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Research Article

Modified Novikov Operators and the Kastler-Kalau-Walze-Type Theorem for Manifolds with Boundary

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In this paper, we give two Lichnerowicz-type formulas for modified Novikov operators. We prove Kastler-Kalau-Walze-type theorems for modified Novikov operators on compact manifolds with (respectively without) a boundary. We also compute the spectral action for Witten deformation on 4-dimensional compact manifolds.

1. Introduction

As has been well known, the noncommutative residue plays a prominent role in noncommutative geometry which is found in [1, 2]. For this reason, it has been studied extensively by geometers. Connes used the noncommutative residue to derive a conformal 4-dimensional Polyakov action analogy in [3]. Connes showed us that the noncommutative residue on a compact manifold M coincided with Dixmier's trace on pseudodifferential operators of order $-\dim M$ in [4]. Connes has also observed that the noncommutative residue of the square of the inverse of the Dirac operator was proportional to the Einstein-Hilbert action, which is called the Kastler-Kalau-Walze-type theorem now. Kastler [5] gave a brute-force proof of this theorem. Kalau and Walze proved this theorem in the normal coordinate system simultaneously in [6]. Ackermann proved that the Wodzicki residue of the square of the inverse of the Dirac operator $\mathrm{Wres}(D^{-2})$ in turn is essentially the second coefficient of the heat kernel expansion of D^2 in [7].

On the other hand, Wang generalized Connes' results to the case of manifolds with a boundary in [8, 9] and proved the Kastler-Kalau-Walze-type theorem for the Dirac operator and the signature operator on lower-dimensional manifolds with a boundary [10]. In [10, 11], Wang computed $\widetilde{\mathrm{Wres}}[\pi^+D^{-1}\circ\pi^+D^{-1}]$ and $\widetilde{\mathrm{Wres}}[\pi^+D^{-2}\circ\pi^+D^{-2}]$, where the two operators are symmetric; in these cases, the boundary term vanished. But for $\widetilde{\mathrm{Wres}}[\pi^+D^{-1}\circ\pi^+D^{-3}]$, Wang got a nonvanishing boundary term [12] and gave a theoretical explanation for gravitational action on the boundary. In others words, Wang provides a kind of method to study the Kastler-Kalau-Walze-type theorem for manifolds with a boundary. In [13], López and his collaborators introduced an elliptic differential operator which is called the Novikov operator. The motivation of this paper is to prove the Kastler-Kalau-Walze-type theorem for Novikov operators on manifolds with a boundary. In [14], Iochum and Levy computed heat kernel coefficients for Dirac operators with one-form perturbations and proved that there are no tadpoles for compact spin manifolds without a boundary. In [15], Sitarz and Zajac investigated the spectral action for scalar perturbations of Dirac operators. In [16], Hanisch and his collaborators derived a formula for the gravitational part of the spectral action for Dirac operators on 4-dimensional spin manifolds with totally antisymmetric torsion. In [17], Zhang introduced an elliptic differential operator which is called the Witten deformation. Motivated by [14–17], we will compute the spectral action for the Witten deformation on 4-dimensional compact manifolds in this paper.

The framework of this paper is organized as follows. Firstly, in Section 2, we give the definition of modified Novikov operators and the Lichnerowicz formulas associated with modified Novikov operators. We study the symbols of some operators associated with modified Novikov operators; by using symbols of operators associated with modified Novikov operators, we can prove the Kastler-Kalau-Walze-type theorem for manifolds with a boundary in Section 3 and in Section 4. In Section 5, we compute the spectral action for the Witten deformation on 4-dimensional compact manifolds.

2. Modified Novikov Operators and Their Lichnerowicz Formula

In this section, we firstly recall the definition of a Novikov operator (see details in [8]). Let M be an n-dimensional ($n \geq 3$) oriented compact Riemannian manifold with a Riemannian metric g^M. The de Rham derivative d is an elliptic differential operator on $C^\infty(M; \wedge^* T^*M)$. Then, we have the de Rham coderivative $\delta = d^*$ and the symmetric operators $D = d + \delta$ and $\Delta = D^2 = d\delta + \delta d$ (the Laplacian).

With more generality, we take any closed $\theta \in C^\infty(M; T^*M)$. For the sake of simplicity, we assume that θ is real. Then, we have the Novikov operators defined by θ, depending on $z \in \mathbb{C}$ in [8],

$$\begin{aligned} d_z &= d + z(\theta\wedge), \delta_z = d_z^* = \delta + \bar{z}(\theta\wedge)^*, \\ D_z &= d_z + \delta_z = (d+\delta) + z(\theta\wedge) + \bar{z}(\theta\wedge)^* \\ &= (d+\delta) + [\mathrm{Rez}(\theta\wedge) + \mathrm{Rez}(\theta\wedge)^*] \\ &\quad + i[\mathrm{Imz}(\theta\wedge) - \mathrm{Imz}(\theta\wedge)^*] \\ &= (d+\delta) + \mathrm{Rez}[\theta\wedge + (\theta\wedge)^*] + i\mathrm{Imz}[\theta\wedge - (\theta\wedge)^*] \\ &= (d+\delta) + \mathrm{Rez}\bar{c}(\theta) + i\mathrm{Imz}c(\theta), \end{aligned} \tag{1}$$

where Rez is the real part of z, Imz is the imaginary part of z, $\bar{c}(\theta) = (\theta)^*\wedge + (\theta\wedge)^*$, and $c(\theta) = (\theta)^*\wedge - (\theta\wedge)^*$. In this paper, we consider the modified Novikov operators; for θ, $\theta' \in \Gamma(\mathrm{TM})$, we define that

$$\widehat{D} = d + \delta + \bar{c}(\theta) + c\left(\theta'\right), \widehat{D}^* = d + \delta + \bar{c}(\theta) - c\left(\theta'\right), \tag{2}$$

where $\bar{c}(\theta) = (\theta)^*\wedge + (\theta\wedge)^*$ and $c(\theta') = (\theta')^* \wedge - (\theta'\wedge)^*$, where $\theta^* = g(\theta, \cdot)$, $(\theta')^* = g(\theta', \cdot)$.

Let ∇^L be the Levi-Civita connection about g^M. In the local coordinates $\{x_i; 1 \leq i \leq n\}$ and the fixed orthonormal frame $\{\widetilde{e_1}, \cdots, \widetilde{e_n}\}$, the connection matrix $(\omega_{s,t})$ is defined by

$$\nabla^L(\widetilde{e_1}, \cdots, \widetilde{e_n}) = (\widetilde{e_1}, \cdots, \widetilde{e_n})(\omega_{s,t}). \tag{3}$$

Let $\epsilon(\widetilde{e_j *})$ and $\iota(\widetilde{e_j *})$ be the exterior and interior multiplications, respectively, and $c(\widetilde{e_j})$ be the Clifford action. Suppose that ∂_i is a natural local frame on TM and $(g^{ij})_{1 \leq i,j \leq n}$ is the inverse matrix associated with the metric matrix $(g_{ij})_{1 \leq i,j \leq n}$ on M. Write

$$c\left(\widetilde{e_j}\right) = \varepsilon\left(\widetilde{e_j *}\right) - \iota\left(\widetilde{e_j *}\right), \bar{c}\left(\widetilde{e_j}\right) = \varepsilon\left(\widetilde{e_j *}\right) + \iota\left(\widetilde{e_j *}\right). \tag{4}$$

The modified Novikov operators $\widehat{D}$ and $\widehat{D}^*$ are defined by

$$\begin{aligned} \widehat{D} &= d + \delta + \bar{c}(\theta) + c\left(\theta'\right) \\ &= \sum_{i=1}^n c(\widetilde{e_i})\left[\widetilde{e_i} + \frac{1}{4}\sum_{s,t}\omega_{s,t}(\widetilde{e_i})[\bar{c}(\widetilde{e_s})\bar{c}(\widetilde{e_t}) - c(\widetilde{e_s})c(\widetilde{e_t})]\right] \\ &\quad + \bar{c}(\theta) + c\left(\theta'\right), \\ \widehat{D}^* &= d + \delta + \bar{c}(\theta) - c\left(\theta'\right) \\ &= \sum_{i=1}^n c(\widetilde{e_i})\left[\widetilde{e_i} + \frac{1}{4}\sum_{s,t}\omega_{s,t}(\widetilde{e_i})[\bar{c}(\widetilde{e_s})\bar{c}(\widetilde{e_t}) - c(\widetilde{e_s})c(\widetilde{e_t})]\right] \\ &\quad + \bar{c}(\theta) - c\left(\theta'\right). \end{aligned} \tag{5}$$

We first establish the main theorem in this section. One has the following Lichnerowicz formulas.

Theorem 1. *The following equalities hold:*

$$\begin{aligned} \widehat{D}^*\widehat{D} &= -\left[g^{ij}\left(\nabla_{\partial_i}\nabla_{\partial_j} - \nabla_{\nabla^L_{\partial_i}\partial_j}\right)\right] \\ &\quad - \frac{1}{8}\sum_{ijkl} R_{ijkl}\bar{c}(\widetilde{e_i})\bar{c}(\widetilde{e_j})c(\widetilde{e_k})c(\widetilde{e_l}) \\ &\quad + \sum_i c(\widetilde{e_i})\bar{c}\left(\nabla^{TM}_{\widetilde{e_i}}\theta\right) + \frac{1}{4}s - c\left(\theta'\right)\bar{c}(\theta) \\ &\quad + \bar{c}(\theta)c\left(\theta'\right) + |\theta|^2 + \left|\theta'\right|^2 \\ &\quad + \frac{1}{4}\sum_i\left[c(\widetilde{e_i})c\left(\theta'\right) - c\left(\theta'\right)c(\widetilde{e_i})\right]^2 \\ &\quad - g\left(\widetilde{e_j}, \nabla^{TM}_{\widetilde{e_j}}\theta'\right), \\ \widehat{D}^2 &= -\left[g^{ij}\left(\nabla_{\partial_i}\nabla_{\partial_j} - \nabla_{\nabla^L_{\partial_i}\partial_j}\right)\right] \\ &\quad - \frac{1}{8}\sum_{ijkl} R_{ijkl}\bar{c}(\widetilde{e_i})\bar{c}(\widetilde{e_j})c(\widetilde{e_k})c(\widetilde{e_l}) \\ &\quad + \sum_i c(\widetilde{e_i})\bar{c}\left(\nabla^{TM}_{\widetilde{e_i}}\theta\right) + \frac{1}{4}s + c\left(\theta'\right)\bar{c}(\theta) \\ &\quad + \bar{c}(\theta)c\left(\theta'\right) + |\theta|^2 - \left|\theta'\right|^2 \\ &\quad + \frac{1}{4}\sum_i\left[c(\widetilde{e_i})c\left(\theta'\right) + c\left(\theta'\right)c(\widetilde{e_i})\right]^2 \\ &\quad - \frac{1}{2}\left[c\left(\nabla^{TM}_{\widetilde{e_j}}\theta'\right)c(\widetilde{e_j}) - c(\widetilde{e_j})c\left(\nabla^{TM}_{\widetilde{e_j}}\theta'\right)\right], \end{aligned} \tag{6}$$

where s is the scalar curvature.

In order to prove Theorem 1, we recall the basic notions of Laplace-type operators. Let M be smooth compact-oriented Riemannian n-dimensional manifolds without a boundary and V' be a vector bundle on M. Any differential operator P of the Laplace type has locally the form

$$P = -\left(g^{ij}\partial_i\partial_j + A^i\partial_i + B\right), \tag{7}$$

where ∂_i is a natural local frame on TM, $(g^{ij})_{1\leq i,j\leq n}$ is the inverse matrix associated with the metric matrix $(g_{ij})_{1\leq i,j\leq n}$ on M, and A^i and B are smooth sections of $\mathrm{End}(V')$ on M (endomorphism). If P is a Laplace-type operator with the form (7), then there is a unique connection ∇ on V' and a unique endomorphism E such that

$$P = -\left[g^{ij}\left(\nabla_{\partial_i}\nabla_{\partial_j} - \nabla_{\nabla^L_{\partial_i}\partial_j}\right) + E\right], \tag{8}$$

where ∇^L is the Levi-Civita connection on M. Moreover (with local frames of T^*M and V'), $\nabla_{\partial_i} = \partial_i + \omega_i$ and E are related to g^{ij}, A^i, and B through

$$\omega_i = \frac{1}{2}g_{ij}\left(A^i + g^{kl}\Gamma^j_{kl}id\right), \tag{9}$$

$$E = B - g^{ij}\left(\partial_i(\omega_j) + \omega_i\omega_j - \omega_k\Gamma^k_{ij}\right), \tag{10}$$

where Γ^j_{kl} is the Christoffel coefficient of ∇^L.

By Proposition 4.6 of [17], we have

$$(d+\delta+\bar{c}(\theta))^2 = (d+\delta)^2 + \sum_i c(\widetilde{e_i})\bar{c}\left(\nabla^{\mathrm{TM}}_{\widetilde{e_i}}\theta\right) + |\theta|^2. \tag{11}$$

By [18], the local expression of $(d+\delta)^2$ is

$$(d+\delta)^2 = -\triangle_0 - \frac{1}{8}\sum_{ijkl}R_{ijkl}\bar{c}(\widetilde{e_i})\bar{c}(\widetilde{e_j})c(\widetilde{e_k})c(\widetilde{e_l}) + \frac{1}{4}s. \tag{12}$$

Let $g^{ij} = g(dx_i, dx_j)$, $\xi = \sum_k \xi_j dx_j$, and $\nabla^L_{\partial_i}\partial_j = \sum_k \Gamma^k_{ij}\partial_k$, we denote that

$$\begin{aligned}
\sigma_i &= -\frac{1}{4}\sum_{s,t}\omega_{s,t}(\widetilde{e_i})c(\widetilde{e_s})c(\widetilde{e_t}),\\
a_i &= \frac{1}{4}\sum_{s,t}\omega_{s,t}(\widetilde{e_i})\bar{c}(\widetilde{e_s})\bar{c}(\widetilde{e_t}),\\
\xi^j &= g^{ij}\xi_i,\\
\Gamma^k &= g^{ij}\Gamma^k_{ij},\\
\sigma^j &= g^{ij}\sigma_i,\\
a^j &= g^{ij}a_i.
\end{aligned} \tag{13}$$

Then, the modified Novikov operators $\widehat{D}$ and $\widehat{D}^*$ can be written as

$$\begin{aligned}
\widehat{D} &= \sum_{i=1}^n c(\widetilde{e_i})[\widetilde{e_i} + a_i + \sigma_i] + \bar{c}(\theta) + c\left(\theta'\right),\\
\widehat{D}^* &= \sum_{i=1}^n c(\widetilde{e_i})[\widetilde{e_i} + a_i + \sigma_i] + \bar{c}(\theta) - c\left(\theta'\right).
\end{aligned} \tag{14}$$

By [7, 18], we have

$$-\triangle_0 = \Delta = -g^{ij}\left(\nabla^L_i\nabla^L_j - \Gamma^k_{ij}\nabla^L_k\right). \tag{15}$$

We note that

$$\begin{aligned}
\widehat{D}^*\widehat{D} &= (d+\delta+\bar{c}(\theta))^2 + (d+\delta)c\left(\theta'\right)\\
&\quad + \bar{c}(\theta)c\left(\theta'\right) - c\left(\theta'\right)(d+\delta) - c\left(\theta'\right)\bar{c}(\theta) + \left|\theta'\right|^2,\\
-c&\left(\theta'\right)(d+\delta) + (d+\delta)c\left(\theta'\right)\\
&= \sum_{i,j} g^{i,j}\left[c(\partial_i)c\left(\theta'\right) - c\left(\theta'\right)c(\partial_i)\right]\partial_j\\
&\quad - \sum_{i,j} g^{i,j}\left[c\left(\theta'\right)c(\partial_i)\sigma_i + c\left(\theta'\right)c(\partial_i)a_i\right.\\
&\quad \left. + c(\partial_i)\partial_i\left(c\left(\theta'\right)\right) + c(\partial_i)\sigma_i c\left(\theta'\right) + c(\partial_i)a_i c\left(\theta'\right)\right],
\end{aligned} \tag{16}$$

then we obtain

$$\begin{aligned}
\widehat{D}^*\widehat{D} &= -\sum_{i,j} g^{i,j}\left[\partial_i\partial_j + 2\sigma_i\partial_j + 2a_i\partial_j - \Gamma^k_{i,j}\partial_k + \left(\partial_i\sigma_j\right)\right.\\
&\quad \left. + \left(\partial_i a_j\right) + \sigma_i\sigma_j + \sigma_i a_j + a_i\sigma_j + a_i a_j - \Gamma^k_{i,j}\sigma_k - \Gamma^k_{i,j}a_k\right]\\
&\quad + \sum_{i,j} g^{i,j}\left[c(\partial_i)c\left(\theta'\right) - c\left(\theta'\right)c(\partial_i)\right]\partial_j\\
&\quad - \sum_{i,j} g^{i,j}\left[c\left(\theta'\right)c(\partial_i)\sigma_i + c\left(\theta'\right)c(\partial_i)a_i\right.\\
&\quad \left. - c(\partial_i)\partial_i\left(c\left(\theta'\right)\right) - c(\partial_i)\sigma_i c\left(\theta'\right) - c(\partial_i)a_i c\left(\theta'\right)\right]\\
&\quad - \frac{1}{8}\sum_{ijkl}R_{ijkl}\bar{c}(\widetilde{e_i})\bar{c}(\widetilde{e_j})c(\widetilde{e_k})c(\widetilde{e_l}) + \frac{1}{4}s\\
&\quad + \sum_i c(\widetilde{e_i})\bar{c}\left(\nabla^{\mathrm{TM}}_{\widetilde{e_i}}\theta\right) + |\theta|^2 + \left|\theta'\right|^2\\
&\quad + \bar{c}(\theta)c\left(\theta'\right) - c\left(\theta'\right)\bar{c}(\theta).
\end{aligned} \tag{17}$$

Similarly, we have

$$\begin{aligned}\widehat{D}^2 = &-\sum_{i,j} g^{i,j}\Big[\partial_i\partial_j + 2\sigma_i\partial_j + 2a_i\partial_j - \Gamma_{i,j}^k\partial_k + (\partial_i\sigma_j) \\ &+ (\partial_i a_j) + \sigma_i\sigma_j + \sigma_i a_j + a_i\sigma_j + a_i a_j - \Gamma_{i,j}^k\sigma_k - \Gamma_{i,j}^k a_k\Big] \\ &+ \sum_{i,j} g^{i,j}\left[c(\partial_i)c(\theta') + c(\theta')c(\partial_i)\right]\partial_j \\ &+ \sum_{i,j} g^{i,j}\Big[c(\theta')c(\partial_i)\sigma_i + c(\theta')c(\partial_i)a_i \\ &+ c(\partial_i)\partial_i\left(c(\theta')\right) + c(\partial_i)\sigma_i c(\theta') + c(\partial_i)a_i c(\theta')\Big] \\ &- \frac{1}{8}\sum_{ijkl} R_{ijkl}\bar{c}(\widetilde{e_i})\bar{c}(\widetilde{e_j})c(\widetilde{e_k})c(\widetilde{e_l}) + \frac{1}{4}s \\ &+ \sum_i c(\widetilde{e_i})\bar{c}\left(\nabla^{TM}_{\tilde{e}_i}\theta\right) + |\theta|^2 - |\theta'|^2 \\ &+ \bar{c}(\theta)c(\theta') + c(\theta')\bar{c}(\theta).\end{aligned} \tag{18}$$

By (8), (9), (10), and (17), we have

$$\begin{aligned}(\omega_i)_{\widehat{D}^*\widehat{D}} &= \sigma_i + a_i - \frac{1}{2}\left[c(\partial_i)c(\theta') - c(\theta')c(\partial_i)\right], \\ E_{\widehat{D}^*\widehat{D}} &= -c(\partial_i)\sigma^i c(\theta') - c(\partial_i)a^i c(\theta') \\ &+ \frac{1}{8}\sum_{ijkl} R_{ijkl}\bar{c}(\widetilde{e_i})\bar{c}(\widetilde{e_j})c(\widetilde{e_k})c(\widetilde{e_l}) \\ &- \sum_i c(\widetilde{e_i})\bar{c}\left(\nabla^{TM}_{\tilde{e}_i}\theta\right) - |\theta|^2 - |\theta'|^2 - \frac{1}{4}s \\ &+ c(\theta')\bar{c}(\theta) + c(\theta')c(\partial_i)\sigma^i \\ &+ c(\theta')c(\partial_i)a^i - c(\partial_i)\partial^i\left(c(\theta')\right) \\ &+ \frac{1}{2}\partial^j\left[c(\partial_j)c(\theta') - c(\theta')c(\partial_j)\right] \\ &- \frac{1}{2}\left[c(\partial_j)c(\theta') - c(\theta')c(\partial_j)\right](\sigma^j + a^j) \\ &- \frac{g^{ij}}{4}\left[c(\partial_i)c(\theta') - c(\theta')c(\partial_i)\right] \\ &\cdot \left[c(\partial_j)c(\theta') - c(\theta')c(\partial_j)\right] \\ &- \frac{1}{2}\Gamma^k\left[c(\partial_k)c(\theta') - c(\theta')c(\partial_k)\right] \\ &- \bar{c}(\theta)c(\theta') - \frac{1}{2}(\sigma^j + a^j) \\ &\cdot \left[c(\partial_j)c(\theta') - c(\theta')c(\partial_j)\right].\end{aligned} \tag{19}$$

For a smooth vector field Y on M, let $c(Y)$ denote the Clifford action. Since E is globally defined on M, taking normal coordinates at x_0, we have $\sigma^i(x_0) = 0$, $a^i(x_0) = 0$, $\partial^j[c(\partial_j)](x_0) = 0$, $\Gamma^k(x_0) = 0$, and $g^{ij}(x_0) = \delta_i^j$, so that

$$\begin{aligned}E_{\widehat{D}^*\widehat{D}}(x_0) = &\frac{1}{8}\sum_{ijkl} R_{ijkl}\bar{c}(\widetilde{e_i})\bar{c}(\widetilde{e_j})c(\widetilde{e_k})c(\widetilde{e_l}) \\ &- \sum_i c(\widetilde{e_i})\bar{c}\left(\nabla^{TM}_{\tilde{e}_i}\theta\right) - \frac{1}{4}s + c(\theta')\bar{c}(\theta) \\ &- \bar{c}(\theta)c(\theta') - |\theta|^2 - |\theta'|^2 \\ &- \frac{1}{4}\sum_i\left[c(\widetilde{e_i})c(\theta') - c(\theta')c(\widetilde{e_i})\right]^2 \\ &- \frac{1}{2}\left[c(\widetilde{e_j})\widetilde{e_j}\left(c(\theta')\right) + \widetilde{e_j}\left(c(\theta')\right)c(\widetilde{e_j})\right] \\ = &\frac{1}{8}\sum_{ijkl} R_{ijkl}\bar{c}(\widetilde{e_i})\bar{c}(\widetilde{e_j})c(\widetilde{e_k})c(\widetilde{e_l}) \\ &- \sum_i c(\widetilde{e_i})\bar{c}\left(\nabla^{TM}_{\tilde{e}_i}\theta\right) - \frac{1}{4}s + c(\theta')\bar{c}(\theta) \\ &- \bar{c}(\theta)c(\theta') - |\theta|^2 - |\theta'|^2 \\ &- \frac{1}{4}\sum_i\left[c(\widetilde{e_i})c(\theta') - c(\theta')c(\widetilde{e_i})\right]^2 \\ &+ g\left(\widetilde{e_j}, \nabla^{TM}_{\tilde{e}_j}\theta'\right).\end{aligned} \tag{20}$$

Similarly, we have

$$\begin{aligned}E_{\widehat{D}^2}(x_0) = &\frac{1}{8}\sum_{ijkl} R_{ijkl}\bar{c}(\widetilde{e_i})\bar{c}(\widetilde{e_j})c(\widetilde{e_k})c(\widetilde{e_l}) \\ &- \sum_i c(\widetilde{e_i})\bar{c}\left(\nabla^{TM}_{\tilde{e}_i}\theta\right) - \frac{1}{4}s - c(\theta')\bar{c}(\theta) \\ &- \bar{c}(\theta)c(\theta') - |\theta|^2 + |\theta'|^2 \\ &- \frac{1}{4}\sum_i\left[c(\widetilde{e_i})c(\theta') + c(\theta')c(\widetilde{e_i})\right]^2 \\ &+ \frac{1}{2}\left[c\left(\nabla^{TM}_{\tilde{e}_j}\theta'\right)c(\widetilde{e_j}) - c(\widetilde{e_j})c\left(\nabla^{TM}_{\tilde{e}_j}\theta'\right)\right],\end{aligned} \tag{21}$$

which, together with (7), yields Theorem 1.

The noncommutative residue of a generalized Laplacian $\tilde{\Delta}$ is expressed as by [7]

$$(n-2)\Phi_2\left(\tilde{\Delta}\right) = (4\pi)^{-(n/2)}\Gamma\left(\frac{n}{2}\right)\widetilde{\operatorname{res}}\left(\tilde{\Delta}^{-(n/2)+1}\right), \tag{22}$$

where $\Phi_2(\tilde{\Delta})$ denotes the integral over the diagonal part of the second coefficient of the heat kernel expansion of $\tilde{\Delta}$. Now let $\tilde{\Delta} = \widehat{D}^*\widehat{D}$. Since $\widehat{D}^*\widehat{D}$ is a generalized Laplacian, we can suppose $\widehat{D}^*\widehat{D} = \Delta - E$, then we have

$$\begin{aligned}&\operatorname{Wres}\left(\widehat{D}^*\widehat{D}\right)^{-((n-2)/2)} \\ &\quad = \frac{(n-2)(4\pi)^{n/2}}{((n/2)-1)!}\int_M \operatorname{tr}\left(\frac{1}{6}s + E_{\widehat{D}^*\widehat{D}}\right)d\mathrm{Vol}_M,\end{aligned} \tag{23}$$

where Wres denotes the noncommutative residue.

Similarly, we have

$$\mathrm{Wres}\left(\widehat{D}^2\right)^{-((n-2)/2)} = \frac{(n-2)(4\pi)^{n/2}}{((n/2)-1)!}\int_M \mathrm{tr}\left(\frac{1}{6}s + E_{\widehat{D}^2}\right)dVol_M, \tag{24}$$

where Wres denotes the noncommutative residue.

Theorem 2. *For even n-dimensional compact-oriented manifolds without a boundary, the following equalities hold:*

$$Wres\left(\widehat{D}^*\widehat{D}\right)^{-((n-2)/2)} = \frac{(n-2)(4\pi)^{n/2}}{((n/2)-1)!}\int_M 2^n\left(-\frac{1}{12}s - |\theta|^2 + (n-2)\left|\theta'\right|^2 + g\left(\widetilde{e_j}, \nabla^{TM}_{\widetilde{e_j}}\theta'\right)\right)dVol_M, \tag{25}$$

$$Wres\left(\widehat{D}^2\right)^{-((n-2)/2)} = \frac{(n-2)(4\pi)^{n/2}}{((n/2)-1)!}\int_M 2^n\left(-\frac{1}{12}s - |\theta|^2\right)dVol_M, \tag{26}$$

where s is the scalar curvature.

3. A Kastler-Kalau-Walze-Type Theorem for 4-Dimensional Manifolds with Boundary

In this section, we prove the Kastler-Kalau-Walze-type theorem for 4-dimensional compact-oriented manifold with a boundary. We firstly give some basic facts and formulas about Boutet de Monvel's calculus and the definition of the noncommutative residue for manifolds with a boundary (see details in Section 2 in [10]).

Let M be a 4-dimensional compact-oriented manifold with boundary ∂M. We assume that the metric g^M on M has the following form near the boundary,

$$g^M = \frac{1}{h(x_n)}g^{\partial M} + dx_n^2, \tag{27}$$

where $g^{\partial M}$ is the metric on ∂M. $h(x_n) \in C^\infty([0,1)) = \{\tilde{h}|_{[0,1)} \mid \tilde{h} \in C^\infty((-\varepsilon,1))\}$ for some $\varepsilon > 0$ and satisfies $h(x_n) > 0$, $h(0) = 1$ where x_n denotes the normal directional coordinate. Let $U \subset M$ be a collar neighborhood of ∂M which is diffeomorphic with $\partial M \times [0,1)$. By the definition of $h(x_n) \in C^\infty([0,1))$ and $h(x_n) > 0$, there exists $\tilde{h} \in C^\infty((-\varepsilon,1))$ such that $\tilde{h}|_{[0,1)} = h$ and $\tilde{h} > 0$ for some sufficiently small $\varepsilon > 0$. Then, there exists a metric $\widehat{g}$ on $\widehat{M} = M\bigcup_{\partial M}\partial M \times (-\varepsilon, 0]$ which has the form on $U\bigcup_{\partial M}\partial M \times (-\varepsilon, 0]$

$$\widehat{g} = \frac{1}{\tilde{h}(x_n)}g^{\partial M} + dx_n^2, \tag{28}$$

such that $\widehat{g}|_M = g$. We fix a metric $\widehat{g}$ on the $\widehat{M}$ such that $\widehat{g}|_M = g$.

Let

$$F : L^2(\mathbf{R}_t) \longrightarrow L^2(\mathbf{R}_v),\ F(u)(v) = \int e^{-ivt}u(t)dt, \tag{29}$$

denote the Fourier transformation and $\Phi(\overline{\mathbf{R}^+}) = r^+\Phi(\mathbf{R})$ (similarly define $\Phi(\overline{\mathbf{R}^-})$), where $\Phi(\mathbf{R})$ denotes the Schwartz space and

$$r^+ : C^\infty(\mathbf{R}) \longrightarrow C^\infty\left(\overline{\mathbf{R}^+}\right), f \longrightarrow f\Big|\overline{\mathbf{R}^+};\ \overline{\mathbf{R}^+} = \{x \geq 0; x \in \mathbf{R}\}. \tag{30}$$

We define $H^+ = F(\Phi(\overline{\mathbf{R}^+}))$ and $H_0^- = F(\Phi(\overline{\mathbf{R}^-}))$ which are orthogonal to each other. We have the following property: $h \in H^+(H_0^-)$ if and only if $h \in C^\infty(\mathbf{R})$ which has an analytic extension to the lower (upper) complex half-plane $\{\mathrm{Im}\,\xi < 0\}(\{\mathrm{Im}\,\xi > 0\})$ such that for all nonnegative integer l,

$$\frac{d^l h}{d\xi^l}(\xi) \sim \sum_{k=1}^{\infty}\frac{d^l}{d\xi^l}\left(\frac{c_k}{\xi^k}\right), \tag{31}$$

as $|\xi| \longrightarrow +\infty$, $\mathrm{Im}\,\xi \leq 0 (Im\xi \geq 0)$.

Let H' be the space of all polynomials and $H^- = H_0^- \oplus H'$ and $H = H^+ \oplus H^-$. Denote by $\pi^+(\pi^-)$, respectively, the projection on $H^+(H^-)$. For calculations, we take $H = \tilde{H} = \{$rational functions having no poles on the real axis$\}$ ($\tilde{H}$ is a dense set in the topology of H). Then, on $\tilde{H}$,

$$\pi^+h(\xi_0) = \frac{1}{2\pi i}\lim_{u\to 0^-}\int_{\Gamma^+}\frac{h(\xi)}{\xi_0 + iu - \xi}d\xi, \tag{32}$$

where Γ^+ is a Jordan close curve including Im $(\xi) > 0$ surrounding all the singularities of h in the upper half-plane and $\xi_0 \in \mathbf{R}$. Similarly, define π' on $\tilde{H}$,

$$\pi'h = \frac{1}{2\pi}\int_{\Gamma^+}h(\xi)d\xi. \tag{33}$$

So, $\pi'(H^-) = 0$. For $h \in H \cap L^1(\mathbf{R})$, $\pi'h = (1/2\pi)\int_{\mathbf{R}}h(v)dv$, and for $h \in H^+ \cap L^1(\mathbf{R})$, $\pi'h = 0$.

An operator of order $m \in \mathbf{Z}$ and type d is a matrix

$$A = \begin{pmatrix} \pi^+P + G & K \\ T & S \end{pmatrix} : \begin{matrix} C^\infty(X, E_1) \\ \oplus \\ C^\infty(\partial X, F_1) \end{matrix} \longrightarrow \begin{matrix} C^\infty(X, E_2) \\ \oplus \\ C^\infty(\partial X, F_2) \end{matrix}. \tag{34}$$

where X is a manifold with boundary ∂X and $E_1, E_2(F_1, F_2)$ are vector bundles over $X(\partial X)$. Here, $P : C_0^\infty(\Omega, \overline{E_1}) \longrightarrow C^\infty(\Omega, \overline{E_2})$ is a classical pseudodifferential operator of order m on Ω, where Ω is an open neighborhood of X and $\overline{E_i}|X$

$= E_i$ $(i = 1, 2)$. P has an extension: $\mathscr{E}'(\Omega, \overline{E_1}) \longrightarrow \mathscr{D}'(\Omega, \overline{E_2})$, where $\mathscr{E}'(\Omega, \overline{E_1})(\mathscr{D}'(\Omega, \overline{E_2}))$ is the dual space of $C^\infty(\Omega, \overline{E_1})(C_0^\infty(\Omega, \overline{E_2}))$. Let $e^+ : C^\infty(X, E_1) \longrightarrow \mathscr{E}'(\Omega, \overline{E_1})$ denote extension by zero from X to Ω and $r^+ : \mathscr{D}'(\Omega, \overline{E_2}) \longrightarrow \mathscr{D}'(\Omega, E_2)$ denote the restriction from Ω to X, then define

$$\pi^+ P = r^+ P e^+ : C^\infty(X, E_1) \longrightarrow \mathscr{D}'(\Omega, E_2). \tag{35}$$

In addition, P is supposed to have the transmission property; this means that, for all j, k, α, the homogeneous component p_j of order j in the asymptotic expansion of the symbol p of P in local coordinates near the boundary satisfies

$$\partial_{x_n}^k \partial_{\xi'}^\alpha p_j\left(x', 0, 0, +1\right) = (-1)^{j-|\alpha|} \partial_{x_n}^k \partial_{\xi'}^\alpha p_j\left(x', 0, 0, -1\right), \tag{36}$$

then $\pi^+ P : C^\infty(X, E_1) \longrightarrow C^\infty(X, E_2)$ by [19]. Let G and T be, respectively, the singular Green operator and the trace operator of order m and type d. K is a potential operator and S is a classical pseudodifferential operator of order m along the boundary (for detailed definition, see [13]). Denote by $B^{m,d}$ the collection of all operators of order m and type d, and $\mathscr{B}$ is the union over all m and d.

Recall $B^{m,d}$ is a Fréchet space. The composition of the above operator matrices yields a continuous map: $B^{m,d} \times B^{m',d'} \longrightarrow B^{m+m', \max\{m'+d, d'\}}$. Write

$$A = \begin{pmatrix} \pi^+ P + G & K \\ T & S \end{pmatrix} \in B^{m,d}, A' = \begin{pmatrix} \pi^+ P' + G' & K' \\ T' & S' \end{pmatrix} \in B^{m',d'}. \tag{37}$$

The composition AA' is obtained by multiplication of the matrices (for more details, see [19]). For example, $\pi^+ P \circ G'$ and $G \circ G'$ are singular Green operators of type d' and

$$\pi^+ P \circ \pi^+ P' = \pi^+\left(PP'\right) + L\left(P, P'\right). \tag{38}$$

Here, PP' is the usual composition of pseudodifferential operators, and $L(P, P')$ called the leftover term is a singular Green operator of type $m' + d$. For our case, P, P' are classical pseudodifferential operators; in other words, $\pi^+ P \in \mathscr{B}^\infty$ and $\pi^+ P' \in \mathscr{B}^\infty$.

Let M be an n-dimensional compact-oriented manifold with boundary ∂M. Denote by $\mathscr{B}$ Boutet de Monvel's algebra, we recall the main theorem in [10, 20].

Theorem 3 ([20], Fedosov-Golse-Leichtnam-Schrohe). *Let X and ∂X be connected,* $\dim X = n \geq 3$,

$$A = \begin{pmatrix} \pi^+ P + G & K \\ T & S \end{pmatrix} \in \mathscr{B}, \tag{39}$$

and denote by p, b, and s the local symbols of P, G, and S, respectively. Define:

$$\begin{aligned} \widetilde{Wres}(A) = & \int_X \int_S tr_E\left[p_{-n}(x, \xi)\right] \sigma(\xi) dx \\ & + 2\pi \int_{\partial X} \int_S{}' \left\{ tr_E\left[(trb_{-n})\left(x', \xi'\right)\right] \right. \\ & \left. + tr_F\left[s_{1-n}\left(x', \xi'\right)\right] \right\} \sigma\left(\xi'\right) dx'. \end{aligned} \tag{40}$$

Then, (a) $\widetilde{Wres}([A, B]) = 0$, for any $A, B \in \mathscr{B}$; (b) it is a unique continuous trace on $\mathscr{B}/\mathscr{B}^{-\infty}$.

Formulas (2.1.4)–(2.1.8) from paper [10] still hold in the case when M is an oriented (not necessarily spin) manifold, since these formulas come from a composition of pseudodifferential operators in Boutet de Monvel algebra (see p.23 in [20] and p.740 in [8]). These formulas hold for general pseudodifferential operators. Thus, these formulas hold for the modified Novikov operator.

By (2.1.4)–(2.1.8) in [10], we get

$$\begin{aligned} & \widetilde{\mathrm{Wres}}\left[\pi^+ D^{-p_1} \circ \pi^+ D^{-p_2}\right] \\ & = \int_M \int_{|\xi|=1} \mathrm{trace}_{\wedge^* T^* M}\left[\sigma_{-n}\left(D^{-p_1-p_2}\right)\right] \sigma(\xi) dx + \int_{\partial M} \Phi, \end{aligned} \tag{41}$$

$$\begin{aligned} \Phi = & \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \sum \frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!} \\ & \times \mathrm{trace}_{\wedge^* T^* M}\left[\partial_{x_n}^j \partial_{\xi'}^\alpha \partial_{\xi_n}^k \sigma_r^+\left(D^{-p_1}\right)\left(x', 0, \xi', \xi_n\right) \right. \\ & \left. \times \partial_{x'}^\alpha \partial_{\xi_n}^{j+1} \partial_{x_n}^k \sigma_l\left(D^{-p_2}\right)\left(x', 0, \xi', \xi_n\right)\right] d\xi_n \sigma\left(\xi'\right) dx', \end{aligned} \tag{42}$$

where the sum is taken over $r + l - k - |\alpha| - j - 1 = -n$, $r \leq -p_1$, $l \leq -p_2$. D denotes the de Rham operator $d + \delta$. In fact, for a general one-order elliptic differential operator, (41) and (42) are also correct.

Since $[\sigma_{-n}(D^{-p_1-p_2})]|_M$ has the same expression as $\sigma_{-n}(D^{-p_1-p_2})$ in the case of manifolds without a boundary, locally, we can use the computations [5, 6, 10, 19] to compute the first term.

For any fixed point $x_0 \in \partial M$, we choose the normal coordinates U of x_0 in ∂M (not in M) and compute $\Phi(x_0)$ in the coordinates $\tilde{U} = U \times [0, 1) \subset M$ and the metric $(1/h(x_n)) g^{\partial M} + dx_n^2$. The dual metric of g^M on $\tilde{U}$ is $h(x_n) g^{\partial M} + dx_n^2$. Write $g_{ij}^M = g^M((\partial/\partial x_i), (\partial/\partial x_j))$ and $g_M^{ij} = g^M(dx_i, dx_j)$, then

$$\begin{aligned} \left[g_{i,j}^M\right] = & \begin{bmatrix} \dfrac{1}{h(x_n)}\left[g_{i,j}^{\partial M}\right] & 0 \\ 0 & 1 \end{bmatrix}, \left[g_M^{i,j}\right] \\ = & \begin{bmatrix} h(x_n)\left[g_{\partial M}^{i,j}\right] & 0 \\ 0 & 1 \end{bmatrix}, \end{aligned} \tag{43}$$

$$\partial_{x_s} g_{ij}^{\partial M}(x_0) = 0, 1 \leq i, j \leq n-1, g_{ij}^M(x_0) = \delta_{ij}.$$

For a general Clifford module, the conclusion of Section 2 and the Appendix in [10] is true. In our case, for $\wedge^* T^* M$, $c(dx_j) = dx_j \wedge - (dx_j\wedge)^*$ is the Clifford module, so we can use the conclusion of Section 2 and the Appendix in [10]. We will give the following three lemmas as computation tools.

Lemma 4 (see [10]). *With the metric g^M on M near the boundary*

$$\partial_{x_j}\left(|\xi|^2_{g^M}\right)(x_0) = \begin{cases} 0, & \text{if } j<n,\\ h'(0)|\xi'|^2_{g^{\partial M}}, & \text{if } j=n,\end{cases}$$
$$\partial_{x_j}[c(\xi)](x_0) = \begin{cases} 0, & \text{if } j<n,\\ \partial x_n\left(c\left(\xi'\right)\right)(x_0), & \text{if } j=n,\end{cases} \tag{44}$$

where $\xi = \xi' + \xi_n dx_n$.

Lemma 5 (see [10]). *With the metric g^M on M near the boundary*

$$\omega_{s,t}(\widetilde{e_i})(x_0) = \begin{cases} \omega_{n,i}(\widetilde{e_i})(x_0) = \frac{1}{2}h'(0), & \text{if } s=n, t=i, i<n,\\ \omega_{i,n}(\widetilde{e_i})(x_0) = -\frac{1}{2}h'(0), & \text{if } s=i, t=n, i<n;\\ \omega_{s,t}(\widetilde{e_i})(x_0) = 0, & \text{other cases},\end{cases} \tag{45}$$

where $(\omega_{s,t})$ denotes the connection matrix of Levi-Civita connection ∇^L.

Lemma 6 (see [10]). *When $i<n$, then*

$$\Gamma^n_{ii}(x_0) = \frac{1}{2}h'(0);\ \Gamma^i_{ni}(x_0) = -\frac{1}{2}h'(0);\ \Gamma^i_{in}(x_0) = -\frac{1}{2}h'(0), \tag{46}$$

in other cases, $\Gamma^i_{st}(x_0) = 0$.

By (41) and (42), we firstly compute

$$\begin{aligned}&\widetilde{\mathrm{Wres}}\left[\pi^+\widehat{D}^{-1}\circ\pi^+\left(\widehat{D}^*\right)^{-1}\right]\\ &\quad = \int_M\int_{|\xi|=1}\mathrm{trace}_{\wedge^*T^*M}\left[\sigma_{-4}\left(\left(\widehat{D}^*\widehat{D}\right)^{-1}\right)\right]\sigma(\xi)dx + \int_{\partial M}\Phi,\end{aligned} \tag{47}$$

where

$$\begin{aligned}\Phi &= \int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{j,k=0}^{\infty}\sum\frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!}\\ &\quad\times\mathrm{trace}_{\wedge^*T^*M}\left[\partial^j_{x_n}\partial^\alpha_{\xi'}\partial^k_{\xi_n}\sigma^+_r\left(\widehat{D}^{-1}\right)\left(x',0,\xi',\xi_n\right)\right.\\ &\quad\left.\times\partial^\alpha_{x'}\partial^{j+1}_{\xi_n}\partial^k_{x_n}\sigma_l\left(\left(\widehat{D}^*\right)^{-1}\right)\left(x',0,\xi',\xi_n\right)\right]d\xi_n\sigma\left(\xi'\right)dx',\end{aligned} \tag{48}$$

and the sum is taken over $r+l-k-j-|\alpha| = -3$, $r\le -1$, $l\le -1$, $\widehat{D}$ denotes the modified Novikov operators.

Locally, we can use Theorem 2 (25) to compute the interior of $\widetilde{\mathrm{Wres}}[\pi^+\widehat{D}^{-1}\circ\pi^+(\widehat{D}^*)^{-1}]$; we have

$$\begin{aligned}&\int_M\int_{|\xi|=1}\mathrm{trace}_{\wedge^*T^*M}\left[\sigma_{-4}\left(\left(\widehat{D}^*\widehat{D}\right)^{-1}\right)\right]\sigma(\xi)dx\\ &\quad = 32\pi^2\int_M\left[16g\left(\widetilde{e_j},\nabla^{TM}_{\widetilde{e_j}}\theta'\right) - \frac{4}{3}s\right.\\ &\quad\left. - 16|\theta|^2 + 32\left|\theta'\right|^2\right]d\mathrm{Vol}_M.\end{aligned} \tag{49}$$

So we only need to compute $\int_{\partial M}\Phi$. Let us now turn to compute the symbols of some operators. By (13)–(18), some operators have the following symbols.

Lemma 7. *The following identities hold:*

$$\begin{aligned}\sigma_1\left(\widehat{D}\right) &= \sigma_1\left(\widehat{D}^*\right) = ic(\xi),\ \sigma_0\left(\widehat{D}\right) = \frac{1}{4}\sum_{i,s,t}\omega_{s,t}(\widetilde{e_i})c(\widetilde{e_i})\bar{c}(\widetilde{e_s})\bar{c}(\widetilde{e_t})\\ &\quad - \frac{1}{4}\sum_{i,s,t}\omega_{s,t}(\widetilde{e_i})c(\widetilde{e_i})c(\widetilde{e_s})c(\widetilde{e_t}) + \bar{c}(\theta)\\ &\quad + c\left(\theta'\right),\ \sigma_0\left(\widehat{D}^*\right) = \frac{1}{4}\sum_{i,s,t}\omega_{s,t}(\widetilde{e_i})c(\widetilde{e_i})\bar{c}(\widetilde{e_s})\bar{c}(\widetilde{e_t})\\ &\quad - \frac{1}{4}\sum_{i,s,t}\omega_{s,t}(\widetilde{e_i})c(\widetilde{e_i})c(\widetilde{e_s})c(\widetilde{e_t}) + \bar{c}(\theta) - c\left(\theta'\right).\end{aligned} \tag{50}$$

Write

$$D^\alpha_x = (-i)^{|\alpha|}\partial^\alpha_x,\ \sigma\left(\widehat{D}\right) = p_1 + p_0,\ \sigma\left(\widehat{D}^{-1}\right) = \sum_{j=1}^{\infty}q_{-j}. \tag{51}$$

By the composition formula of pseudodifferential operators, we have

$$\begin{aligned}1&=\sigma\left(\widehat{D}\circ\widehat{D}^{-1}\right)=\sum_{\alpha}\frac{1}{\alpha!}\partial_{\xi}^{\alpha}\left[\sigma\left(\widehat{D}\right)\right]D_{x}^{\alpha}\left[\sigma\left(\widehat{D}^{-1}\right)\right]\\&=(p_1+p_0)(q_{-1}+q_{-2}+q_{-3}+\cdots)\\&\quad+\sum_{j}\left(\partial_{\xi_j}p_1+\partial_{\xi_j}p_0\right)\left(D_{x_j}q_{-1}+D_{x_j}q_{-2}+D_{x_j}q_{-3}+\cdots\right)\\&=p_1q_{-1}+\left(p_1q_{-2}+p_0q_{-1}+\sum_{j}\partial_{\xi_j}p_1D_{x_j}q_{-1}\right)+\cdots,\end{aligned}\tag{52}$$

so

$$q_{-1}=p_1^{-1},\ q_{-2}=-p_1^{-1}\left[p_0p_1^{-1}+\sum_{j}\partial_{\xi_j}p_1D_{x_j}\left(p_1^{-1}\right)\right].\tag{53}$$

By Lemma 7, we have some symbols of operators.

Lemma 8. *The following identities hold:*

$$\begin{aligned}\sigma_{-1}\left(\widehat{D}^{-1}\right)&=\sigma_{-1}\left(\left(\widehat{D}^{*}\right)^{-1}\right)=\frac{ic(\xi)}{|\xi|^{2}},\ \sigma_{-2}\left(\widehat{D}^{-1}\right)\\&=\frac{c(\xi)\sigma_{0}\left(\widehat{D}^{-1}\right)c(\xi)}{|\xi|^{4}}+\frac{c(\xi)}{|\xi|^{6}}\sum_{j}c\left(dx_j\right)\\&\quad\cdot\left[\partial_{x_j}(c(\xi))|\xi|^{2}-c(\xi)\partial_{x_j}\left(|\xi|^{2}\right)\right],\ \sigma_{-2}\left(\left(\widehat{D}^{*}\right)^{-1}\right)\\&=\frac{c(\xi)\sigma_{0}\left(\left(\widehat{D}^{*}\right)^{-1}\right)c(\xi)}{|\xi|^{4}}+\frac{c(\xi)}{|\xi|^{6}}\sum_{j}c\left(dx_j\right)\\&\quad\cdot\left[\partial_{x_j}(c(\xi))|\xi|^{2}-c(\xi)\partial_{x_j}\left(|\xi|^{2}\right)\right].\end{aligned}\tag{54}$$

From the remark above, we can now compute Φ (see formula (48) for the definition of Φ). We use tr as shorthand of trace. Since $n=4$, then $\mathrm{tr}_{\wedge*T*M}[\mathrm{id}]=\dim\left(\wedge^{*}(4)\right)=16$, since the sum is taken over $r+l-k-j-|\alpha|=-3$, $r\le-1$, $l\le-1$, then we have the following five cases:

Case 1. (i) $r=-1$, $l=-1$, $k=j=0$, and $|\alpha|=1$.

By (48), we get

$$\begin{aligned}\text{Case 1 (i)}=&-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{tr}\left[\partial_{\xi'}^{\alpha}\pi_{\xi_n}^{+}\sigma_{-1}\left(\widehat{D}^{-1}\right)\right.\\&\left.\times\partial_{x'}^{\alpha}\partial_{\xi_n}\sigma_{-1}\left(\left(\widehat{D}^{*}\right)^{-1}\right)\right](x_0)d\xi_n\sigma\left(\xi'\right)dx'.\end{aligned}\tag{55}$$

By Lemma 4, for $i<n$, then

$$\partial_{x_i}\left(\frac{ic(\xi)}{|\xi|^{2}}\right)(x_0)=\frac{i\partial_{x_i}[c(\xi)](x_0)}{|\xi|^{2}}-\frac{ic(\xi)\partial_{x_i}\left(|\xi|^{2}\right)(x_0)}{|\xi|^{4}}=0,\tag{56}$$

so Case 1 (i) vanishes.

Case 1. (ii) $r=-1$, $l=-1$, $k=|\alpha|=0$, and $j=1$.

By (48), we get

$$\begin{aligned}\text{Case 1 (ii)}=&-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\left[\partial_{x_n}\pi_{\xi_n}^{+}\sigma_{-1}\left(D\wedge^{-1}\right)\right.\\&\left.\times\partial_{\xi_n}^{2}\sigma_{-1}\left(\left(D\wedge^{*}\right)^{-1}\right)\right](x_0)d\xi_n\sigma\left(\xi'\right)dx'.\end{aligned}\tag{57}$$

By Lemma 8, we have

$$\partial_{\xi_n}^{2}\sigma_{-1}\left(\left(\widehat{D}^{*}\right)^{-1}\right)(x_0)=i\left(-\frac{6\xi_nc(dx_n)+2c\left(\xi'\right)}{|\xi|^{4}}+\frac{8\xi_n^{2}c(\xi)}{|\xi|^{6}}\right),\tag{58}$$

$$\partial_{x_n}\sigma_{-1}\left(\widehat{D}^{-1}\right)(x_0)=\frac{i\partial_{x_n}c\left(\xi'\right)(x_0)}{|\xi|^{2}}-\frac{ic(\xi)|\xi'|^{2}h'(0)}{|\xi|^{4}}.\tag{59}$$

By (32), (33), and the Cauchy integral formula, we have

$$\begin{aligned}&\pi_{\xi_n}^{+}\left[\frac{c(\xi)}{|\xi|^{4}}\right](x_0)\Bigg|_{|\xi'|=1}\\&=\pi_{\xi_n}^{+}\left[\frac{c\left(\xi'\right)+\xi_nc(dx_n)}{\left(1+\xi_n^{2}\right)^{2}}\right]=\frac{1}{2\pi i}\lim_{u\longrightarrow0^{-}}\int_{\Gamma^{+}}\\&\quad\cdot\frac{\left(\left(c\left(\xi'\right)+\eta_nc(dx_n)\right)/\left((\eta_n+i)^{2}(\xi_n+iu-\eta_n)\right)\right)}{(\eta_n-i)^{2}}d\eta_n\\&=-\frac{(i\xi_n+2)c\left(\xi'\right)+ic(dx_n)}{4(\xi_n-i)^{2}}.\end{aligned}\tag{60}$$

Similarly, we have,

$$\pi_{\xi_n}^{+}\left[\frac{i\partial_{x_n}c\left(\xi'\right)}{|\xi|^{2}}\right](x_0)\Bigg|_{|\xi'|=1}=\frac{\partial_{x_n}\left[c\left(\xi'\right)\right](x_0)}{2(\xi_n-i)}.\tag{61}$$

By (59), then

$$\pi^+_{\xi_n}\partial_{x_n}\sigma_{-1}\left(\widehat{D}^{-1}\right)\Big|_{|\xi'|=1}=\frac{\partial_{x_n}\left[c\left(\xi'\right)\right](x_0)}{2(\xi_n-i)}+ih'(0)\cdot\left[\frac{(i\xi_n+2)c\left(\xi'\right)+ic(dx_n)}{4(\xi_n-i)^2}\right]. \tag{62}$$

By the relation of the Clifford action and $\mathrm{tr}AB=\mathrm{tr}BA$, we have the equalities:

$$\begin{aligned}&\mathrm{tr}\left[c\left(\xi'\right)c(dx_n)\right]=0,\ \mathrm{tr}\left[c(dx_n)^2\right]=-16,\ \mathrm{tr}\left[c\left(\xi'\right)^2\right](x_0)\Big|_{|\xi'|=1}\\&=-16,\ \mathrm{tr}\left[\partial_{x_n}c\left(\xi'\right)c(dx_n)\right]\\&=0,\ \mathrm{tr}\left[\partial_{x_n}c\left(\xi'\right)c\left(\xi'\right)\right](x_0)|_{|\xi'|=1}\\&=-8h'(0),\ \mathrm{tr}\left[\bar{c}(\widetilde{e_i})\bar{c}(\widetilde{e_j})c(\widetilde{e_k})c(\widetilde{e_l})\right]=0(i\neq j).\end{aligned} \tag{63}$$

By (61) and a direct computation, we have

$$\begin{aligned}&h'(0)\mathrm{tr}\left[\frac{(i\xi_n+2)c\left(\xi'\right)+ic(dx_n)}{4(\xi_n-i)^2}\times\left(\frac{6\xi_nc(dx_n)+2c\left(\xi'\right)}{\left(1+\xi_n^2\right)^2}-\frac{8\xi_n^2\left[c\left(\xi'\right)+\xi_nc(dx_n)\right]}{\left(1+\xi_n^2\right)^3}\right)\right](x_0)\Bigg|_{|\xi'|=1}\\&=-16h'(0)\frac{-2i\xi_n^2-\xi_n+i}{(\xi_n-i)^4(\xi_n+i)^3}.\end{aligned} \tag{64}$$

Similarly, we have

$$\begin{aligned}&-i\mathrm{tr}\left[\left(\frac{\partial_{x_n}\left[c\left(\xi'\right)\right](x_0)}{2(\xi_n-i)}\right)\times\left(\frac{6\xi_nc(dx_n)+2c\left(\xi'\right)}{\left(1+\xi_n^2\right)^2}-\frac{8\xi_n^2\left[c\left(\xi'\right)+\xi_nc(dx_n)\right]}{\left(1+\xi_n^2\right)^3}\right)\right](x_0)\Bigg|_{|\xi'|=1}\\&=-8ih'(0)\frac{3\xi_n^2-1}{(\xi_n-i)^4(\xi_n+i)^3}.\end{aligned} \tag{65}$$

Then,

$$\begin{aligned}\text{Case 1 (ii)}&=-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{4ih'(0)(\xi_n-i)^2}{(\xi_n-i)^4(\xi_n+i)^3}d\xi_n\sigma\left(\xi'\right)dx'\\&=-4ih'(0)\Omega_3\int_{\Gamma^+}\frac{1}{(\xi_n-i)^2(\xi_n+i)^3}d\xi_ndx'\\&=-4ih'(0)\Omega_32\pi i\left[\frac{1}{(\xi_n+i)^3}\right]^{(1)}\Bigg|_{\xi_n=i}dx'\\&=-\frac{3}{2}\pi h'(0)\Omega_3dx',\end{aligned} \tag{66}$$

where Ω_3 is the canonical volume of S^3.

Case 1. (iii) $r=-1$, $l=-1$, $j=|\alpha|=0$, and $k=1$.

By (48), we get

$$\begin{aligned}\text{Case 1 (iii)}=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}&\mathrm{trace}\left[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}\left(\widehat{D}^{-1}\right)\right.\\&\left.\times\partial_{\xi_n}\partial_{x_n}\sigma_{-1}\left(\left(\widehat{D}^*\right)^{-1}\right)\right](x_0)d\xi_n\sigma\left(\xi'\right)dx'.\end{aligned} \tag{67}$$

By Lemma 8, we have

$$\begin{aligned}&\partial_{\xi_n}\partial_{x_n}\sigma_{-1}\left(\left(\widehat{D}^*\right)^{-1}\right)(x_0)\Big|_{|\xi'|=1}\\&=-ih'(0)\left[\frac{c(dx_n)}{|\xi|^4}-4\xi_n\frac{c\left(\xi'\right)+\xi_nc(dx_n)}{|\xi|^6}\right]-\frac{2\xi_ni\partial_{x_n}c\left(\xi'\right)(x_0)}{|\xi|^4},\end{aligned} \tag{68}$$

$$\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}\left(\widehat{D}^{-1}\right)(x_0)\Big|_{|\xi'|=1}=-\frac{c\left(\xi'\right)+ic(dx_n)}{2(\xi_n-i)^2}. \tag{69}$$

Similar to Case 1 (ii), we have

$$\begin{aligned}&\mathrm{tr}\left\{\frac{c\left(\xi'\right)+ic(dx_n)}{2(\xi_n-i)^2}\times ih'(0)\left[\frac{c(dx_n)}{|\xi|^4}-4\xi_n\frac{c\left(\xi'\right)+\xi_nc(dx_n)}{|\xi|^6}\right]\right\}=8h'(0)\frac{i-3\xi_n}{(\xi_n-i)^4(\xi_n+i)^3},\\&\mathrm{tr}\left[\frac{c\left(\xi'\right)+ic(dx_n)}{2(\xi_n-i)^2}\times\frac{2\xi_ni\partial_{x_n}c\left(\xi'\right)(x_0)}{|\xi|^4}\right]=\frac{-8ih'(0)\xi_n}{(\xi_n-i)^4(\xi_n+i)^2}.\end{aligned} \tag{70}$$

So we have

$$\begin{aligned}\text{Case 1 (iii)} &= -\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{h'(0)4(i-3\xi_n)}{(\xi_n-i)^4(\xi_n+i)^3}d\xi_n\sigma\left(\xi'\right)dx' \\ &\quad -\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{h'(0)4i\xi_n}{(\xi_n-i)^4(\xi_n+i)^2}d\xi_n\sigma\left(\xi'\right)dx' \\ &= -h'(0)\Omega_3\frac{2\pi i}{3!}\left[\frac{4(i-3\xi_n)}{(\xi_n+i)^3}\right]^{(3)}\Bigg|_{\xi_n=i}dx' \\ &\quad + h'(0)\Omega_3\frac{2\pi i}{3!}\left[\frac{4i\xi_n}{(\xi_n+i)^2}\right]^{(3)}\Bigg|_{\xi_n=i}dx' \\ &= \frac{3}{2}\pi h'(0)\Omega_3 dx'.\end{aligned} \tag{71}$$

Case 2. $r=-2$, $l=-1$, and $k=j=|\alpha|=0$.

By (48), we get

$$\begin{aligned}\text{Case 2} &= -i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{trace}\left[\pi^+_{\xi_n}\sigma_{-2}\left(\widehat{D}^{-1}\right)\right. \\ &\quad \left.\times\partial_{\xi_n}\sigma_{-1}\left(\left(\widehat{D}^*\right)^{-1}\right)\right](x_0)d\xi_n\sigma\left(\xi'\right)dx'.\end{aligned} \tag{72}$$

By Lemma 8, we have

$$\begin{aligned}\sigma_{-2}\left(\widehat{D}^{-1}\right)(x_0) &= \frac{c(\xi)\sigma_0\left(\widehat{D}\right)(x_0)c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}c(dx_n) \\ &\quad \cdot\left[\partial_{x_n}\left[c\left(\xi'\right)\right](x_0)|\xi|^2-c(\xi)h'(0)|\xi|^2_{\partial M}\right],\end{aligned} \tag{73}$$

where

$$\begin{aligned}\sigma_0\left(\widehat{D}\right)(x_0) &= \frac{1}{4}\sum_{s,t,i}\omega_{s,t}(\tilde{e}_i)(x_0)c(\tilde{e}_i)\bar{c}(\tilde{e}_s)\bar{c}(\tilde{e}_t) \\ &\quad -\frac{1}{4}\sum_{s,t,i}\omega_{s,t}(\tilde{e}_i)(x_0)c(\tilde{e}_i)c(\tilde{e}_s)c(\tilde{e}_t)) \\ &\quad +\bar{c}(\theta)+c\left(\theta'\right).\end{aligned} \tag{74}$$

We denote

$$\begin{aligned}b_0^1(x_0) &= \frac{1}{4}\sum_{s,t,i}\omega_{s,t}(\tilde{e}_i)(x_0)c(\tilde{e}_i)\bar{c}(\tilde{e}_s)\bar{c}(\tilde{e}_t), \\ b_0^2(x_0) &= -\frac{1}{4}\sum_{s,t,i}\omega_{s,t}(\tilde{e}_i)(x_0)c(\tilde{e}_i)c(\tilde{e}_s)c(\tilde{e}_t)).\end{aligned} \tag{75}$$

Then,

$$\begin{aligned}&\pi^+_{\xi_n}\sigma_{-2}\left(\widehat{D}^{-1}(x_0)\right)\Big|_{|\xi'|=1} \\ &= \pi^+_{\xi_n}\left[\frac{c(\xi)b_0^1(x_0)c(\xi)}{\left(1+\xi_n^2\right)^2}\right] \\ &\quad +\pi^+_{\xi_n}\left[\frac{c(\xi)\left(\bar{c}(\theta)+c\left(\theta'\right)\right)(x_0)c(\xi)}{\left(1+\xi_n^2\right)^2}\right] \\ &\quad +\pi^+_{\xi_n}\left[\frac{c(\xi)b_0^2(x_0)c(\xi)+c(\xi)c(dx_n)\partial_{x_n}\left[c\left(\xi'\right)\right](x_0)}{\left(1+\xi_n^2\right)^2}\right. \\ &\quad \left.-h'(0)\frac{c(\xi)c(dx_n)c(\xi)}{\left(1+\xi_n^2\right)^3}\right].\end{aligned} \tag{76}$$

By direct calculation, we have

$$\begin{aligned}\pi^+_{\xi_n}\left[\frac{c(\xi)b_0^1(x_0)c(\xi)}{\left(1+\xi_n^2\right)^2}\right] &= \pi^+_{\xi_n}\left[\frac{c\left(\xi'\right)p_0^1(x_0)c\left(\xi'\right)}{\left(1+\xi_n^2\right)^2}\right] \\ &\quad +\pi^+_{\xi_n}\left[\frac{\xi_n c\left(\xi'\right)b_0^1(x_0)c(dx_n)}{\left(1+\xi_n^2\right)^2}\right] \\ &\quad +\pi^+_{\xi_n}\left[\frac{\xi_n c(dx_n)b_0^1(x_0)c\left(\xi'\right)}{\left(1+\xi_n^2\right)^2}\right] \\ &\quad +\pi^+_{\xi_n}\left[\frac{\xi_n^2 c(dx_n)b_0^1(x_0)c(dx_n)}{\left(1+\xi_n^2\right)^2}\right] \\ &= -\frac{c\left(\xi'\right)b_0^1(x_0)c\left(\xi'\right)(2+i\xi_n)}{4(\xi_n-i)^2} \\ &\quad +\frac{ic\left(\xi'\right)b_0^1(x_0)c(dx_n)}{4(\xi_n-i)^2} \\ &\quad +\frac{ic(dx_n)b_0^1(x_0)c\left(\xi'\right)}{4(\xi_n-i)^2} \\ &\quad +\frac{-i\xi_n c(dx_n)b_0^1(x_0)c(dx_n)}{4(\xi_n-i)^2}.\end{aligned} \tag{77}$$

Since

$$c(dx_n)b_0^1(x_0) = -\frac{1}{4}h'(0)\sum_{i=1}^{n-1}c(\tilde{e}_i)\bar{c}(\tilde{e}_i)c(\tilde{e}_n)\bar{c}(\tilde{e}_n), \tag{78}$$

then by the relation of the Clifford action and $\mathrm{tr}AB=\mathrm{tr}BA$, we have the equalities:

$$\begin{aligned}\mathrm{tr}[c(\widetilde{e_i})\bar{c}(\widetilde{e_i})c(\widetilde{e_n})\bar{c}(\widetilde{e_n})] &= 0(i<n), \mathrm{tr}\left[b_0^1 c(dx_n)\right]\\ &= 0, \mathrm{tr}[\bar{c}(\theta)c(dx_n)]\\ &= 0, \mathrm{tr}\left[c\left(\theta'\right)c(dx_n)\right]\\ &= -16g\left(\theta', dx_n\right), \mathrm{tr}\left[\bar{c}\left(\xi'\right)\bar{c}(dx_n)\right]\\ &= 0.\end{aligned} \tag{79}$$

Since

$$\begin{aligned}\partial_{\xi_n}\sigma_{-1}\left(\left(\widehat{D}^*\right)^{-1}\right) &= \partial_{\xi_n} q_{-1}(x_0)|_{|\xi'|=1}\\ &= i\left[\frac{c(dx_n)}{1+\xi_n^2} - \frac{2\xi_n c\left(\xi'\right)+2\xi_n^2 c(dx_n)}{\left(1+\xi_n^2\right)^2}\right].\end{aligned} \tag{80}$$

By (77) and (80), we have

$$\begin{aligned}&\mathrm{tr}\pi_{\xi_n}^+\left[\frac{c(\xi)b_0^1(x_0)c(\xi)}{\left(1+\xi_n^2\right)^2}\right]\times\partial_{\xi_n}\sigma_{-1}\left(\left(\widehat{D}^*\right)^{-1}\right)(x_0)\Bigg|_{|\xi'|=1}\\ &= \frac{1}{2\left(1+\xi_n^2\right)^2}\mathrm{tr}\left[c\left(\xi'\right)b_0^1(x_0)\right]\\ &\quad + \frac{i}{2\left(1+\xi_n^2\right)^2}\mathrm{tr}\left[c(dx_n)b_0^1(x_0)\right]\\ &= \frac{1}{2\left(1+\xi_n^2\right)^2}\mathrm{tr}\left[c\left(\xi'\right)b_0^1(x_0)\right].\end{aligned} \tag{81}$$

We note that $\int_{|\xi'|=1}\{\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2d+1}}\}\sigma(\xi')=0$, $i<n$, so $\mathrm{tr}[c(\xi')b_0^1(x_0)]$ has no contribution for computing Case 2.

By direct calculation, we have

$$\begin{aligned}&\pi_{\xi_n}^+\left[\frac{c(\xi)b_0^2(x_0)c(\xi)+c(\xi)c(dx_n)\partial_{x_n}\left[c\left(\xi'\right)\right](x_0)}{\left(1+\xi_n^2\right)^2}\right]\\ &\quad - h'(0)\pi_{\xi_n}^+\left[\frac{c(\xi)c(dx_n)c(\xi)}{(1+\xi_n)^3}\right] := B_1 - B_2,\end{aligned} \tag{82}$$

where

$$\begin{aligned}B_1 = \frac{-1}{4(\xi_n-i)^2}&\left[(2+i\xi_n)c\left(\xi'\right)b_0^2(x_0)c\left(\xi'\right)\right.\\ &+ i\xi_n c(dx_n)b_0^2(x_0)c(dx_n)\\ &+ (2+i\xi_n)c\left(\xi'\right)c(dx_n)\partial_{x_n}c\left(\xi'\right)\\ &+ ic(dx_n)b_0^2(x_0)c\left(\xi'\right) + ic\left(\xi'\right)b_0^2(x_0)c(dx_n)\\ &\left.- i\partial_{x_n}c\left(\xi'\right)\right],\end{aligned} \tag{83}$$

$$\begin{aligned}B_2 = \frac{h'(0)}{2}&\left[\frac{c(dx_n)}{4i(\xi_n-i)} + \frac{c(dx_n)-ic\left(\xi'\right)}{8(\xi_n-i)^2}\right.\\ &\left.+ \frac{3\xi_n-7i}{8(\xi_n-i)^3}\left[ic\left(\xi'\right)-c(dx_n)\right]\right].\end{aligned} \tag{84}$$

By (80) and (84), we have

$$\begin{aligned}&\mathrm{tr}\left[B_2\times\partial_{\xi_n}\sigma_{-1}\left(\left(\widehat{D}^*\right)^{-1}\right)\right]\Bigg|_{|\xi'|=1}\\ &= \frac{i}{2}h'(0)\frac{-i\xi_n^2-\xi_n+4i}{4(\xi_n-i)^3(\xi_n+i)^2}\mathrm{tr}[\mathrm{id}]\\ &= 8ih'(0)\frac{-i\xi_n^2-\xi_n+4i}{4(\xi_n-i)^3(\xi_n+i)^2}.\end{aligned} \tag{85}$$

By (80) and (83), we have

$$\begin{aligned}&\mathrm{tr}B_1\times\partial_{\xi_n}\sigma_{-1}\left(\left(\widehat{D}^*\right)^{-1}\right)\Bigg|_{|\xi'|=1}\\ &= \frac{-8ic_0}{\left(1+\xi_n^2\right)^2} + 2h'(0)\frac{\xi_n^2-i\xi_n-2}{(\xi_n-i)\left(1+\xi_n^2\right)^2},\end{aligned} \tag{86}$$

where $b_0^2=c_0c(dx_n)$ and $c_0=-3/4h'(0)$.

By (86) and (85), we have

$$\begin{aligned}&-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\left[(B_1-B_2)\times\partial_{\xi_n}\sigma_{-1}\left(\left(\widehat{D}^*\right)^{-1}\right)\right]\\ &\quad\cdot(x_0)d\xi_n\sigma\left(\xi'\right)dx'\\ &= -\Omega_3\int_{\Gamma^+}\frac{8c_0(\xi_n-i)+ih'(0)}{(\xi_n-i)^3(\xi_n+i)^2}d\xi_n dx'\\ &= \frac{9}{2}\pi h'(0)\Omega_3 dx'.\end{aligned} \tag{87}$$

Similar to (81), we have

$$\begin{aligned}
&\operatorname{tr}\pi^+_{\xi_n}\left[\frac{c(\xi)\bar{c}(\theta)(x_0)c(\xi)}{\left(1+\xi_n^2\right)^2}\right]\times\partial_{\xi_n}\sigma_{-1}\left(\left(\widehat{D}^*\right)^{-1}\right)(x_0)\Bigg|_{|\xi'|=1}\\
&\quad=\frac{1}{2\left(1+\xi_n^2\right)^2}\operatorname{tr}\left[c\left(\xi'\right)\bar{c}(\theta)(x_0)\right]\\
&\qquad+\frac{i}{2\left(1+\xi_n^2\right)^2}\operatorname{tr}\left[c(dx_n)\bar{c}(\theta)(x_0)\right]\\
&\quad=\frac{1}{2\left(1+\xi_n^2\right)^2}\operatorname{tr}\left[c\left(\xi'\right)\bar{c}(\theta)(x_0)\right].
\end{aligned}\tag{88}$$

Similar to (83), we have

$$\begin{aligned}
&\operatorname{tr}\left[\pi^+_{\xi_n}\left[\frac{c(\xi)c\left(\theta'\right)(x_0)c(\xi)}{\left(1+\xi_n^2\right)^2}\right]\times\partial_{\xi_n}\sigma_{-1}\left(\left(\widehat{D}^*\right)^{-1}\right)(x_0)\right]\Bigg|_{|\xi'|=1}\\
&\quad=\frac{i}{2\left(1+\xi_n^2\right)^2}\operatorname{tr}\left[c(dx_n)c\left(\theta'\right)(x_0)\right].
\end{aligned}\tag{89}$$

By (88) and (89), we have

$$\begin{aligned}
&-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{trace}\left[\pi^+_{\xi_n}\left[\frac{c(\xi)\left(\bar{c}(\theta)+c\left(\theta'\right)\right)c(\xi)}{\left(1+\xi_n^2\right)^2}\right]\right.\\
&\left.\quad\times\partial_{\xi_n}\sigma_{-1}\left(\left(\widehat{D}^*\right)^{-1}\right)\right](x_0)d\xi_n\sigma\left(\xi'\right)dx'\\
&\quad=\frac{\pi}{4}\operatorname{tr}\left[c(dx_n)c\left(\theta'\right)\right]\Omega_3dx'=-4\pi g\left(\theta',dx_n\right)\Omega_3dx'.
\end{aligned}\tag{90}$$

By (87) and (90), we have

$$\text{Case 2}=\frac{9}{2}\pi h'(0)\Omega_3dx'-4\pi g\left(\theta',dx_n\right)\Omega_3dx'.\tag{91}$$

Case 3. $r=-1$, $l=-2$, and $k=j=|\alpha|=0$.

By (48), we get

$$\begin{aligned}
\text{Case 3}=&-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{trace}\left[\pi^+_{\xi_n}\sigma_{-1}\left(\widehat{D}^{-1}\right)\right.\\
&\left.\times\partial_{\xi_n}\sigma_{-2}\left(\left(\widehat{D}^*\right)^{-1}\right)\right](x_0)d\xi_n\sigma\left(\xi'\right)dx'.
\end{aligned}\tag{92}$$

By (32) and (33) and Lemma 8, we have

$$\pi^+_{\xi_n}\sigma_{-1}\left(\widehat{D}^{-1}\right)\Big|_{|\xi'|=1}=\frac{c\left(\xi'\right)+ic(dx_n)}{2(\xi_n-i)}.\tag{93}$$

Since

$$\begin{aligned}
\sigma_{-2}\left(\left(\widehat{D}^*\right)^{-1}\right)(x_0)=&\frac{c(\xi)\sigma_0\left(\widehat{D}^*\right)(x_0)c(\xi)}{|\xi|^4}\\
&+\frac{c(\xi)}{|\xi|^6}c(dx_n)\left[\partial_{x_n}\left[c\left(\xi'\right)\right](x_0)|\xi|^2\right.\\
&\left.-c(\xi)h'(0)|\xi|^2_{\partial M}\right],
\end{aligned}\tag{94}$$

where

$$\begin{aligned}
\sigma_0\left(\widehat{D}^*\right)(x_0)=&\frac{1}{4}\sum_{s,t,i}\omega_{s,t}(\widetilde{e_i})(x_0)c(\widetilde{e_i})\bar{c}(\widetilde{e_s})\bar{c}(\widetilde{e_t})\\
&-\frac{1}{4}\sum_{s,t,i}\omega_{s,t}(\widetilde{e_i})(x_0)c(\widetilde{e_i})c(\widetilde{e_s})c(\widetilde{e_t}))\\
&+\left(\bar{c}(\theta)-c\left(\theta'\right)\right)(x_0)\\
=&b_0^1(x_0)+b_0^2(x_0)+\left(\bar{c}(\theta)-c\left(\theta'\right)\right)(x_0),
\end{aligned}\tag{95}$$

then

$$\begin{aligned}
&\partial_{\xi_n}\sigma_{-2}\left(\left(\widehat{D}^*\right)^{-1}\right)(x_0)\Big|_{|\xi'|=1}\\
&\quad=\partial_{\xi_n}\left\{\frac{c(\xi)\left[b_0^1(x_0)+b_0^2(x_0)+\left(\bar{c}(\theta)-c\left(\theta'\right)\right)(x_0)\right]c(\xi)}{|\xi|^4}\right.\\
&\qquad\left.+\frac{c(\xi)}{|\xi|^6}c(dx_n)\left[\partial_{x_n}\left[c\left(\xi'\right)\right](x_0)|\xi|^2-c(\xi)h'(0)\right]\right\}\\
&\quad=\partial_{\xi_n}\left\{\frac{c(\xi)b_0^1(x_0)]c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}c(dx_n)\left[\partial_{x_n}\left[c\left(\xi'\right)\right](x_0)|\xi|^2\right.\right.\\
&\qquad\left.\left.-c(\xi)h'(0)\right]\right\}+\partial_{\xi_n}\frac{c(\xi)b_0^2(x_0)c(\xi)}{|\xi|^4}\\
&\qquad+\partial_{\xi_n}\frac{c(\xi)\left(\bar{c}(\theta)-c\left(\theta'\right)\right)(x_0)c(\xi)}{|\xi|^4}.
\end{aligned}\tag{96}$$

By direct calculation, we have

$$\partial_{\xi_n}\frac{c(\xi)b_0^1(x_0)c(\xi)}{|\xi|^4}=\frac{c(dx_n)b_0^1(x_0)c(\xi)}{|\xi|^4}+\frac{c(\xi)b_0^1(x_0)c(dx_n)}{|\xi|^4}-\frac{4\xi_n c(\xi)b_0^1(x_0)c(\xi)}{|\xi|^6}, \tag{97}$$

$$\begin{aligned}\partial_{\xi_n}&\frac{c(\xi)\left(\bar{c}(\theta)-c\left(\theta'\right)\right)(x_0)c(\xi)}{|\xi|^4}\\&=\frac{c(dx_n)\left(\bar{c}(\theta)-c\left(\theta'\right)\right)(x_0)c(\xi)}{|\xi|^4}\\&\quad+\frac{c(\xi)\left(\bar{c}(\theta)-c\left(\theta'\right)\right)(x_0)c(dx_n)}{|\xi|^4}\\&\quad-\frac{4\xi_n c(\xi)\left(\bar{c}(\theta)-c\left(\theta'\right)\right)(x_0)c(\xi)}{|\xi|^4}.\end{aligned} \tag{98}$$

We denote

$$\begin{aligned}q_{-2}^1=&\frac{c(\xi)b_0^2(x_0)c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}c(dx_n)\\&\cdot\left[\partial_{x_n}\left[c\left(\xi'\right)\right](x_0)|\xi|^2-c(\xi)h'(0)\right],\end{aligned} \tag{99}$$

then

$$\begin{aligned}\partial_{\xi_n}\left(q_{-2}^1\right)=&\frac{1}{\left(1+\xi_n^2\right)^3}\left[\left(2\xi_n-2\xi_n^3\right)c(dx_n)b_0^2c(dx_n)\right.\\&+\left(1-3\xi_n^2\right)c(dx_n)b_0^2c\left(\xi'\right)\\&+\left(1-3\xi_n^2\right)c\left(\xi'\right)b_0^2c(dx_n)-4\xi_n c\left(\xi'\right)b_0^2c\left(\xi'\right)\\&+\left(3\xi_n^2-1\right)\partial_{x_n}c\left(\xi'\right)-4\xi_n c\left(\xi'\right)c(dx_n)\partial_{x_n}c\left(\xi'\right)\\&\left.+2h'(0)c\left(\xi'\right)+2h'(0)\xi_n c(dx_n)\right]\\&+6\xi_n h'(0)\frac{c(\xi)c(dx_n)c(\xi)}{\left(1+\xi_n^2\right)^4}.\end{aligned} \tag{100}$$

By (93) and (97), we have

$$\begin{aligned}\operatorname{tr}&\left[\pi_{\xi_n}^+\sigma_{-1}\left(\widehat{D}^{-1}\right)\times\partial_{\xi_n}\frac{c(\xi)b_0^1c(\xi)}{|\xi|^4}\right](x_0)\Bigg|_{|\xi'|=1}\\&=\frac{-1}{(\xi-i)(\xi+i)^3}\operatorname{tr}\left[c\left(\xi'\right)b_0^1(x_0)\right]\\&\quad+\frac{i}{(\xi-i)(\xi+i)^3}\operatorname{tr}\left[c(dx_n)b_0^1(x_0)\right].\end{aligned} \tag{101}$$

By (79), we have

$$\begin{aligned}\operatorname{tr}&\left[\pi_{\xi_n}^+\sigma_{-1}\left(\widehat{D}^{-1}\right)\times\partial_{\xi_n}\frac{c(\xi)b_0^1c(\xi)}{|\xi|^4}\right](x_0)|_{|\xi'|=1}\\&=\frac{-1}{(\xi-i)(\xi+i)^3}\operatorname{tr}\left[c\left(\xi'\right)b_0^1(x_0)\right].\end{aligned} \tag{102}$$

We note that $\int_{|\xi'|=1}\{\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2d+1}}\}\sigma(\xi')=0$, $i<n$, so $\operatorname{tr}[c(\xi')b_0^1(x_0)]$ has no contribution for computing Case 3.

By (93) and (100), we have

$$\begin{aligned}\operatorname{tr}&\left[\pi_{\xi_n}^+\sigma_{-1}\left(\widehat{D}^{-1}\right)\times\partial_{\xi_n}\left(q_{-2}^1\right)\right](x_0)|_{|\xi'|=1}\\&=\frac{12h'(0)\left(i\xi_n^2+\xi_n-2i\right)}{(\xi-i)^3(\xi+i)^3}+\frac{48h'(0)i\xi_n}{(\xi-i)^3(\xi+i)^4},\end{aligned} \tag{103}$$

then

$$\begin{aligned}-i\Omega_3&\int_{\Gamma_+}\left[\frac{12h'(0)\left(i\xi_n^2+\xi_n-2i\right)}{(\xi_n-i)^3(\xi_n+i)^3}+\frac{48h'(0)i\xi_n}{(\xi_n-i)^3(\xi_n+i)^4}\right]d\xi_n dx'\\&=-\frac{9}{2}\pi h'(0)\Omega_3 dx'.\end{aligned} \tag{104}$$

By (93) and (98), we have

$$\begin{aligned}\operatorname{tr}&\left[\pi_{\xi_n}^+\sigma_{-1}\left(\widehat{D}^{-1}\right)\times\partial_{\xi_n}\frac{c(\xi)\left(\bar{c}(\theta)-c\left(\theta'\right)\right)c(\xi)}{|\xi|^4}\right](x_0)\Bigg|_{|\xi'|=1}\\&=\frac{-1}{(\xi-i)(\xi+i)^3}\operatorname{tr}\left[c\left(\xi'\right)\left(\bar{c}(\theta)-c\left(\theta'\right)\right)(x_0)\right]\\&\quad+\frac{i}{(\xi-i)(\xi+i)^3}\operatorname{tr}\left[c(dx_n)\left(\bar{c}(\theta)-c\left(\theta'\right)\right)(x_0)\right].\end{aligned} \tag{105}$$

By $\int_{|\xi'|=1}\{\xi_1\cdots\xi_{2d+1}\}\sigma(\xi')=0$ and (81), we have

$$\begin{aligned}-i&\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{tr}\left[\pi_{\xi_n}^+\sigma_{-1}\left(\widehat{D}^{-1}\right)\right.\\&\left.\times\partial_{\xi_n}\frac{c(\xi)\left(\bar{c}(\theta)-c\left(\theta'\right)\right)c(\xi)}{|\xi|^4}\right](x_0)d\xi_n\sigma\left(\xi'\right)dx'\\&=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{i}{(\xi-i)(\xi+i)^3}\operatorname{tr}\left[c(dx_n)\left(\bar{c}(\theta)\right.\right.\\&\quad\left.\left.-c\left(\theta'\right)\right)\right](x_0)d\xi_n\sigma\left(\xi'\right)dx'\\&=-\frac{\pi}{4}\operatorname{tr}\left[c(dx_n)\left(\bar{c}(\theta)-c\left(\theta'\right)\right)\right]\Omega_3dx'\\&=-4\pi g\left(\theta',dx_n\right)\Omega_3dx'.\end{aligned} \tag{106}$$

So we have

$$\text{Case 3} = -\frac{9}{2}\pi h'(0)\Omega_3 dx' - 4\pi g\left(\theta', dx_n\right)\Omega_3 dx'. \quad (107)$$

Since Φ is the sum of Cases 1–3, so $\Phi = -8\pi g(\theta', dx_n)\Omega_3 dx'$.

Theorem 9. *Let M be 4-dimensional compact-oriented manifolds with the boundary ∂M and the metric g^M as above, $\widehat{D}$ and $\widehat{D}^*$ be modified Novikov operators on $\widehat{M}$, then*

$$\begin{aligned}\widetilde{Wres}\left[\pi^+\widehat{D}^{-1}\circ\pi^+\left(\widehat{D}^*\right)^{-1}\right] &= 32\pi^2\int_M\left[16g\left(\tilde{e}_j, \nabla^{TM}_{\tilde{e}_j}\theta'\right) - \frac{4}{3}s - 16|\theta|^2\right.\\ &\left.+ 32\left|\theta'\right|^2\right]dVol_M - 8\pi\int_{\partial M} g\left(\theta', dx_n\right)\Omega_3 dx'.\end{aligned} \quad (108)$$

where s is the scalar curvature.

On the other hand, we also prove the Kastler-Kalau-Walze-type theorem for 4-dimensional manifolds with a boundary associated to $\widehat{D}^2$. By (41) and (42), we will compute

$$\begin{aligned}\widetilde{\mathrm{Wres}}\left[\pi^+\widehat{D}^{-1}\circ\pi^+\widehat{D}^{-1}\right] = \int_M\int_{|\xi|=1}\mathrm{trace}_{\wedge^*T^*M}\left[\sigma_{-4}\left(\widehat{D}^{-2}\right)\right]\sigma(\xi)dx + \int_{\partial M}\widehat{\Phi},\end{aligned} \quad (109)$$

where

$$\begin{aligned}\widehat{\Phi} = \int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{j,k=0}^{\infty}\sum\frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!} &\times \mathrm{trace}_{\wedge^*T^*M}\left[\partial^j_{x_n}\partial^\alpha_{\xi'}\partial^k_{\xi_n}\sigma^+_r\left(\widehat{D}^{-1}\right)\left(x', 0, \xi', \xi_n\right)\right.\\ &\left.\times\partial^\alpha_{x'}\partial^{j+1}_{\xi_n}\partial^k_{x_n}\sigma_l\left(\widehat{D}^{-1}\right)\left(x', 0, \xi', \xi_n\right)\right]d\xi_n\sigma\left(\xi'\right)dx',\end{aligned} \quad (110)$$

and the sum is taken over $r+l-k-j-|\alpha| = -3$, $r \le -1$, $l \le -1$.

Locally, we can use Theorem 2 (26) to compute the interior of $\widetilde{\mathrm{Wres}}[\pi^+\widehat{D}^{-1}\circ\pi^+\widehat{D}^{-1}]$; we have

$$\begin{aligned}\int_M\int_{|\xi|=1} trace_{\wedge^*T^*M}\left[\sigma_{-4}\left(\widehat{D}^{-2}\right)\right]\sigma(\xi)dx = 32\pi^2\int_M\left[-\frac{4}{3}s - 16|\theta|^2\right]dVol_M.\end{aligned} \quad (111)$$

So we only need to compute $\int_{\partial M}\widehat{\Phi}$. From the remark above, now we can compute $\widehat{\Phi}$ (see formula (110) for the definition of $\widehat{\Phi}$). We use tr as shorthand of trace. Since $n=4$, then $\mathrm{tr}_{\wedge^*T^*M}[\mathrm{id}] = \dim(\wedge^*(4)) = 16$, since the sum is taken over $r+l-k-j-|\alpha| = -3$, $r \le -1$, $l \le -1$, then we have the following five cases:

Case 1. (i) $r=-1$, $l=-1$, $k=j=0$, and $|\alpha|=1$.

By (110), we get

$$\begin{aligned}\text{Case 1 (i)} = -\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{tr}\left[\partial^\alpha_{\xi'}\pi^+_{\xi_n}\sigma_{-1}\left(\widehat{D}^{-1}\right)\right.\\ \left.\times\partial^\alpha_{x'}\partial_{\xi_n}\sigma_{-1}\left(\widehat{D}^{-1}\right)\right](x_0)d\xi_n\sigma\left(\xi'\right)dx'.\end{aligned} \quad (112)$$

Case 1. (ii) $r=-1$, $l=-1$, $k=|\alpha|=0$, and $j=1$.

By (110), we get

$$\begin{aligned}\text{Case 1 (ii)} = -\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\left[\partial_{x_n}\pi^+_{\xi_n}\sigma_{-1}\left(\widehat{D}^{-1}\right)\right.\\ \left.\times\partial^2_{\xi_n}\sigma_{-1}\left(\widehat{D}^{-1}\right)\right](x_0)d\xi_n\sigma\left(\xi'\right)dx'.\end{aligned} \quad (113)$$

Case 1. (iii) $r=-1$, $l=-1$, $j=|\alpha|=0$, and $k=1$.

By (110), we get

$$\begin{aligned}\text{Case 1 (iii)} = -\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\left[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}\left(\widehat{D}^{-1}\right)\right.\\ \left.\times\partial_{\xi_n}\partial_{x_n}\sigma_{-1}\left(\widehat{D}^{-1}\right)\right](x_0)d\xi_n\sigma\left(\xi'\right)dx'.\end{aligned} \quad (114)$$

By Lemma 8, we have $\sigma_{-1}(\widehat{D}^{-1}) = \sigma_{-1}((\widehat{D}^*)^{-1})$. By (55)-(71), so Case 1 vanishes.

Case 2. $r=-2$, $l=-1$, and $k=j=|\alpha|=0$.

By (110), we get

$$\begin{aligned}\text{Case 2} = -i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\left[\pi^+_{\xi_n}\sigma_{-2}\left(\widehat{D}^{-1}\right)\right.\\ \left.\times\partial_{\xi_n}\sigma_{-1}\left(\widehat{D}^{-1}\right)\right](x_0)d\xi_n\sigma\left(\xi'\right)dx'.\end{aligned} \quad (115)$$

By Lemma 8, we have $\sigma_{-1}(\widehat{D}^{-1}) = \sigma_{-1}((\widehat{D}^*)^{-1})$. By (72)–(91), we have

$$\text{Case 2} = \frac{9}{2}\pi h'(0)\Omega_3 dx' - 4\pi g\left(\theta', dx_n\right)\Omega_3 dx', \quad (116)$$

where Ω_4 is the canonical volume of S^4.

Case 3. $r=-1$, $l=-2$, and $k=j=|\alpha|=0$.

By (110), we get

$$\text{Case 3} = -i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\text{trace}\left[\pi_{\xi_n}^{+}\sigma_{-1}\left(\widehat{D}^{-1}\right)\times\partial_{\xi_n}\sigma_{-2}\left(\widehat{D}^{-1}\right)\right](x_0)d\xi_n\sigma\left(\xi'\right)dx'. \tag{117}$$

By (33) and (32) and Lemma 8, we have

$$\pi_{\xi_n}^{+}\sigma_{-1}\left(\widehat{D}^{-1}\right)\Big|_{|\xi'|=1} = \frac{c\left(\xi'\right)+ic(dx_n)}{2(\xi_n-i)}. \tag{118}$$

Since

$$\sigma_{-2}\left(\widehat{D}^{-1}\right)(x_0) = \frac{c(\xi)\sigma_0\left(\widehat{D}\right)(x_0)c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}c(dx_n)\cdot\left[\partial_{x_n}\left[c\left(\xi'\right)\right](x_0)|\xi|^2-c(\xi)h'(0)|\xi|^2_{\partial M}\right], \tag{119}$$

where

$$\begin{aligned}\sigma_0\left(\widehat{D}\right)(x_0) &= \frac{1}{4}\sum_{s,t,i}\omega_{s,t}(\widetilde{e_i})(x_0)c(\widetilde{e_i})\bar{c}(\widetilde{e_s})\bar{c}(\widetilde{e_t}) \\ &\quad-\frac{1}{4}\sum_{s,t,i}\omega_{s,t}(\widetilde{e_i})(x_0)c(\widetilde{e_i})c(\widetilde{e_s})c(\widetilde{e_t})) \\ &\quad+\left(\bar{c}(\theta)+c\left(\theta'\right)\right)(x_0) \\ &= b_0^1(x_0)+b_0^2(x_0)+\left(\bar{c}(\theta)+c\left(\theta'\right)\right)(x_0),\end{aligned} \tag{120}$$

then

$$\begin{aligned}&\partial_{\xi_n}\sigma_{-2}\left(\widehat{D}^{-1}\right)(x_0)\Big|_{|\xi'|=1} \\ &= \partial_{\xi_n}\left\{\frac{c(\xi)\left[b_0^1(x_0)+b_0^2(x_0)+\left(\bar{c}(\theta)+c\left(\theta'\right)\right)(x_0)\right]c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}c(dx_n)\left[\partial_{x_n}\left[c\left(\xi'\right)\right](x_0)|\xi|^2-c(\xi)h'(0)\right]\right\} \\ &= \partial_{\xi_n}\left\{\frac{c(\xi)b_0^1(x_0)]c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}c(dx_n)\cdot\left[\partial_{x_n}\left[c\left(\xi'\right)\right](x_0)|\xi|^2-c(\xi)h'(0)\right]\right\} \\ &\quad+\partial_{\xi_n}\frac{c(\xi)b_0^2(x_0)c(\xi)}{|\xi|^4}+\partial_{\xi_n}\frac{c(\xi)\left(\bar{c}(\theta)+c\left(\theta'\right)\right)(x_0)c(\xi)}{|\xi|^4}.\end{aligned} \tag{121}$$

By direct calculation, we have

$$\partial_{\xi_n}\frac{c(\xi)b_0^1(x_0)c(\xi)}{|\xi|^4} = \frac{c(dx_n)b_0^1(x_0)c(\xi)}{|\xi|^4}+\frac{c(\xi)b_0^1(x_0)c(dx_n)}{|\xi|^4}-\frac{4\xi_nc(\xi)b_0^1(x_0)c(\xi)}{|\xi|^6}, \tag{122}$$

$$\begin{aligned}&\partial_{\xi_n}\frac{c(\xi)\left(\bar{c}(\theta)+c\left(\theta'\right)\right)(x_0)c(\xi)}{|\xi|^4} \\ &= \frac{c(dx_n)\left(\bar{c}(\theta)+c\left(\theta'\right)\right)(x_0)c(\xi)}{|\xi|^4}+\frac{c(\xi)\left(\bar{c}(\theta)+c\left(\theta'\right)\right)(x_0)c(dx_n)}{|\xi|^4} \\ &\quad-\frac{4\xi_nc(\xi)\left(\bar{c}(\theta)+c\left(\theta'\right)\right)(x_0)c(\xi)}{|\xi|^4}.\end{aligned} \tag{123}$$

We denote

$$q_{-2}^1 = \frac{c(\xi)b_0^2(x_0)c(\xi)}{|\xi|^4}+\frac{c(\xi)}{|\xi|^6}c(dx_n)\cdot\left[\partial_{x_n}\left[c\left(\xi'\right)\right](x_0)|\xi|^2-c(\xi)h'(0)\right], \tag{124}$$

then

$$\begin{aligned}\partial_{\xi_n}\left(q_{-2}^1\right) &= \frac{1}{\left(1+\xi_n^2\right)^3}\left[\left(2\xi_n-2\xi_n^3\right)c(dx_n)b_0^2c(dx_n)\right. \\ &\quad+\left(1-3\xi_n^2\right)c(dx_n)b_0^2c\left(\xi'\right) \\ &\quad+\left(1-3\xi_n^2\right)c\left(\xi'\right)b_0^2c(dx_n)-4\xi_nc\left(\xi'\right)b_0^2c\left(\xi'\right) \\ &\quad+\left(3\xi_n^2-1\right)\partial_{x_n}c\left(\xi'\right)-4\xi_nc\left(\xi'\right)c(dx_n)\partial_{x_n}c\left(\xi'\right) \\ &\quad\left.+2h'(0)c\left(\xi'\right)+2h'(0)\xi_nc(dx_n)\right] \\ &\quad+6\xi_nh'(0)\frac{c(\xi)c(dx_n)c(\xi)}{\left(1+\xi_n^2\right)^4}.\end{aligned} \tag{125}$$

By (118) and (122), we have

$$\begin{aligned}&\text{tr}\left[\pi_{\xi_n}^{+}\sigma_{-1}\left(\widehat{D}^{-1}\right)\times\partial_{\xi_n}\frac{c(\xi)b_0^1c(\xi)}{|\xi|^4}\right](x_0)\Big|_{|\xi'|=1} \\ &= \frac{-1}{(\xi-i)(\xi+i)^3}\text{tr}\left[c\left(\xi'\right)b_0^1(x_0)\right]+\frac{i}{(\xi-i)(\xi+i)^3}\text{tr}\left[c(dx_n)b_0^1(x_0)\right].\end{aligned} \tag{126}$$

By (79), we have

$$\operatorname{tr}\left[\pi^+_{\xi_n}\sigma_{-1}\left(\widehat{D}^{-1}\right)\times\partial_{\xi_n}\frac{c(\xi)b_0^1c(\xi)}{|\xi|^4}\right](x_0)|_{|\xi'|=1} = \frac{-1}{(\xi-i)(\xi+i)^3}\operatorname{tr}\left[c\left(\xi'\right)b_0^1(x_0)\right]. \tag{127}$$

We note that $\int_{|\xi'|=1}\{\xi_{i_1}\xi_{i_2}\cdots\xi_{i_{2d+1}}\}\sigma(\xi')=0,\ i<n,$ so $\operatorname{tr}[c(\xi')b_0^1(x_0)]$ has no contribution for computing Case 3. By (118) and (125), we have

$$\operatorname{tr}\left[\pi^+_{\xi_n}\sigma_{-1}\left(\widehat{D}^{-1}\right)\times\partial_{\xi_n}\left(q^1_{-2}\right)\right](x_0)|_{|\xi'|=1} = \frac{12h'(0)\left(i\xi_n^2+\xi_n-2i\right)}{(\xi-i)^3(\xi+i)^3}+\frac{48h'(0)i\xi_n}{(\xi-i)^3(\xi+i)^4}, \tag{128}$$

then

$$-i\Omega_3\int_{\Gamma_+}\left[\frac{12h'(0)\left(i\xi_n^2+\xi_n-2i\right)}{(\xi_n-i)^3(\xi_n+i)^3}+\frac{48h'(0)i\xi_n}{(\xi_n-i)^3(\xi_n+i)^4}\right]d\xi_ndx' = -\frac{9}{2}\pi h'(0)\Omega_3dx'. \tag{129}$$

By (118) and (123), we have

$$\operatorname{tr}\left[\pi^+_{\xi_n}\sigma_{-1}\left(\widehat{D}^{-1}\right)\times\partial_{\xi_n}\frac{c(\xi)\left(\bar{c}(\theta)+c\left(\theta'\right)\right)c(\xi)}{|\xi|^4}\right](x_0)|_{|\xi'|=1} = \frac{-1}{(\xi-i)(\xi+i)^3}\operatorname{tr}\left[c\left(\xi'\right)\left(\bar{c}(\theta)+c\left(\theta'\right)\right)(x_0)\right] + \frac{i}{(\xi-i)(\xi+i)^3}\operatorname{tr}\left[c(dx_n)\left(\bar{c}(\theta)+c\left(\theta'\right)\right)(x_0)\right]. \tag{130}$$

By $\int_{|\xi'|=1}\{\xi_1\cdots\xi_{2d+1}\}\sigma(\xi')=0$ and (79), we have

$$\begin{aligned}&-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{tr}\left[\pi^+_{\xi_n}\sigma_{-1}\left(\widehat{D}^{-1}\right)\times\partial_{\xi_n}\frac{c(\xi)\left(\bar{c}(\theta)+c\left(\theta'\right)\right)c(\xi)}{|\xi|^4}\right](x_0)d\xi_n\sigma\left(\xi'\right)dx' \\ &= -i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{i}{(\xi-i)(\xi+i)^3}\operatorname{tr}\left[c(dx_n)\left(\bar{c}(\theta)+c\left(\theta'\right)\right)\right](x_0)d\xi_n\sigma\left(\xi'\right)dx' \\ &= -\frac{\pi}{4}\operatorname{tr}\left[c(dx_n)\left(\bar{c}(\theta)+c\left(\theta'\right)\right)\right]\Omega_3dx' \\ &= 4\pi g\left(\theta',dx_n\right)\Omega_3dx'.\end{aligned} \tag{131}$$

So we have

$$\text{Case 3} = -\frac{9}{2}\pi h'(0)\Omega_3dx' + 4\pi g\left(\theta',dx_n\right)\Omega_3dx'. \tag{132}$$

Since $\widehat{\Phi}$ is the sum of Cases 1–3, so $\widehat{\Phi}=0$.

Theorem 10. *Let M be a 4-dimensional compact-oriented manifold with the boundary ∂M and the metric g^M as above and $\widehat{D}$ be a modified Novikov operator on $\widehat{M}$, then*

$$\widetilde{Wres}\left[\pi^+\widehat{D}^{-1}\circ\pi^+\widehat{D}^{-1}\right] = 32\pi^2\int_M\left(-\frac{4}{3}s-16|\theta|^2\right)dVol_M. \tag{133}$$

where s is the scalar curvature.

4. A Kastler-Kalau-Walze-Type Theorem for 6-Dimensional Manifolds with Boundary

In this section, we prove the Kastler-Kalau-Walze-type theorems for 6-dimensional manifolds with a boundary. An application of (2.1.4) in [12] shows that

$$\widetilde{\text{Wres}}\left[\pi+\widehat{D}^{-1}\circ\pi+\left(\widehat{D}^*\widehat{D}\widehat{D}^*\right)^{-1}\right] = \int_M\int_{|\xi|=1}\text{trace}_{\wedge^*T^*M}\left[\sigma_{-4}\left(\left(\widehat{D}^*\widehat{D}\right)^{-2}\right)\right]\sigma(\xi)dx+\int_{\partial M}\Psi, \tag{134}$$

where

$$\begin{aligned}\Psi = \int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{j,k=0}^{\infty}\sum\frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!} &\times\text{trace}_{\wedge^*T^*M}\left[\partial^j_{x_n}\partial^\alpha_{\xi'}\partial^k_{\xi_n}\sigma^+_r\left(\widehat{D}^{-1}\right)\left(x',0,\xi',\xi_n\right)\right. \\ &\times\partial^\alpha_{x'}\partial^{j+1}_{\xi_n}\partial^k_{x_n}\sigma_l\left(\left(\widehat{D}^*\widehat{D}\widehat{D}^*\right)^{-1}\right) \\ &\left.\cdot\left(x',0,\xi',\xi_n\right)\right]d\xi_n\sigma\left(\xi'\right)dx',\end{aligned} \tag{135}$$

and the sum is taken over $r+\ell-k-j-|\alpha|-1=-6$, $r\leq-1$, $\ell\leq-3$.

Locally, we can use Theorem 2 (25) to compute the interior term of (134); we have

$$\begin{aligned}&\int_M\int_{|\xi|=1}\text{trace}_{\wedge^*T^*M}\left[\sigma_{-4}\left(\left(\widehat{D}^*\widehat{D}\right)^{-2}\right)\right]\sigma(\xi)dx \\ &= 128\pi^3\int_M\left[64g\left(\widetilde{e_j},\nabla^{TM}_{\widetilde{e_j}}\theta'\right)-\frac{16}{3}s - 64|\theta|^2+256\left|\theta'\right|^2\right]d\text{Vol}_M.\end{aligned} \tag{136}$$

So we only need to compute $\int_{\partial M}\Psi$. Let us now turn to compute the specification of $\widehat{D}^*\widehat{D}\widehat{D}^*$.

$$\begin{aligned}\widehat{D}^*\widehat{D}\widehat{D}^* = &\sum_{i=1}^n c(\widetilde{e_i})\langle \widetilde{e_i}, dx_l\rangle \left(-g^{ij}\partial_l\partial_i\partial_j\right) + \sum_{i=1}^n c(\widetilde{e_i})\langle \widetilde{e_i}, dx_l\rangle \\ &\cdot \left\{-\left(\partial_l g^{ij}\right)\partial_i\partial_j - g^{ij}\left(4(\sigma_i + a_i)\partial_j - 2\Gamma_{ij}^k\partial_k\right)\partial_l\right\} \\ &+ \sum_{i=1}^n c(\widetilde{e_i})\langle \widetilde{e_i}, dx_l\rangle \left\{-2\left(\partial_l g^{ij}\right)(\sigma_i + a_i)\partial_j\right. \\ &+ g^{ij}\left(\partial_l\Gamma_{ij}^k\right)\partial_k - 2g^{ij}[(\partial_l\sigma_i) + (\partial_l a_i)]\partial_j \\ &+ \left(\partial_l g^{ij}\right)\Gamma_{ij}^k\partial_k + \sum_{j,k}\left[\partial_l\left(c\left(\theta'\right)c(\widetilde{e_j}) - c(\widetilde{e_j})c\left(\theta'\right)\right)\right] \\ &\cdot \left\langle \widetilde{e_j}, dx^k\right\rangle\partial_k + \sum_{j,k}\left(c\left(\theta'\right)c(\widetilde{e_j}) - c(\widetilde{e_j})c\left(\theta'\right)\right) \\ &\cdot \left.\left[\partial_l\left\langle \widetilde{e_j}, dx^k\right\rangle\right]\partial_k\right\} + \sum_{i=1}^n c(\widetilde{e_i})\langle \widetilde{e_i}, dx_l\rangle\partial_l \\ &\cdot \left\{-g^{ij}\left[\left(\partial_i\sigma_j\right) + \left(\partial_i a_j\right) + \sigma_i\sigma_j + \sigma_i a_j + a_i\sigma_j + a_i a_j\right.\right. \\ &- \Gamma_{i,j}^k\sigma_k - \Gamma_{i,j}^k a_k + \sum_{i,j} g^{i,j}\left[c\left(\theta'\right)c(\partial_i)\sigma_i\right. \\ &+ c\left(\theta'\right)c(\partial_i)a_i - c(\partial_i)\partial_i\left(c\left(\theta'\right)\right) - c(\partial_i)\sigma_i c\left(\theta'\right) \\ &\left. - c(\partial_i)a_i c\left(\theta'\right)\right] + \frac{1}{4}s - \frac{1}{8}\sum_{ijkl}R_{ijkl}\bar{c}(\widetilde{e_i})\bar{c}(\widetilde{e_j})c(\widetilde{e_k})c(\widetilde{e_l}) \\ &+ \sum_i c(\widetilde{e_i})\bar{c}\left(\nabla_{\widetilde{e_i}}^{\mathrm{TM}}\theta\right) + |\theta|^2 + \left|\theta'\right|^2 - \bar{c}(\theta)c\left(\theta'\right) \\ &\left. + c\left(\theta'\right)\bar{c}(\theta)\right\} + \left[(\sigma_i + a_i) + \left(\bar{c}(\theta) - c\left(\theta'\right)\right)\right] \\ &\cdot \left(-g^{ij}\partial_i\partial_j\right) + \sum_{i=1}^n c(\widetilde{e_i})\langle \widetilde{e_i}, dx_l\rangle\left\{2\sum_{j,k}\left[c\left(\theta'\right)c(\widetilde{e_j})\right.\right. \\ &\left.\left. - c(\widetilde{e_j})c\left(\theta'\right)\right] \times \langle \widetilde{e_i}, dx_k\rangle\right\}\partial_l\partial_k + \left[(\sigma_i + a_i)\right. \\ &\left. + \left(\bar{c}(\theta) - c\left(\theta'\right)\right)\right]\left\{-\sum_{i,j} g^{i,j}\left[2\sigma_i\partial_j + 2a_i\partial_j - \Gamma_{i,j}^k\partial_k\right.\right. \\ &+ \left(\partial_i\sigma_j\right) + \left(\partial_i a_j\right) + \sigma_i\sigma_j + \sigma_i a_j + a_i\sigma_j + a_i a_j - \Gamma_{i,j}^k\sigma_k \\ &\left. - \Gamma_{i,j}^k a_k\right] - \sum_{i,j} g^{i,j}\left[c(\partial_i)c\left(\theta'\right) - c\left(\theta'\right)c(\partial_i)\right]\partial_j \\ &+ \sum_{i,j} g^{i,j}\left[c\left(\theta'\right)c(\partial_i)\sigma_i + c\left(\theta'\right)c(\partial_i)a_i - c(\partial_i)\partial_i\right. \\ &\cdot \left.\left(c\left(\theta'\right)\right) - c(\partial_i)\sigma_i c\left(\theta'\right) - c(\partial_i)a_i c\left(\theta'\right)\right] + \frac{1}{4}s \\ &+ \left|\theta'\right|^2 - \frac{1}{8}\sum_{ijkl}R_{ijkl}\bar{c}(\widetilde{e_i})\bar{c}(\widetilde{e_j})c(\widetilde{e_k})c(\widetilde{e_l}) + \sum_i c(\widetilde{e_i}) \\ &\cdot \left.\bar{c}\left(\nabla_{\widetilde{e_i}}^{\mathrm{TM}}\theta\right) + |\theta|^2 - \bar{c}(\theta)c\left(\theta'\right) + c\left(\theta'\right)\bar{c}(\theta)\right\}.\end{aligned} \tag{137}$$

Then, we obtain

Lemma 11. *The following identities hold:*

$$\begin{aligned}\sigma_2\left(\widehat{D}^*\widehat{D}\widehat{D}^*\right) = &\sum_{i,j,l} c(dx_l)\partial_l\left(g^{i,j}\right)\xi_i\xi_j \\ &+ c(\xi)\left(4\sigma^k + 4a^k - 2\Gamma^k\right)\xi_k \\ &- 2\left[c(\xi)c\left(\theta'\right)c(\xi) + |\xi|^2 c\left(\theta'\right)\right] \\ &+ \frac{1}{4}|\xi|^2\sum_{s,t,l}\omega_{s,t}(\widetilde{e_l})[c(\widetilde{e_l})\bar{c}(\widetilde{e_s})\bar{c}(\widetilde{e_t}) \\ &- c(\widetilde{e_l})c(\widetilde{e_s})c(\widetilde{e_t})] + |\xi|^2\left(\bar{c}(\theta)\right. \\ &\left. - c\left(\theta'\right)\right), \quad \sigma_3\left(\widehat{D}^*\widehat{D}\widehat{D}^*\right) = ic(\xi)|\xi|^2.\end{aligned} \tag{138}$$

Write

$$\begin{aligned}\sigma\left(\widehat{D}^*\widehat{D}\widehat{D}^*\right) &= p_3 + p_2 + p_1 + p_0, \sigma\left(\left(\widehat{D}^*\widehat{D}\widehat{D}^*\right)^{-1}\right) \\ &= \sum_{j=3}^{\infty} q_{-j}.\end{aligned} \tag{139}$$

By the composition formula of pseudodifferential operators, we have

$$\begin{aligned}1 &= \sigma\left(\left(\widehat{D}^*\widehat{D}\widehat{D}^*\right) \circ \left(\widehat{D}^*\widehat{D}\widehat{D}^*\right)^{-1}\right) \\ &= \sum_{\alpha}\frac{1}{\alpha!}\partial_{\xi}^{\alpha}\left[\sigma\left(\widehat{D}^*\widehat{D}\widehat{D}^*\right)\right]D_x^{\alpha}\left[\left(\widehat{D}^*\widehat{D}\widehat{D}^*\right)^{-1}\right] \\ &= (p_3 + p_2 + p_1 + p_0)(q_{-3} + q_{-4} + q_{-5} + \cdots) \\ &\quad + \sum_j\left(\partial_{\xi_j}p_3 + \partial_{\xi_j}p_2 + \partial_{\xi_j}p_1 + \partial_{\xi_j}p_0\right) \\ &\quad \cdot \left(D_{x_j}q_{-3} + D_{x_j}q_{-4} + D_{x_j}q_{-5} + \cdots\right) \\ &= p_3q_{-3} + \left(p_3q_{-4} + p_2q_{-3} + \sum_j\partial_{\xi_j}p_3D_{x_j}q_{-3}\right) + \cdots,\end{aligned} \tag{140}$$

by (140), we have

$$q_{-3} = p_3^{-1}, q_{-4} = -p_3^{-1}\left[p_2p_3^{-1} + \sum_j\partial_{\xi_j}p_3D_{x_j}\left(p_3^{-1}\right)\right]. \tag{141}$$

By Lemma 11, we have some symbols of operators.

Lemma 12. *The following identities hold:*

$$\sigma_{-3}\left(\left(\widehat{D}^*\widehat{D}\widehat{D}^*\right)^{-1}\right)=\frac{ic(\xi)}{|\xi|^4},$$
$$\sigma_{-4}\left(\left(\widehat{D}^*\widehat{D}\widehat{D}^*\right)^{-1}\right)=\frac{c(\xi)\sigma_2\left(\widehat{D}^*\widehat{D}\widehat{D}^*\right)c(\xi)}{|\xi|^8}+\frac{ic(\xi)}{|\xi|^8}\Big(|\xi|^4c(dx_n)\partial_{x_n}c\left(\xi'\right)-2h'(0)c(dx_n)c(\xi)+2\xi_nc(\xi)\partial_{x_n}c\left(\xi'\right)+4\xi_nh'(0)\Big). \quad (142)$$

From the remark above, now we can compute Ψ (see formula (135) for the definition of Ψ). We use tr as shorthand of trace. Since $n=6$, then $\mathrm{tr}_{\wedge^*T^*M}[\mathrm{id}]=64$. Since the sum is taken over $r+\ell-k-j-|\alpha|-1=-6$, $r\le -1$, $\ell\le -3$, then we have the $\int_{\partial_M}\Psi$ as the sum of the following five cases:

Case 1. (i) $r=-1$, $l=-3$, $j=k=0$, and $|\alpha|=1$.

By (135), we get

$$\text{Case 1 (i)}=-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\mathrm{trace}\Big[\partial_{\xi'}^{\alpha}\pi_{\xi_n}^{+}\sigma_{-1}\left(\widehat{D}^{-1}\right)\times\partial_{x'}^{\alpha}\partial_{\xi_n}\sigma_{-3}\left(\left(\widehat{D}^*\widehat{D}\widehat{D}^*\right)^{-1}\right)\Big](x_0)d\xi_n\sigma\left(\xi'\right)dx'. \quad (143)$$

By Lemma 12, for $i<n$, we have

$$\partial_{x_i}\sigma_{-3}\left(\left(\widehat{D}^*\widehat{D}\widehat{D}^*\right)^{-1}\right)(x_0)=\partial_{x_i}\left[\frac{ic(\xi)}{|\xi|^4}\right](x_0)=i\partial_{x_i}[c(\xi)]|\xi|^{-4}(x_0)-2ic(\xi)\partial_{x_i}\left[|\xi|^2\right]|\xi|^{-6}(x_0)=0. \quad (144)$$

so Case 1 (i) vanishes.

Case 1. (ii) $r=-1$, $l=-3$, $|\alpha|=k=0$, and $j=1$.

By (135), we have

$$\text{Case 1 (ii)}=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\Big[\partial_{x_n}\pi_{\xi_n}^{+}\sigma_{-1}\left(\widehat{D}^{-1}\right)\times\partial_{\xi_n}^2\sigma_{-3}\left(\left(\widehat{D}^*\widehat{D}\widehat{D}^*\right)^{-1}\right)\Big](x_0)d\xi_n\sigma\left(\xi'\right)dx'. \quad (145)$$

By Lemma 12 and direct calculations, we have

$$\partial_{\xi_n}^2\sigma_{-3}\left(\left(\widehat{D}^*\widehat{D}\widehat{D}^*\right)^{-1}\right)=i\left[\frac{\left(20\xi_n^2-4\right)c\left(\xi'\right)+12\left(\xi_n^3-\xi_n\right)c(dx_n)}{\left(1+\xi_n^2\right)^4}\right]. \quad (146)$$

Since $n=6$, $\mathrm{tr}[-\mathrm{id}]=-64$. By the relation of the Clifford action and $\mathrm{tr}AB=\mathrm{tr}BA$, then

$$\mathrm{tr}\left[c\left(\xi'\right)c(dx_n)\right]=0,\ \mathrm{tr}\left[c(dx_n)^2\right]=-64,\ \mathrm{tr}\left[c\left(\xi'\right)^2\right](x_0)\big|_{|\xi'|=1}=-64, \quad (147)$$

$$\mathrm{tr}\left[\partial_{x_n}\left[c\left(\xi'\right)\right]c(dx_n)\right]=0,\ \mathrm{tr}\left[\partial_{x_n}c\left(\xi'\right)c\left(\xi'\right)\right](x_0)\Big|_{|\xi'|=1}=-32h'(0). \quad (148)$$

By (62), (146), and (147), we get

$$\mathrm{trace}\left[\partial_{x_n}\pi_{\xi_n}^{+}\sigma_{-1}\left(\widehat{D}^{-1}\right)\times\partial_{\xi_n}^2\sigma_{-3}\left(\left(\widehat{D}^*\widehat{D}\widehat{D}^*\right)^{-1}\right)\right](x_0)=64h'(0)\frac{-1-3\xi_ni+5\xi_n^2+3i\xi_n^3}{(\xi_n-i)^6(\xi_n+i)^4}. \quad (149)$$

Then, we obtain

$$\begin{aligned}\text{Case 1 (ii)}&=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}h'(0)\dim F\cdot\frac{-8-24\xi_ni+40\xi_n^2+24i\xi_n^3}{(\xi_n-i)^6(\xi_n+i)^4}d\xi_n\sigma\left(\xi'\right)dx'\\&=8h'(0)\Omega_4\int_{\Gamma^+}\frac{4+12\xi_ni-20\xi_n^2-122i\xi_n^3}{(\xi_n-i)^6(\xi_n+i)^4}d\xi_ndx'\\&=h'(0)\Omega_4\frac{\pi i}{5!}\left[\frac{8+24\xi_ni-40\xi_n^2-24i\xi_n^3}{(\xi_n+i)^4}\right]^{(5)}\Bigg|_{\xi_n=i}dx'\\&=-\frac{15}{2}\pi h'(0)\Omega_4dx',\end{aligned} \quad (150)$$

where Ω_4 is the canonical volume of S^4.

Case 1. (iii) $r=-1$, $l=-3$, $|\alpha|=j=0$, and $k=1$.

By (135), we have

$$\text{Case 1 (iii)} = -\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{trace}\left[\partial_{\xi_n}\pi_{\xi_n}^{+}\sigma_{-1}\left(\widehat{D}^{-1}\right)\right.$$
$$\left.\times\,\partial_{\xi_n}\partial_{x_n}\sigma_{-3}\left(\left(\widehat{D}^{*}\widehat{D}\widehat{D}^{*}\right)^{-1}\right)\right] \tag{151}$$
$$\cdot\,(x_0)d\xi_n\sigma\left(\xi'\right)dx'.$$

By Lemma 12 and direct calculations, we have

$$\partial_{\xi_n}\partial_{x_n}\sigma_{-3}\left(\left(\widehat{D}^{*}\widehat{D}\widehat{D}^{*}\right)^{-1}\right) = -\frac{4i\xi_n\partial_{x_n}c\left(\xi'\right)(x_0)}{\left(1+\xi_n^2\right)^3} + i\frac{12h'(0)\xi_n c\left(\xi'\right)}{\left(1+\xi_n^2\right)^4} - i\frac{\left(2-10\xi_n^2\right)h'(0)c(dx_n)}{\left(1+\xi_n^2\right)^4}. \tag{152}$$

Combining (69) and (152), we have

$$\operatorname{trace}\left[\partial_{\xi_n}\pi_{\xi_n}^{+}\sigma_{-1}\left(\widehat{D}^{-1}\right)\times\partial_{\xi_n}\partial_{x_n}\sigma_{-3}\left(\left(\widehat{D}^{*}\widehat{D}\widehat{D}^{*}\right)^{-1}\right)\right](x_0)|_{|\xi'|=1} = 8h'(0)\frac{8i-32\xi_n-8i\xi_n^2}{(\xi_n-i)^5(\xi+i)^4}. \tag{153}$$

Then,

$$\begin{aligned}\text{Case 1 (iii)} &= -\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}8h'(0)\cdot\frac{8i-32\xi_n-8i\xi_n^2}{(\xi_n-i)^5(\xi+i)^4}d\xi_n\sigma\left(\xi'\right)dx' \\ &= -\frac{1}{2}h'(0)8\Omega_4\int_{\Gamma^+}\frac{8i-32\xi_n-8i\xi_n^2}{(\xi_n-i)^5(\xi+i)^4}d\xi_n dx' \\ &= -8h'(0)\Omega_4\frac{\pi i}{4!}\left[\frac{8i-32\xi_n-8i\xi_n^2}{(\xi+i)^4}\right]^{(4)}\Bigg|_{\xi_n=i}dx' \\ &= \frac{25}{2}\pi h'(0)\Omega_4 dx'.\end{aligned} \tag{154}$$

Case 2. $r=-1$, $l=-4$, and $|\alpha|=j=k=0$.

By (135), we have

$$\begin{aligned}\text{Case 2} &= -i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{trace}\left[\pi_{\xi_n}^{+}\sigma_{-1}\left(\widehat{D}^{-1}\right)\times\partial_{\xi_n}\sigma_{-4}\left(\left(\widehat{D}^{*}\widehat{D}\widehat{D}^{*}\right)^{-1}\right)\right](x_0)d\xi_n\sigma\left(\xi'\right)dx' \\ &= i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{trace}\left[\partial_{\xi_n}\pi_{\xi_n}^{+}\sigma_{-1}\left(\widehat{D}^{-1}\right)\times\sigma_{-4}\left(\left(\widehat{D}^{*}\widehat{D}\widehat{D}^{*}\right)^{-1}\right)\right](x_0)d\xi_n\sigma\left(\xi'\right)dx'.\end{aligned} \tag{155}$$

In the normal coordinate, $g^{ij}(x_0)=\delta_i^j$ and $\partial_{x_j}(g^{\alpha\beta})(x_0)=0$, if $j<n$; $\partial_{x_j}(g^{\alpha\beta})(x_0)=h'(0)\delta_\beta^\alpha$, if $j=n$. So by Lemma A.2 in [10], we have $\Gamma^n(x_0)=(5/2)h'(0)$ and $\Gamma^k(x_0)=0$ for $k<n$. By the definition of δ^k and Lemma 2.3 in [10], we have $\delta^n(x_0)=0$ and $\delta^k=(1/4)h'(0)c(\widetilde{e_k})c(\widetilde{e_n})$ for $k<n$. By Lemma 12, we obtain

$$\begin{aligned}\sigma_{-4}&\left(\left(\widehat{D}^{*}\widehat{D}\widehat{D}^{*}\right)^{-1}\right)(x_0)|_{|\xi'|=1} \\ &= \frac{c(\xi)\sigma_2\left(\left(\widehat{D}^{*}\widehat{D}\widehat{D}^{*}\right)^{-1}\right)(x_0)|_{|\xi'|=1}c(\xi)}{|\xi|^8} - \frac{c(\xi)}{|\xi|^4}\sum_j\partial_{\xi_j}\left(c(\xi)|\xi|^2\right)D_{x_j}\left(\frac{ic(\xi)}{|\xi|^4}\right) \\ &= \frac{1}{|\xi|^8}c(\xi)\left(\frac{1}{2}h'(0)c(\xi)\sum_{k<n}\xi_k c(\widetilde{e_k})c(\widetilde{e_n}) - \frac{1}{2}h'(0)c(\xi)\sum_{k<n}\xi_k\bar{c}(\widetilde{e_k})\bar{c}(\widetilde{e_n}) - \frac{5}{2}h'(0)\xi_n c(\xi)\right. \\ &\quad - \frac{1}{4}h'(0)|\xi|^2c(dx_n) - 2\left[c(\xi)c\left(\theta'\right)c(\xi)+|\xi|^2c\left(\theta'\right)\right] \\ &\quad\left.+|\xi|^2\left(\widehat{c}(\theta)-c\left(\theta'\right)\right)\right)c(\xi) + \frac{ic(\xi)}{|\xi|^8}\left(|\xi|^4c(dx_n)\partial_{x_n}c\left(\xi'\right)\right. \\ &\quad\left.-2h'(0)c(dx_n)c(\xi)+2\xi_n c(\xi)\partial_{x_n}c\left(\xi'\right)+4\xi_n h'(0)\right).\end{aligned} \tag{156}$$

By (69) and (156), we have

$$\begin{aligned}\operatorname{tr}&\left[\partial_{\xi_n}\pi_{\xi_n}^{+}\sigma_{-1}\left(\widehat{D}^{-1}\right)\times\sigma_{-4}\left(\widehat{D}^{*}\widehat{D}\widehat{D}^{*}\right)^{-1}\right](x_0)|_{|\xi'|=1} \\ &= \frac{1}{2(\xi_n-i)^2\left(1+\xi_n^2\right)^4}\left(\frac{3}{4}i+2+(3+4i)\xi_n+(-6+2i)\xi_n^2+3\xi_n^3+\frac{9i}{4}\xi_n^4\right)h'(0)\operatorname{tr}[id] \\ &\quad+\frac{1}{2(\xi_n-i)^2\left(1+\xi_n^2\right)^4}\left(-1-3i\xi_n-2\xi_n^2-4i\xi_n^3\right.\end{aligned}$$

$$- \xi_n^4 - i\xi_n^5\Big)\mathrm{tr}\left[c\left(\xi'\right)\partial_{x_n}c\left(\xi'\right)\right] - \frac{1}{2(\xi_n - i)^2\left(1+\xi_n^2\right)^4} \cdot \left(\frac{1}{2}i + \frac{1}{2}\xi_n + \frac{1}{2}\xi_n^2 + \frac{1}{2}\xi_n^3\right)\mathrm{tr}\left[c\left(\xi'\right)\bar{c}\left(\xi'\right)c(dx_n)\bar{c}(dx_n)\right] + \frac{-\xi_n i + 3}{2(\xi_n - i)^4(i+\xi_n)^3}\mathrm{tr}\left[c\left(\theta'\right)c(dx_n)\right] - \frac{3\xi_n + i}{2(\xi_n - i)^4(i+\xi_n)^3}\mathrm{tr}\left[c\left(\theta'\right)c\left(\xi'\right)\right]. \tag{157}$$

By direct calculation and the relation of the Clifford action and $\mathrm{tr}AB = \mathrm{tr}BA$, we then have equalities:

$$\mathrm{tr}\left[c\left(\theta'\right)(x_0)c(dx_n)\right] = -64g\left(\theta', dx_n\right), \mathrm{tr}\left[c\left(\theta'\right)(x_0)c\left(\xi'\right)\right] = -64g\left(\theta', \xi'\right), \tag{158}$$

$$\mathrm{tr}[c(\widetilde{e_i})\bar{c}(\widetilde{e_i})c(\widetilde{e_n})\bar{c}(\widetilde{e_n})] = 0 (i < n). \tag{159}$$

Then,

$$\mathrm{tr}\left[c\left(\xi'\right)\bar{c}\left(\xi'\right)c(dx_n)\bar{c}(dx_n)\right] = \sum_{i<n, j<n}\mathrm{tr}\left[\xi_i\xi_j c(\widetilde{e_i})\bar{c}(\widetilde{e_j})c(dx_n)\bar{c}(dx_n)\right] = 0. \tag{160}$$

So, we have

$$\begin{aligned}\text{Case 2} &= ih'(0)\int_{|\xi'|=1}\int_{-\infty}^{+\infty}64 \times \frac{(3/4)i + 2 + (3+4i)\xi_n + (-6+2i)\xi_n^2 + 3\xi_n^3 + (9i/4)\xi_n^4}{2(\xi_n - i)^5(\xi_n + i)^4} \\ &\quad \cdot d\xi_n\sigma\left(\xi'\right)dx' + ih'(0)\int_{|\xi'|=1}\int_{-\infty}^{+\infty}32 \times \frac{1 + 3i\xi_n + 2\xi_n^2 + 4i\xi_n^3 + \xi_n^4 + i\xi_n^5}{2(\xi_n - i)^2\left(1+\xi_n^2\right)^4}d\xi_n\sigma\left(\xi'\right)dx' \\ &\quad + i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{\xi_n - i - 2\xi_n i + 1}{2(\xi_n - i)^4(i+\xi_n)^3}\mathrm{tr}\left[c\left(\theta'\right)c(dx_n)\right] \\ &\quad \cdot d\xi_n\sigma\left(\xi'\right)dx' - i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{3\xi_n + i}{2(\xi_n - i)^4(i+\xi_n)^3} \cdot \mathrm{tr}\left[c\left(\theta'\right)c\left(\xi'\right)\right]d\xi_n\sigma\left(\xi'\right)dx' \\ &= \left(-\frac{19}{4}i - 15\right)\pi h'(0)\Omega_4 dx' + \left(-\frac{3}{8}i - \frac{75}{8}\right) \cdot \pi h'(0)\Omega_4 dx' + 120i\pi g\left(dx_n, \theta'\right)\Omega_4 dx' \\ &= \left(-\frac{41}{8}i - \frac{195}{8}\right)\pi h'(0)\Omega_4 dx' + 120i\pi g\left(dx_n, \theta'\right)\Omega_4 dx'.\end{aligned} \tag{161}$$

Case 3. $r = -2$, $l = -3$, and $|\alpha| = j = k = 0$.

By (135), we have

$$\text{Case 3} = -i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\mathrm{trace}\left[\pi_{\xi_n}^{+}\sigma_{-2}\left(\widehat{D}^{-1}\right) \times \partial_{\xi_n}\sigma_{-3}\left(\left(\widehat{D}^*\widehat{D}\widehat{D}^*\right)^{-1}\right)\right](x_0)d\xi_n\sigma\left(\xi'\right)dx'. \tag{162}$$

By Lemmas 11 and 12, we have

$$\sigma_{-2}\left(\widehat{D}^{-1}\right)(x_0) = \frac{c(\xi)\sigma_0\left(\widehat{D}\right)c(\xi)}{|\xi|^4}(x_0) + \frac{c(\xi)}{|\xi|^6}\sum_j c(dx_j) \cdot \left[\partial_{x_j}(c(\xi))|\xi|^2 - c(\xi)\partial_{x_j}\left(|\xi|^2\right)\right](x_0), \tag{163}$$

where

$$\sigma_0\left(\widehat{D}\right) = \frac{1}{4}\sum_{i,s,t}\omega_{s,t}(\widetilde{e_i})c(\widetilde{e_i})\bar{c}(\widetilde{e_s})\bar{c}(\widetilde{e_t}) - \frac{1}{4}\sum_{i,s,t}\omega_{s,t}(\widetilde{e_i})c(\widetilde{e_i})c(\widetilde{e_s})c(\widetilde{e_t}) + \bar{c}(\theta) + c\left(\theta'\right). \tag{164}$$

On the other hand,

$$\partial_{\xi_n}\sigma_{-3}\left(\left(\widehat{D}^*\widehat{D}\widehat{D}^*\right)^{-1}\right) = \frac{-4i\xi_n c\left(\xi'\right)}{\left(1+\xi_n^2\right)^3} + \frac{i\left(1 - 3\xi_n^2\right)c(dx_n)}{\left(1+\xi_n^2\right)^3}. \tag{165}$$

By (163), (28), and (32), we have

$$\pi_{\xi_n}^{+}\left(\sigma_{-2}\left(\widehat{D}^{-1}\right)\right)(x_0)\Big|_{|\xi'|=1} = \pi_{\xi_n}^{+}\left[\frac{c(\xi)\sigma_0\left(\widehat{D}\right)(x_0)c(\xi) + c(\xi)c(dx_n)\partial_{x_n}\left[c\left(\xi'\right)\right](x_0)}{\left(1+\xi_n^2\right)^2}\right] - h'(0)\pi_{\xi_n}^{+}\left[\frac{c(\xi)c(dx_n)c(\xi)}{\left(1+\xi_n^2\right)^3}\right]. \tag{166}$$

We denote

$$\sigma_0\left(\widehat{D}\right)(x_0)|_{\xi_n=i}=b_0(x_0)=b_0^1(x_0)+b_0^2(x_0)+\bar{c}(\theta)+c\left(\theta'\right). \tag{167}$$

Then, we obtain

$$\begin{aligned}&\pi^+_{\xi_n}\left(\sigma_{-2}\left(\widehat{D}^{-1}\right)\right)(x_0)|_{|\xi'|=1}\\&=\pi^+_{\xi_n}\left[\frac{c(\xi)b_0^2(x_0)c(\xi)+c(\xi)c(dx_n)\partial_{x_n}\left[c\left(\xi'\right)\right](x_0)}{\left(1+\xi_n^2\right)^2}\right.\\&\quad\left.-h'(0)\frac{c(\xi)c(dx_n)c(\xi)}{\left(1+\xi_n^2\right)^3}\right]+\pi^+_{\xi_n}\left[\frac{c(\xi)\left[b_0^1(x_0)\right]c(\xi)(x_0)}{\left(1+\xi_n^2\right)^2}\right]\\&\quad+\pi^+_{\xi_n}\left[\frac{c(\xi)\left[\bar{c}(\theta)+c\left(\theta'\right)\right]c(\xi)(x_0)}{\left(1+\xi_n^2\right)^2}\right].\end{aligned} \tag{168}$$

Furthermore,

$$\begin{aligned}&\pi^+_{\xi_n}\left[\frac{c(\xi)\left[\bar{c}(\theta)+c\left(\theta'\right)\right](x_0)c(\xi)}{\left(1+\xi_n^2\right)^2}\right]\\&=\pi^+_{\xi_n}\left[\frac{c\left(\xi'\right)\left[\bar{c}(\theta)+c\left(\theta'\right)\right](x_0)c\left(\xi'\right)}{\left(1+\xi_n^2\right)^2}\right]\\&\quad+\pi^+_{\xi_n}\left[\frac{\xi_nc\left(\xi'\right)\left[\bar{c}(\theta)+c\left(\theta'\right)\right](x_0)c(dx_n)}{\left(1+\xi_n^2\right)^2}\right]\\&\quad+\pi^+_{\xi_n}\left[\frac{\xi_nc(dx_n)\left[\bar{c}(\theta)+c\left(\theta'\right)\right](x_0)c\left(\xi'\right)}{\left(1+\xi_n^2\right)^2}\right]\\&\quad+\pi^+_{\xi_n}\left[\frac{\xi_n^2c(dx_n)\left[\bar{c}(\theta)+c\left(\theta'\right)\right](x_0)c(dx_n)}{\left(1+\xi_n^2\right)^2}\right]\\&=-\frac{c\left(\xi'\right)\left[\bar{c}(\theta)+c\left(\theta'\right)\right](x_0)c\left(\xi'\right)(2+i\xi_n)}{4(\xi_n-i)^2}\\&\quad+\frac{ic\left(\xi'\right)\left[\bar{c}(\theta)+c\left(\theta'\right)\right](x_0)c(dx_n)}{4(\xi_n-i)^2}\\&\quad+\frac{ic(dx_n)\left[\bar{c}(\theta)+c\left(\theta'\right)\right](x_0)c\left(\xi'\right)}{4(\xi_n-i)^2}\\&\quad+\frac{-i\xi_nc(dx_n)\left[\bar{c}(\theta)+c\left(\theta'\right)\right](x_0)c(dx_n)}{4(\xi_n-i)^2},\end{aligned}$$

$$\begin{aligned}&\pi^+_{\xi_n}\left[\frac{c(\xi)b_0^1(x_0)c(\xi)}{\left(1+\xi_n^2\right)^2}\right]=\pi^+_{\xi_n}\left[\frac{c\left(\xi'\right)p_0^1(x_0)c\left(\xi'\right)}{\left(1+\xi_n^2\right)^2}\right]\\&\quad+\pi^+_{\xi_n}\left[\frac{\xi_nc\left(\xi'\right)b_0^1(x_0)c(dx_n)}{\left(1+\xi_n^2\right)^2}\right]\\&\quad+\pi^+_{\xi_n}\left[\frac{\xi_nc(dx_n)b_0^1(x_0)c\left(\xi'\right)}{\left(1+\xi_n^2\right)^2}\right]\\&\quad+\pi^+_{\xi_n}\left[\frac{\xi_n^2c(dx_n)b_0^1(x_0)c(dx_n)}{\left(1+\xi_n^2\right)^2}\right]\\&=-\frac{c\left(\xi'\right)b_0^1(x_0)c\left(\xi'\right)(2+i\xi_n)}{4(\xi_n-i)^2}\\&\quad+\frac{ic\left(\xi'\right)b_0^1(x_0)c(dx_n)}{4(\xi_n-i)^2}\\&\quad+\frac{ic(dx_n)b_0^1(x_0)c\left(\xi'\right)}{4(\xi_n-i)^2}\\&\quad+\frac{-i\xi_nc(dx_n)b_0^1(x_0)c(dx_n)}{4(\xi_n-i)^2}.\end{aligned} \tag{169}$$

By the relation of the Clifford action and $\mathrm{tr}AB=\mathrm{tr}BA$, we then have equalities:

$$\mathrm{tr}\left[b_0^1c(dx_n)\right]=0,\ \mathrm{tr}\left[\bar{c}\left(\xi'\right)\bar{c}(dx_n)\right]=0,\ \mathrm{tr}\left[\bar{c}(\theta)c\left(\xi'\right)\right]=0. \tag{170}$$

Then, we have

$$\begin{aligned}&\mathrm{tr}\left[\pi^+_{\xi_n}\left(\frac{c(\xi)b_0^1(x_0)c(\xi)}{\left(1+\xi_n^2\right)^2}\right)\times\partial_{\xi_n}\sigma_{-3}\left(\left(\widehat{D}^*\widehat{D}\widehat{D}^*\right)^{-1}\right)(x_0)\right]\Bigg|_{|\xi'|=1}\\&=\frac{2-8i\xi_n-6\xi_n^2}{4(\xi_n-i)^2\left(1+\xi_n^2\right)^3}\mathrm{tr}\left[b_0^1(x_0)c\left(\xi'\right)\right],\end{aligned} \tag{171}$$

By direct calculation, we have

$$\begin{aligned}&\pi^+_{\xi_n}\left[\frac{c(\xi)b_0^2(x_0)c(\xi)+c(\xi)c(dx_n)\partial_{x_n}\left(c\left(\xi'\right)\right)(x_0)}{\left(1+\xi_n^2\right)^2}\right]\\&-h'(0)\pi^+_{\xi_n}\left[\frac{c(\xi)c(dx_n)c(\xi)}{(1+\xi_n)^3}\right]:=B_1-B_2,\end{aligned} \tag{172}$$

where

$$B_1 = \frac{-1}{4(\xi_n - i)^2}\Big[(2+i\xi_n)c(\xi')b_0^2c(\xi') + i\xi_n c(dx_n)b_0^2c(dx_n) + (2+i\xi_n)c(\xi')c(dx_n)\partial_{x_n}c(\xi') + ic(dx_n)b_0^2c(\xi') + ic(\xi')b_0^2c(dx_n) - i\partial_{x_n}c(\xi')\Big] = \frac{1}{4(\xi_n - i)^2}\Big[\frac{5}{2}h'(0)c(dx_n) - \frac{5i}{2}h'(0)c(\xi') - (2+i\xi_n)c(\xi')c(dx_n)\partial_{\xi_n}c(\xi') + i\partial_{\xi_n}c(\xi')\Big], \tag{173}$$

$$B_2 = \frac{h'(0)}{2}\left[\frac{c(dx_n)}{4i(\xi_n - i)} + \frac{c(dx_n) - ic(\xi')}{8(\xi_n - i)^2} + \frac{3\xi_n - 7i}{8(\xi_n - i)^3}\left(ic(\xi') - c(dx_n)\right)\right]. \tag{174}$$

By (165) and (174), we have

$$\begin{aligned}\operatorname{tr}\left[B_2 \times \partial_{\xi_n}\sigma_{-3}\left(\left(\widehat{D}^*\widehat{D}\widehat{D}^*\right)^{-1}\right)(x_0)\right]\Big|_{|\xi'|=1} &= \operatorname{tr}\left\{\frac{h'(0)}{2}\left[\frac{c(dx_n)}{4i(\xi_n - i)} + \frac{c(dx_n) - ic(\xi')}{8(\xi_n - i)^2} + \frac{3\xi_n - 7i}{8(\xi_n - i)^3}\left[ic(\xi') - c(dx_n)\right]\right] \times \frac{-4i\xi_n c(\xi') + (i - 3i\xi_n^2)c(dx_n)}{(1+\xi_n^2)^3}\right\} \\ &= 8h'(0)\frac{4i - 11\xi_n - 6i\xi_n^2 + 3\xi_n^3}{(\xi_n - i)^5(\xi_n + i)^3}.\end{aligned} \tag{175}$$

Similarly, we have

$$\begin{aligned}\operatorname{tr}\left[B_1 \times \partial_{\xi_n}\sigma_{-3}\left(\left(\widehat{D}^*\widehat{D}\widehat{D}^*\right)^{-1}\right)(x_0)\right]\Big|_{|\xi'|=1} &= \operatorname{tr}\left\{\frac{1}{4(\xi_n - i)^2}\left[\frac{5}{2}h'(0)c(dx_n) - \frac{5i}{2}h'(0)c(\xi') - (2+i\xi_n)c(\xi')c(dx_n)\partial_{\xi_n}c(\xi') + i\partial_{\xi_n}c(\xi')\right] \times \frac{-4i\xi_n c(\xi') + (i - 3i\xi_n^2)c(dx_n)}{(1+\xi_n^2)^3}\right\} \\ &= 8h'(0)\frac{3 + 12i\xi_n + 3\xi_n^2}{(\xi_n - i)^4(\xi_n + i)^3},\end{aligned}$$

$$\begin{aligned}&\operatorname{tr}\left[\pi^+_{\xi_n}\left(\frac{c(\xi)\left[\bar{c}(\theta) + c(\theta')\right](x_0)c(\xi)}{(1+\xi_n^2)^2}\right) \times \partial_{\xi_n}\sigma_{-3}\left(\left(\widehat{D}^*\widehat{D}\widehat{D}^*\right)^{-1}\right)(x_0)\right]\Big|_{|\xi'|=1} \\ &= \frac{2 - 8i\xi_n - 6\xi_n^2}{4(\xi_n - i)^2(1+\xi_n^2)^3}\operatorname{tr}\left[\left[\bar{c}(\theta) + c(\theta')\right](x_0)c(\xi')\right] \\ &= \frac{2 - 8i\xi_n - 6\xi_n^2}{4(\xi_n - i)^2(1+\xi_n^2)^3}\operatorname{tr}\left[c(\theta')(x_0)c(\xi')\right] \\ &= \frac{2 - 8i\xi_n - 6\xi_n^2}{4(\xi_n - i)^2(1+\xi_n^2)^3}\left[-g(\theta', \xi')\right]\operatorname{tr}[\mathrm{id}].\end{aligned} \tag{176}$$

By $\int_{|\xi'|=1}\xi_1\cdots\xi_{2q+1}\sigma(\xi') = 0$, we have

$$\begin{aligned}\text{Case 3} &= -ih'(0)\int_{|\xi'|=1}\int_{-\infty}^{+\infty} 8 \times \frac{-7i + 26\xi_n + 15i\xi_n^2}{(\xi_n - i)^5(\xi_n + i)^3}d\xi_n\sigma(\xi')dx' - i\int_{|\xi'|=1}\int_{-\infty}^{+\infty} \cdot \left[\frac{2 - 8i\xi_n - 6\xi_n^2}{4(\xi_n - i)^2(1+\xi_n^2)^3}\left[-g(\theta', \xi')\right]\operatorname{tr}[\mathrm{id}]\right]\Bigg|_{|\xi'|=1} \cdot d\xi_n\sigma(\xi')dx' \\ &= -8ih'(0) \times \frac{2\pi i}{4!}\left|\left[\frac{-7i + 26\xi_n + 15i\xi_n^2}{(\xi_n + i)^3}\right]^{(5)}\right|_{\xi_n = i}\Omega_4 dx' \\ &= \frac{55}{2}\pi h'(0)\Omega_4 dx'.\end{aligned} \tag{177}$$

Now Ψ is the sum of Cases 1–3, then

$$\Psi = \left(\frac{65}{8} - \frac{41}{8}i\right)\pi h'(0)\Omega_4 dx' + 120i\pi g(dx_n, \theta')\Omega_4 dx'. \tag{178}$$

Theorem 13. *Let M be a 6-dimensional compact-oriented manifold with the boundary ∂M and the metric g^M as above and $\widehat{D}$ and $\widehat{D}^*$ be modified Novikov operators on $\widehat{M}$, then*

$$\widetilde{Wres}\left[\pi^+\widehat{D}^{-1}\circ\pi^+\left(\widehat{D}^*\widehat{D}\widehat{D}^*\right)^{-1}\right] = 128\pi^3\int_M\left[64g\left(\widetilde{e_j},\nabla^{TM}_{\widetilde{e_j}}\theta'\right)-\frac{16}{3}s-64|\theta|^2 + 256\left|\theta'\right|^2\right]dVol_M + \int_{\partial M}\left[\left(\frac{65}{8}-\frac{41}{8}i\right)\pi h'(0) + 120i\pi g\left(dx_n,\theta'\right)\right]\Omega_4 dx'. \quad (179)$$

where s is the scalar curvature.

On the other hand, we prove the Kastler-Kalau-Walze-type theorem for a 6-dimensional manifold with a boundary associated with $\widehat{D}^3$. An application of (2.1.4) in [12] shows that

$$\widetilde{\mathrm{Wres}}\left[\pi^+\widehat{D}^{-1}\circ\pi^+\widehat{D}^{-3}\right] = \int_M\int_{|\xi|=1}\mathrm{trace}_{\wedge^*T^*M}\left[\sigma_{-4}\left(\widehat{D}^{-4}\right)\right]\sigma(\xi)dx + \int_{\partial M}\widehat{\Psi}, \quad (180)$$

where $\widetilde{\mathrm{Wres}}$ denotes a noncommutative residue on manifolds with a boundary,

$$\widehat{\Psi} = \int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{j,k=0}^{\infty}\sum\frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!} \times \mathrm{trace}_{\wedge^*T^*M}\left[\partial^j_{x_n}\partial^\alpha_{\xi'}\partial^k_{\xi_n}\sigma^+_r\left(\widehat{D}^{-1}\right)\left(x',0,\xi',\xi_n\right) \times \partial^\alpha_{x'}\partial^{j+1}_{\xi_n}\partial^k_{x_n}\sigma_l\left(\widehat{D}^{-3}\right)\left(x',0,\xi',\xi_n\right)\right]d\xi_n\sigma\left(\xi'\right)dx', \quad (181)$$

and the sum is taken over $r+\ell-k-j-|\alpha|-1=-6$, $r\le -1$, $\ell\le -3$.

Locally, we can use Theorem 2 (26) to compute the interior term of (181); we have

$$\int_M\int_{|\xi|=1}\mathrm{trace}_{\wedge^*T^*M}\left[\sigma_{-4}\left(\widehat{D}^{-4}\right)\right]\sigma(\xi)dx = 128\pi^3\int_M\left[-\frac{16}{3}s-64|\theta|^2\right]d\mathrm{Vol}_M. \quad (182)$$

So we only need to compute $\int_{\partial M}\widehat{\Psi}$. Let us now turn to compute the specification of $\widehat{D}^3$.

$$\begin{aligned}
\widehat{D}^3 = {} & \sum_{i=1}^n c(\widetilde{e_i})\langle\widetilde{e_i},dx_l\rangle\left(-g^{ij}\partial_l\partial_i\partial_j\right) + \sum_{i=1}^n c(\widetilde{e_i})\langle\widetilde{e_i},dx_l\rangle \\
& \cdot\left\{-\left(\partial_l g^{ij}\right)\partial_i\partial_j - g^{ij}\left(4(\sigma_i+a_i)\partial_j - 2\Gamma^k_{ij}\partial_k\right)\partial_l\right\} \\
& + \sum_{i=1}^n c(\widetilde{e_i})\langle\widetilde{e_i},dx_l\rangle\Bigg\{-2\left(\partial_l g^{ij}\right)(\sigma_i+a_i)\partial_j \\
& + g^{ij}\left(\partial_l\Gamma^k_{ij}\right)\partial_k - 2g^{ij}[(\partial_l\sigma_i)+(\partial_l a_i)]\partial_j + \left(\partial_l g^{ij}\right)\Gamma^k_{ij}\partial_k \\
& + \sum_{j,k}\left[\partial_l\left(c\left(\theta'\right)c(\widetilde{e_j}) + c(\widetilde{e_j})c\left(\theta'\right)\right)\right]\left\langle\widetilde{e_j},dx^k\right\rangle\partial_k \\
& + \sum_{j,k}\left(c\left(\theta'\right)c(\widetilde{e_j}) + c(\widetilde{e_j})c\left(\theta'\right)\right)\left[\partial_l\left\langle\widetilde{e_j},dx^k\right\rangle\right]\partial_k\Bigg\} \\
& + \sum_{i=1}^n c(\widetilde{e_i})\langle\widetilde{e_i},dx_l\rangle\partial_l\Bigg\{-g^{ij}\left[\left(\partial_i\sigma_j\right)+\left(\partial_i a_j\right)+\sigma_i\sigma_j\right. \\
& \left.+\sigma_i a_j + a_i\sigma_j + a_i a_j - \Gamma^k_{i,j}\sigma_k - \Gamma^k_{i,j}a_k\right] \\
& + \sum_{i,j}g^{i,j}\left[c\left(\theta'\right)c(\partial_i)\sigma_i + c\left(\theta'\right)c(\partial_i)a_i\right. \\
& \left.+ c(\partial_i)\partial_i\left(c\left(\theta'\right)\right) + c(\partial_i)\sigma_i c\left(\theta'\right) + c(\partial_i)a_i c\left(\theta'\right)\right] \\
& + \frac{1}{4}s - \frac{1}{8}\sum_{ijkl}R_{ijkl}\bar{c}(\widetilde{e_i})\bar{c}(\widetilde{e_j})c(\widetilde{e_k})c(\widetilde{e_l}) + \sum_i c(\widetilde{e_i})\bar{c}\left(\nabla^{\mathrm{TM}}_{\widetilde{e_i}}\right) \\
& + |\theta|^2 - \left|\theta'\right|^2 + \bar{c}(\theta)c\left(\theta'\right) + c\left(\theta'\right)\bar{c}(\theta)\Bigg\} \\
& + \left[(\sigma_i+a_i) + \left(\bar{c}(\theta)+c\left(\theta'\right)\right)\right]\left(-g^{ij}\partial_i\partial_j\right) \\
& + \sum_{i=1}^n c(\widetilde{e_i})\langle\widetilde{e_i},dx_l\rangle\Bigg\{2\sum_{j,k}\left[c\left(\theta'\right)c(\widetilde{e_j}) + c(\widetilde{e_j})c\left(\theta'\right)\right] \\
& \times\langle\widetilde{e_i},dx_k\rangle\Bigg\}\partial_l\partial_k + \left[(\sigma_i+a_i) + \left(\bar{c}(\theta)+c\left(\theta'\right)\right)\right] \\
& \cdot\Bigg\{-\sum_{i,j}g^{i,j}\left[2\sigma_i\partial_j + 2a_i\partial_j - \Gamma^k_{i,j}\partial_k + \left(\partial_i\sigma_j\right) + \left(\partial_i a_j\right)\right. \\
& \left.+\sigma_i\sigma_j + \sigma_i a_j + a_i\sigma_j + a_i a_j - \Gamma^k_{i,j}\sigma_k - \Gamma^k_{i,j}a_k\right] \\
& + \sum_{i,j}g^{i,j}\left[c(\partial_i)c\left(\theta'\right) + c\left(\theta'\right)c(\partial_i)\right]\partial_j \\
& + \sum_{i,j}g^{i,j}\left[c\left(\theta'\right)c(\partial_i)\sigma_i + c\left(\theta'\right)c(\partial_i)a_i\right. \\
& \left.+ c(\partial_i)\partial_i\left(c\left(\theta'\right)\right) + c(\partial_i)\sigma_i c\left(\theta'\right) + c(\partial_i)a_i c\left(\theta'\right)\right] \\
& + \frac{1}{4}s - \left|\theta'\right|^2 - \frac{1}{8}\sum_{ijkl}R_{ijkl}\bar{c}(\widetilde{e_i})\bar{c}(\widetilde{e_j})c(\widetilde{e_k})c(\widetilde{e_l}) \\
& + \sum_i c(\widetilde{e_i})\bar{c}\left(\nabla^{\mathrm{TM}}_{\widetilde{e_i}}\theta\right) + |\theta|^2 + \bar{c}(\theta)c\left(\theta'\right) + c\left(\theta'\right)\bar{c}(\theta)\Bigg\}.
\end{aligned} \quad (183)$$

Then, we obtain

Lemma 14. *The following identities hold:*

$$\sigma_2\left(\widehat{D}^3\right)=\sum_{i,j,l}c(dx_l)\partial_l\left(g^{i,j}\right)\xi_i\xi_j+c(\xi)\left(4\sigma^k+4a^k-2\Gamma^k\right)\xi_k-2\left[c(\xi)c\left(\theta'\right)c(\xi)-|\xi|^2c\left(\theta'\right)\right]+\frac{1}{4}|\xi|^2\sum_{s,t,l}\omega_{s,t}(\widetilde{e_l})[c(\widetilde{e_l})\bar{c}(\widetilde{e_s})\bar{c}(\widetilde{e_t})-c(\widetilde{e_l})c(\widetilde{e_s})c(\widetilde{e_t})]+|\xi|^2\left(\bar{c}(\theta)+c\left(\theta'\right)\right),\sigma_3\left(\widehat{D}^3\right)=ic(\xi)|\xi|^2. \tag{184}$$

Write

$$\sigma\left(\widehat{D}^3\right)=p_3+p_2+p_1+p_0,\sigma\left(\widehat{D}^{-3}\right)=\sum_{j=3}^{\infty}q_{-j}. \tag{185}$$

By the composition formula of pseudodifferential operators, we have

$$\begin{aligned}1&=\sigma\left(\widehat{D}^3\circ\widehat{D}^{-3}\right)=\sum_{\alpha}\frac{1}{\alpha!}\partial_{\xi}^{\alpha}\left[\sigma\left(\widehat{D}^3\right)\right]D_x^{\alpha}\left[\sigma\left(\widehat{D}^{-3}\right)\right]\\&=(p_3+p_2+p_1+p_0)(q_{-3}+q_{-4}+q_{-5}+\cdots)\\&\quad+\sum_j\left(\partial_{\xi_j}p_3+\partial_{\xi_j}p_2+\partial_{\xi_j}p_1+\partial_{\xi_j}p_0\right)\\&\quad\cdot\left(D_{x_j}q_{-3}+D_{x_j}q_{-4}+D_{x_j}q_{-5}+\cdots\right)\\&=p_3q_{-3}+\left(p_3q_{-4}+p_2q_{-3}+\sum_j\partial_{\xi_j}p_3D_{x_j}q_{-3}\right)+\cdots,\end{aligned} \tag{186}$$

by (186), we have

$$q_{-3}=p_3^{-1},q_{-4}=-p_3^{-1}\left[p_2p_3^{-1}+\sum_j\partial_{\xi_j}p_3D_{x_j}\left(p_3^{-1}\right)\right]. \tag{187}$$

By (183)–(187), we have some symbols of operators.

Lemma 15. *The following identities hold:*

$$\sigma_{-3}\left(\widehat{D}^{-3}\right)=\frac{ic(\xi)}{|\xi|^4},\sigma_{-4}\left(\widehat{D}^{-3}\right)=\frac{c(\xi)\sigma_2\left(\widehat{D}^3\right)c(\xi)}{|\xi|^8}+\frac{ic(\xi)}{|\xi|^8}\left(|\xi|^4c(dx_n)\partial_{x_n}c\left(\xi'\right)-2h'(0)c(dx_n)c(\xi)+2\xi_nc(\xi)\partial_{x_n}c\left(\xi'\right)+4\xi_nh'(0)\right). \tag{188}$$

From the remark above, we can now compute $\widehat{\Psi}$ (see formula (181) for the definition of $\widehat{\Psi}$). We use tr as shorthand of trace. Since $n=6$, then $\mathrm{tr}_{\wedge^*T^*M}[\mathrm{id}]=64$. Since the sum is taken over $r+\ell-k-j-|\alpha|-1=-6$, $r\le-1$, $\ell\le-3$, then we have the $\int_{\partial_M}\widehat{\Psi}$ as the sum of the following five cases:

Case 1. (i) $r=-1$, $l=-3$, $j=k=0$, and $|\alpha|=1$.

By (181), we get

$$\text{Case 1 (i)}=-\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\sum_{|\alpha|=1}\operatorname{trace}\left[\partial_{\xi'}^{\alpha}\pi_{\xi_n}^{+}\sigma_{-1}\left(\widehat{D}^{-1}\right)\times\partial_{x'}^{\alpha}\partial_{\xi_n}\sigma_{-3}\left(\widehat{D}^{-3}\right)\right](x_0)d\xi_n\sigma\left(\xi'\right)dx'. \tag{189}$$

Case 1. (ii) $r=-1$, $l=-3$, $|\alpha|=k=0$, and $j=1$.

By (181), we have

$$\text{Case 1 (ii)}=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{trace}\left[\partial_{x_n}\pi_{\xi_n}^{+}\sigma_{-1}\left(\widehat{D}^{-1}\right)\times\partial_{\xi_n}^{2}\sigma_{-3}\left(\widehat{D}^{-3}\right)\right](x_0)d\xi_n\sigma\left(\xi'\right)dx'. \tag{190}$$

Case 1. (iii) $r=-1$, $l=-3$, $|\alpha|=j=0$, and $k=1$.

By (181), we have

$$\text{Case 1 (iii)}=-\frac{1}{2}\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{trace}\left[\partial_{\xi_n}\pi_{\xi_n}^{+}\sigma_{-1}\left(\widehat{D}^{-1}\right)\times\partial_{\xi_n}\partial_{x_n}\sigma_{-3}\left(\widehat{D}^{-3}\right)\right](x_0)d\xi_n\sigma\left(\xi'\right)dx'. \tag{191}$$

By Lemmas 12 and 15, we have $\sigma_{-3}((\widehat{D}^*\widehat{D}\widehat{D}^*)^{-1})=\sigma_{-3}(\widehat{D}^{-3})$; by (143)–(154), we obtain

$$\text{Case 1}=5\pi h'(0)\Omega_4dx', \tag{192}$$

where Ω_4 is the canonical volume of S^4.

Case 2. $r=-1$, $l=-4$, $|\alpha|=j=k=0$.

By (181), we have

$$\begin{aligned}\text{Case 2}&=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{trace}\left[\pi_{\xi_n}^{+}\sigma_{-1}\left(\widehat{D}^{-1}\right)\times\partial_{\xi_n}\sigma_{-4}\left(\widehat{D}^{-3}\right)\right](x_0)d\xi_n\sigma\left(\xi'\right)dx'\\&=i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{trace}\left[\partial_{\xi_n}\pi_{\xi_n}^{+}\sigma_{-1}\left(\widehat{D}^{-1}\right)\times\sigma_{-4}\left(\widehat{D}^{-3}\right)\right](x_0)d\xi_n\sigma\left(\xi'\right)dx'.\end{aligned} \tag{193}$$

In the normal coordinate, $g^{ij}(x_0)=\delta_i^j$ and $\partial_{x_j}(g^{\alpha\beta})(x_0)=0$, if $j<n$; $\partial_{x_j}(g^{\alpha\beta})(x_0)=h'(0)\delta_\beta^\alpha$, if $j=n$. So by Lemma A.2 in [10], we have $\Gamma^n(x_0)=(5/2)h'(0)$ and $\Gamma^k(x_0)=0$ for $k<n$. By the definition of δ^k and Lemma 2.3 in [10], we have

$\delta^n(x_0)=0$ and $\delta^k=(1/4)h'(0)c(\widetilde{e_k})c(\widetilde{e_n})$ for $k<n$. By Lemma 15, we obtain

$$\begin{aligned}
&\sigma_{-4}\left(\widehat{D}^{-3}\right)(x_0)\Big|_{|\xi'|=1}\\
&=\frac{c(\xi)\sigma_2\left(\widehat{D}^{-3}\right)(x_0)\Big|_{|\xi'|=1}c(\xi)}{|\xi|^8}\\
&\quad-\frac{c(\xi)}{|\xi|^4}\sum_j\partial_{\xi_j}\left(c(\xi)\,|\,\xi|^2\right)D_{x_j}\left(\frac{ic(\xi)}{|\xi|^4}\right)\\
&=\frac{1}{|\xi|^8}c(\xi)\left(\frac{1}{2}h'(0)c(\xi)\sum_{k<n}\xi_k c(\widetilde{e_k})c(\widetilde{e_n})\right.\\
&\quad-\frac{1}{2}h'(0)c(\xi)\sum_{k<n}\xi_k\bar{c}(\widetilde{e_k})\bar{c}(\widetilde{e_n})-\frac{5}{2}h'(0)\xi_n c(\xi)\\
&\quad-\frac{1}{4}h'(0)|\xi|^2c(dx_n)-2\left[c(\xi)c\left(\theta'\right)c(\xi)-|\xi|^2c\left(\theta'\right)\right]\\
&\quad\left.+|\xi|^2\left(\widehat{c}(\theta)+c\left(\theta'\right)\right)\right)c(\xi)+\frac{ic(\xi)}{|\xi|^8}\\
&\quad\cdot\left(|\xi|^4c(dx_n)\partial_{x_n}c\left(\xi'\right)-2h'(0)c(dx_n)c(\xi)\right.\\
&\quad\left.+2\xi_n c(\xi)\partial_{x_n}c\left(\xi'\right)+4\xi_n h'(0)\right).
\end{aligned}\tag{194}$$

By (69) and (194), we have

$$\begin{aligned}
&\operatorname{tr}\left[\partial_{\xi_n}\pi^+_{\xi_n}\sigma_{-1}\left(\widehat{D}^{-1}\right)\times\sigma_{-4}\left(\widehat{D}^{-3}\right)\right](x_0)\Big|_{|\xi'|=1}\\
&=\frac{1}{2(\xi_n-i)^2\left(1+\xi_n^2\right)^4}\left(\frac{3}{4}i+2+(3+4i)\xi_n\right.\\
&\quad\left.+(-6+2i)\xi_n^2+3\xi_n^3+\frac{9i}{4}\xi_n^4\right)h'(0)\operatorname{tr}[\mathrm{id}]\\
&\quad+\frac{1}{2(\xi_n-i)^2\left(1+\xi_n^2\right)^4}\left(-1-3i\xi_n-2\xi_n^2-4i\xi_n^3\right.\\
&\quad\left.-\xi_n^4-i\xi_n^5\right)\operatorname{tr}\left[c\left(\xi'\right)\partial_{x_n}c\left(\xi'\right)\right]-\frac{1}{2(\xi_n-i)^2\left(1+\xi_n^2\right)^4}\\
&\quad\cdot\left(\frac{1}{2}i+\frac{1}{2}\xi_n+\frac{1}{2}\xi_n^2+\frac{1}{2}\xi_n^3\right)\\
&\quad\cdot\operatorname{tr}\left[c\left(\xi'\right)\bar{c}\left(\xi'\right)c(dx_n)\bar{c}(dx_n)\right]\\
&\quad+\frac{-3\xi_n i+1}{2(\xi_n-i)^4(i+\xi_n)^3}\operatorname{tr}\left[c\left(\theta'\right)c(dx_n)\right]\\
&\quad-\frac{\xi_n+3i}{2(\xi_n-i)^4(i+\xi_n)^3}\operatorname{tr}\left[c\left(\theta'\right)c\left(\xi'\right)\right].
\end{aligned}\tag{195}$$

By (158) and (160), we have

$$\begin{aligned}
\text{Case 2}&=ih'(0)\int_{|\xi'|=1}\int_{-\infty}^{+\infty}64\\
&\quad\times\frac{(3/4)i+2+(3+4i)\xi_n+(-6+2i)\xi_n^2+3\xi_n^3+(9i/4)\xi_n^4}{2(\xi_n-i)^5(\xi_n+i)^4}\\
&\quad\cdot d\xi_n\sigma\left(\xi'\right)dx'+ih'(0)\int_{|\xi'|=1}\int_{-\infty}^{+\infty}32\\
&\quad\times\frac{1+3i\xi_n+2\xi_n^2+4i\xi_n^3+\xi_n^4+i\xi_n^5}{2(\xi_n-i)^2\left(1+\xi_n^2\right)^4}d\xi_n\sigma\left(\xi'\right)dx'\\
&\quad+i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\frac{-3\xi_n i+1}{2(\xi_n-i)^4(i+\xi_n)^3}\\
&\quad\cdot\operatorname{tr}\left[c\left(\theta'\right)c(dx_n)\right]d\xi_n\sigma\left(\xi'\right)dx'-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\\
&\quad\cdot\frac{\xi_n+3i}{2(\xi_n-i)^4(i+\xi_n)^3}\operatorname{tr}\left[c\left(\theta'\right)c\left(\xi'\right)\right]d\xi_n\sigma\left(\xi'\right)dx'\\
&=\left(-\frac{41}{8}i-\frac{195}{8}\right)\pi h'(0)\Omega_4dx'.
\end{aligned}\tag{196}$$

Case 3. $r=-2$, $l=-3$, $|\alpha|=j=k=0$.

By (181), we have

$$\begin{aligned}
\text{Case 3}=-i\int_{|\xi'|=1}\int_{-\infty}^{+\infty}\operatorname{trace}\left[\pi^+_{\xi_n}\sigma_{-2}\left(\widehat{D}^{-1}\right)\right.\\
\left.\times\partial_{\xi_n}\sigma_{-3}\left(\widehat{D}^{-3}\right)\right](x_0)d\xi_n\sigma\left(\xi'\right)dx'.
\end{aligned}\tag{197}$$

By Lemmas 12 and 15, we have $\sigma_{-3}((\widehat{D}^*\widehat{D}\widehat{D}^*)^{-1})=\sigma_{-3}(\widehat{D}^{-3})$; by (162)–(177), we obtain

$$\text{Case 3}=\frac{55}{2}\pi h'(0)\Omega_4dx'.\tag{198}$$

Now $\widehat{\Psi}$ is the sum of Cases 1–3, then

$$\widehat{\Psi}=\left[\left(\frac{65}{8}-\frac{41}{8}i\right)\pi h'(0)\right]\Omega_4dx'.\tag{199}$$

Theorem 16. *Let M be a 6-dimensional compact-oriented manifold with the boundary ∂M and the metric g^M as above and $\widehat{D}$ be a modified Novikov operator on $\widehat{M}$, then*

$$\begin{aligned}
&\widetilde{Wres}\left[\pi^+\widehat{D}^{-1}\circ\pi^+\left(\widehat{D}^{-3}\right)\right]\\
&=128\pi^3\int_M\left[-\frac{16}{3}s-64|\theta|^2\right]dVol_M\\
&\quad+\int_{\partial M}\left[\left(\frac{65}{8}-\frac{41}{8}i\right)\pi h'(0)\right]\Omega_4dx'.
\end{aligned}\tag{200}$$

where s is the scalar curvature.

5. The Spectral Action for Witten Deformation

In this section, we will compute the spectral action for the Witten deformation. Let (M, g^M) be an n-dimensional compact-oriented Riemannian manifold. Now we will recall the definition of the Witten deformation D_θ (see details in [17]).

Let ∇^L denote the Levi-Civita connection about g^M which is a Riemannian metric of M. In the local coordinates $\{x_i, 1 \le i \le n\}$ and the fixed orthonormal frame $\{\widetilde{e_1}, \cdots, \widetilde{e_n}\}$, the connection matrix $(\omega_{s,t})$ is defined by

$$\nabla^L(\widetilde{e_1}, \cdots, \widetilde{e_n}) = (\widetilde{e_1}, \cdots, \widetilde{e_n})(\omega_{s,t}). \tag{201}$$

Let $\varepsilon(\widetilde{e_j *})$ and $\iota(\widetilde{e_j *})$ be the exterior and interior multiplications, respectively. The Witten deformation is defined by

$$D_\theta = d + \delta + \bar{c}(\theta) = \sum_{i=1}^{n} c(\widetilde{e_i}) \cdot \left[\widetilde{e_i} + \frac{1}{4}\sum_{s,t}\omega_{s,t}(\widetilde{e_i})[\bar{c}(\widetilde{e_s})\bar{c}(\widetilde{e_t}) - c(\widetilde{e_s})c(\widetilde{e_t})]\right] + \bar{c}(\theta). \tag{202}$$

By Proposition 4.6 of [17], we have

$$D_\theta^2 = (d+\delta)^2 + \sum_i c(\widetilde{e_i})\bar{c}\left(\nabla^{\mathrm{TM}}_{\widetilde{e_i}}\theta\right) + |\theta|^2. \tag{203}$$

Let $g^{ij} = g(dx_i, dx_j)$, $\xi = \sum_k \xi_j dx_j$, and $\nabla^L_{\partial_i}\partial_j = \sum_k \Gamma_{ij}^k \partial_k$, we denote

$$\sigma_i = -\frac{1}{4}\sum_{s,t}\omega_{s,t}(\widetilde{e_i})c(\widetilde{e_s})c(\widetilde{e_t}), a_i = \frac{1}{4}\sum_{s,t}\omega_{s,t}(\widetilde{e_i})\bar{c}(\widetilde{e_s})\bar{c}(\widetilde{e_t}), \xi^j = g^{ij}\xi_i, \Gamma^k = g^{ij}\Gamma_{ij}^k, \sigma^j = g^{ij}\sigma_i, a^j = g^{ij}a_i. \tag{204}$$

For a smooth vector field X on M, let $c(X)$ denote the Clifford action. Since E is globally defined on M, we can perform computations of E in normal coordinates. Taking normal coordinates about x_0, then $\sigma^i(x_0) = 0$, $a^i(x_0) = 0$, $\partial^j[c(\partial_j)](x_0) = 0$, $\Gamma^k(x_0) = 0$, and $g^{ij}(x_0) = \delta_i^j$, so that

$$E(x_0) = \frac{1}{8}\sum_{ijkl} R_{ijkl}\bar{c}(\widetilde{e_i})\bar{c}(\widetilde{e_j})c(\widetilde{e_k})c(\widetilde{e_l}) - \frac{1}{4}s - \sum_i c(\widetilde{e_i})\bar{c}\left(\nabla^{\mathrm{TM}}_{\widetilde{e_i}}\theta\right) - |\theta|^2. \tag{205}$$

For the Witten deformation D_θ, we will compute the spectral action for it on a 4-dimensional compact manifold. We will calculate the bosonic part of the spectral action for the Witten deformation. It is defined to be the number of eigenvalues of D_θ in the interval $[-\wedge, \wedge]$ with $\wedge \in \mathbb{R}^+$. It is expressed as

$$I = \mathrm{tr}\widehat{F}\left(\frac{D_\theta^2}{\wedge^2}\right). \tag{206}$$

Here, tr denotes the operator trace in the L^2 completion of $\Gamma(M, S(\mathrm{TM}))$ and $\widehat{F} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a cut-off function with support in the interval $[0, 1]$ which is constant near the origin. By Lemma 1.7.4 in [21], we have the heat trace asymptotics, for $t \longrightarrow 0$,

$$\mathrm{tr}\left(e^{-tD_\theta^2}\right) \sim \sum_{m\ge 0} t^{m-(n/2)} a_{2m}\left(D_\theta^2\right). \tag{207}$$

One uses the Seeley-DeWitt coefficients $a_{2m}(D_\theta^2)$ and $t = \wedge^{-2}$ to obtain asymptotics for the spectral action when $\dim M = 4$,

$$I = \mathrm{tr}\widehat{F}\left(\frac{D_\theta^2}{\wedge^2}\right) \sim \overset{4}{\wedge} F_4 a_0\left(D_\theta^2\right) + \overset{2}{\wedge} F_2 a_2\left(D_\theta^2\right) + \overset{0}{\wedge} F_0 a_4\left(D_\theta^2\right) as \wedge \longrightarrow \infty, \tag{208}$$

with the first three moments of the cut-off function which are given by $F_4 = \int_0^\infty s\widehat{F}(s)ds$, $F_2 = \int_0^\infty \widehat{F}(s)ds$, and $F_0 = \widehat{F}(0)$.

We use Theorem 4.1.6 in [17] to obtain the first three coefficients of the heat trace asymptotics:

$$\begin{aligned} a_0\left(D_\theta^2\right) &= (4\pi)^{-(4/2)}\int_M \mathrm{tr}(\mathrm{id})dvol, \\ a_2\left(D_\theta^2\right) &= (4\pi)^{-(4/2)}\int_M \mathrm{tr}\left[\frac{s}{6} + E\right]dvol, \\ a_4\left(D_\theta^2\right) &= \frac{(4\pi)^{-(4/2)}}{360}\int_M \mathrm{tr}\left[-12R_{ijij,kk} + 5R_{ijij}R_{klkl} - 2R_{ijik}R_{ljlk} + 2R_{ijkl}R_{ijkl} - 60R_{ijij}E + 180E^2 + 60E_{,kk} + 30\Omega_{ij}\Omega_{ij}\right]dvol. \end{aligned} \tag{209}$$

By the Clifford action and cyclicity of the trace, we have

$$\mathrm{tr}(c(e_i)) = 0, \mathrm{tr}\left(c(e_i)c(e_j)\right) = 0 (i \ne j), \mathrm{tr}\left[\bar{c}(\widetilde{e_i})\bar{c}(\widetilde{e_j})c(\widetilde{e_k})c(\widetilde{e_l})\right] = 0 (i \ne j), \mathrm{tr}\left[\sum_i c(\widetilde{e_i})\bar{c}\left(\nabla^{\mathrm{TM}}_{\widetilde{e_i}}\theta\right)\right] = 0. \tag{210}$$

So we obtain

$$\begin{aligned} a_0\left(D_\theta^2\right) &= (4\pi)^{-(4/2)}\int_M \operatorname{tr}(id)dvol = \pi^{-2}\int_M dvol, a_2\left(D_\theta^2\right) \\ &= (4\pi)^{-(4/2)}\int_M \operatorname{tr}\left[\frac{s}{6}+E\right]dvol \\ &= (4\pi)^{-(4/2)}\int_M \operatorname{tr}\left[\frac{s}{6}+\frac{1}{8}\sum_{ijkl}R_{ijkl}\bar{c}(\widetilde{e_i})\bar{c}(\widetilde{e_j})c(\widetilde{e_k})c(\widetilde{e_l})\right. \\ &\quad \left. -\frac{1}{4}s-\sum_i c(\widetilde{e_i})\bar{c}\left(\nabla_{\tilde{e}_i}^{\mathrm{TM}}\theta\right)-|\theta|^2\right]dvol \\ &= -\int_M\left(\frac{s}{12\pi^2}+\frac{|\theta|^2}{\pi^2}\right)dvol. \end{aligned} \tag{211}$$

And we have

$$\begin{aligned} &\int_M \operatorname{tr}\left(5R_{ijij}R_{klkl}-60R_{ijij}E+180E^2-12R_{ijij,kk}+60E_{kk}\right)dvol \\ &= \int_M\left[\operatorname{tr}\left(5s^2+60sE+180E^2\right)\right. \\ &\quad \left. -\operatorname{tr}[12\Delta s]+60[-\Delta(\operatorname{tr}E)]\right]dvol \\ &= \int_M \operatorname{trace}\left[\frac{5}{4}s^2+180\left(\sum_i c(\widetilde{e_i})\bar{c}\left(\nabla_{\tilde{e}_i}^{\mathrm{TM}}\theta\right)\right)^2+\frac{45}{16}\right. \\ &\quad \cdot\left(\sum_{ijkl}R_{ijkl}\bar{c}(\widetilde{e_i})\bar{c}(\widetilde{e_j})c(\widetilde{e_k})c(\widetilde{e_l})\right)^2+30s|\theta|^2+180|\theta|^4 \\ &\quad -45\sum_{ijkl}R_{ijkl}\bar{c}(\widetilde{e_i})\bar{c}(\widetilde{e_j})c(\widetilde{e_k})c(\widetilde{e_l})\sum_p c(\widetilde{e_p})\bar{c}\left(\nabla_{\tilde{e}_p}^{\mathrm{TM}}\theta\right) \\ &\quad \left.+48\Delta s+960\Delta\left(|\theta|^2\right)\right]dvol \\ &= \int_M\left[20s^2+2880\left(\sum_i\left|\nabla_{\tilde{e}_i}^{\mathrm{TM}}\theta\right|^2+|\theta|^4\right)\right. \\ &\quad \left.+180\sum_{ijkl}R_{ijkl}^2+480s|\theta|^2\right]dvol. \end{aligned} \tag{212}$$

And $\operatorname{tr}[\Omega_{ij}\Omega_{ij}]$ is globally defined, so we only compute it in normal coordinates about x_0 and the local orthonormal frame e_i obtained by parallel transport along geodesics from x_0. Then,

$$\omega_{s,t}(x_0)=0,\quad \partial_i\left(c\left(\widetilde{e_j}\right)\right)=0,\quad \left[\widetilde{e_i},\widetilde{e_j}\right](x_0)=0. \tag{213}$$

Then, we have

$$\begin{aligned} \Omega\left(\widetilde{e_i},\widetilde{e_j}\right)(x_0) &= \nabla_{\tilde{e}_i}^{\wedge^* T^*M}\nabla_{\tilde{e}_j}^{\wedge^* T^*M}-\nabla_{\tilde{e}_j}^{\wedge^* T^*M}\nabla_{\tilde{e}_i}^{\wedge^* T^*M}-\nabla_{[\tilde{e}_i,\tilde{e}_j]}^{\wedge^* T^*M} \\ &= -\frac{1}{4}\sum_{s,t=1}^{n}R_{ijst}^M\left[\bar{c}(\widetilde{e_s})\bar{c}(\widetilde{e_t})-c(\widetilde{e_s})c(\widetilde{e_t})\right]. \end{aligned} \tag{214}$$

So we have

$$\begin{aligned} \operatorname{tr}\left[\Omega_{ij}\Omega_{ij}\right](x_0) &= \operatorname{tr}\left[\frac{1}{16}\sum_{s,t,s_1,t_1=1}R_{ijst}^M R_{ijs_1t_1}^M\left[\bar{c}(\widetilde{e_s})\bar{c}(\widetilde{e_t})\right.\right. \\ &\quad \left.\left.-c(\widetilde{e_s})c(\widetilde{e_t})\right]\left[\bar{c}\left(\widetilde{e_{s_1}}\right)\bar{c}\left(\widetilde{e_{t_1}}\right)-c\left(\widetilde{e_{s_1}}\right)c\left(\widetilde{e_{t_1}}\right)\right]\right] \\ &= \operatorname{tr}\left[\frac{1}{16}\sum_{s,t,s_1,t_1=1}R_{ijst}^M R_{ijs_1t_1}^M\left[\bar{c}(\widetilde{e_s})\bar{c}(\widetilde{e_t})\bar{c}\left(\widetilde{e_{s_1}}\right)\bar{c}\left(\widetilde{e_{t_1}}\right)\right.\right. \\ &\quad -\bar{c}(\widetilde{e_s})\bar{c}(\widetilde{e_t})c\left(\widetilde{e_{s_1}}\right)c\left(\widetilde{e_{t_1}}\right)-c(\widetilde{e_s})c(\widetilde{e_t})\bar{c}\left(\widetilde{e_{s_1}}\right)\bar{c}\left(\widetilde{e_{t_1}}\right) \\ &\quad \left.\left.+c(\widetilde{e_s})c(\widetilde{e_t})c\left(\widetilde{e_{s_1}}\right)c\left(\widetilde{e_{t_1}}\right)\right]\right] = -4\sum_{ijst}\left(R_{ijst}^M\right)^2. \end{aligned} \tag{215}$$

Proposition 17. *The following equality holds:*

$$\begin{aligned} a_4\left(D_\theta^2\right) = \frac{1}{5760\pi^2}\int_M&\left[20s^2+2880\left(\sum_i\left|\nabla_{\tilde{e}_i}^{TM}\theta\right|^2+|\theta|^4\right)\right. \\ &+180\sum_{ijkl}R_{ijkl}^2+480s|\theta|^2-32R_{ijik}R_{ljlk} \\ &\left.+32R_{ijik}^2-1920\sum_{i,j,s,t}\left(R_{ijst}^M\right)^2\right]dvol, \end{aligned} \tag{216}$$

where s is the scalar curvature.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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