*Research Article*

# On the Existence and Uniqueness of Solution to Fractional-Order Langevin Equation

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In this work, we give sufficient conditions to investigate the existence and uniqueness of solution to fractional-order Langevin equation involving two distinct fractional orders with unprecedented conditions (three-point boundary conditions including two nonlocal integrals). The problem is introduced to keep track of the progress made on exploring the existence and uniqueness of solution to the fractional-order Langevin equation. As a result of employing the so-called Krasnoselskii and Leray-Schauder alternative fixed point theorems and Banach contraction mapping principle, some novel results are presented in regarding to our main concern. These results are illustrated through providing three examples for completeness.

## 1. Introduction

Unquestionably, the fractional calculus, which requires the order of the functional operator of calculus to be turned into the fractional case, is a forceful mathematical argument for providing much and more flexibility in dealing with many real-world applications [1, 2]. More particularly, the fractional-order differential equations (FODEs), which are considered a cornerstone of this field, have been extensively employed in modeling several phenomena and numerous scientific applications such as diffusion modeling [3], robot manipulators [4], economics [5], and many more [6–9]. In view of its outstanding importance, lots of mathematicians have paid their attentions to deeply explore the initial and/or boundary value problems (BVPs) established based on such equations (see [10–13]). Actually, these problems, which may have different boundary conditions, describe several physical real-world issues like thermoelasticity, underground water flow, and heat conduction. One of the most significant of those problems is known as the Langevin equation which was formulated for the first time in 1908 by Langevin, who still much earlier described the progress of some physical matters in fluctuating settings [14, 16]. It came out, later

on, that such equation had assisted physicians successfully to identify some techniques related to the abnormal diffusion. It also, afterward, succeeded in describing some other techniques in economic field that contain the cost index operating [17]. Besides, its generalization played, and still plays, a significant duty in describing the noise causes in the field of critical dynamics [18]. In 1996, Mainardi and Pironi fractionalized such equation and hence turned it into the fractional-order case to be named later the fractional-order Langevin equation (FoLE) [19]. Several articles then made this equation up at the top of their works (see, e.g., [20–22]). It has been recently shown that such equation, for example, but not limited to, plays notable role in outlining the reaction-diffusion models [23, 24] and describes some processes associated with porous media [25, 26].

In the same framework, many efforts have been underway to explore the two most main aspects associated with the solutions of the FoLEs: the existence and uniqueness [27]. In view of studying such two aspects, several recent articles have been published. In particular, through using some special boundary conditions, some applicable results related to the existence of solutions for the FoLE have been reported in [28]. Both aspects have been investigated when the

coupled incommensurate fractional-order systems were handled by Salem and Alnegga and Salem et al. in [15, 29]. However, in more recent times, numerous outcomes associated with those aspects have, moreover, been informed widely (see [1, 14, 21, 30–34]). In more concrete terms, some useful theoretical tools, like the so-called Banach contraction principle and Krasnoselskii's fixed point theorem, have been employed to explore the existence and uniqueness of solutions for several nonlinear FODEs such as the implicit FODEs [35], the fractional-order Cauchy problems [36].

As per the best of our knowledge, there are few published works that address the FoLE with nonlocal integral boundary conditions as informed in [14]. In view of this motivation, this work examines the existence and uniqueness of solution for the nonlinear FoLE with three-point boundary conditions involving two nonlocal integrals. Actually, this boundary value problem, with two different values of fractional order, has the following form:

$$^{c}D^{r}\left(^{c}D^{s}+\ell\right)z(t)=f(t,z(t),^{c}D^{\rho}z(t)),t\in[0,T],\qquad(1)$$

where $^{c}D^{r}$ is the Caputo derivative operator of order $r$ in which $k-1<r\leq k$ and $k\in\mathbb{N}$. In this work, we assume that the function $f:[0,T]\times\mathbb{R}\times\mathbb{R}\rightarrow\mathbb{R}$ is a continuous function, and $1<r<2$, $0<\rho\leq s\leq1$, and $\ell\in\mathbb{R}$ are the dissipative parameters. We take here the function $f$, in general form, constitutes from position (displacement) $z(t)$ and fractional velocity $^{c}D^{\rho}z(t)$ of the particle at time $t\in[0,T]$, $T>0$. This function perhaps contains external force field, position-dependent phenomenological fluid friction coefficient, intensity of the stochastic force, or zero-mean Gaussian white noise term.

The equation (1) subjects to the following boundary conditions:

$$z(0)=0,\;\;z(T)=\alpha\int_{0}^{T}z(u)du,z(\zeta)=\beta\int_{0}^{\zeta}z(u)du,\qquad(2)$$

where $\alpha,\beta\in\mathbb{R}$ and $0<\zeta<T$. It is remarkable to see that the first condition indicates the particle begins its motion from the origin point. Also, all conditions can be interpreted informally as "the displacement of the particle at a certain point proportionates the signed area of the region in the $tz$-plane that is bounded by the graph of $z$ and the $t$-axis from the origin to this certain point".

One more observation should be recorded; all hypotheses and conditions in this work are assumed on the basis of the various views inspired by the aforementioned cited articles. Anyhow, here is how the rest of this paper is arranged. Section 2 introduces some primary preliminaries associated with fractional calculus and some fixed point theorems, while Section 3 deduces some auxiliary lemmas that pave the way for establishing novel theorems related to the existence and uniqueness of the solution for our main problem. Such theorems are illustrated and shown in Section 4, while Section 5 involves one illustrative example to verify all findings, followed by the last section that summarizes the achievements of the whole work.

## 2. Preliminaries

In the whole of this work, the Caputo derivative operator and its inverse operator, the Riemann-Liouville (R-L) integral, are adopted since they suite most physical systems. These two operators as well as some basic concepts and essential preliminaries about fractional calculus are briefly presented here together with some theoretical results associated with the fixed point theorem.

*Definition 1.* [37]. *The R-L integral operator of order $r$ for a continuous function $z:[0,\infty)\rightarrow\mathbb{R}$ is outlined as*

$$I^{r}z(t)=\frac{1}{\Gamma(r)}\int_{0}^{t}(t-v)^{r-1}z(v)dv,\qquad(3)$$

*where $r>0$.*

*Definition 2.* [37]. *The R-L derivative operator of order $r$ for a $k$-times absolutely continuous function $z:[0,\infty)\rightarrow\mathbb{R}$ is outlined as*

$$D^{r}z(t)=\frac{1}{\Gamma(k-r)}\left(\frac{d}{dt}\right)^{k}\int_{0}^{t}(t-v)^{k-r-1}z(v)dv=\left(\frac{d}{dt}\right)^{k}I^{k-r}z(t),\qquad(4)$$

*where $k-1\leq r<k$ and $k\in\mathbb{N}$.*

*Definition 3.* [37]. *The Caputo derivative operator of order $r$ for a $k$-times absolutely continuous function $z:[0,\infty)\rightarrow\mathbb{R}$ is outlined as*

$$^{c}D^{r}z(t)=\frac{1}{\Gamma(k-r)}\int_{0}^{t}(t-v)^{k-r-1}z^{(k)}(v)dv,\qquad(5)$$

*where $k-1<r\leq k$ and $k\in\mathbb{N}$.*

The following remark states some important relations chosen from the famous books [37]. Actually, such relations will be used during the whole of this work.

*Remark 4.* Suppose that $z:[0,\infty)\rightarrow\mathbb{R}$ *is a continuous function, $r$ is a positive real. Then,*
   (i) $I^{r}I^{s}z(t)=I^{r+s}z(t),s>0$
   (ii) $^{c}D^{r}I^{s}z(t)=I^{s-r}z(t)$, for $r\in[0,s]$, while $^{c}D^{r}I^{r}z(t)=z(t)$
   (iii) $I^{r}t^{\rho}=\Gamma(\rho+1)\Gamma(\rho+r+1)t^{\rho+r},\rho>-1$
   (iv) $^{c}D^{r}t^{\rho}=\Gamma(\rho+1)\Gamma(\rho-r+1)t^{\rho-r},\rho>-1,\rho\neq0,1,\cdots,[r]$
   (v) $^{c}D^{r}t^{\rho}=0,\rho=0,1,\cdots,[r]$

**Lemma 5.** *Suppose $^{c}D^{r}z$ is a continuous function, then*

$$I^{rc}D^{r}z(t)=z(t)+c_{0}+c_{1}t+\cdots+c_{k-1}t^{k-1}\qquad(6)$$

*for $k\in\mathbb{N}$, where $k-1<r\leq k$.*

Next, some primary keys and essential theorems related to the fixed point theorem are stated for the purpose of getting novel results.

*Definition 6.* [38, 39]. *A contraction of a normed space $\mathbb{X}$ is a mapping $S : \mathbb{X} \to \mathbb{X}$ that meets*

$$\|Sz_1 - Sz_2\| \leq \delta \|z_1 - z_2\|, \forall z_1, z_2 \in \mathbb{X} \tag{7}$$

*for some $\delta < 1$.*

**Theorem 7.** (Banach fixed point theorem, Granas and Dugundji [38, 39]). *There is a unique fixed point for every contraction mapping on a complete metric space.*

**Theorem 8.** (Krasnoselskii fixed point theorem [38]). *For a convex set $\mathcal{B}_\varepsilon$ of a Banach space $\mathbb{X}$ and a mapping $Sz = S_1 z + S_2 z$ such that*

(1) *$S_1 z_1 + S_2 z_2 \in \mathcal{B}_\varepsilon$, for each $z_1, z_2 \in \mathcal{B}_\varepsilon$*

(2) *$S_2$ represents a contraction mapping*

(3) *$S_1$ is continuous and compact*

Then, *$S$ has a fixed point.*

**Theorem 9.** [39]. *Let $\mathcal{O}$ be an open set of the Banach space $\mathbb{X} = C([0, T], \mathbb{R}^+)$ such that $\mathcal{O} \neq \phi$. If the operator $S : \bar{\mathcal{O}} \to \mathbb{X}$ is relatively compact, then either*

(1) *$S$ has a fixed point $z^* \in \bar{\mathcal{O}}$, or*

(2) *There exists $\gamma \in (0, 1)$ and $z^* \in \partial\mathcal{O}$ such that $\gamma S(z) = z$, where $\partial\mathcal{O}$ is the boundary of $\mathcal{O}$ itself*

## 3. Some Deduced Auxiliary Lemmas

In this part, some auxiliary lemmas are introduced in order to pave the way for exploring the existence and uniqueness of the solution for our main problem. In other words, such lemmas will play a major role in transforming the main issue to a fixed point problem. For this purpose, let us introduce the following two results, with noting that the function $f(t, z(t), {}^cD^p z(t))$ given in (1) will be indicated by $f(t)$ for simplicity.

**Lemma 10.** *Let $f \in C([0, T], \mathbb{R})$. Then, the unique root of the problem given in (1)-(2) can be expressed as*

$$z(t) = I^{s+r} f(t) - \ell I^s z(t) + \chi_1(t) \Lambda_\alpha(T) + \chi_2(t) \Lambda_\beta(\zeta), \tag{8}$$

*where*

$$\chi_1(t) = \frac{t^s(t - \zeta)}{(T - \zeta)T^s}, \tag{9}$$

$$\chi_2(t) = \frac{t^s(T - t)}{(T - \zeta)\zeta^s}, \tag{10}$$

$$\Lambda_\alpha(t) = \alpha I z(t) - I^{s+r} f(t) + \ell I^s z(t). \tag{11}$$

*Proof.* Applying the integral operator $I^r$ to both sides of (1) and then using Lemma 5. yield

$${}^cD^s z(t) = -\ell I z(t) + I^r f(t) + c_2 t + c_1. \tag{12}$$

Again, applying $I^r$ to both sides of (12) gives

$$z(t) = I^{s+r} f(t) - \ell I^s z(t) + \frac{c_2 t^{s+1}}{\Gamma(s+2)} + \frac{c_1 t^s}{\Gamma(s+1)} + c_0, \tag{13}$$

where $c_i{}'$s are arbitrary constants, and $i = 1, 2, 3$. Applying the first condition of (2) yields $c_0 = 0$, and hence

$$z(t) = I^{s+r} f(t) - \ell I^s z(t) + \frac{c_2 t^{s+1}}{\Gamma(s+2)} + \frac{c_1 t^s}{\Gamma(s+1)}. \tag{14}$$

By using the two other conditions given in (2), we obtain the following:

$$\frac{c_1 T^s}{\Gamma(s+1)} + \frac{c_2 T^{s+1}}{\Gamma(s+2)} = \Lambda_\alpha(T), \tag{15}$$

$$\frac{c_1 \zeta^s}{\Gamma(s+1)} + \frac{c_2 \zeta^{s+1}}{\Gamma(s+2)} = \Lambda_\beta(\zeta). \tag{16}$$

Solving the two equations (15) and (16) gives the following:

$$\frac{c_1}{\Gamma(s+1)} = \frac{T}{(T - \zeta)\zeta^s} \Lambda_\beta(\zeta) - \frac{\zeta}{(T - \zeta)T^s} \Lambda_\alpha(T), \tag{17}$$

$$\frac{c_2}{\Gamma(s+2)} = \frac{1}{(T - \zeta)T^s} \Lambda_\alpha(T) - \frac{1}{(T - \zeta)\zeta^s} \Lambda_\beta(\zeta). \tag{18}$$

Substituting (17) and (18) into (14) yields the result. Conversely, it is easy by aiding with Remark 4. and Lemma 5. to see that the solution (8) satisfies the fractional differential equation (1) and the boundary conditions in (2).

**Lemma 11.** *Let $\chi_1$ and $\chi_2$ be defined as given in (9) and (10), respectively. Then,*

$$\chi_1 = \max_{t \in [0,T]} |\chi_1(t)| = \max \left\{ \frac{\zeta^{s+1} s^s}{(s+1)^{s+1}(T - \zeta)T^s}, 1 \right\}, \tag{19}$$

$$\chi_2 = \max_{t \in [0,T]} |\chi_2(t)| = \frac{T^{s+1} s^s}{(s+1)^{s+1}(T - \zeta)\zeta^s}. \tag{20}$$

*Proof.* Let $t, \zeta \in [0, T]$. Define the function $u(t)$ as follows:

$$u(t) = t^s |t - \zeta| = \{ t^s(\zeta - t), 0 \leq t \leq \zeta, t^s(t - \zeta), \zeta \leq t \leq T. \tag{21}$$

One observes that when $0 \le t \le \zeta$, we have $u'(t) = t^{s-1}(s\zeta - (s+1)t)$, which means that $u(t)$ is increasing on $(0, t_0)$ and it is decreasing on $(t_0, \zeta)$, where $t_0 = s\zeta/(s+1)$. Hence, we can deduce that $u(t)$ has a maximum value at $t = t_0$. On the other hand, when $\zeta < t \le T$, we obtain $u'(t) = t^{s-1}((s+1)t - s\zeta)$, which implies that $u(t)$ is increasing on $(\zeta, T)$, and so $u(t)$ has a maximum value at $t = T$. Therefore, the result holds due to the following:

$$\chi_1 = \max \left\{ \frac{u(t_0)}{(T-\zeta)T^s}, \frac{u(T)}{(T-\zeta)T^s} \right\}. \quad (22)$$

In similar manner, one can deduce the second statement of this lemma.

To follow up on the progress made in exploring the existence and uniqueness of solutions for the FoLEs, some further results are established below, which will surely serve us in the next sections. In this regard, let $\mathscr{C} = C([0, T], \mathbb{R})$ be Banach space of continuous real-valued functions equipped with the usual supremum norm

$$\|z\| = \sup_{0 \le t \le T} |z(t)|, z \in \mathscr{C}. \quad (23)$$

Define the following set

$$\mathscr{P} = \{z(t): z(t) \in \mathscr{C}, {}^c D^\rho z(t) \in \mathscr{C}\}, t \in [0, T]. \quad (24)$$

One can easily verify that $\mathscr{P}$ is a Banach subspace equipped with the following norm:

$$\begin{aligned} \|z\|_{\mathscr{P}} &= \max \{\|z\|, \|{}^c D^\rho z\|\} \\ &= \max \left\{ \sup_{0 \le t \le T} |z(t)|, \sup_{0 \le t \le T} \left| {}^c D^\rho(z(t)) \right| \right\}. \end{aligned} \quad (25)$$

In view of Lemma 10., the main problem given in (1)-(2) will be transformed into its equivalent fixed point problem. In other words, the function $z(t)$ should satisfy the two statements; $Sz(t) = z(t)$ and ${}^c D^\rho Sz(t) = {}^c D^\rho z(t)$, for all $t \in [0, T]$, where $S : \mathscr{C} \to \mathscr{C}$ is an operator defined on the Banach space $\mathscr{C}$. In order to achieve this goal, consider the operator $S$ as defined a little while ago. That is,

$$(Sz)(t) = I^{s+r} f(t) - \ell I^s z(t) + \chi_1(t) \Lambda_\alpha(T) + \chi_2(t) \Lambda_\beta(\zeta), \quad (26)$$

$$\begin{aligned} ({}^c D^\rho Sz)(t) = I^{s+r-\rho} f(t) - \ell I^{s-\rho} z(t) \\ + \chi_3(t) \Lambda_\alpha(T) + \chi_4(t) \Lambda_\beta(\zeta), \end{aligned} \quad (27)$$

where $\Lambda_\alpha(\cdot)$ is defined in (11) and

$$\chi_3(t) = {}^c D^\rho \chi_1 = \frac{\Gamma(s+2)t^{s-\rho}}{\Gamma(s-\rho+2)(T-\zeta)T^s} \left[ t - \frac{\zeta(s+1-\rho)}{s+1} \right], \quad (28)$$

$$\chi_4(t) = {}^c D^\rho \chi_2(t) = -\frac{\Gamma(s+2)t^{s-\rho}}{\Gamma(s-\rho+2)(T-\zeta)\zeta^s} \left[ t - \frac{T(s+1-\rho)}{s+1} \right]. \quad (29)$$

Next, we introduce the following auxiliary result based on all provided aforementioned principles.

**Lemma 12.** *Let $\chi_3(t)$ and $\chi_4(t)$ are defined as in (28) and (29), respectively. Then,*

$$\begin{aligned} \chi_3 &= \|\chi_3\| \\ &= \frac{\Gamma(s+1)}{\Gamma(s-\rho+2)(T-\zeta)} \max \\ &\cdot \left\{ \frac{\zeta^{s-\rho+1}}{T^s} \left( \frac{s-\rho}{s+1} \right)^{s-\rho}, \frac{(s+1)T - (s-\rho+1)\zeta}{T^\rho} \right\}, \end{aligned} \quad (30)$$

$$\chi_4 = \|\chi_4\| = \frac{\Gamma(s+1)T^{s-\rho+1}}{\Gamma(s-\rho+2)(T-\zeta)\zeta^s} \max \left\{ \left( \frac{s-\rho}{s+1} \right)^{s-\rho}, \rho \right\}. \quad (31)$$

*Proof.* Let $t_0 = \zeta(s+1-\rho)/(s+1)$. Define the function

$$u(t) = t^{s-\rho}|t - t_0|, t \in [0, T]. \quad (32)$$

If $0 \le t \le t_0$, then $u(t) = t^{s-\rho}(t_0 - t)$ and

$$u'(t) = t^{s-\rho-1}[(s-\rho)t_0 - (s-\rho+1)t] \quad (33)$$

which leads to $u(t)$ is increasing on $(0, t_1)$ and is decreasing on $(t_1, t_0)$ where

$$t_1 = \frac{s-\rho}{s-\rho+1}t_0 = \frac{\zeta(s-\rho)}{s+1} < t_0. \quad (34)$$

This concludes that $u(t_1)$ is a maximum value.
If $t_0 < t \le T$, then $u(t) = t^{s-\rho}(t - t_0)$ and

$$u'(t) = t^{s-\rho-1}[(s-\rho+1)t - (s-\rho)t_0] > t^{s-\rho-1}t_0 > 0 \quad (35)$$

which leads to $u(t)$ is increasing on $(t_0, T)$. This concludes that $u(T)$ is a maximum value. In similar manner, one can deduce the second statement of this lemma.

## 4. Existence and Uniqueness Results

This section proposes some theoretical results related to the existence and uniqueness of the solution for the main target problem, the FoLE given in (1)-(2). First of all, we establish some mathematical assumptions that will be used from now on. Such assumptions are given below.

$$\Delta = |\ell|\Delta(s) + |\alpha|\chi_1 T + |\beta|\chi_2 \zeta, \quad (36)$$

$$\Delta' = |\ell|\Delta'(s) + |\alpha|\chi_3 T + |\beta|\chi_4 \zeta. \quad (37)$$

where

$$\Delta(s) = (1 + \chi_1)T^s + \chi_2 \zeta^s \Gamma(s+1), \quad (38)$$

$$\Delta'(s) = T^{s-\rho}\Gamma(s-\rho+1) + \chi_3 T^s + \chi_4 \zeta^s \Gamma(s+1), \quad (39)$$

and $\chi_i$, $i = 1, 2, 3, 4$ are given in (19), (20), (30), and (31), respectively.

In the same vein, we also introduce some further conditions below to establish our proposed results. In fact, these conditions have been recently introduced in [40] for the purpose of investigating the same issue of ours, but they were adopted for handling another problem.

$\mathbf{H}_1$ The function $f(t, z(t), {}^cD^\rho z(t))$ is continuous on $[0, T]$

$\mathbf{H}_2$ There exist a positive function $\psi : [0, T] \to \mathbb{R}^+$ such that

$$|f(t, z(t), {}^cD^\rho z(t))| q(t) \quad (40)$$

$\mathbf{H}_3$ There exist two constants $M_1, M_2 > 0$ such that

$$\begin{aligned}
&|f(t, z_1(t), {}^cD^\rho z_1) - f(t, z_2, {}^cD^\rho z_2)| \\
&\quad \le M_1|z_1 - z_2| + M_2|c_{D^\rho z_1} - {}^cD^\rho z_2|, t \in [0, T]
\end{aligned} \quad (41)$$

$\mathbf{H}_4$ There exist a positive function $h : [0, T] \to \mathbb{R}^+$ and other nondecreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|f(t, z(t), {}^cD^\rho z(t))| \le h(t)\Psi(\|z\|_{\mathscr{P}}), t \in [0, T] \quad (42)$$

On the other side, by going back to (26) and (27), one can rewrite each of $Sz(t)$ and ${}^cD^\rho Sz(t)$, as a sum of two operators $S_1$ and $S_2$. In other words, $Sz(t) = (S_1 z + S_2 z)(t)$ and ${}^cD^\rho Sz(t) = ({}^cD^\rho S_1 z + {}^cD^\rho S_2 z)(t)$, where

$$(S_1 z)(t) = I^{s+r} f(t) - \chi_1(t) I^{s+r} f(T) - \chi_2(t) I^{s+r} f(\zeta),$$

$$\begin{aligned}
(S_2 z)(t) = &-\ell I^s z(t) + \chi_1(t)[\alpha I z(T) + \ell I^s z(T)] \\
&+ \chi_2(t)[\beta I z(\zeta) + \ell I^s z(\zeta)],
\end{aligned}$$

$$({}^cD^\rho S_1 z)(t) = I^{s+r-\rho} f(t) - \chi_3(t) I^{s+r} f(T) - \chi_4(t) I^{s+r} f(\zeta),$$

$$\begin{aligned}
({}^cD^\rho S_2 z)(t) = &-\ell I^{s-\rho} z(t) + \chi_3(t)[\alpha I z(T) + \ell I^s z(T)] \\
&+ \chi_4(t)[\beta I z(\zeta) + \ell I^s z(\zeta)].
\end{aligned} \quad (43)$$

After this preparation, we find ourselves ready to present the results that are at the heart of the matter. Here is the first one that deduced on the basis of the Krasnoselskii theorem.

**Theorem 13.** *Suppose $\mathbf{H}_1$ and $\mathbf{H}_2$ hold. If*

$$\max\{\Delta, \Delta'\} < 1, \quad (44)$$

*where $\Delta$ and $\Delta'$ are defined in (36) and (37), respectively, then the problem given in (1)-(2) has at least one solution on $[0, T]$.*

*Proof.* To prove this theorem, define the closed ball, $\mathscr{B}_\varepsilon$, as follows:

$$\mathscr{B}_\varepsilon = \{z(t) \in \mathscr{P} : {}^cD^\rho z(t) \in \mathscr{P}, \|z(t)\|_{\mathscr{P}} \le \varepsilon\}, \quad (45)$$

where $\mathscr{P}$ and $\|z\|_{\mathscr{P}}$ are defined in Section 3, with radius

$$\varepsilon \ge \|q\| \max\{\Delta(s+r), \Delta'(s+r)\} 1 - \max\{\Delta, \Delta'\}, \quad (46)$$

where $\Delta(\cdot)$ and $\Delta'(\cdot)$ are defined in (38) and (39), respectively.

One can easily verify that $\mathscr{B}_\varepsilon$ is a convex closed set. However, for the purpose of implementing on the Krasnoselskii theorem given in 2.2, we need firstly to show that $S_1 z_1 + S_2 z_2 \in \mathscr{B}_\varepsilon$, for each $z_1(t), z_2(t) \in \mathscr{B}_\varepsilon$. To carry out this step, we begin with

$$\begin{aligned}
|S_1 z_1| \le & |I^{s+r} f(t, z_1(t), D^\rho z_1(t))| \\
& + |\chi_1(t)| |I^{s+r} f(T, z_1(T), D^\rho z_1(T))| \\
& + |\chi_2(t)| |I^{s+r} f(\zeta, z_1(\zeta), D^\rho z_1(\zeta))| \\
\le & q(t)\Gamma(s+r)\left[\int_0^t (t-v)^{s+r-1} dv \right. \\
& + |\chi_1(t)| \int_0^T (T-v)^{s+r-1} dv + |\chi_2(t)| \left. \int_0^\zeta (\zeta-v)^{s+r-1} dv\right] \\
= & [t^{s+r} + |\chi_1(t)| T^{s+r} + |\chi_2(t)| \zeta^{s+r}]\Gamma(s+r+1) \|q(t)\| \\
\le & (1 + |\chi_1(t)|) |T^{s+r} + |\chi_2(t)| \zeta^{s+r} \Gamma(s+r+1) |q(t)|,
\end{aligned}$$

$$\begin{aligned}
|S_2 z_2| \le & |\ell| I^s |z_2(t)| + |\chi_1(t)| [|\alpha| I | z_2(T)| + |\ell| I^s | z_2(T)|] \\
& + |\chi_2(t)| [|\beta| I | z_2(\zeta)| + |\ell| I^s | z_2(\zeta)|] \\
\le & [(1 + |\chi_1(t)|) T^s + |\chi_2(t)| \zeta^s \Gamma(s+1)] |\ell| \\
& + |\alpha| |\chi_1(t)| T + |\beta| |\chi_2(t)| \zeta] |z_2|.
\end{aligned} \quad (47)$$

Due to $z_2(t) \in \mathscr{B}_\varepsilon$, then $|z_2(t)| \le \|z_2(t)\| \le \|z_2(t)\|_{\mathscr{P}} \le \varepsilon$. This matter implies to

$$\begin{aligned}
|S_1 z_1 + S_2 z_2| \le & (1 + \chi_1) |T^{s+r} + \chi_2 \zeta^{s+r} \Gamma(s+r+1) \|q\| + [(1 + \chi_1) T^s \\
& + \chi_2 \zeta^s \Gamma(s+1) | \ell| + |\alpha| \chi_1 T + |\beta| \chi_2 \zeta] \varepsilon \\
= & \|q\| \Delta(s+r) + (|\ell| \Delta(s) + |\alpha| \chi_1 T + |\beta| \chi_2 \zeta) \varepsilon \\
= & \|q\| \Delta(s+r) + \Delta \varepsilon.
\end{aligned} \quad (48)$$

Thus, the last inequality implies $\|S_1 z_1 + S_2 z_2\| \le \varepsilon$. In a similar manner, we can show that

$$\begin{aligned}
|{}^cD^\rho S_1 z_1| \le & (T^{s+r-\rho} \Gamma(s+r-\rho+1) + \chi_3 T^{s+r} \\
& + \chi_4 \zeta^{s+r}] \Gamma(s+r+1)) \|q\|,
\end{aligned}$$

$$\begin{aligned}
|{}^cD^\rho S_2 z_2| \le & [(T^{s-\rho} \Gamma(s-\rho+1) + \chi_3 T^s \\
& + \chi_4 \zeta^s \Gamma(s+1)) |\ell| + |\alpha| \chi_3 T + |\beta| \chi_4 \zeta] \varepsilon,
\end{aligned} \quad (49)$$

which implies that

$$\begin{aligned}
|{}^cD^\rho S_1 z_1 + {}^cD^\rho S_2 z_2| \le & \|q\| \Delta'(s+r) + (|\ell| \Delta'(s) + |\alpha| \chi_3 T + |\beta| \chi_4 \zeta) \varepsilon \\
= & \|q\| \Delta'(s+r) + \Delta' \varepsilon.
\end{aligned} \quad (50)$$

These conclude that

$$
\begin{aligned}
\|S_1 z_1 + S_2 z_2\|_{\mathscr{P}} &= \max \left\{ \sup_{t \in [0,T]} |S_1 z_1 + S_2 z_2|, \sup_{t \in [0,T]} \left|{}^c D^\rho S_1 z_1 + {}^c D^\rho S_2 z_2 \right| \right\} \\
&= \max \left\{ \|q\| \Delta(s+r) + \Delta\varepsilon, \|q\| \Delta'(s+r) + \Delta'\varepsilon \right\} \\
&= \|q\| \max \left\{ \Delta(s+r), \Delta'(s+r) \right\} + \max \left\{ +\Delta, \Delta' \right\} \varepsilon \leq \varepsilon,
\end{aligned}
\tag{51}
$$

which means that $S_1 z_1 + S_2 z_2 \in \mathscr{B}_\varepsilon$ for each $z_1(t), z_2(t) \in \mathscr{B}_\varepsilon$. Thus, the first part of Krasnoselskii Theorem 8 has been shown. The next step is to show that the operator $S_2$ is contraction. For this purpose, we estimate $|S_2 z_1 - S_2 z_2|$ as follows:

$$
\begin{aligned}
|S_2 z_1 - S_2 z_2| &\leq \ell I^s |z_1(t) - z_2(t)| + |\chi_1(t)|(|\alpha| I |z_1(T) - z_2(T)| \\
&\quad + \ell I^s |z_1(T) - z_2(T)|) + |\chi_2(t)|((|\beta| I |z_1(\zeta) \\
&\quad - z_2(\zeta)| + \ell I^s |z_1(\zeta) - z_2(\zeta)|)) \\
&\leq \left[ (1 + \chi_1) T^s + \chi_2 \zeta^s \Gamma(s+1) |\ell| \right. \\
&\quad + |\alpha| \chi_1 T + |\beta| \chi_2 \zeta |z_1 - z_2| \\
&\leq \left[ |\ell| \Delta(s) + |\alpha| \chi_1 T + |\beta| \chi_2 \zeta \right] \|z_1 - z_2\|_{\mathscr{P}} \\
&= \Delta \|z_1 - z_2\|_{\mathscr{P}}.
\end{aligned}
\tag{52}
$$

Similarly, we can prove that

$$
\begin{aligned}
|{}^c D^\rho S_2 z_1 - {}^c D^\rho S_2 z_2| \\
\leq \left[ |\ell| \Delta'(s) + |\alpha| \chi_3 T + |\beta| \chi_4 \zeta \right] \|z_1 \\
- z_2\|_{\mathscr{P}} = \Delta' \|z_1 - z_2\|_{\mathscr{P}}.
\end{aligned}
\tag{53}
$$

These lead to

$$
\|S_2 z_1 - S_2 z_2\|_{\mathscr{P}} \leq \max \left\{ \Delta, \Delta' \right\} \|z_1 - z_2\|_{\mathscr{P}},
\tag{54}
$$

which implies that $S_2$ is contraction if $\max \left\{ \Delta, \Delta' \right\} < 1$. At the same time, as the function $f$ is assumed to be continuous according to the assumption $\mathbf{H}_1$, the operator $S_1$ is also continuous on $\mathscr{B}_\varepsilon$. To complete the proof of this theorem, it remains to show that $S_1$ is compact. In regard to Arzela-Ascoli theorem, it is enough to show that $S_1 \mathscr{B}_\varepsilon$ is uniformly bounded and equicontinuous. For the boundedness, one can easily verify that $S_1 z$ satisfies the following perspective.

$$
\|S_1 z\|_{\mathscr{P}} \leq \|q\| \max \left\{ \Delta(s+r), \Delta'(s+r) \right\}.
\tag{55}
$$

In the meantime, to show that $S_1$ is an equicontinuous operator, let $t_1, t_2 \in [0, T]$ such that $t_1 < t_2$. Then, we have

$$
\begin{aligned}
|S_1 z(t_2) - S_1 z(t_1)| &\leq \frac{1}{\Gamma(s+r)} \int_0^{t_1} \left| (t_2 - \tau)^{s+r-1} - (t_1 - \tau)^{s+r-1} \right| \\
&\quad \times |f(\tau, z(\tau), D^\rho z(\tau))| d\tau \\
&\quad + \frac{1}{\Gamma(s+r)} \int_{t_1}^{t_2} (t_2 - \tau)^{s+r-1} |f(\tau, z(\tau), D^\rho z(\tau))| d\tau \\
&\quad + \frac{|\chi_1(t_1) - \chi_1(t_2)|}{\Gamma(s+r)} \int_0^T (T - \tau)^{s+r-1} |f(\tau, z(\tau), D^\rho z(\tau))| d\tau \\
&\quad + \frac{|\chi_2(t_1) - \chi_2(t_2)|}{\Gamma(s+r)} \int_0^\zeta (\zeta - \tau)^{s+r-1} |f(\tau, z(\tau), D^\rho z(\tau))| d\tau.
\end{aligned}
\tag{56}
$$

Again, using condition $\mathbf{H}_2$ to turn the inequality above to be in the form

$$
\begin{aligned}
|S_1 z(t_2) - S_1 z(t_1)| &\leq \frac{\|q\|}{\Gamma(s+r)} \left[ \int_0^{t_1} \left( (t_2 - \tau)^{s+r-1} \right. \right. \\
&\quad \left. - (t_1 - \tau)^{s+r-1} \right) d\tau + \int_{t_1}^{t_2} (t_2 - \tau)^{s+r-1} d\tau \\
&\quad + \frac{|t_2^s(t_2 - \zeta) - t_1^s(t_1 - \zeta)|}{T^s(T - \zeta)} \int_0^T (T - \tau)^{s+r-1} d\tau \\
&\quad \left. + \frac{|t_2^s(T - t_2) - t_1^s(T - t_1)|}{\zeta^s(T - \zeta)} \int_0^\zeta (\zeta - \tau)^{s+r-1} d\tau \right].
\end{aligned}
\tag{57}
$$

Consequently, the above inequality leads us to another one of the form

$$
\begin{aligned}
&|S_1 z(t_2) - S_1 z(t_1)| \\
&\leq \frac{\|q\|}{\Gamma(s+r+1)} \\
&\cdot \left[ t_2^{s+r} - t_1^{s+r} + \frac{\left( t_2^{s+1} - t_1^{s+1} \right) \left( T^r + \zeta^r \right) + \zeta T(t_2 - t_1) \left( T^{r-1} + \zeta^{r-1} \right)}{T - \zeta} \right].
\end{aligned}
\tag{58}
$$

Observe that as $t_2 \to t_1$, the right-hand side goes to zero uniformly. This means that such side does not depend never on $z$. Therefore, as per Arzela-Ascoli theorem, the operator $S_1$ is equicontinuous. Hence, in view of Krasnoselskii theorem, we conclude that (1)-(2) has at least one solution.

In the next argument, we will investigate the uniqueness of solution for the problem under consideration. Of course, Banach fixed point theorem, reported in Theorem 7, will be utilized in this investigation. However, the following result is established for achieving that purpose.

**Theorem 14.** *Suppose $\mathbf{H}_1$ and $\mathbf{H}_3$ hold. If*

$$
\begin{aligned}
\Psi &= (M_1 + M_2) \max \left\{ \Delta(s+r), \Delta'(s+r) \right\} \\
&\quad + \max \left\{ \Delta, \Delta' \right\} < 1.
\end{aligned}
\tag{59}
$$

Then, the problem given in (1)-(2) has a unique solution on $[0, T]$.

*Proof.* Consider the set $\mathscr{B}_\varepsilon$ as given in (45) with

$$\varepsilon \geq \frac{W \max \left\{ \Delta(s+r), \Delta'(s+r) \right\}}{1 - \Psi}, \qquad (60)$$

where $W = \sup_{0t\,T} | f(t, 0, 0) |$. For each $t \in [0, T]$ and $z \in \mathscr{B}_\varepsilon$, one might estimate the term $| f(t, z(t), {}^cD^\rho z(t)) |$ as follows:

$$
\begin{aligned}
| f(t, z(t), {}^cD^\rho z(t)) | &= | f(t, z(t), {}^cD^\rho z(t)) - f(t, 0, 0) + f(t, 0, 0) | \\
&\leq | f(t, z(t), {}^cD^\rho z(t)) - f(t, 0, 0) | + | f(t, 0, 0) | \\
&\leq M_1 |z(t)| + M_2 |{}^cD^\rho z(t)| + W \\
&\leq (M_1 + M_2) \|z\|_{\mathscr{P}} + W \leq (M_1 + M_2)\varepsilon + W.
\end{aligned}
\qquad (61)
$$

The next step is now to show that $S\mathscr{B}_\varepsilon \subseteq \mathscr{B}_\varepsilon$, $\forall z \in \mathscr{B}_\varepsilon$. Based on the arguments given in the previous theorem, one can obtain the following assertions.

$$
\begin{aligned}
|S_1 z(t)| &\leq ((M_1 + M_2)\varepsilon + W)\Delta(s+r), \\
|{}^cD^\rho S_1 z(t)| &\leq ((M_1 + M_2)\varepsilon + W)\Delta'(s+r), \\
|S_2 z(t)| &\leq \Delta\varepsilon, \\
|{}^cD^\rho S_2 z(t)| &\leq \Delta'\varepsilon.
\end{aligned}
\qquad (62)
$$

Applying the norm over $t$ gives

$$\|S_1 z\|_{\mathscr{P}} \leq ((M_1 + M_2)\varepsilon + W) \max \left\{ \Delta(s+r), \Delta'(s+r) \right\},$$

$$\|S_2 z\|_{\mathscr{P}} \leq \varepsilon \max \left\{ \Delta, \Delta' \right\}. \qquad (63)$$

Therefore,

$$
\begin{aligned}
\|Sz\|_{\mathscr{P}} &\leq \|S_1 z\|_{\mathscr{P}} + \|S_2 z\|_{\mathscr{P}} \\
&\leq ((M_1 + M_2)\varepsilon + W) \max \\
&\quad \cdot \left\{ \Delta(s+r), \Delta'(s+r) \right\} + \varepsilon \max \left\{ \Delta, \Delta' \right\} \\
&= \Psi\varepsilon + W \max \left\{ \Delta(s+r), \Delta'(s+r) \right\} \leq \varepsilon.
\end{aligned}
\qquad (64)
$$

Now, in accordance to Theorem 7, we need to show that the mapping $S$ is contraction. In order to achieve this goal, we should notice that the operator $S_2$ is contraction as shown in the previous theorem. Hence, it remains to verify that $S_1$ is also contraction. For this purpose, let $z_1, z_2 \in \mathscr{B}_\varepsilon$, $t \in [0, T]$ and consider $|S_1 z_1(t) - S_1 z_2(t)|$ as follows:

$$
\begin{aligned}
|S_1 z_1(t) - S_1 z_2(t)| \leq\ & I^{s+r} | f(t, z_1(t), {}^cD^\rho z_1(t)) \\
& - f(t, z_2(t), {}^cD^\rho z_2(t)) | \\
& + |\chi_1(t)| I^{s+r} | f(T, z_1(T), {}^cD^\rho z_1(T)) \\
& - f(T, z_2(T), {}^cD^\rho z_2(T)) | \\
& + |\chi_2(t)| I^{s+r} | f(\zeta, z_1(\zeta), {}^cD^\rho z_1(\zeta)) \\
& - f(\zeta, z_2(\zeta), {}^cD^\rho z_2(\zeta)) |.
\end{aligned}
\qquad (65)
$$

Using condition $\mathbf{H}_3$ yields

$$
\begin{aligned}
|S_1 z_1(t) - S_1 z_2(t)| \leq\ & \frac{1}{\Gamma(s+r)} \int_0^t (t-\tau)^{s+r-1} \\
& \cdot (M_1 | z_1 - z_2 | + M_2 |{}^cD^\rho z_1 - {}^cD^\rho z_2 |) d\tau \\
& + \frac{|\chi_1(t)|}{\Gamma(s+r)} \int_0^T (T-\tau)^{s+r-1} \\
& \cdot (M_1 | z_1 - z_2 | + M_2 |{}^cD^\rho z_1 - {}^cD^\rho z_2 |) d\tau \\
& + \frac{|\chi_2(t)|}{\Gamma(s+r)} \int_0^\zeta (\zeta-\tau)^{s+r-1} \\
& \cdot (M_1 | z_1 - z_2 | + M_2 |{}^cD^\rho z_1 - {}^cD^\rho z_2 |) d\tau, \\
\leq\ & (M_1 | z_1(t) - z_2(t) | + M_2 |{}^cD^\rho z_1(t) \\
& - {}^cD^\rho z_2(t) |)\Delta(s+r).
\end{aligned}
\qquad (66)
$$

Similarly, we can obtain the following inequality:

$$
\begin{aligned}
|{}^cD^\rho S_1 z_1(t) - {}^cD^\rho S_1 z_2(t)| & \\
\leq (M_1 | z_1(t) - z_2(t) | + M_2 |{}^cD^\rho z_1(t) & \\
- {}^cD^\rho z_2(t) |)\Delta'(s+r). &
\end{aligned}
\qquad (67)
$$

Taking under the norm $\|\cdot\|_{\mathscr{P}}$ implies

$$\|S_1 z_1 - S_1 z_2\|_{\mathscr{P}} \leq (M_1 + M_2) \max \left\{ \Delta(s+r), \Delta'(s+r) \right\} \|z_1 - z_2\|_{\mathscr{P}}. \qquad (68)$$

Using the two deduced assertions in (54) and (68) implies

$$
\begin{aligned}
\|Sz_1 - Sz_2\|_{\mathscr{P}} &\leq \|S_1 z_1 - S_1 z_2\|_{\mathscr{P}} + \|S_2 z_1 - S_2 z_2\|_{\mathscr{P}} \\
&\leq \left[ (M_1 + M_2) \max \left\{ \Delta(s+r), \Delta'(s+r) \right\} \right. \\
&\quad \left. + \max \left\{ \Delta, \Delta' \right\} \right] \|z_1 - z_2\|_{\mathscr{P}} \leq \Psi \|z_1 - z_2\|_{\mathscr{P}},
\end{aligned}
\qquad (69)
$$

which shows that $S$ is a contraction mapping, and this finishes the proof.

In the same context, we introduce the following further result which also explores the existence matter of our main problem by taking into account, this time, Theorem 9.

**Theorem 15.** *Suppose* $\mathbf{H}_1$ *and* $\mathbf{H}_4$ *hold. If*

$$\frac{\lambda}{\max\left\{\left(\|h\|\Psi(\|z\|_{\mathscr{P}})\Delta(s+r)+\Delta\right),\left(\|h\|\Psi(\|z\|_{\mathscr{P}})\Delta'(s+r)+\Delta'\right)\right\}} > 1.$$
(70)

*Then, the problem given in (1)-(2) has at least one solution on* $[0, T]$, *for some* $\lambda, \varepsilon > 0$, *where* $\Delta, \Delta, \Delta(\cdot)$ *and* $\Delta'(\cdot)$ *are defined, respectively, in (36), (37), (38) and (39).*

*Proof.* Suppose $\mathcal{O} = \{z \in \mathscr{C}: {}^c D^\rho z \in \mathscr{C}, \|z\|_{\mathscr{P}} < \lambda\}$ is an open subset of the Banach space in $\mathscr{C} = C([0, T], \mathbb{R}^+)$. Assume that $S : \overline{\mathcal{O}} \to \varepsilon\mathscr{C}$ is an operator defined as given in (26) with its fractional derivatives in (27), which are also previously defined as a sum of two operators $S_1$ and $S_2$. Observe that $S$ is continuous since $S_1$ and $S_2$ are so. From this point, we find that, it is the right time, to apply on Theorem 9. For that purpose and according to Arzela-Ascoli theorem, we need to show that $S$ is compact. In other words, we need to show that such operator is bounded and also equicontinuous. For the boundedness, let $t \in [0, T]$ and consider the term $|Sz(t)|$ as follows:

$$
\begin{aligned}
|Sz(t)| \leq{} & I^{s+r}|f(t, z(t), D^\rho z(t))| + |\ell|I^s|z(t)| \\
& + |\chi_1(t)|[|\alpha|I|z(T)| + I^{s+r}|f(T, z(T), D^\rho z(T))| \\
& + |\ell|I^s|z(T)|] + |\chi_2(t)|[|\beta|I|z(\zeta)| \\
& + I^{s+r}|f(\zeta, z(\zeta), D^\rho z(\zeta))| + |\ell|I^s|z(\zeta)|].
\end{aligned}
$$
(71)

Using condition $\mathbf{H}_4$ yields

$$
\begin{aligned}
|Sz(t)| \leq{} & \frac{\|h\|\Psi(\|z\|_{\mathscr{P}})}{\Gamma(s+r)}\int_0^t (t-\tau)^{s+r-1}d\tau \\
& + \frac{|\ell|\|z\|}{\Gamma(s+1)}t^s + |\chi_1(t)|[|\alpha|T\|z\|_{\mathscr{P}} \\
& + \frac{\|h\|\Psi(\|z\|_{\mathscr{P}})}{\Gamma(s+r)}\int_0^T (T-\tau)^{s+r-1}d\tau \\
& + \frac{|\ell|\|z\|}{\Gamma(s+1)}T^s] + |\chi_2(t)|[|\beta|\zeta\|z\|_{\mathscr{P}} \\
& + \frac{\|h\|\Psi(\|z\|_{\mathscr{P}})}{\Gamma(s+r)}\int_0^\zeta (\zeta-\tau)^{s+r-1}d\tau + \frac{|\ell|\|z\|}{\Gamma(s+1)}\zeta^s].
\end{aligned}
$$
(72)

Consequently, the above inequality leads to the following assertion.

$$\|Sz(t)\| = \sup_{0 t T}|Sz(t)| \leq \|h\|\Psi(\|z\|_{\mathscr{P}})\Delta(s+r) + \|z\|_{\mathscr{P}}\Delta.$$
(73)

Similarly, we can easily obtain the following other assertion.

$$\|{}^c D^\rho Sz\| = \sup_{0 t T}\left|{}^c D^\rho Sz(t)\right| \leq \|h\|\Psi(\|z\|_{\mathscr{P}})\Delta'(s+r) + \|z\|_{\mathscr{P}}\Delta'.$$
(74)

By the last two inequalities, one can deduce that the operator $S$ is bounded. On the other hand, to prove that this operator is also equicontinuous, we should recall that it has been previously formulated as a sum of two operators $S_1$ and $S_2$, with noting that the first one, $S_1$, has appeared as an equicontinuous operator through the proof of Theorem 13.. Therefore, to complete the proof of this theorem, it remains to show that $S_2$ is also equicontinuous. In order to achieve this objective, let $t_1, t_2 \in [0, T]$ such that $t_1 < t_2$. Then, we have

$$
\begin{aligned}
|S_2 z(t_2) - S_2 z(t_1)| \leq{} & \frac{|\ell|}{\Gamma(s)} \times \left[\int_0^{t_1}\left((t_1-\tau)^{s-1}\right.\right. \\
& \left. - (t_2-\tau)^{s-1}\right)|z(\tau)|\,d\tau \\
& + \int_{t_1}^{t_2}(t_2-\tau)^{s-1}|z(\tau)|\,d\tau \\
& + |\chi_1(t_2) - \chi_1(t_1)|\left[|\alpha|\int_0^T |z(\tau)|\,d\tau\right. \\
& \left. + \frac{|\ell|}{\Gamma(s)}\int_0^T (T-\tau)^{s-1}|z(\tau)|\,d\tau\right] \\
& + |\chi_2(t_2) - \chi_2(t_1)|\left[|\beta|\int_0^\zeta |z(\tau)|\,d\tau\right. \\
& \left.\left. + \frac{|\ell|}{\Gamma(s)}\int_0^\zeta (\zeta-\tau)^{s-1}|z(\tau)|\,d\tau\right].\right.
\end{aligned}
$$
(75)

Actually, the above inequality leads to the following other one

$$
\begin{aligned}
|S_2 z(t_2) - S_2 z(t_1)| \leq{} & \|z\|\left[\frac{|\ell|}{\Gamma(s+1)}(2(t_2-t_1)^s\right. \\
& + t_1^s - t_2^s) + \frac{|t_2^s(t_2-\zeta) - t_1^s(t_1-\zeta)|}{T^s(T-\zeta)} \\
& \cdot \left(|\alpha|T + \frac{|\ell|T^s}{\Gamma(s+1)}\right) \\
& + \frac{|t_2^s(T-t_2) - t_1^s(T-t_1)|}{\zeta^s(T-\zeta)} \\
& \left. \cdot \left(|\beta|\zeta + \frac{|\ell|\zeta^s}{\Gamma(s+1)}\right)\right] \\
\leq{} & \|z\|\left[\frac{2|\ell|}{\Gamma(s+1)}(t_2-t_1)^s\right. \\
& + \frac{t_2^{s+1} - t_1^{s+1} + \zeta(t_2^s - t_1^s)}{T^s(T-\zeta)} \\
& \cdot \left(|\alpha|T + \frac{|\ell|T^s}{\Gamma(s+1)}\right) \\
& + \frac{t_2^{s+1} - t_1^{s+1} + T(t_2^s - t_1^s)}{\zeta^s(T-\zeta)} \\
& \left. \cdot \left(|\beta|\zeta + \frac{|\ell|\zeta^s}{\Gamma(s+1)}\right)\right].
\end{aligned}
$$
(76)

One observes that the right-hand side of inequality above goes to zero uniformly as $t_2 \to t_1$. This, however, implies that $S_2$ is definitely equicontinuous via Arzela-Ascoli theorem. The last step in this proof that should be considered is showing the boundedness of the solution of the equation $z = \gamma S z$, where $\gamma \in [0, 1]$. In this regard, let $t \in [0, T]$ and $z$ be such solution, then by considering (4.12, 4.12), we have

$$
\begin{aligned}
\|z\|_{\mathscr{P}} &= \gamma \|Sz\|_{\mathscr{P}} \\
&\leq \|Sz\|_{\mathscr{P}} \\
&= \max \left\{ \|Sz\|, \|{}^c D^\rho S z\| \right\} \\
&\leq \|h\| \Psi(\|z\|_{\mathscr{P}}) \max \\
&\quad \cdot \left\{ \Delta(s+r), \Delta'(s+r) \right\} + \|z\|_{\mathscr{P}} \max \left\{ \Delta, \Delta' \right\}.
\end{aligned}
$$
(77)

Using the previous inequality allows one to reveal the following inequality.

$$
\frac{\|z(t)\|_{\mathscr{P}}}{\|h\| \Psi(\|z\|_{\mathscr{P}}) \max \left\{ \Delta(s+r), \Delta'(s+r) \right\} + \|z\|_{\mathscr{P}} \max \left\{ \Delta, \Delta' \right\}} \leq 1.
$$
(78)

Obviously, due to $\|z(t)\|_{\mathscr{P}} < \lambda$ as previously defined in $\mathcal{O}$, then the last assertion contradicts assumption (70). Hence, by Theorem 9, one can conclude that $S$ has a fixed point $z \in \bar{\mathcal{O}}$, which finishes the proof.

## 5. Illustrative Examples

This section introduces three examples to demonstrate the proposed results related to the main issue in this work. For simplicity in calculations, these three examples are unified under the same values of our problem's parameters (1)–(2). In other words, we take $r = 8/5$, $s = 2/3$, $\rho = 1/2$, $\zeta = 1/2$, $T = 1$, $\ell = 1/75$, $\alpha = 1/150$ and $\beta = 3/200$. Setting these values in the terms (19), (20), (30) and (31) simplifies their expressions to be in the following forms $\chi_1 = 1$, $\chi_2 = 1.03413$, $\chi_3 = 1.80715$ and $\chi_4 = 1.80456$. Accordingly, the following terms are also estimated via (36)-(39) as $\Delta(s = 2/3) = 2.937$, $\Delta(s + r = 34/15) = 0.854$, $\Delta = 0.054$, $\Delta'(s = 2/3) = 4.339$, $\Delta'(s + r = 34/15) = 1.455$ and $\Delta' = 0.083$.

*Example 1.* Consider the following problem.

$$
D^{8/5} \left( D^{2/3} + \frac{1}{75} \right) z(t) = \frac{e^{-t}}{(7 + e^t)} \left( \frac{|z(t) D^{1/2} z(t)|}{1 + |z(t) D^{1/2} z(t)|} \right), t \in [0, 1],
$$
(79)

subject to the following conditions

$$
z(0) = 0, \quad z(1) = \frac{1}{150} \int_0^1 z(u) du, \quad z\left(\frac{1}{2}\right) = \frac{3}{200} \int_0^{1/2} z(u) du.
$$
(80)

Comparing (79)-(80) with (1)-(2) yields the following continuous function on $[0, 1]$

$$
f\left(t, z(t), D^{1/2} z(t)\right) = \frac{e^{-t}}{(7 + e^t)} \left( \frac{|z(t) D^{1/2} z(t)|}{1 + |z(t) D^{1/2} z(t)|} \right).
$$
(81)

Note that $\mathbf{H}_1$ is satisfied, and, on the other hand, we have

$$
|f\left(t, z(t), D^{1/2} z(t)\right)| \leq \left| \frac{e^{-t}}{(7 + e^t)} \right| \left| \frac{|z(t) D^{1/2} z(t)|}{1 + |z(t) D^{1/2} z(t)|} \right|,
$$
(82)

which implies

$$
|f\left(t, z(t), D^{1/2} z(t)\right)| \leq q(t) = \frac{e^{-t}}{7 + e^t}.
$$
(83)

Therefore, condition $\mathbf{H}_2$ is also satisfied with $\|q\| = 1/8$. Now, based on the above estimations, one can verify the following statement:

$$
\max \left\{ \Delta, \Delta' \right\} = 0.083 < 1.
$$
(84)

This means that the problem given in (79)-(80) has a unique solution on $[0, 1]$, as per Theorem 13..

*Example 2.* Consider the following problem:

$$
\begin{aligned}
D^{8/5} &\left( D^{2/3} + \frac{1}{75} \right) z(t) \\
&= \frac{\ln t}{2t} \cdot \frac{|z(t)|}{2 + |z(t)|} \\
&\quad + \frac{|D^{1/2} z(t)|}{10(1 + t)^2 (|D^{1/2} z(t)| + 1)},
\end{aligned}
$$
(85)

subject to the same conditions given in (80). One can notice that the function $f$ has the form

$$
f\left(t, z(t), D^{1/2} z(t)\right) = \frac{\ln t}{2t} \cdot \frac{|z(t)|}{2 + |z(t)|} + \frac{|D^{1/2} z(t)|}{10(1 + t)^2 (|D^{1/2} z(t)| + 1)},
$$
(86)

which satisfies $\mathbf{H}_1$ and implies

$$
\begin{aligned}
|f\left(t, z_1(t), D^{1/2} z_1(t)\right) &- f\left(t, z_2(t), D^{1/2} z_2(t)\right)| \\
&\leq \frac{1}{4} |z_1 - z_2| + \frac{1}{10} |D^{1/2} z_1 - D^{1/2} z_2|.
\end{aligned}
$$
(87)

This definitely means that condition $\mathbf{H}_3$ is hold, with $M_1 = 1/4$ and $M_2 = 1/10$. Besides, one can have the following assertion.

$$
\begin{aligned}
\Psi &= (M_1 + M_2) \max \left\{ \Delta(s + r), \Delta'(s + r) \right\} \\
&\quad + \max \left\{ \Delta, \Delta' \right\} = 0.5926 < 1,
\end{aligned}
$$
(88)

which implies that the problem (85) has a unique solution on $[0, 1]$ according to Theorem 14.

*Example 3.* Consider the following problem.

$$
\begin{aligned}
&D^{8/5}\left(D^{2/3} + \frac{1}{75}\right)z(t) \\
&= \frac{e^{-t^4}\max\left\{\displaystyle\sup_{0\le t\le 1}|z(t)|, \sup_{0\le t\le 1}\left|{}^c D^{1/2}(z(t))\right|\right\}}{5\sqrt{1+t^2}},
\end{aligned}
\tag{89}
$$

subject to the same conditions given in (80). In this example, we have the function $f$ is of the form

$$
\begin{aligned}
&f\left(t, z(t), D^{1/2}z(t)\right) \\
&= \frac{e^{-t^4}\max\left\{\displaystyle\sup_{0\le t\le 1}|z(t)|, \sup_{0\le t\le 1}\left|{}^c D^{1/2}(z(t))\right|\right\}}{5\sqrt{1+t^2}}.
\end{aligned}
\tag{90}
$$

In fact, the above function satisfies $\mathbf{H}_1$ and implies

$$
\left|f\left(t, z(t), D^{1/2}z(t)\right)\right| \le h(t)\Psi\left(\|z\|_{\mathscr{P}}\right),
\tag{91}
$$

where $h(t) = e^{-t^4}/5\sqrt{1+t^2}$ and $\Psi\left(\|z\|_{\mathscr{P}}\right) = \|z\|_{\mathscr{P}} < \lambda$, in which $\lambda$ is assumed as given in (70). In other words, condition $\mathbf{H}_4$ is satisfied with $\|h\| = 1/5$, while the assumption given in (70) will be hold if

$$
\max\left\{\left(0.1708\|z\|_{\mathscr{P}} + 0.054\right), \left(0.291\|z\|_{\mathscr{P}} + 0.083\right)\right\} < \lambda.
\tag{92}
$$

This, consequently, implies $\lambda > 0.117$. It means that the problem (89), in view of Theorem 15., has at least one solution on $[0, 1]$, for $\lambda > 0.117$.

## 6. Conclusion

The existence and uniqueness of solution for the fractional-order Langevin equation have been studied and explored in this article. This problem has been addressed subject to novel three-point boundary conditions involving two nonlocal integrals. In view of this study, some new applicable conditions and results have been established associated with our main target. These results have been illustrated through three numerical examples.

## Data Availability

There is no data available.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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