. (see [22]). The Hausdorff measure of noncompactness on a bounded subset Ω_X of Banach space X is the mapping $\mu : \mathscr{B}\subset\Omega_X\rightarrow[0,\infty)$ defined by

$$\mu(\mathscr{B})=\inf\{\varepsilon>0 : \mathscr{B}=\cup\mathscr{B}_i \text{ with radius of } \mathscr{B}_i\le\varepsilon \text{ for } i=1,2,..,m\}. \quad (6)$$

We are going to look back on some properties of the measure of noncompactness.

Lemma 5 (see [22, 23]). *The measure of noncompactness μ defined on bounded subsets $\mathscr{A}$ and $\mathscr{B}$ of a Banach space X has the properties:*

(1) $\mu(\mathscr{A}) = 0$ *if and only if* $\mathscr{A}$ *is a relatively compact set*

(2) $\mathscr{A} \subset \mathscr{B}$ *implies that* $\mu(\mathscr{A}) \leq \mu\mathscr{B}$

$$\mu(\bar{\mathscr{A}}) = \mu(\mathscr{A}) \tag{7}$$

(3) $\mu(\mathscr{A} \cup \mathscr{B}) = \max\{\mu(\mathscr{A}), \mu(\mathscr{B})\}$

(4) $\mu(\lambda\mathscr{A}) = |\lambda|\mu(\mathscr{A}), \forall \lambda \in \mathbb{R}$

Lemma 6 (see [22, 23]). *For a bounded set $D \subset X$, there is a countable set $D_0 \subset D$ such that $\mu(D_0) \leq \mu(D)$.*

Lemma 7 (see [24]). *For a bounded and equicontinuous function $\mathscr{G} \in C(J, X)$, the Hausdorff measure of noncompactness $\mu(\mathscr{G}(t))$ is continuous on J, and $\mu(\mathscr{G}) = \max_{t\in J}\mu(\mathscr{G}(t))$.*

Lemma 8 (see [23]). *Let $D = \{x_n\} \subset C(J, X)$ be a bounded countable subset of X. Then, $\mu(D(t))$ is Lebesque integrable on X, and*

$$\mu\left(\int x_n(t)dt : n \in \mathbb{N}\right) \leq \int \mu(x_n(t))dt. \tag{8}$$

Lemma 9. (see [21]). *We apply lemmas 2 and 3; then, we obtain an equivalent system of equations to the system 1 as follows:*

$$x_1(t) = \begin{cases} \dfrac{t^{\alpha_1-1}x_{01}}{\Gamma(\alpha_1)} - f_1(t, x_1(t), x_2(t)) + \dfrac{1}{\Gamma(\beta_1)}\displaystyle\int_0^t (t-s)^{\beta_1-1}\left(A_1x_1(s) + g_1\left(s, x_1(s), x_2(s), \int_0^\tau u_1(s,\tau)\varphi_1(s,\tau,x_1(\tau),x_2(\tau))dt\right)\right)ds, t \in (0, t_1], \\ m_i(t, x_1(t)), t \in (t_i, s_i], i = 1, 2, \cdots, m \\ m_i(t, x_1(t)) - f_1(t, x_1(t), x_2(t)) + \dfrac{1}{\Gamma(\beta_1)}\displaystyle\int_0^t (t-s)^{\beta_1-1}\left(A_1x_1(s) + g_1\left(s, x_1(s), x_2(s), \int_0^\tau u_1(s,\tau)\varphi_1(s,\tau,x_1(\tau),x_2(\tau))dt\right)\right)ds, t \in (s_i, t_{i+1}), i = 1, 2, \cdots, m, \end{cases}$$
$$x_2(t) = \begin{cases} \dfrac{t^{\alpha_2-1}x_{02}}{\Gamma(\alpha_1)} - f_2(t, x_1(t), x_2(t)) + \dfrac{1}{\Gamma(\beta_2)}\displaystyle\int_0^t (t-s)^{\beta_2-1}\left(A_2x_2(s) + g_2\left(s, x_1(s), x_2(s), \int_0^\tau u_2(s,\tau)\varphi_2(s,\tau,x_1(\tau),x_2(\tau))dt\right)\right)ds, t \in (0, t_1], \\ n_i(t, x_2(t)), t \in (t_i, s_i], i = 1, 2, \cdots, m \\ n_i(t, x_2(t)) - f_2(t, x_1(t), x_2(t)) + \dfrac{1}{\Gamma(\beta_2)}\displaystyle\int_0^t (t-s)^{\beta_2-1}\left(A_2x_2(s) + g_2\left(s, x_1(s), x_2(s), \int_0^\tau u_2(s,\tau)\varphi_2(s,\tau,x_1(\tau),x_2(\tau))dt\right)\right)ds, t \in (s_i, t_{i+1}), i = 1, 2, \cdots, m, \end{cases} \tag{9}$$

Remark 10. The Laplace transform of the Hilfer fractional derivative of a function $f(t)$ of order $0 < \beta < 1$ and $0 < \gamma < 1$ is given in [13] by

$$L\left\{D_{a^+}^{\beta,\gamma}f(t)\right\}(s) = s^\beta L\{f(t)\}(s) - s^{\gamma(\beta-1)}I_{0^+}^{(1-\beta)(1-\gamma)}f(0^+), \tag{10}$$

where $I_{0^+}^{(1-\beta)(1-\gamma)}f(0^+)$ is the Riemann-Liouville derivative of order $(1-\beta)(1-\gamma)$.

Now, we give a definition of a pair of mild solution to the problem 1, which is obtained by applying the Laplace transform of the Hilfer fractional derivative.

Definition 11 (see [15, 21]). A pair $(x, y) \in PC_{1-\alpha 1}(J, X) \times PC_{1-\alpha 2}(J, X) = \mathbb{X}$ is said to be a pair of mild solutions of the system (1) if the couple (x, y) satisfies the following coupled system:

$$x_1(t) = \begin{cases} R_{\beta_1,\gamma_1}(t)x_{01} - f_1(t, x_1(t), x_2(t)) - \displaystyle\int_0^t S_{\beta_1}(t-s)A_1f_1(s, x_1(s), x_2(s))ds + \int_0^t S_{\beta_1}(t-s)g_1(s, x_1(s), x_2(s), \int_0^\tau u_1(s,\tau)\varphi_1(s,\tau,x_1(\tau),x_2(\tau))d\tau)ds, t \in (0, t_1], \\ m_i(t, x_1(t)), t \in (t_i, s_i], i = 1, 2, \cdots, m \\ R_{\beta_1,\gamma_1}(t)m_i(s_i, x_1(s_i)) - f_1(t, x_1(t), x_2(t)) - \displaystyle\int_{S_i}^t S_{\beta_1}(t-s)A_1f_1(s, x_1(s), x_2(s))ds + \int_{s_i}^t S_{\beta_1}(t-s)g_1(s, x_1(s), x_2(s), \int_0^\tau u_1(s,\tau)\varphi_1(s,\tau,x_1(\tau),x_2(\tau))d\tau)ds, t \in (s_i, t_{i+1}), i = 1, 2, \cdots, m, \end{cases} \tag{11}$$

$$x_2(t) = \begin{cases} R_{\beta_2,\gamma_2}(t)x_{02} - f_2(t, x_1(t), x_2(t)) - \displaystyle\int_0^t S_{\beta_2}(t-s)A_2f_2(s, x_1(s), x_2(s))ds + \int_0^t S_{\beta_2}(t-s)g_2(s, x_1(s), x_2(s), \int_0^\tau u_2(s,\tau)\varphi_2(s,\tau,x_1(\tau),x_2(\tau))d\tau)ds, t \in (0, t_1], \\ n_i(t, x_2(t)), t \in (t_i, s_i], i = 1, 2, \cdots, m \\ R_{\beta_2,\gamma_2}(t)n_i(s_i, x_2(s_i)) - f_2(t, x_1(t), x_2(t)) - \displaystyle\int_{s_i}^t S_{\beta_2}(t-s)A_2f_2(s, x_1(s), x_2(s))ds + \int_{s_i}^t S_{\beta_2}(t-s)g_2(s, x_1(s), x_2(s), \int_0^\tau u_2(s,\tau)\varphi_2(s,\tau,x_1(\tau),x_2(\tau))d\tau)ds, t \in (s_i, t_{i+1}), i = 1, 2, \cdots, m, \end{cases} \tag{12}$$

for $j=1,2$, we have

$$R_{\beta_j,\gamma_j}(t)=I_{v}^{\gamma_j(1-\beta_j}S_{\beta_j}(t),$$
$$S_{\beta_j}(t)=t^{\alpha_j-1}Z_{\beta_j}(t),$$
$$Z_{\beta_j}(t)=\int_0^\infty \beta_j v_j W_{\beta_j}(v_j)T_j\left(t^{\beta_j}v_j\right)dv_j, \quad (13)$$

where $W_{\beta_j}(v_j)$ are the Wright functions defined as follows:

$$W_{\beta_j}(v_j)=\sum_{n=1}^{\infty}\frac{(-v_j)^{n-1}}{(n-1)\Gamma\left(1-\mu_j n\right)},\ 0<\mu_j<1,\ v_j\in\mathbb{C}, \quad (14)$$

and satisfying

$$\int_0^\infty v_j^{\sigma_j}W_{\beta_j}(v_j)dv_j=\frac{\Gamma(1+\sigma_j)}{\Gamma\left(1+\beta_j\sigma_j\right)},\ v_j,\sigma_j\geq 0. \quad (15)$$

From [15, 25], we can assume that for $j=1,2$

(i) The linear operators $\{W_{\beta_j,\gamma_j}(t)\}_{t>0}$ and $\{S_\beta(t)\}_{t>0}$ are strongly continuous and verify:

$$\left\|W_{\beta_j,\gamma_j}(t)\right\|\leq\frac{M_j t^{\alpha_j-1}}{\Gamma(\alpha_j)}\ \text{and}\ \left\|S_{\beta_j}(t)\right\|\leq\frac{M_j t^{\beta_j-1}}{\Gamma\left(\beta_j\right)}\text{for}\ t>0, \quad (16)$$

(ii) The norm continuity of the family $\{T_j(t)\}$ for $t>0$

3. Existence Results

In this section, we make some assumptions that are necessary to obtain our results:

(H_1) For $j=1,2$, the functions $f_j: J\times X\times X\rightarrow X$ are bounded and Lipschitz continuous, that is, there exist $L_{fj}>0$ and $M_{fj}\in(0,1)$ such that

$$\left\|f_j(t,u,v)\right\|\leq L_{f_j}$$
$$\left\|f_j(t,u_1,v_1)-f_j(t,u_2,v_2)\right\|\leq M_{f_j}\|u_1-u_2\|+N_{f_j}\|v_1-v_2\|. \quad (17)$$

(H_2) For $j=1,2$, the functions $g_j: J\times X\times X\times X\rightarrow D(\mathscr{B})\subset X$ are Caratheodory, that is, $g_j(\cdot,x_1,x_2,x_3)$: $J\rightarrow D(\mathscr{B})\subset X$ is measurable for all $(x_1,x_2,x_3)\in X\times X\times X$, $g_j(t,\cdot,\cdot,\cdot)$: $X\times X\times X\rightarrow D(\mathscr{B})\subset X$ is continuous a.e for $t\in J$, and there exist $\psi_1,\psi_2\in L_{1/\delta}(J,\mathbb{R}^+)$, $1/\delta>1$, and a continuous function ψ_3 such that

$$\left\|g_j(\cdot,x_1,x_2,x_3)\right\|\leq\psi_{j1}(t)\left\|x_1\right\|+\psi_{j2}(t)\|x_2\|+\psi_{j3}(t)\|x_3\|, \quad (18)$$

for almost all $t\in J$.

(H_3) For $j=1,2$, there exist functions $\widehat{\psi}_{j1},\widehat{\psi}_{j2}\in L_{1/\delta}(J,\mathbb{R}^+)$ and constants $M_{gj1},M_{gj2}>0$ such that $\mu(g_j(t,D_1,D_2))\leq M_{gj1}\widehat{\psi}_{j1}\mu(D_1)+M_{gj2}\widehat{\psi}_{j2}\mu(D_2)$, $t\in J$, for any bounded, equicontinuous, and countable sets $D_i\subset X$, $i=1,2$.

(H_4) The impulsive functions $m_i,n_i:[t_i,s_i]\times X\rightarrow X$ are Lipschitz continuous, that is, there exist $K_{mi},K_{ni}>0$, $i=1,2,\cdots,m$, such that for all $x,y\in X$, we have:

$$\|m_i(t,x)-m_i(t,y)\|\leq K_{m_i}\|x-y\|,$$
$$\text{and}\left\|n_i(t,x)-n_i(t,y)\right\|\leq K_{n_i}\left\|x-y\right\|. \quad (19)$$

(H_5) For $j=1,2$, $\phi_j(t,s,\cdot,\cdot)$: $X\rightarrow X$ are caratheodory functions, and there exist $\xi_j:[0,T]\times[0,T]\rightarrow\mathbb{R}$ with

$$\xi_j^*=\sup_{t\in J}\int_0^t\xi_j(t,s)<\infty\ \textit{such that}\ \left\|\varphi(t,s,x_1,x_2)\right\|\leq\xi_j(t,s)\left\|(x_1,x_2)\right\|. \quad (20)$$

(H_6) For $j=1,2$, and for any bounded set $D_1\subset X$ and $0\leq s\leq t\leq T$, there exist functions $\widehat{v}_j:[0,T]\times[0,T]\rightarrow\mathbb{R}$ such that

$$\mu\left(\varphi_j(t,s,D_1)\leq\widehat{v}_j(t,s)\mu(D_1)\text{where}\ \widehat{v}_j^*=\sup\int_0^t\widehat{v}_j(t,s)ds<\infty.\right. \quad (21)$$

(H_7) For $j=1,2$, $u_j^*=\sup\{u_j(t,s),0\leq s\leq t\}$ is bounded and measurable on J along with the continuity of $u_{jt}:J\rightarrow L^\infty(J,\mathbb{R})$ defined by $u_{jt}=u_j(t,s)$.

Theorem 12. *The system (1) has a pair of solutions $(x_1(t),x_2(t))$ in the space $X=PC_{1-\alpha 1}\times PC_{1-\alpha_2}$ if the assumptions (H_1)–(H_7) hold and the following conditions are verified, for $j=1,2$:*

$$M_j\left[\frac{K_j}{\Gamma(\alpha_j)}+\frac{T^{1-\alpha_j+\beta_j+\delta}}{\Gamma\left(\beta_j\right)}\left(\frac{1-\delta}{\beta_j-\delta}\right)^{1-\delta}\left(\left\|\psi_{j1}\right\|+\left\|\psi_{j2}\right\|\right)+\frac{\left\|\psi_{j3}\right\|T^{\beta_j+1-\alpha_j}u_j^*\xi_j^*}{\beta_j\Gamma\left(\beta_j\right)}\right]<1,\ \textit{and}$$

$$\max_{j=1,2}\left\{M_jK_j+M_{fj}+N_{fj}\right\}+\max_{j=1,2}\left\{\frac{M_jT^{\beta_j-\delta}}{\Gamma\left(\beta_j\right)}\left(\frac{1-\delta}{\beta_j-\delta}\right)^{1-\delta}\cdot\left[M_{g1}\left\|\widehat{\psi}_{j1}\right\|_{L^{\frac{1}{\delta}}[0,T]}+M_{g_{j2}}u_j^*\widehat{v}_j^*\left\|\psi_{j2}\right\|_{L^{\frac{1}{\delta}}[0,T]}\right]+\frac{M_jL_{fj}T^{\delta}}{\Gamma\left(\beta_j+1\right)}\right\}<1 \tag{22}$$

Note that $K_1=K_{mi}$ and $K_2=K_{ni}$.

Proof. To prove the existence of solutions for system (1), we only have to prove the existence of solutions for the system (11) and (12) because they are equivalent.

Let us define $\Omega_r=\{(x_1,x_2)\in PC_{1-\alpha_1}(J,X)\times PC_{1-\alpha_2}(J,X)=\mathbb{X}:\|(x_1,x_2)\|\le r\}$ with fix radius r,Ω_r is a nonempty closed convex bounded subset of $\mathbb{X}$.

Define the operator $S:\mathbb{X}\to\mathbb{X}$ such that $S(x_1,x_2)(t)=(P(x_1,x_2)(t),Q(x_1,x_2)(t))^T,\forall(x_1,x_2)\in\mathbb{X},t\in[0,T]$, where

$$P(x_1,x_2)(t)=\begin{cases}R_{\beta_1,\gamma_1}(t)x_{01}-f_1(t,x_1(t),x_2(t))-\int_0^t S_{\beta_1}(t-s)A_1f_1(s,x_1(s),x_2(s))ds+\int_0^t S_{\beta_1}(t-s)g_1(s,x_1(s),x_2(s),\int_0^\tau u_1(s-\tau)\varphi_1(s,\tau,x_1(\tau),x_2(\tau))d\tau)ds, & t\in(0,t_1],\\ m_i(t,x_1(t)), \quad t\in(t_i,s_i],i=1,2,\cdots,m\\ R_{\beta_1,\gamma_1}(t)m_1(s_i,x_1(s_i))-f_1(t,x_1(t),x_2(t))-\int_{s_i}^t S_{\beta_1}(t-s)A_1f_1(s,x_1(s),x_2(s))ds+\int_{s_i}^t S_{\beta_1}(t-s)g_1(s,x_1(s),x_2(s),\int_0^\tau u_1(s-\tau)\varphi_1(s,\tau,x_1(\tau),x_2(\tau))d\tau)ds, & t\in(s_i,t_{i+1}),i=1,2,\cdots,m,\end{cases} \tag{23}$$

$$Q(x_1,x_2)(t)=\begin{cases}R_{\beta_2,\gamma_2}(t)x_{02}-f_2(t,x_1(t),x_2(t))-\int_0^t S_{\beta_2}(t-s)A_2f_2(s,x_1(s),x_2(s))ds+\int_0^t S_{\beta_2}(t-s)g_2(s,x_1(s),x_2(s),\int_0^\tau u_2(s-\tau)\varphi_2(s,\tau,x_1(\tau),x_2(\tau))d\tau)ds, & t\in(0,t_1],\\ n_i(t,x_2(t)), \quad t\in(t_i,s_i],i=1,2,\cdots,m\\ R_{\beta_2,\gamma_2}(t)n_1(s_i,x_2(s_i))-f_2(t,x_1(t),x_2(t))-\int_{s_i}^t S_{\beta_2}(t-s)A_2f_2(s,x_1(s),x_2(s))ds+\int_{s_i}^t S_{\beta_2}(t-s)g_2(s,x_1(s),x_2(s),\int_0^\tau u_2(s-\tau)\varphi_2(s,\tau,x_1(\tau),x_2(\tau))d\tau)ds, & t\in(s_i,t_{i+1}),i=1,2,\cdots,m,\end{cases} \tag{24}$$

by splitting both (23) and (24), we have:

$$P_1(x_1,x_2)(t)=\begin{cases}R_{\beta_1,\gamma_1}(t)x_{01}-f_1(t,x_1(t),x_2(t)),t\in(0,t_1],\\ m_i(t,x_1(t)), \quad t\in(t_i,s_i],i=1,2,\cdots,m\\ R_{\beta_1,\gamma_1}(t)m_1(s_i,x_1(s_i))-f_1(t,x_1(t),x_2(t)),t\in(s_i,t_{i+1}),i=1,2,\cdots,m,\end{cases}$$

$$P_2(x_1,x_2)(t)=\begin{cases}\int_0^t S_{\beta_1}(t-s)g_1(s,x_1(s),x_2(s),\int_0^\tau u_1(s-\tau)\varphi_1(s,\tau,x_1(\tau),x_2(\tau))d\tau)ds-\int_0^t S_{\beta_1}(t-s)A_1f_1(s,x_1(s),x_2(s))ds, & t\in(0,t_1],\\ \int_{s_i}^t S_{\beta_1}(t-s)g_1(s,x_1(s),x_2(s),\int_0^\tau u_1(s-\tau)\varphi_1(s,\tau,x_1(\tau),x_2(\tau))d\tau)ds-\int_{s_i}^t S_{\beta_1}(t-s)A_1f_1(s,x_1(s),x_2(s))ds, & t\in(s_i,t_{i+1}),i=1,2,\cdots m,\end{cases}$$

$$Q_1(x_1,x_2)(t)=\begin{cases}R_{\beta_2,\gamma_2}(t)x_{02}-f_2(t,x_1(t),x_2(t)),t\in(0,t_1],\\ n_i(t,x_2(t)), \quad t\in(t_i,s_i],i=1,2,\cdots,m\\ R_{\beta_2,\gamma_2}(t)n_1(s_i,x_2(s_i))-f_2(t,x_1(t),x_2(t)),t\in(s_i,t_{i+1}),i=1,2,\cdots,m,\end{cases}$$

$$Q_2(x_1,x_2)(t)=\begin{cases}\int_0^t S_{\beta_2}(t-s)g_2(s,x_1(s),x_2(s),\int_0^\tau u_2(s-\tau)\varphi_2(s,\tau,x_1(\tau),x_2(\tau))d\tau)ds-\int_0^t S_{\beta_2}(t-s)A_2f_2(s,x_1(s),x_2(s))ds, & t\in(0,t_1],\\ \int_{s_i}^t S_{\beta_2}(t-s)g_2(s,x_1(s),x_2(s),\int_0^\tau u_2(s-\tau)\varphi_2(s,\tau,x_1(\tau),x_2(\tau))d\tau)ds-\int_{s_i}^t S_{\beta_2}(t-s)A_2f_2(s,x_1(s),x_2(s))ds, & t\in(s_i,t_{i+1}),i=1,2,\cdots m,\end{cases} \tag{25}$$

The upcoming part of the proof needs us to rewrite the operator S as follows:

$$S(x_1, x_2)(t) = S_1(x_1, x_2)(t) + S_2(x_1, x_2)(t), \tag{26}$$

where

$$\begin{aligned} S_1(x_1, x_2)(t) &= (P_1(x_1, x_2), Q_1(x_1, x_2))(t), \\ S_2(x_1, x_2)(t) &= (P_2(x_1, x_2), Q_2(x_1, x_2))(t). \end{aligned} \tag{27}$$

Step 1. We first show that $S(x_1, x_2) \in \mathbb{X}$ for $(x_1, x_2) \in \mathbb{X}$, that is, $t^{1-\alpha_1}P(x_1, x_2)(t)$ and $t^{1-\alpha_2}Q(x_1, x_2)(t)$ are continuous functions for $t \in (t_i t_{i+1}]$, $i = 0, 1, 2, \cdots, m$.

For $0 \le t_2 \le t \le t_1$, we have:

$$\begin{aligned} &\left\| t^{1-\alpha_1}(P(x_1, x_2)(t) - P(x_1, x_2)(t_2)) \right\| \\ &\quad \le \left\| R_{\beta_1,\gamma_1}(t)x_{01} - R_{\beta_1,\gamma_1}(t_2)x_{01} \right\| \\ &\quad + \| f_1(t, x_1(t), x_2(t)) - f_1(t_2, x_1(t_2), x_2(t_2)) \| \\ &\quad + \int_0^{t_2} t^{1-\alpha_1} \left\| S_{\beta_1}(t-s) - S_{\beta_1}(t_2 - s) \right\| \\ &\quad \times \left\| g_1\left(s, x_1(s), x_2(s), \int_0^\tau u_1(s,\tau)\varphi_1(s,\tau,x_1(\tau),x_2(\tau))d\tau\right) \right\| ds \\ &\quad + \int_{t_2}^t t^{1-\alpha_1} \left\| S_{\beta_1}(t-s) \right\| \\ &\quad \times \left\| g_1\left(s, x_1(s), x_2(s), \int_0^\tau u_1(s,\tau)\varphi_1(s,\tau,x_1(\tau),x_2(\tau))d\tau\right) \right\| ds \\ &\quad + \int_0^{t_2} t^{1-\alpha_1} \left\| S_{\beta_1}(t-s) - S_{\beta_1}(t_2 - s) \right\| \|A_1\| \\ &\quad \cdot \| f_1(s, x_1(s), x_2(s)) \| ds + \int_{t_2}^t t^{1-\alpha_1} \left\| S_{\beta_1}(t-s) \right\| \|A_1\| \\ &\quad \cdot \| f_1(s, x_1(s), x_2(s)) \| ds. \end{aligned} \tag{28}$$

We make the substitution $t_2 - s = s_1$ in the third and fifth terms, we get
$\le \| R_{\beta_1,\gamma_1}(t)x_{01} - R_{\beta_1,\gamma_1}(t_2)x_{01} \| + L\|t - t_2\| + \int_0^{t_2} T^{1-\alpha_1} \| S_{\beta_1}(t - t_2 + s_1) - S_{\beta_1}(s_1) \| \times \| g_1(t_2 - s_1, x_1(t_2 - s_1), x_2(t_2 - s_1), \int_0^\tau u_1(t_2 - s_1, \tau)\varphi_1(t_2 - s_1, \tau, x_1(\tau), x_2(\tau))d\tau) \| ds_1 + \int_{t_2}^t T^{1-\alpha_1} \| S_{\beta_1}(t-s) \| \times \| g_1(s, x_1(s), x_2(s), \int_0^\tau u_1(s,\tau)\varphi_1(s,\tau,x_1(\tau),x_2(\tau))d\tau) \| ds + \int_0^{t_2} T^{1-\alpha_1} \| S_{\beta_1}(t - t_2 + s_1) - S_{\beta_1}(s_1) \| \|A_1\| \| f_1(t_2 - s_1, x_1(t_2 - s_1), x_2(t_2 - s_1)) \| ds_1 + \int_{t_2}^t T^{1-\alpha_1} \| S_{\beta_1}(t-s) \| \|A_1\| \| f_1(s - x_1(s), x_2(s)) \| ds. \to 0$ as $t \to t_2$.

This proves the continuity of $t^{1-\alpha 1}P(x_1, x_2)(t)$ for $t \in [0, t_1]$, and similarly, we prove the continuity of $t^{1-\alpha 2}Q(x_1, x_2)(t)$.

To show that the operator S is continuous on the intervals $(t_i s_i]$ and $(s_i, t_{i+1}]$ for $i = 1, 2, \cdots, m$, we use the continuity of noninstantaneous impulsive functions $m_i(t, x_1(t))$ and $n_i(t, x_2(t))$. Thus, we conclude that $S(x_1, x_2) \in \mathbb{X}$.

Step 2. We show that $S : \Omega_r \to \Omega_r$, that is, $S(x_1, x_2) \in \Omega_r$, for $(x_1, x_2) \in \Omega_r$.

We first show that the operator P is bounded, which means, $P(x_1, x_2) \in \Omega_r$, for $(x_1, x_2) \in \Omega_r$. Suppose the opposite, so there exist $(x_1, x_2) \in \Omega_r$ and $t \in J$ such that $\|P(x_1, x_2)\|C_{1-\alpha 1} > r$.

For $t \in [0, t_1]$, we have

$$\begin{aligned} &\|P(x_1, x_2)(t)\|_{C_{1-\alpha_1}} = \left\| t^{1-\alpha_1}P(x_1, x_2)(t) \right\| \le \left\| t^{1-\alpha_1}R_{\beta_1,\gamma_1}(t)x_{01} \right\| \\ &\quad + \left\| t^{1-\alpha_1} f_1(t, x_1(t), x_2(t)) \right\| + \int_0^t \left\| S_{\beta_1}(t-s) \right\| \\ &\quad \times \left\| t^{1-\alpha_1} g_1\left(s, x_1(s), x_2(s), \int_0^\tau u_1(s,\tau)\varphi_1(s,\tau,x_1(\tau),x_2(\tau))d\tau\right) \right\| ds \\ &\quad + \int_0^t \left\| S_{\beta_1}(t-s) \right\| \|A_1\| \left\| t^{1-\alpha_1} f_1(s, x_1(s), x_2(s)) \right\| ds \le L_{f_1} + \frac{M_1\|x_{01}\|}{\Gamma(\alpha_1)} \\ &\quad + \frac{M_1 L_{f_1}\|A_1\|T^{\beta_1}}{\Gamma(\beta_1 + 1)} + \frac{M_1 t^{1-\alpha_1}}{\Gamma(\beta_1)} \int_0^t (t-s)^{\beta_1 - 1} \| g_1(s, x_1(s), x_2(s), \\ &\quad \left. \int_0^\tau u_1(s,\tau)\varphi_1(s,\tau,x_1(\tau),x_2(\tau))d\tau\right) \Bigg\| ds \le L_{f_1} + \frac{M_1\|x_{01}\|}{\Gamma(\alpha_1)} \\ &\quad + \frac{M_1 L_{f_1}\|A_1\|T^{\beta_1}}{\Gamma(\beta_1 + 1)} + \frac{M_1 t^{1-\alpha_1}}{\Gamma(\beta_1)} \int_0^t (t-s)^{\beta_1 - 1}(\psi_{11}(s)r + \psi_{12}(s)r \\ &\quad + \psi_{13}(s)u_1^* r \int_0^s \xi_1(s-\tau)ds \le L_{f_1} + \frac{M_1\|x_{01}\|}{\Gamma(\alpha_1)} + \frac{M_1 L_{f_1}\|A_1\|T^{\beta_1}}{\Gamma(\beta_1 + 1)} \\ &\quad + \frac{M_1\|\psi_{13}\|u_1^* r T^{\beta+1-\alpha_1}\xi_1^*}{\beta_1\Gamma(\beta_1)} + \frac{r M_1 T^{1-\alpha_1}}{\Gamma(\beta_1)} \left(\int_0^t (t-s)^{\frac{\beta_1 - 1}{1-\delta}} ds \right)^{1-\delta} \\ &\quad \cdot \left(\int_0^t (\psi_{11}(s))^{\frac{1}{\delta}} ds \right)^\delta + \frac{r M_1 T^{1-\alpha_1}}{\Gamma(\beta_1)} \left(\int_0^t (t-s)^{\frac{\beta_1 - 1}{1-\delta}} ds \right)^{1-\delta} \\ &\quad \cdot \left(\int_0^t (\psi_{12}(s))^{\frac{1}{\delta}} ds \right)^\delta = L_{f_1} + \frac{M_1\|x_{01}\|}{\Gamma(\alpha_1)} + \frac{M_1 L_{f_1}\|A_1\|T^{\beta_1}}{\Gamma(\beta_1 + 1)} \\ &\quad + \frac{r M_1 T^{1-\alpha_1}}{\Gamma(\beta_1)} \left[\left(\frac{1-\delta}{\beta_1 - \delta} \right)^{1-\delta} T^{\beta_1 - \delta} \|\psi_{11}\|_{L^1_{\frac{1}{\delta}[0,T]}} \right. \\ &\quad \left. + \left(\frac{1-\delta}{\beta_1 - \delta} \right)^{1-\delta} T^{\beta_1 - \delta} \|\psi_{12}\|_{L^1_{\frac{1}{\delta}[0,T]}} + \frac{\|\psi_{13}\| T^{\beta_1} u_1^* \xi_1^*}{\beta_1} \right]. \end{aligned} \tag{29}$$

which implies that

$$\begin{aligned} \left\| t^{1-\alpha_1}P(x_1, x_2)(t) \right\| &\le Lf_1 + \frac{M_1\|x_{01}\|}{\Gamma(\alpha_1)} + \frac{M_1 Lf_1\|A_1\|T^{\beta_1}}{\Gamma(\beta_1 + 1)} \\ &+ \frac{r M_1 T^{1-\alpha_1}}{\Gamma(\beta_1)} \left[\left(\frac{1-\delta}{\beta_1 - \delta} \right)^{1-\delta} T^{\beta_1 - \delta} \|\psi_{11}\|_{L^1_{\frac{1}{\delta}[0,T]}} \right. \\ &\left. + \left(\frac{1-\delta}{\beta_1 - \delta} \right)^{1-\delta} T^{\beta_1 - \delta} \|\psi_{12}\|_{L^1_{\frac{1}{\delta}[0,T]}} + \frac{\|\psi_{13}\| T^{\beta_1} u_1^* \xi_1^*}{\beta_1} \right]. \end{aligned} \tag{30}$$

For $t \in (t_i, s_i]$, $i = 1, 2, \cdots, m$, we have

$$\begin{aligned} \left\| t^{1-\alpha_1}P(x_1, x_2)(t) \right\| &= \| m_i(t, x_1(t)) \|_{C_{1-\alpha_1}} \\ &= \| m_i(t, x_1(t)) - m_i(t, 0) + m_i(t, 0) \|_{C_{1-\alpha_1}} \\ &\le K_{m_i} \| x_1(t) \|_{C_{1-\alpha_1}} + \| m_i(t, 0) \|_{C_{1-\alpha_1}} \\ &\le K_{m_i} r + V, \\ &\quad \text{where} \quad V = \sup \| m_i(t, 0) \|_{C_{1-\alpha_1}}. \end{aligned} \tag{31}$$

which implies that

$$\left\|t^{1-\alpha_1}P(x_1,x_2)(t)\right\| \le K_{m_i}r + V. \quad (32)$$

For $t \in (s_i, t_{i+1}]$,

$$\begin{aligned}\left\|t^{1-\alpha_1}P(x_1,x_2)(t)\right\| &= \left\|t^{1-\alpha_1}\Big[R_{\beta_1,\gamma_1}(t)m_i(s_i,x_1(s_i)) - f_1(t,x_1(t),x_2(t)) - \int_{s_i}^t S_{\beta_1}(t-s)A_1 f_1(s,x_1(s),x_2(s))ds \right.\\&\quad \left. + \int_{s_i}^t S_{\beta_1}(t-s)g_1\Big(s,x_1(s),x_2(s),\int_0^\tau u_1(s,\tau)\varphi_1(s,\tau,x_1(\tau),x_2(\tau))d\tau\Big)ds\Big]\right\| \le \frac{M_1\left(K_{m_i}r+V\right)}{\Gamma(\alpha_1)} + L_{f_1}\\&\quad + \frac{M_1 L_{f_1}\|A_1\|\|t-s_i\|^{\beta_1}}{\Gamma(\beta_1+1)} + \frac{rM_1T^{1-\alpha_1}}{\Gamma(\beta_1)}\left[\left(\frac{1-\delta}{\beta_1-\delta}\right)^{1-\delta}\right.\\&\quad \cdot T^{\beta_1-\delta}\|\psi_{11}\|_{L^1_{\frac{1}{\delta}[0,T]}} + \left(\frac{1-\delta}{\beta_1-\delta}\right)^{1-\delta}T^{\beta_1-\delta}\|\psi_{12}\|_{L^1_{\frac{1}{\delta}[0,T]}}\\&\quad \left.+\frac{\|\psi_{13}\|T^{\beta_1}u_1^*\xi_1^*}{\beta_1}\right] \end{aligned} \quad (33)$$

which implies that

$$\begin{aligned}\left\|t^{1-\alpha_1}P(x_1,x_2)(t)\right\| &\le L_{f1} + \frac{M_1\left(K_{m_i}r+V\right)}{\Gamma(\alpha_1)} + \frac{M_1L_{f1}\|A_1\|\|t-s_i\|^{\beta_1}}{\Gamma(\beta_1+1)}\\&\quad + \frac{rM_1T^{1-\alpha_1}}{\Gamma(\beta_1)}\left[\left(\frac{1-\delta}{\beta_1-\delta}\right)^{1-\delta}T^{\beta_1-\delta}\|\psi_{11}\|_{L^1_{\frac{1}{\delta}[0,T]}}\right.\\&\quad \left.+\left(\frac{1-\delta}{\beta_1-\delta}\right)^{1-\delta}T^{\beta_1-\delta}\|\psi_{12}\|L^1_{\frac{1}{\delta}[0,T]} + \frac{\|\psi_{13}\|T^{\beta_1}u_1^*\xi_1^*}{\beta_1}\right].\end{aligned} \quad (34)$$

Combining the expressions (30), (32) and (34), we obtain

$$\begin{aligned}\|P(x_1,x_2)(t)\|_{C_{1-\alpha_1}} &\le L_{f_1} + \frac{M_1\|x_{01}\|}{\Gamma(\alpha_1)} + \frac{M_1\left(K_{m_i}r+V\right)}{\Gamma(\alpha_1)}\\&\quad + \frac{M_1L_{f_1}\|A_1\|T^{\beta_1}}{\Gamma(\beta_1+1)} + \frac{rM_1}{\Gamma(\beta_1)}\left[\left(\frac{1-\delta}{\beta_1-\delta}\right)^{1-\delta}T^{1-\alpha_1+\beta_1-\delta}\|\psi_{11}\|_{L^1_{\frac{1}{\delta}[0,T]}}\right.\\&\quad \left.+\left(\frac{1-\delta}{\beta_1-\delta}\right)^{1-\delta}T^{\beta_1-\delta+1-\alpha_1}\|\psi_{12}\|_{L^1_{\frac{1}{\delta}[0,T]}} + \frac{\|\psi_{13}\|T^{\beta_1-\delta+1-\alpha_1}u_1^*\xi_1^*}{\beta_1}\right].\end{aligned} \quad (35)$$

By our assumptions, we have $\|P(x_1,x_2)(t)\|C_{1-\alpha_1} > r$ which implies that

$$\begin{aligned}&L_{f_1} + \frac{M_1\|x_{01}\|}{\Gamma(\alpha_1)} + \frac{M_1\left(K_{m_i}r+V\right)}{\Gamma(\alpha_1)} + \frac{M_1L_{f_1}\|A_1\|T^{\beta_1}}{\Gamma(\beta_1+1)}\\&\quad + \frac{rM_1}{\Gamma(\beta_1)}\left[\left(\frac{1-\delta}{\beta_1-\delta}\right)^{1-\delta}T^{1-\alpha_1+\beta_1-\delta}\|\psi_{11}\|_{L^1_{\frac{1}{\delta}[0,T]}}\right.\\&\quad \left.+\left(\frac{1-\delta}{\beta_1-\delta}\right)^{1-\delta}T^{\beta_1-\delta+1-\alpha_1}\|\psi_{12}\|_{L^1_{\frac{1}{\delta}[0,T]}} + \frac{\|\psi_{13}\|T^{\beta_1+1-\alpha_1}u_1^*\xi_1^*}{\beta_1}\right]\\&\quad \ge \|P(x_1,x_2)(t)\|C_{1-\alpha_1} > r.\end{aligned} \quad (36)$$

Dividing both sides by r and taking $\delta \to \infty$, we obtain

$$\begin{aligned}&M_1\left[\frac{K_{m_i}}{\Gamma(\alpha_1)} + \frac{T^{1-\alpha_1+\beta_1+\delta}}{\Gamma(\beta_1)}\left(\frac{1-\delta}{\beta_1-\delta}\right)^{1-\delta}(\|\psi_{11}\|+\|\psi_{12}\|)\right.\\&\quad \left.+\frac{\|\psi_{13}\|T^{\beta_1+1-\alpha_1}u_1^*\xi_1^*}{\beta_1\Gamma(\beta_1)}\right] > 1\end{aligned} \quad (37)$$

which is a contradiction. Hence, $\|P(x_1,x_2)(t)\|C_{1-\alpha_1} < r$.

Similarly, we show that $\|Q(x_1,x_2)(t)\|C_{1-\alpha_2} < r$.

Finally, $\|S(x_1,x_2)(t)\| = \max\left(\|P(x_1,x_2)(t)\|_{C_{1-\alpha_1}}, \|Q(x_1,x_2)(t)\|_{C_{1-\alpha_2}}\right) < r$.

That shows that the operator S maps bounded sets to bounded sets.

Step 3. We prove that the operator S_1 is Lipschitz continuous.

For $t \in [0, t_1]$, we have

$$\|P_1(x_1,x_2) - P_1(y_1,y_2)\|C_{1-\alpha_1} \le \left(M_{f_1} + N_{f1}\right)\|(x_1-y_1, x_2-y_2)\|. \quad (38)$$

For $t \in (t_i, s_i]$, $i = 1, 2, \cdots, m$, we have

$$\begin{aligned}&\|P_1(x_1,x_2) - P_1(y_1,y_2)\|C_{1-\alpha_1}\\&\quad = \|m_i(t,x_1(t)) - m_i(t,y_1(t))\|C_{1-\alpha_1} \le K_{mi}\|x_1-y_1\|C_{1-\alpha_1},\end{aligned} \quad (39)$$

so we get

$$\|P_1(x_1,x_2) - P_1(y_1,y_2)\|C_{1-\alpha_1} \le K_{m_i}\|(x_1-y_1, x_2-y_2)\|, \quad (40)$$

and for $t \in (t_i, s_i]$, we have

$$\begin{aligned}&\|P_1(x_1,x_2) - P_1(y_1,y_2)\|C_{1-\alpha_1}\\&\quad \le \left(M_1K_{mi} + M_{f_1} + N_{f_1}\right)\|(x_1-y_1, x_2-y_2)\|.\end{aligned} \quad (41)$$

From (38), (40) and (41), we can say that the operator P_1 is Lipschitz continuous with constant $(M_1K_{mi} + M_{f1} + N_{f1})$, and similarly, we show that the operator Q_1 is Lipschitz continuous with constant $(M_2K_{ni}+M_{f2}+N_{f2})$.

Finally, the operator S_1 is Lipschitz continuous with the constant $\max\{M_1K_{mi}+M_{f_1}+N_{f_1};M_2K_{ni}+M_{f2}+N_{f2}\}$.

Step 4. We show that S_2 is a continuous operator.

Let $\{x_{1n},x_{2n}\}$ be a sequence in Ω_r such that $(x_{1n},x_{2n})\to(x_1,x_2)$ as $n\to\infty$.

The functions f_1,g_1 are continuous with respect to the second, third, and fourth variables, it follows that

$$\lim_{t\to\infty} f_1(t,x_{1n}(t),x_{2n}(t))=f_1(t,x_1(t),x_2(t))$$

$$\text{and}\lim_{t\to\infty} g_1\left(t,x_{1n}(t),x_{2n}(t),\int_0^\tau u_1(s,\tau)\phi_1(t,\tau,x_{1n}(\tau),x_{2n}(\tau))d\tau\right)$$
$$=g_1\left(t,x_1(t),x_2(t),\int_0^\tau u_1(s,\tau)\phi_1(t,\tau,x_1(\tau),x_2(\tau))d\tau\right) \tag{42}$$

by (H_1) and (H_2), we have

$$\begin{aligned}&\|f_1(t,x_{1n}(t),x_{2n}(t))-f_1(t,x_1(t),x_2(t))\|\\&\le 2L_{f_1}\left\|g_1\left(t,x_{1n}(t),x_{2n}(t)\int_0^\tau u_1(s,\tau)\phi_1(t,\tau,x_{1n}(\tau),x_{2n}(\tau))d\tau\right)\right.\\&\left.-g_1\left(,t,x_1(t),x_2(t),\int_0^\tau u_1(s,\tau)\phi_1(t,\tau,x_1(\tau),x_2(\tau))d\tau\right)\right\|\\&\le 2r\left[\psi_{11}(t)+\psi_{12}(t)+\psi_{13}(t)u_1^*\xi^*\right].\end{aligned} \tag{43}$$

Since $\psi_{11},\psi_{12}\in L(1/\delta)[0,T]$ and $\psi_{13}(t)$ is continuous, the functions on the right hand side are integrable.

For all $t\in J$, $(x_{1n},x_{2n}),(x_1,x_2)\in\Omega_r$, we have

$$\begin{aligned}&\|t^{1-\alpha_1}(P_2(x_{1n},x_{2n})(t)-P_2(x_1,x_2)(t_2))\|\le\left\|\int_{S_i}^t t^{1-\alpha_1}S_{\beta_1}(t-s)\right.\\&\times\left[g_1\left(s,x_{1n}(s),x_{2n}(s),\int_0^\tau u_1(s,\tau)\phi_1(s,\tau,x_{1n}(\tau),x_{2n}(\tau))d\tau\right)\right.\\&\left.\left.-g_1\left(s,x_1(s),x_2(s),\int_0^\tau u_1(s,\tau)\phi_1(s,\tau,x_{1n}(\tau),x_{2n}(\tau))d\tau\right)\right]ds\right\|\\&+\left\|\int_{S_i}^t t^{1-\alpha_1}S_{\beta_1}(t-s)A_1(f_1(s,x_{1n}(s),x_{2n}(s)-f_1(s,x_1(s),x_2(s))ds\right\|\\&\le\frac{M_1T^{1-\alpha_1}}{\Gamma(\beta_1)}\int_{s_i}^t(t-s)^{\beta_1-1}\times\|g_1(s,x_{1n}(s),x_{2n}(s),\\&\int_0^\tau u_1(s,\tau)\phi_1(s,\tau,x_{1n}(\tau),x_{2n}(\tau))d\tau)\\&-g_1\left(s,x_1(s),x_2(s),\int_0^\tau u_1(s,\tau)\phi_1(s,\tau,x_{1n}(\tau),x_{2n}(\tau))d\tau\right)\|ds\\&+\frac{M_1\|A_1\|T^{1-\alpha_1}}{\Gamma(\beta_1)}\times\int_{s_i}^t(t-s)^{\beta_1-1}\|f_1(s,x_{1n}(s),x_{2n}(s)-f_1(s,x_1(s),x_2(s))\|ds\end{aligned} \tag{44}$$

By the Lebesgue dominated convergence theorem, we have

$$\left\|t^{1-\alpha_1}(P_2(x_{1n},x_{2n})(t)-P_2(x_1,x_2)(t_2))\right\|\to 0\text{ as }n\to\infty, \tag{45}$$

and by the same method, we show that

$$\left\|t^{1-\alpha_2}(Q_2(x_{1n},x_{2n})(t)-Q_2(x_1,x_2)(t_2))\right\|\to 0\text{ as }n\to\infty, \tag{46}$$

consequently, the operator S_2 is continuous.

Step 5. We show that the operator S_2 is equicontinuous.

For any $(x_1,x_2)\in\Omega_r$ and $s_i<t_1<t_2<t_{i+1}$, $i=0,1,\cdots,m$, we have

$$\begin{aligned}&\left\|t^{1-\alpha_1}(P_2(x_1,x_2)(t_2)-P_2(x_1,x_2)(t_1))\right\|=\left\|\int_{s_i}^{t_2}t^{1-\alpha_1}S_{\beta_1}(t-s)\right.\\&\times g_1\left(s,x_1(s),x_2(s),\int_0^\tau u_1(s,\tau)\varphi_1(s,\tau,x_1(\tau),x_2(\tau))d\tau\right)ds\\&-\int_{s_i}^{t_1}t^{1-\alpha_1}S_{\beta_1}(t-s)\times g_1(s,x_1(s),x_2(s),\\&\cdot\int_0^\tau u_1(s,\tau)\varphi_1(s,\tau,x_1(\tau),x_2(\tau))d\tau)ds\\&-\int_{s_i}^{t_2}t^{1-\alpha_1}S_{\beta_1}(t-s)A_1f_1(s,x_1(s),x_2(s))ds\\&\left.+\int_{s_i}^{s_i}t^{1-\alpha_1}S_{\beta_1}(t-s)A_1f_1(s,x_1(s),x_2(s))ds\right\|\\&\le\left\|\int_{s_i}^{t_1}\left(S_{\beta_1}(t_2-s)-S_{\beta_1}(t_1-s)\right)t^{1-\alpha_1}\right.\\&\left.\times g_1\left(s,x_1(s),x_2(s),\int_0^\tau u_1(s,\tau)\varphi_1(s,\tau,x_1(\tau),x_2(\tau))d\tau\right)ds\right\|\\&+\left\|\int_{s_i}^{t_1}\left(S_{\beta_1}(t_2-s)-S_{\beta_1}(t_1-s)\right)t^{1-\alpha_1}A_1f_1(s,x_1(s),x_2(s))ds\right\|\\&+\left\|\int_{t_1}^{t_2}S_{\beta_1}(t_2-s)t^{1-\alpha_1}A_1f_1(s,x_1(s),x_2(s))ds\right\|=I_1+I_2+I_3+I_4.\end{aligned} \tag{47}$$

We substitute $(t_1-s)=s_1$ in I_1 and I_3, we obtain

$$\begin{aligned}I_1&\le\int_0^{t_1}T^{1-\alpha_1}\left\|S_{\beta_1}(t_2-t_1+s_1)-S_{\beta_1}(s_1)\right\|\times\|g_1(t_1-s_1,x_1(t_1-s_1),\\&\cdot\int_0^\tau u_1(t_1-s_1,\tau)\varphi_1(t_1-s_1,\tau,x_1(\tau),x_2(\tau))d\tau)\|ds_1,\\I_3&\le\int_0^{t_1}T^{1-\alpha_1}\left\|T^{1-\alpha_1}S_{\beta_1}(t_2-t_1+s_1)-S_{\beta_1}(s_1)\right\|\\&\cdot\|A_1f_1(t_1-s_1,x_1(t_1-s_1),x_2(t_1-s_1))\|ds_1,\end{aligned} \tag{48}$$

by the equicontinuity of $(\beta_1-\gamma_1)$-resolvent operator nad Lebesgue dominated convergence theorem, the integrals I_1, $I_3\to 0$ as $t_2\to t_1$.

And we have $I_2, I_4 \to 0$ as $t_2 \to t_1$; it follows that

$$\left\| t^{1-\alpha_1}(P_2(x_1,x_2)(t_2) - P_2(x_1,x_2)(t_1)) \right\| \to 0 \text{ as } t_2 \to t_1, \tag{49}$$

we show with a similar method that

$$\left\| t^{1-\alpha_2}(Q_2(x_1,x_2)(t_2) - Q_2(x_1,x_2)(t_1)) \right\| \to 0 \text{ as } t_2 \to t_1. \tag{50}$$

This proves the equicontinuity of the operator S_2.

Step 6. We prove that the operator S is condensing. Hence, we have to show that for any bounded subset $D \subset \Omega_r$, $\mu(S(D)) < \mu(D)$.

Since S_2 is continuous, for any bounded set $D \subset \Omega_r$, there exists a countable set $D_0 = \{(x_{1n}, x_{2n})\} \subset D$ such that $\mu(S_2(D)) = \mu(S_2(D_0))$.

We know that S_2 is bounded and equicontinuous; it follows that

$$\mu(s_2(D_0)) = \max_{t\in(s_i,t_i+1]} \mu(S_2(D_0(t)), i = 0, 1, \cdots, m, \tag{51}$$

we recall that $S_2(x_1,x_2)(t) = (P_2(x_1,x_2), Q_2(x_1,x_2))(t)$we have

$$\begin{aligned}
\mu(P_2(D_0)(t)) &= \mu\left(\int_{s_i}^t S_{\beta_1}(t-s) g_1(s, x_{1n}(s), x_{2n}(s), \right.\\
&\qquad \left.\int_0^\tau u_1(s,\tau)\varphi_1(s,\tau,x_{1n}(\tau),x_{2n}(\tau))d\tau\right)\\
&\qquad - \int_{s_i}^t S_{\beta_1}(t-s) A_1 f_1(s, x_{1n}(s), x_{2n}(s))ds\\
&\le \frac{M_1}{\Gamma(\beta_1)}\int_{s_i}^t (t-s)^{\beta_1-1}\mu(g_1(s,x_{1n}(s),x_{2n}(s),\\
&\qquad \int_0^\tau u_1(s,\tau)\varphi_1(s,\tau,x_{1n}(\tau),x_{2n}(\tau))d\tau))\\
&\qquad + \frac{M_1}{\Gamma(\beta_1)}\int_{s_i}^t (t-s)^{\beta_1-1}\mu(A_1, f_1(s,x_{1n}(s),x_{2n}(s)))ds\\
&\le \frac{M_1}{\Gamma(\beta_1)}\mu(D_0(s))\int_{s_i}^t (t-s)^{\beta_1-1}\Big(M_{g11}\widehat{\Psi}_{11}(s)\\
&\qquad + M_{g12}(s)u_1^*\widehat{v}_1^*\Big)ds + \frac{M}{\Gamma(\beta_1)}\mu(D_0(s))\\
&\qquad \cdot \int_{s_i}^t (t-s)^{\beta_1-1}\|A_1\|L_{f_1}ds \le \frac{M_1 T^{\beta_1-\delta}}{\Gamma(\beta_1)}\left(\frac{1-\delta}{\beta_1-\delta}\right)^{1-\delta}\\
&\qquad \cdot \left[M_{g11}\|\widehat{\psi}_{11}\|_{L^{\frac{1}{\delta}}[0,T]} + M_{g12}u_1^*\widehat{v}_1^*\|\widehat{\psi}_{12}\|_{L^{\frac{1}{\delta}}[0,T]}\right]\mu(D)\\
&\qquad + \frac{M_1 L_{f1} T^{\beta_1-\delta}}{\Gamma(\beta_1+1)}\mu(D),
\end{aligned} \tag{52}$$

which implies that

$$\begin{aligned}
\mu(P_2(D)) \le{}& \frac{M_1 T^{\beta_1-\delta}}{\Gamma(\beta_1)}\left(\frac{1-\delta}{\beta_1-\delta}\right)^{1-\delta}\Big[M_{g_{11}}\|\widehat{\psi}_{11}\|_{L^{\frac{1}{\delta}}[0,T]}\\
&+ M_{g_{12}}u_1^*\widehat{v}_1^*\|\widehat{\psi}_{12}\|_{L^{\frac{1}{\delta}}[0,T]}\Big]\mu(D) + \frac{M_1 L_{f_1} T^{\beta_1-\delta}}{\Gamma(\beta_1+1)}\mu(D).
\end{aligned} \tag{53}$$

Similarly, we show that

$$\begin{aligned}
\mu(Q_2(D)) \le{}& \frac{M_2 T^{\beta_2-\delta}}{\Gamma(\beta_2)}\left(\frac{1-\delta}{\beta_2-\delta}\right)^{1-\delta}\Big[M_{g_{21}}\|\widehat{\psi}_{21}\|_{L^{\frac{1}{\delta}}[0,T]}\\
&+ M_{g_{22}}u_2^*\widehat{v}_2^*\|\widehat{\psi}_{22}\|_{L^{\frac{1}{\delta}}[0,T]}\Big]\mu(D) + \frac{M_2 L_{f_2} T^{\beta_2-\delta}}{\Gamma(\beta_2+1)}\mu(D).
\end{aligned} \tag{54}$$

Hence, we get

$$\begin{aligned}
\mu(S_2(D)) &\le \max\{\mu(P_2(D)), \mu(Q_2(D))\}\\
&\le \max_{j=1,2}\left\{\frac{M_j T^{\beta_j-\delta}}{\Gamma(\beta_j)}\left(\frac{1-\delta}{\beta_j-\delta}\right)^{1-\delta}\left[M_{g_{j1}}\left\|\widehat{\psi}_{j1}\right\|_{L^{\frac{1}{\delta}}[0,T]}\right.\right.\\
&\qquad \left.\left. + M_{g_{j2}}u_j^*\widehat{v}_j^*\left\|\widehat{\psi}_{j2}\right\|_{L^{\frac{1}{\delta}}[0,T]}\right] + \frac{M_j L_{f_j} T^{\beta_j-\delta}}{\Gamma(\beta_j+1)}\right\}\mu(D).
\end{aligned} \tag{55}$$

Since the operator S_1 is a Lipschitz operator with constant $\max\{M_1K_{m_i} + M_{f_i} + N_{f_i}; M_2K_{n_i} + M_{f_2} + N_{f_2}\}$ for any bounded set $D \subset \Omega_r$, we have

$$\mu(S_1(D_0)) \le \max\left\{M_1K_{m_i} + M_{f_i} + N_{f_i}; M_2K_{n_i} + M_{f_2} + N_{f_2}\right\}\mu(D). \tag{56}$$

As the operator $S = S_1 + S_2$, we obtain $\mu(S(D)) \le \mu(S_1(D)) + \mu(S_2(D)) \le (\max\{M_1K_{m_i} + M_{f_1} + N_{f_1}; M_2K_{n_i} + M_{f_2} + N_{f_2}\} + \max_{j=1,2}\{(M_j T^{\beta_j-\delta}/\Gamma(\beta_j))(1-\delta/\beta_j-\delta)^{1-\delta}[M_{g_{j1}} \|\widehat{\psi}_{j1}\|_{L^{1\delta/1\delta}[0,T]} + M_{g_{j2}}\|\widehat{\psi}_{j2}\|_{L^{1\delta/1\delta}[0,T]}] + (M_j L_{f_j} T^{\beta_j-\delta}/\Gamma(\beta_j+1))\})\mu(D) < \mu(D)$

Thus, $S: \Omega_r \to \Omega_r$ is a condensing operator. Hence, by Sadovskii's fixed point theorem [26], the operator S has at least a pair of solutions $(x_1(t), x_2(t))$. Therefore, the problem (1) has a pair of solutions $(x_1(t), x_2(t))$. This completes the proof.

4. Example

We consider in this example the following problem on $J = [0, 1]$

$$\begin{cases} {}^{H}D_{0}^{\frac{1}{2},\frac{1}{8}}\left[x_1(t,w) + \dfrac{\sin(x_1(t,w)) + \sin(x_2(t,w))}{40}\right] = \dfrac{\partial x_1(t,w)}{\partial w^2} + \dfrac{x_1(t,w)}{10(1+e^t)^{1/4}} + \dfrac{x_2(t,w)}{10(1+e^t)^{1/4}}, + \dfrac{e^{-t/2}}{5}\displaystyle\int_0^t \dfrac{e^{-t}}{t^2}(\sin(x_1(s,w)) + \sin(x_2(t,w)))ds, \in \left(0, \dfrac{1}{3}\right] \cup \left(\dfrac{2}{3}, 1\right] \\ {}^{H}D_{0}^{\frac{1}{2},\frac{1}{8}}\left[x_2(t,w) + \dfrac{\sin(x_1(t,w)) + \sin(x_2(t,w))}{40}\right] = \dfrac{\partial x_2(t,w)}{\partial w^2} + \dfrac{x_1(t,w)}{10(1+e^t)^{1/4}} + \dfrac{x_2(t,w)}{10(1+e^t)^{1/4}}, + \dfrac{e^{-t/2}}{5}\displaystyle\int_0^t \dfrac{e^{-t}}{t^2}(\sin(x_1(s,w)) + \sin(x_2(t,w)))ds, t \in \left(0, \dfrac{1}{3}\right] \cup \left(\dfrac{2}{3}, 1\right] \\ x_1(t,w) = \dfrac{\cos t|x_1(t,w)|}{25 + |x_1(t,w)|}, t \in \left(\dfrac{1}{3}, \dfrac{2,}{3}\right], \\ x_2(t,w) = \dfrac{\cos t|x_2(t,w)|}{25 + |x_2(t,w)|}, t \in \left(\dfrac{1}{3}, \dfrac{2,}{3}\right], \\ x_1(t,0) = x_1(t,1) = 0, t \in [0,1], \\ x_2(t,0) = x_2(t,1) = 0, t \in [0,1], \\ I_{0+}^{1-\alpha_1}(x_1(0,w) + f_1(0, x_1(0,w), x_2(0,w))) = x_{01}, \\ I_{0+}^{1-\alpha_2}(x_2(0,w) + f_2(0, x_1(0,w), x_2(0,w))) = x_{02}, \end{cases} \tag{57}$$

Let $X = L^2[0,1]$ and for $j = 1, 2$, $A_j x_j = x_j''$ with

$D(A_j) = \{x_j \in X : x_j, x_j'$ are absolutely continuous, and $x_j(0) = x_j(1) = 0\}$.

The operators A_j generates equicontinuous C_0-semigroups $T_j(t)$, $(t \geq 0)$ on X with $\|T_j(t)\| \leq 1$ for $t \geq 0$.

We have the following for $j = 1, 2$

$$\beta_j = \frac{1}{2}, \gamma_j = \frac{1}{8}, \alpha_j = \frac{9}{16}, T = 1$$

$$\begin{aligned} g_j\left(t, x_1(t), x_2(t), \int_0^t u_j(t,s)\varphi_j(t,s,x_1(s,w)x_2(s,w))ds\right) &= \frac{x_1(t,w)}{10(1+e^t)^{1/4}} + \frac{x_2(t,w)}{10(1+e^t)^{1/4}} \\ &\quad + \frac{e^{-t/2}}{5}\int_0^t \frac{e^{-t}}{t^2}(\sin(x_1(s,w)) + \sin(x_2(t,w)))ds, \end{aligned}$$

$$\varphi_j(t,s,x_1(s,w),x_2(s,w)) = \frac{e^{-t}}{t^2}(\sin(x_1(s,w)) + \sin(x_2(t,w))), u_j(t,s) = 1$$

$$f_j(t, x_1(t,w), x_2(t,w)) = \frac{\sin(x_1(t,w)) + \sin(x_2(t,w))}{40}$$

$$\begin{aligned} m_i(t, x_1(t,w)) &= \frac{\cos t|x_1(t,w)|}{25 + |x_1(t,w)|}, n_i(t, x_2(t,w)) \\ &= \frac{\cos t|x_2(t,w)|}{25 + |x_2(t,w)|}. \end{aligned} \tag{58}$$

Taking $x_j(t,w) = x_j(t)$, we can see that

$$\begin{aligned} &\left\| g_j\left(t, x_1(t), x_2(t), \int_0^t u_j(t,s)\varphi_j(t,s,x_1(s,w),x_2(s,w))ds\right)\right\| \\ &\quad \leq \psi_{j1}(t)\|x_1(t)\| + \psi_{j2}(t)\|x_2(t)\| \\ &\quad + \psi_{j3}(t)\left\|\int_0^t \varphi_j(t,s,x_1(s),x_2(s))ds\right\| \end{aligned}$$

$$\psi_{j1}(t) = \psi_{j2}(t) = \frac{1}{10(1+e^t)^{1/4}}, \psi_{j3}(t) = \frac{e^{-t/2}}{5}. \tag{59}$$

Similarly, we have

$$\begin{aligned} &\mu\left(g_j\left(t, x_1(t), x_2(t), \int_0^t u_j(t,s)\varphi_j(t,s,x_1(s,w),x_2(s,w))ds\right)\right) \\ &\quad \leq \widehat{\psi}_{j1}(t)\mu((x_1(t), x_2(t))) + \widehat{\psi}_{j2}(t)\mu\left(\int_0^t \varphi_j(t,s,x_1(s),x_2(s))ds\right), \end{aligned}$$

$$\widehat{\psi}_{j1}(t) = \frac{1}{10(1+e^t)^{1/4}}, \widehat{\psi}_{j2}(t) = \frac{e^{-t/2}}{5}.$$

$$\left\|\varphi_j(t,s,x_1(s,w),x_2(s,w))\right\| \leq \xi_j(t,s)\|(x_1(s,w),x_2(s,w))\| \tag{60}$$

with $\xi_j(t,s) = e^{-s}/t^2$ and $\xi_j^* = \sup_{t\in[0,1]} \int_0^t \xi_j(t,s)ds = 0.63212$, We take $\delta = 1/4$

$$\left\|\psi_{j1}\right\|_{L^{\frac{1}{\delta}}[0,1]} = \left\|\psi_{j2}\right\|_{L^{\frac{1}{\delta}}[0,1]} = \left(\int_0^1 \left[\frac{1}{10(1+e^t)^{1/4}}dt\right]^4\right)^{\frac{1}{4}} = 0.063$$

$$\text{and}\left\|\psi_{j3}(t)\right\|_{L^{\frac{1}{\delta}}[0,1]} = \left(\int_0^1 \left[\frac{e^{-t/2}}{5}dt\right]^4\right)^{\frac{1}{4}} = 0.102. \tag{61}$$

For $j = 1, 2, i = 1, 2, \cdots, m$, the functions $f_j(t, x_1(t), x_2(t))$, $m_i(t, x_1(t))$ and $n_i(t, x_2(t))$ are Lipschitz functions with constants:

$L_{f_j} = 1/20, M_{f_j} = N_{f_j} = 1/40$ and $K_{m_i} = K_{n_i} = 1/25$. Thus, we have

$$\begin{aligned} &\beta_j = \frac{1}{2}, \gamma_j = \frac{1}{8}, \alpha_j = \frac{9}{16}, \delta = \frac{1}{4}, M_j = 1, u_j^* = 1, \\ &M_{g_{j1}} = M_{g_{j2}} = 1, L_{f_j} = \frac{1}{40}, L_{m_i} = L_{n_i} = \frac{1}{25}, \\ &\left\|\psi_{j1}\right\| = \left\|\psi_{j2}\right\| = \left\|\widehat{\psi}_{j1}\right\| = 0.063, \\ &\left\|\psi_{j3}\right\| = \left\|\widehat{\psi}_{j2}\right\| = 0.102, \widehat{v}_j^* = \xi_j^* = 0.63212. \end{aligned} \tag{62}$$

For these values, the condition (1) of theorem 12 is satisfied:

we have for $j = 1, 2$:

$$\begin{aligned} &M_j\left[\frac{K_j}{\Gamma(\alpha_j)} + \frac{T^{1-\alpha_j+\beta_j+\delta}}{\Gamma\left(\beta_j\right)}\left(\frac{1-\delta}{\beta_j-\delta}\right)^{1-\delta}\left(\left\|\psi_{j1}\right\| + \left\|\psi_{j2}\right\|\right)\right. \\ &\left. + \frac{\left\|\psi_{j3}\right\| T^{\beta_j+1-\alpha_j} u_j^* \xi_j^*}{\beta_j \Gamma\left(\beta_j\right)}\right] \simeq 0.260 < 1. \end{aligned} \tag{63}$$

The second condition is also verified:

$$\begin{aligned} &\max_{j=1,2}\left\{M_j K_j + M_{f_j} + N_{f_j}\right\} + \max_{j=1,2}\left\{\frac{M_j T^{\beta_j-\delta}}{\Gamma\left(\beta_j\right)}\left(\frac{1-\delta}{\beta_j-\delta}\right)^{1-\delta}\right. \\ &\cdot\left[M_{g_{j1}}\left\|\widehat{\psi}_{j1}\right\|_{L^{\frac{1}{\delta}}[0,T]} + M_{g_{j2}} u_j^* \widehat{v}_j^* \left\|\widehat{\psi}_{j2}\right\|_{L^{\frac{1}{\delta}}[0,T]}\right] \\ &\left. + \frac{M_j L_{f_j} T^{\beta_j-\delta}}{\Gamma\left(\beta_j+1\right)}\right\} \simeq 0.3103 < 1 \end{aligned} \tag{64}$$

Consequently, both conditions are satisfied, which means that the problem (1) has a couple of solutions $(x_1(t), x_2(t))$ in the space $\mathbb{X} = PC_{1-\alpha 1} \times PC_{1-\alpha 2}$.

5. Conclusion

In this paper, we achieved the existence of solutions for a class of impulsive Hilfer fractional coupled systems by converting the problem to an integral form and then using the Sadovskii's fixed point theorem. For future works, we can consider other fractional operators for example the ψ-Hilfer fractional operator for its new results and applications.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

The authors received no specific funding for this work.

References

[1] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[2] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Amsterdam, 1987.

[3] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[4] F. Jiao and Y. Zhou, "Existence results for fractional boundary value problem via critical point theory," *International Journal of Bifurcation and Chaos*, vol. 22, no. 4, article 1250086, 2012.

[5] E. C. de Oliveira and J. A. T. Machado, "A review of definitions for fractional derivatives and integral," *Mathematical Problems in Engineering*, vol. 2014, Article ID 238459, 6 pages, 2014.

[6] J. V. C. Sousa and E. C. de Oliveira, "On the ψ -Hilfer fractional derivative," *Communications in Nonlinear Science and Numerical Simulation*, vol. 60, pp. 72–91, 2018.

[7] J. V. C. Sousa and E. C. de Oliveira, "On the ψ-fractional integral and applications," *Computational and Applied Mathematics*, vol. 38, p. 4, 2019.

[8] J. V. C. Sousa and E. C. de Oliveira, "Leibniz type rule: ψ-Hilfer fractional operator," *Communications in Nonlinear Science and Numerical Simulation*, vol. 60, pp. 72–91, 2018.

[9] R. Almeida, "A Caputo fractional derivative of a function with respect to another function," *Communications in Nonlinear Science and Numerical Simulation*, vol. 44, pp. 460–481, 2017.

[10] K. M. Furati and M. D. Kassim, "Non-existence of global solutions for a differential equation involving Hilfer fractional derivative," *Electronic Journal of Differential Equations*, vol. 2013, no. 235, pp. 1–10, 2013.

[11] R. Kamocki and C. Obczyński, "On fractional Cauchy-type problems containing Hilfer's derivative," *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 2016, no. 50, pp. 1–12, 2016.

[12] J. R. Wang and Y. Zhang, "Nonlocal initial value problems for differential equations with Hilfer fractional derivative,"

Applied Mathematics and Computation, vol. 266, pp. 850–859, 2015.

[13] R. Hilfer, “Fractional time evolution,” *Applications of Fractional Calculus in Physics*, pp. 87–130, World Scientific, 2000.

[14] K. M. Furati, M. D. Kassim, and N. Tatar, “Existence and uniqueness for a problem involving Hilfer fractional derivative,” *Computers & Mathematics with Applications*, vol. 64, no. 6, pp. 1616–1626, 2012.

[15] H. Gu and J. J. Trujillo, “Existence of mild solution for evolution equation with Hilfer fractional derivative,” *Applied Mathematics and Computation*, vol. 257, pp. 344–354, 2015.

[16] E. Hernandez and D. O’Regan, “On a new class of abstract impulsive differential equations,” *Proceedings of American Mathematical Society*, vol. 141, no. 5, pp. 1641–1649, 2013.

[17] P. Chen, X. Zhang, and Y. Li, “Existence of mild solutions of partial differential equations with non-instantaneous impulses,” *Electronic Journal of Differential EquationsElectron. Journal of Difference Equations*, vol. 2016, no. 241, pp. 1–11, 2016.

[18] M. Pierri, D. O’Regan, and V. Rolnik, “Existence of solutions for semi-linear abstract differential equations with not instantaneous impulses,” *Applied Mathematics and Computation*, vol. 219, no. 12, pp. 6743–6749, 2013.

[19] K. Hilal, K. Guida, L. Ibnelazyz, and M. Oukessou, “Existence results for an impulsive fractional integro-differential equations with a non-compact semigroup: theoretical aspects and applications,” in *Recent Advances in Intuitionistic Fuzzy Logic Systems. Studies in Fuzziness and Soft Computing, vol 372*, S. Melliani and O. Castillo, Eds., pp. 191–211, Springer, Cham, 2019.

[20] K. Hilal, L. Ibnelazyz, K. Guida, and S. Melliani, “Existence of Mild Solutions for an Impulsive Fractional Integro-differential Equations with Non-local Condition,” in *Recent Advances in Intuitionistic Fuzzy Logic Systems. Studies in Fuzziness and Soft Computing, vol 372*, S. Melliani and O. Castillo, Eds., pp. 251–271, Springer, Cham, 2019.

[21] J. Sousa, “Existence of mild solutions to Hilfer fractional evolution equations in Banach Space,” 2018, http://arxiv.org/abs/1812.02213.

[22] J. Banas, “Measures of noncompactness in the study of solutions of nonlinear differential and integral equations,” *Open Mathematics*, vol. 10, no. 6, 2012.

[23] M. Mursaleen and A. K. Noman, “The Hausdorff measure of noncompactness of matrix operators on some BK spaces,” *Operators and Matrices*, vol. 5, pp. 473–486, 2007.

[24] R. Ye, “Existence of solutions for impulsive partial neutral functional differential equation with infinite delay,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 1, pp. 155–162, 2010.

[25] C. Chen and M. Li, “On fractional resolvent operator functions,” *Semigroup Forum*, vol. 80, no. 1, pp. 121–142, 2010.

[26] S. J. Daher, “On a fixed point principle of Sadovskii,” *Nonlinear Analysis: Theory Methods & Applications*, vol. 2, no. 5, pp. 643–645, 1978.