

Research Article

Rational Solutions and Their Interaction Solutions of the $(3 + 1)$ -Dimensional Jimbo-Miwa Equation

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In this paper, we gave a form of rational solution and their interaction solution to a nonlinear evolution equation. The rational solution contained lump solution, general lump solution, high-order lump solution, lump-type solution, etc. Their interaction solution contained the classical interaction solution, such as the lump-kink solution and the lump-soliton solution. As the example, by using the generalized bilinear method and symbolic computation Maple, we obtained abundant high-order lump-type solutions and their interaction solutions between lumps and other function solutions under certain constraints of the $(3 + 1)$ -dimensional Jimbo-Miwa equation. Via three-dimensional plots, contour plots and density plots with the help of Maple, the physical characteristics and structures of these waves are described very well. These solutions have greatly enriched the exact solutions of the $(3 + 1)$ -dimensional Jimbo-Miwa equation on the existing literature.

1. Introduction

Nonlinear phenomena have a lot of significant applications in different sides of physics with natural and engineering fields. Basically, all the fundamental equations of physics are nonlinear and, generally, such types of nonlinear evolution equations (NLEEs) are often very tough to solve clearly. The exact solutions of NLEEs play a crucial role in the study of nonlinear physical or natural phenomena. In the recent decade, several direct methods for finding the exact solutions to NLEEs have been proposed [1–9]. Thousands of examples have shown that these methods are powerful for obtaining exact solutions of NLEEs, such as soliton [10–14], rogue wave [15, 16], breather solution [17], periodic wave solution [18–21], and optical solution [22, 23].

The lump solution has attracted a great deal of attention since lump solutions were firstly discovered [24]. The research to lump solution has not been well developed, because it is very complex to solve the lump solution of NLEEs. Recently, based on the Hirota bilinear method, Ma and Zhou introduced a new way to get the lump solution of NLEEs by using symbolic computation and gave a theoretical

testimony [25, 26]. By using this method, researchers successfully obtained the lump solutions and interaction solutions of NLEEs [27–57]. In the present paper, we will propose the form of rational solution and their interaction solution to NLEE. The rational solution contains lump solution, general lump solution, high-order lump solutions, lump-type solution, etc. Their interaction solution contains the classical interaction solution, such as the lump-kink solution and the lump-soliton solution.

The rest of the paper is organized as follows. In Section 2, we will give the form of rational solution and their interaction solution to NLEE. In Section 3, by using the generalized bilinear method and symbolic computation Maple, we will obtain the high-order lump-type solutions of the $(3 + 1)$ -dimensional Jimbo-Miwa equation. In Section 4, by using the symbolic computation Maple, we will get abundant interaction solutions between the high-order lump-type solution and other function solutions. Via three-dimensional plots, contour plots, and density plots with the help of Maple, the physical characteristics and structures of these waves are described very well. In Section 5, a few of the conclusions and outlook will be given.

2. Rational Solution and Their Interaction Solution

Consider a K th order NLEE ($K \geq 2$)

$$F(x, u, \partial u, \partial^2 u, \dots, \partial^K u) = 0, \quad (1)$$

where $x = (x_1, x_2, \dots, x_n)$ are n independent variables and $x_i (i \neq 1)$ contain time variable t . u is the dependent variable.

2.1. Rational Solution. In order to get the rational solution of NLEE (1), we take its main steps as follows.

Step 1. Under dependent variable transformation,

$$\begin{aligned} u(x) &= 2[\ln f(x)]_{x_1}, \\ \text{or } u(x) &= 2[\ln f(x)]_{x_1, x_1}. \end{aligned} \quad (2)$$

Equation (1) is transformed into the following bilinear form:

$$G(x, f, \partial f, \partial^2 f, \dots, \partial^l f) = 0. \quad (3)$$

Step 2. We suppose that Equation (3) has the following general positive quadratic function solution:

$$f = a_0 + \sum_{i=1}^N \xi_i^{2n_i}, \quad (4)$$

where

$$\xi_i = a_{i0} + \sum_{j=1}^n a_{ij} x_j, \quad (5)$$

where $a_0, a_{ij} (i = 1, \dots, N; j = 0, 1, \dots, n)$ are arbitrary real constants.

Step 3. By substituting (4) and (5) into Equation (3), collecting all terms with the same order of x_i together, the left-hand side of Equation (3) is converted into another polynomial in x_i . Equating each coefficient of this different power terms to zero yields a set of nonlinear algebraic equations for a_0, a_{ij} . With the aid of Maple (or Mathematica), we solve the above nonlinear algebraic equations.

Step 4. By substituting a_0, a_{ij} into expression (4) and using bilinear transformation (2), we can obtain rational solution (4) of Equation (1).

Remark 1. When choosing $N = 2, n_i = 1$ in expression (4), the rational solution is reduced to the lump solution [24–40, 45–57].

Remark 2. When choosing $N = 3, n_i = 1$ in expression (4), we obtain the lump-type solution of Refs. [41–43]. When $N = 3,$

$n_1 = 2, n_2 = 1, n_3 = 1$, we obtain the high-order lump-type of Ref. [50].

2.2. General Interaction Solution. In order to obtain the general interaction solution, we take its main steps as follows:

Step 1. By using transformation (2), Equation (1) is transformed into bilinear form (3).

Step 2. We suppose that Equation (3) has the following solution:

$$f = a_0 + \sum_{i=1}^N \xi_i^{2n_i} + \sum_{j=1}^M g_j(\eta_j), \quad (6)$$

where ξ_i is given in (5), and

$$\eta_j = b_{j0} + \sum_{k=1}^n b_{jk} x_k, \quad (7)$$

where $b_{jk} (j = 1, \dots, M; k = 0, 1, \dots, n)$ are arbitrary real constants.

Step 3. By substituting (6) and (7) into Equation (3), collecting all terms with the same order of $x_i, g_j(\eta_j), g'_j(\eta_j), g''_j(\eta_j), \dots$ together, the left-hand side of Equation (3) is converted into another polynomial in $x_i, g_j(\eta_j), g'_j(\eta_j), g''_j(\eta_j), \dots$. Equating each coefficient of this different power terms to zero yields a set of nonlinear algebraic equations for a_0, a_{ij}, b_{jk} . With the aid of Maple (or Mathematica), we solve the above nonlinear algebraic equations.

Step 4. By substituting a_0, a_{ij}, b_{jk} into (6) and using bilinear transformation (2), we can obtain the general interaction solution (6) of Equation (1).

Remark 3. When choosing $N = 2, n_1 = n_2 = 1$, and $M = 1, g_1(\eta_1) = e^{\eta_1}$ or $g_1(\eta_1) = \cosh(\eta_1)$, interaction solution (6) is reduced to the lump-kink solution and the lump-soliton solution [37–40, 42, 47, 48, 52–56].

Remark 4. When choosing $N = 2, n_1 = n_2 = 1$ and $M = 1, g_1(\eta_1) = \cos \eta_1$ or $g_1(\eta_1) = c_1 e^{\eta_1} + c_2 e^{-\eta_1}$ or $g_1(\eta_1) = \sinh(\eta_1)$ or $g_1(\eta_1) = \sin(\eta_1)$, we obtain the interaction solutions of Refs. [42, 43, 47, 57], respectively.

Remark 5. In Step 3, the connection between $g_j(\eta_j)$ and $g'_j(\eta_j), g''_j(\eta_j), \dots$ must be considered when we take a coefficient of different power terms to $g_j(\eta_j), g'_j(\eta_j), g''_j(\eta_j), \dots$

3. High-Order Lump-Type Solutions of the (3 + 1)-Dimensional Jimbo-Miwa Equation

We consider the (3 + 1)-dimensional Jimbo-Miwa equation [58]:

$$u_{xxxxy} + 3u_y u_{xx} + 3u_x u_{xy} + 2u_{yt} - 3u_{xz} = 0. \tag{8}$$

Equation (8) is the second equation in the well known KP-hierarchy of integrable systems [11, 12, 58], which are used to describe certain interesting (3 + 1)-dimensional waves in physics. Although Equation (8) is nonintegrable, the exact solutions of the Jimbo-Miwa equation have been investigated by using various methods [6, 7]. Recently, researchers studied the solitary wave solutions of Equation (8) in [13, 14]. Based on the bilinear method, we obtained several interaction solutions and the periodic lump wave solutions for Equation (8) [20, 21]. The classes of lump solutions, lump-type solutions, general lump-type solutions, and interaction solutions for Equation (8) were presented in [36–41].

3.1. *Bilinear Form.* Under the Cole-Hopf transformation,

$$u(x, y, z, t) = 2[\ln f(x, y, z, t)]_x. \tag{9}$$

Equation (8) becomes the generalized Hirota bilinear equation:

$$GB_{JM}(f) := (D_{p,x}^3 D_{p,y} + 2D_{p,y} D_{p,t} - 3D_{p,x} D_{p,z})f \cdot f, \tag{10}$$

where p , being an arbitrarily natural number, is often a prime number. D is a generalized bilinear differential operator as follows [3]:

$$\begin{aligned} D_{p,x_1}^m D_{p,x_2}^n f \cdot f &= \left(\partial_{x_1} + \alpha_p \partial_{x_1'}\right)^m \left(\partial_{x_2} + \alpha_p \partial_{x_2'}\right)^n f(x_1, x_2) f \\ &\cdot \left(x_1', x_2'\right) \Big|_{x_1'=x_1, x_2'=x_2} = \sum_{i=0}^m \sum_{j=0}^n \binom{m}{i} \binom{n}{j} \\ &\cdot \alpha_p^i \alpha_p^j \frac{\partial^{m+n-i-j} f(x_1, x_2)}{\partial x_1^{m-i} \partial x_2^{n-j}} \frac{\partial x^{i+j} f(x_1, x_2)}{\partial x_1^i \partial x_2^j}, \end{aligned} \tag{11}$$

where $m, n \geq 0$, $\alpha_p^s = (-1)^{r_p(s)}$, if $s \equiv r_p(s) \pmod p$.

When taking $p = 2$, we obtain the Hirota bilinear equation:

$$\begin{aligned} GB_{JM}(f) &:= (D_x^3 D_y + 2D_y D_t - 3D_x D_z)f \cdot f \\ &= 2 \left[ff_{xxxxy} - f_y f_{xxx} + 2(f_{ty} f - f_t f_y) \right. \\ &\quad \left. + 3(f_{xy} f_{xx} - f_x f_{xxy} - f_{xz} f + f_x f_z) \right] = 0. \end{aligned} \tag{12}$$

When taking $p = 3$, we can obtain the generalized bilinear

Jimbo-Miwa equation:

$$\begin{aligned} GB_{JM}(f) &:= (D_{3,x}^3 D_{3,y} + 2D_{3,y} D_{3,t} - 3D_{3,x} D_{3,z})f \cdot f \\ &= 2 \left[3f_{xx} f_{xy} + 2(f_{yt} f - f_y f_t) + 3(f_x f_z - f_{xz} f) \right] = 0. \end{aligned} \tag{13}$$

By using transformation (9), generalized bilinear Jimbo-Miwa equation (13) is transformed into the following form:

$$\begin{aligned} GP_{JM}(u) &:= \frac{9}{8} u^2 u_x v + \frac{3}{8} u^3 u_y + \frac{3}{4} uv u_{xx} + \frac{3}{4} u_x^2 v + \frac{3}{4} u^2 u_{xy} \\ &\quad + \frac{9}{4} uu_x u_y + \frac{3}{2} u_{xx} u_y + \frac{3}{2} u_x u_{xy} + 2u_{yt} - 3u_{xz} = 0, \end{aligned} \tag{14}$$

where $u_y = v_x$. Transformation (9) is also a characteristic one in establishing Bell polynomial theories of soliton equations [59], and an accurate relation is

$$GP_{JM}(u) = \left[\frac{GB_{JM}(f)}{f^2} \right]_x. \tag{15}$$

Hence, if f solves generalized bilinear Jimbo-Miwa equation (13), Jimbo-Miwa equation (14) will be solved.

3.2. *High-Order Lump-Type Solutions.* In the section, we will study the high-order lump-type solutions of (3 + 1)-dimensional Jimbo-Miwa equation (8) by constructing positive quadratic function solutions to the corresponding generalized bilinear equation (13).

Step 1. By using the Cole-Hopf transformation (9), Equation (8) is transformed into generalized bilinear equation (13).

Step 2. To get the positive quadratic function solution of generalized bilinear equation (13), we take $N = 3$, $n_1 = 2$, $n_2 = 1$, $n_3 = 1$ in expression (4),

$$f = a_0 + \xi_1^4 + \xi_2^2 + \xi_3^2, \tag{16}$$

where

$$\xi_i = a_{i0} + a_{i1}x + a_{i2}y + a_{i3}z + a_{i4}t, \quad i = 1, 2, 3. \tag{17}$$

where $a_{ij} (i = 1, 2, 3; j = 0, 1, 2, 3, 4)$ are arbitrary real constants, and

$$\begin{aligned} \sum_{j=1}^4 a_{ij}^2 &\neq 0 (i = 1, 2, 3), \\ \sum_{i=1}^3 a_{ij}^2 &\neq 0 (j = 1, 2, 3, 4). \end{aligned} \tag{18}$$

Step 3. By substituting (16) and (17) into Equation (13),

collecting all terms with the same order of x, y, z, t together, the left-hand side of Equation (13) is converted into another polynomial in x, y, z, t . Equating each coefficient of this different power terms to zero yields a set of nonlinear algebraic equations for a_0, a_{ij} .

Solving the algebraic equations by Maple yields the following sets of solutions,

Case 1.

$$\begin{aligned}
 a_{11} &= 0, \\
 a_{13} &= \frac{2a_{12}a_{34}}{3a_{31}}, \\
 a_{14} &= 0, \\
 a_{21} &= 0, \\
 a_{22} &= \frac{3a_{23}a_{31}}{2a_{34}}, \\
 a_{24} &= 0, \\
 a_{32} &= 0, \\
 a_{33} &= 0, \quad a_{31}a_{34} \neq 0
 \end{aligned} \tag{19}$$

Case 2.

$$\begin{aligned}
 a_{11} &= 0, \\
 a_{12} &= \frac{a_{32}a_{13}}{a_{33}}, \\
 a_{14} &= 0, \\
 a_{22} &= -\frac{a_{31}a_{32}}{a_{21}}, \\
 a_{23} &= -\frac{a_{31}a_{33}}{a_{21}}, \\
 a_{24} &= \frac{3a_{21}a_{33}}{2a_{32}}, \\
 a_{30} &= \frac{a_{20}a_{31}}{a_{21}}, \\
 a_{34} &= \frac{3a_{31}a_{33}}{2a_{32}}, \quad a_{21}a_{32}a_{33} \neq 0
 \end{aligned} \tag{20}$$

Case 3.

$$\begin{aligned}
 a_{11} &= 0, \\
 a_{14} &= 0, \\
 a_{22} &= -\frac{a_{31}a_{32}}{a_{21}}, \\
 a_{23} &= -\frac{a_{13}a_{31}a_{32}}{a_{12}a_{21}}, \\
 a_{24} &= \frac{3a_{13}a_{21}}{2a_{12}},
 \end{aligned}$$

$$\begin{aligned}
 a_{33} &= \frac{a_{32}a_{13}}{a_{12}}, \\
 a_{34} &= \frac{3a_{13}a_{31}}{2a_{12}}, \quad a_{12}a_{21} \neq 0
 \end{aligned} \tag{21}$$

where other parameters in Cases 1–3 are arbitrary real constants.

If we consider the special solutions $a_{11} = a_{12} = a_{13} = a_{14} = 0$, we obtain the following solutions which are different from those solutions given in [36–41],

Case 4.

$$\begin{aligned}
 a_{21} &= 0, \\
 a_{23} &= -\frac{2a_{22}(2a_{10}^4a_{24}^2 - 3a_{31}^3a_{34} + 2a_0a_{24}^2)}{9a_{31}^4}, \\
 a_{32} &= -\frac{2a_{22}a_{24}(a_{10}^4 + a_0)}{3a_{31}^3}, \\
 a_{33} &= -\frac{2a_{22}a_{24}(2a_{10}^4a_{34} + 3a_{31}^3 + 2a_0a_{34})}{9a_{31}^4}, \quad a_{31} \neq 0
 \end{aligned} \tag{22}$$

Case 5.

$$\begin{aligned}
 a_{22} &= -\frac{2a_{32}a_{34}(a_{10}^4 + a_0)}{3a_{21}^3}, \\
 a_{23} &= -\frac{2a_{32}a_{34}(2a_{10}^4a_{24} + 3a_{21}^3 + 2a_{24}a_0)}{9a_{21}^4}, \\
 a_{30} &= 0, \\
 a_{31} &= 0, \\
 a_{33} &= -\frac{2a_{32}(2a_{10}^4a_{34}^2 - 3a_{21}^3a_{24} + 2a_0a_{34}^2)}{9a_{21}^4}, \quad a_{21} \neq 0
 \end{aligned} \tag{23}$$

Case 6.

$$\begin{aligned}
 a_{23} &= \frac{a_{22}a_{33}}{a_{32}} + \frac{(a_{22}^2 + a_{32}^2)(a_{21}a_{22} + a_{31}a_{32})(a_{21}^2 + a_{31}^2)}{a_{32}(a_{10}^4 + a_0)(a_{21}a_{32} - a_{22}a_{31})}, \\
 a_{24} &= \frac{3a_{21}a_{33}}{2a_{32}} + \frac{3(a_{21}^2 + a_{31}^2)(a_{21}a_{22} + a_{31}a_{32})^2}{2a_{32}(a_{10}^4 + a_0)(a_{21}a_{32} - a_{22}a_{31})}, \\
 a_{30} &= \frac{a_{20}a_{31}}{a_{21}}, \\
 a_{34} &= \frac{3a_{31}a_{33}}{2a_{32}} - \frac{3(a_{21}^2 + a_{31}^2)(a_{21}a_{22} + a_{31}a_{32})}{2a_{32}(a_{10}^4 + a_0)}, \\
 &\quad (a_{32}(a_{10}^4 + a_0)(a_{21}a_{32} - a_{22}a_{31}) \neq 0)
 \end{aligned} \tag{24}$$

Case 7.

$$\begin{aligned}
 a_0 &= -a_{10}^4 + \frac{a_{21}a_{22}(a_{21}^2 + a_{31}^2)}{a_{31}a_{33}}, \\
 a_{23} &= -\frac{3a_{21}a_{33} - 2a_{22}a_{34}}{3a_{31}}, \\
 a_{24} &= -\frac{3a_{21}^2a_{33} - 2a_{21}a_{22}a_{34} + 3a_{31}^2a_{33}}{2a_{22}a_{31}}, \\
 a_{30} &= \frac{a_{20}a_{31}}{a_{21}}, \\
 a_{32} &= 0, \quad a_{21}a_{22}a_{31} \neq 0
 \end{aligned} \tag{25}$$

where other parameters in Cases 4–7 are arbitrary real constants.

Step 4. By substituting a_0, a_{ij} in Cases 1–3 into (16) and (17) and using transformation (9), we obtain the following high-order lump-type solutions for Jimbo-Miwa equation (8).

$$u_i(x, y, z, t) = \frac{2f_{ix}(x, y, z, t)}{f_i(x, y, z, t)}, \quad i = 1, 2, 3 \tag{26}$$

where $f_i(x, y, z, t)$ are the positive quadratic function solutions to the generalized bilinear Jimbo-Miwa equation (13), and

$$\begin{aligned}
 f_1(x, y, z, t) &= \left(a_{12}y + \frac{2a_{12}a_{34}}{3a_{31}}z + a_{10} \right)^4 + \left(\frac{3a_{23}a_{31}}{2a_{34}}y + a_{23}z + a_{20} \right)^2 \\
 &\quad + (a_{31}x + a_{34}t + a_{30})^2 + a_0, \\
 f_2(x, y, z, t) &= \left(\frac{a_{13}a_{32}}{a_{33}}y + a_{13}z + a_{10} \right)^4 + \left(a_{21}x - \frac{a_{31}a_{32}}{a_{21}}y \right. \\
 &\quad \left. + \frac{3a_{21}a_{33}}{2a_{32}}t - \frac{a_{31}a_{33}}{a_{21}}z + a_{20} \right)^2 \\
 &\quad + \left(a_{31}x + a_{32}y + \frac{3a_{31}a_{33}}{2a_{32}}t + a_{33}z + a_{30} \right)^2 + a_0, \\
 f_3(x, y, z, t) &= (a_{12}y + a_{13}z + a_{10})^4 + \left(a_{21}x - \frac{a_{31}a_{32}}{a_{21}}y + \frac{3a_{13}a_{21}}{2a_{12}}t \right. \\
 &\quad \left. - \frac{a_{13}a_{31}a_{32}}{a_{12}a_{21}}z + a_{20} \right)^2 + \left(a_{31}x + a_{32}y + \frac{3a_{13}a_{31}}{2a_{12}}t \right. \\
 &\quad \left. + \frac{a_{32}a_{13}}{a_{12}}z + a_{30} \right)^2 + a_0.
 \end{aligned} \tag{27}$$

It is also readily observed that at any given time t , the above high-order lump-type solutions $u_i \rightarrow 0$ if and only if the corresponding sum of squares $\xi_1^4 + \xi_2^2 + \xi_3^2 \rightarrow \infty$, namely,

$$\lim_{x^2+y^2+z^2 \rightarrow +\infty} u_i(x, y, z, t) = 0. \tag{28}$$

In order to exhibit the dynamical characteristics of these waves, we plot various three-dimensional, contour, and density plots as follows. We choose the following parameters to illustrate the high-order lump-type solution $u_2(x, y, z, t)$ for Jimbo-Miwa equation (8),

$$\begin{aligned}
 a_0 &= 2, \\
 a_{10} &= 1, \\
 a_{13} &= 2, \\
 a_{20} &= -1, \\
 a_{21} &= 3, \\
 a_{30} &= 2, \\
 a_{31} &= 4, \\
 a_{32} &= -5, \\
 a_{33} &= 3, \\
 z &= y.
 \end{aligned} \tag{29}$$

The physical properties and structures for the high-order lump-type solution $u_2(x, y, z, t)$ are described in Figure 1. Figure 1 shows the three-dimensional dynamic graphs A_1, B_1, C_1 , corresponding contour maps A_2, B_2, C_2 , and density plots A_3, B_3, C_3 in the (x, y) plane when $t = -6, 0, 6$, respectively. The three-dimensional graphs reflect the localized structures, and the density plots show the energy distribution.

Remark 6. By substituting a_0, a_{ij} in Cases 4–7 to (16) and (17) and using bilinear transformation (9), we obtain the new lump solutions for Jimbo-Miwa equation (8). Due to the lack of space, we omit the expressions of lump solutions.

4. Interaction Solutions between Lump and Soliton Solutions of the (3 + 1)-Dimensional Jimbo-Miwa Equation

In this section, we will study the general interaction solutions between the high-order lump-type solutions and other function solutions of (3 + 1)-dimensional Jimbo-Miwa equation (8).

Step 1. We use generalize bilinear Jimbo-Miwa equation (13).

Step 2. To get general interaction solutions, we take $N = 3, M = 4, n_1 = 2, n_2 = 1, n_3 = 1$ in (6), namely,

$$f = a_0 + \sum_{i=1}^3 \xi_i^{2n_i} + \sum_{j=1}^4 m_j g_j(\eta_j), \tag{30}$$

where $\xi_i (i = 1, 2, 3)$ are given in (17), and

$$\eta_j = b_{j0} + b_{j1}x + b_{j2}y + b_{j3}z + b_{j4}t, \quad j = 1, 2, 3, 4, \tag{31}$$

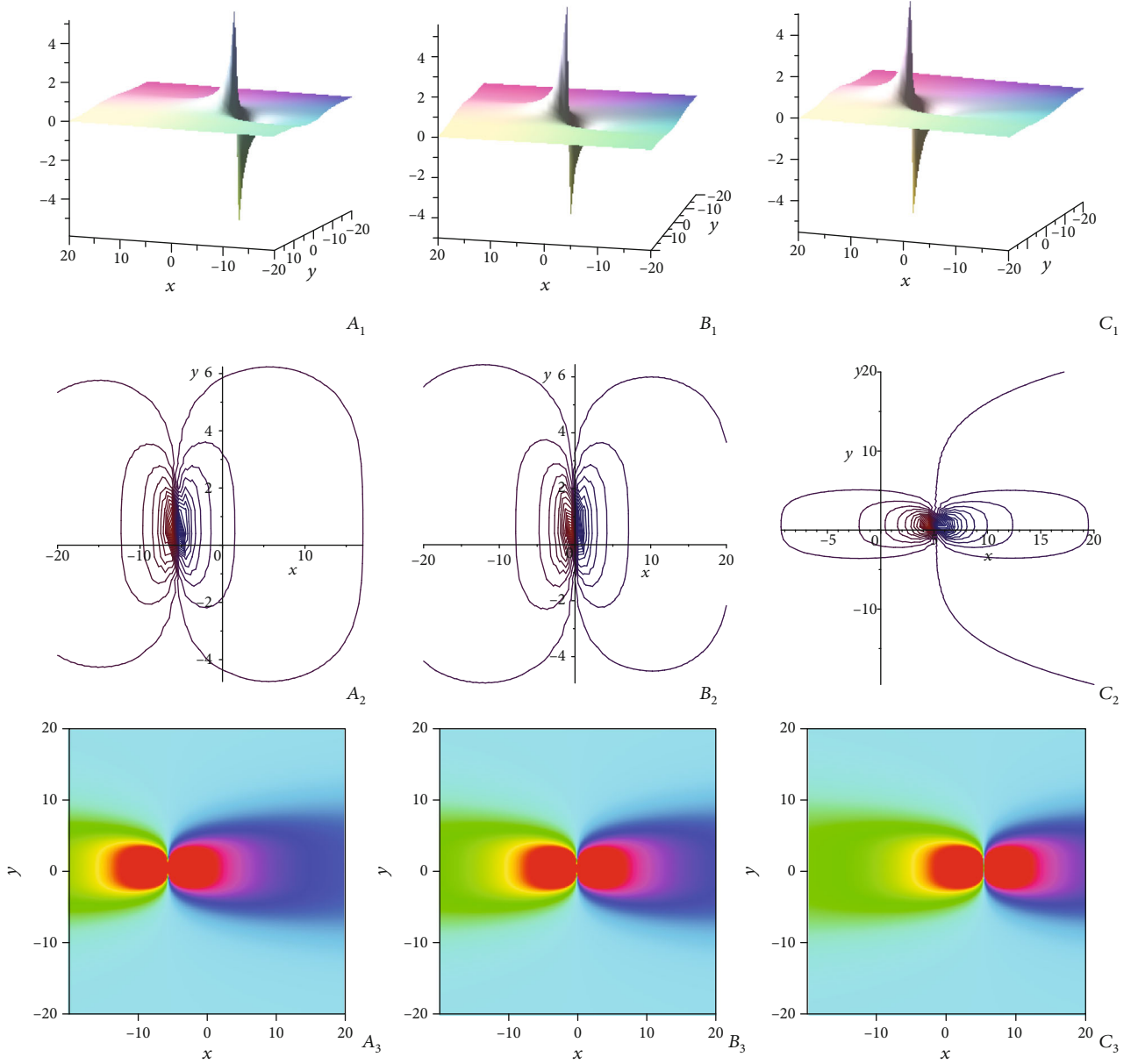


FIGURE 1: Three-dimensional plots, contour plots, and density plots of the wave with the parameters (29) at times $t = -6$ (A_1, A_2, A_3), $t = 0$ (B_1, B_2, B_3), and $t = 6$ (C_1, C_2, C_3).

where m_j, b_{jk} ($j = 1, 2, 3, 4; k = 0, 1, 2, 3, 4$) are arbitrary real constants. In order to obtain the interaction solution between the high-order lump-type solution and the double exponential function, the trigonometric function, and the hyperbolic function of (3 + 1)-dimensional Jimbo-Miwa equation (8), we suppose

$$g_1(\eta_1) = e^{\eta_1}, g_2(\eta_2) = e^{-\eta_2}, g_3(\eta_3) = \tan \eta_3, g_4(\eta_4) = \tanh \eta_4. \quad (32)$$

The interaction solution of generalized bilinear equation (13) is written the following form:

$$f = a_0 + \xi_1^4 + \xi_2^2 + \xi_3^2 + m_1 e^{\eta_1} + m_2 e^{-\eta_2} + m_3 \tan \eta_3 + m_4 \tanh \eta_4. \quad (33)$$

Step 3. By substituting (33) into Equation (13), collecting all terms with the same order of $x, y, z, t, e^{\eta_1}, e^{-\eta_2}, \tan \eta_3, \tanh \eta_4$ together, the left-hand side of Equation (13) is converted into another polynomial in $x, y, z, t, e^{\eta_1}, e^{-\eta_2}, \tan \eta_3, \tanh \eta_4$. Equating each coefficient of this different power terms to zero yields a set of nonlinear algebraic equations for a_0, a_{ij}, b_{jk}, m_j . Solving the algebraic equations by Maple, yields the following sets of solutions.

4.1. Between Lump and a Pair of Line-Soliton Solutions.

When $m_2 = m_1(m_1 \neq 0)$, $m_3 = m_4 = 0$ in (33), solution (33) represents the interaction solutions between the high-order lump-type solution and a pair of line-soliton solution $f = a_0 + \xi_1^4 + \xi_2^2 + \xi_3^2 + m_1 e^{\eta_1} + m_1 e^{-\eta_2}$,

Case 1.

$$\begin{aligned}
 a_{11} &= 0, \\
 a_{13} &= \frac{2a_{12}a_{24}}{3a_{21}}, \\
 a_{14} &= 0, \\
 a_{22} &= 0, \\
 a_{23} &= 0, \\
 a_{31} &= 0, \\
 a_{33} &= \frac{2a_{24}a_{32}}{3a_{21}}, \\
 a_{34} &= 0, \\
 b_{12} &= 0, \\
 b_{13} &= 0, \\
 b_{14} &= \frac{a_{24}b_{11}}{a_{21}}, \\
 b_{22} &= 0, \\
 b_{23} &= 0, \\
 b_{24} &= \frac{a_{24}b_{21}}{a_{21}}, \quad a_{21} \neq 0
 \end{aligned} \tag{34}$$

Case 2.

$$\begin{aligned}
 a_{11} &= 0, \\
 a_{14} &= 0, \\
 a_{21} &= \frac{2a_{12}a_{24}}{3a_{13}}, \\
 a_{22} &= 0, \\
 a_{23} &= 0, \\
 a_{31} &= \frac{2a_{12}a_{34}}{3a_{13}}, \\
 a_{32} &= 0, \\
 a_{33} &= 0, \\
 b_{12} &= 0, \\
 b_{13} &= 0, \\
 b_{14} &= \frac{3a_{13}b_{11}}{2a_{12}}, \\
 b_{22} &= 0, \\
 b_{23} &= 0, \\
 b_{24} &= \frac{3a_{13}b_{21}}{2a_{12}}, \quad a_{12}a_{13} \neq 0
 \end{aligned} \tag{35}$$

Case 3.

$$\begin{aligned}
 a_{10} &= 0, \\
 a_{11} &= 0, \\
 a_{12} &= \frac{3a_{13}a_{31}}{2a_{34}}, \\
 a_{14} &= 0, \\
 a_{21} &= 0, \\
 a_{22} &= 0, \\
 a_{23} &= 0, \\
 a_{24} &= 0, \\
 a_{32} &= 0, \\
 a_{33} &= 0, \\
 b_{12} &= 0, \\
 b_{13} &= 0, \\
 b_{14} &= \frac{a_{34}b_{11}}{a_{31}}, \\
 b_{22} &= 0, \\
 b_{23} &= 0, \\
 b_{24} &= \frac{a_{34}b_{21}}{a_{31}}, \quad a_{31}a_{34} \neq 0
 \end{aligned} \tag{36}$$

Case 4.

$$\begin{aligned}
 a_{11} &= 0, \\
 a_{14} &= 0, \\
 a_{21} &= 0, \\
 a_{22} &= 0, \\
 a_{23} &= 0, \\
 a_{24} &= 0, \\
 a_{31} &= 0, \\
 a_{33} &= \frac{a_{13}a_{32}}{a_{12}}, \\
 a_{34} &= 0, \\
 b_{12} &= 0, \\
 b_{13} &= 0, \\
 b_{14} &= \frac{3a_{13}b_{11}}{2a_{12}}, \\
 b_{22} &= 0, \\
 b_{23} &= 0, \\
 b_{24} &= \frac{3a_{13}b_{21}}{2a_{12}}, \quad a_{12} \neq 0
 \end{aligned} \tag{37}$$

Case 5.

$$\begin{aligned}
a_{11} &= 0, \\
a_{13} &= \frac{2a_{12}a_{34}}{3a_{31}}, \\
a_{14} &= 0, \\
a_{21} &= -\frac{a_{31}a_{32}}{a_{22}}, \\
a_{23} &= \frac{2a_{22}a_{34}}{3a_{31}}, \\
a_{24} &= -\frac{a_{32}a_{34}}{a_{22}}, \\
a_{33} &= \frac{2a_{32}a_{34}}{3a_{31}}, \\
b_{12} &= 0, \\
b_{13} &= 0, \\
b_{14} &= \frac{b_{11}a_{34}}{a_{31}}, \\
b_{20} &= b_{10}, \\
b_{21} &= b_{11}, \\
b_{22} &= 0, \\
b_{23} &= 0, \\
b_{24} &= \frac{b_{11}a_{34}}{a_{31}}, \quad a_{22}a_{31} \neq 0
\end{aligned} \tag{38}$$

Case 6.

$$\begin{aligned}
a_{11} &= 0, \\
a_{14} &= 0, \\
a_{21} &= 0, \\
a_{23} &= \frac{a_{13}a_{22}}{a_{12}}, \\
a_{24} &= 0, \\
a_{31} &= 0, \\
a_{32} &= \frac{a_{12}a_{33}}{a_{13}}, \\
a_{34} &= 0, \\
b_{12} &= 0, \\
b_{13} &= 0, \\
b_{14} &= \frac{3a_{13}b_{11}}{2a_{12}}, \\
b_{20} &= b_{10}, \\
b_{21} &= b_{11}, \\
b_{22} &= 0, \\
b_{23} &= 0, \\
b_{24} &= \frac{3a_{13}b_{11}}{2a_{12}}, \quad a_{12}a_{13} \neq 0
\end{aligned} \tag{39}$$

Case 7.

$$\begin{aligned}
a_{11} &= 0, \\
a_{13} &= \frac{2a_{12}a_{34}}{3a_{31}}, \\
a_{14} &= 0, \\
a_{21} &= 0, \\
a_{22} &= \frac{3a_{23}a_{31}}{2a_{34}}, \\
a_{24} &= 0, \\
a_{32} &= 0, \\
a_{33} &= 0, \\
b_{11} &= 0, \\
b_{13} &= \frac{2a_{34}b_{12}}{3a_{31}}, \\
b_{14} &= 0, \\
b_{20} &= b_{10}, \\
b_{21} &= 0, \\
b_{22} &= b_{12}, \\
b_{23} &= \frac{2a_{34}b_{12}}{3a_{31}}, \\
b_{24} &= 0, \quad a_{31}a_{34} \neq 0
\end{aligned} \tag{40}$$

where other parameters in Cases 1–7 are arbitrary real constants.

4.2. Between Lump and One Line-Soliton Solutions. When $m_1 = 0$ or $m_2 = 0$ and $m_3 = m_4 = 0$ in (33), solution (33) represents the interaction solutions between the high-order lump-type solution and one line-soliton solution $f = a_0 + \xi_1^4 + \xi_2^2 + \xi_3^2 + m_1 e^{\eta_1}$,

Case 1.

$$\begin{aligned}
m_2 &= 0, \\
a_{11} &= 0, \\
a_{14} &= 0, \\
a_{22} &= -\frac{a_{31}a_{32}}{a_{21}}, \\
a_{23} &= -\frac{a_{13}a_{31}a_{32}}{a_{12}a_{21}}, \\
a_{24} &= \frac{3a_{13}a_{21}}{2a_{12}}, \\
a_{33} &= \frac{a_{13}a_{32}}{a_{12}}, \\
a_{34} &= \frac{3a_{13}a_{31}}{2a_{12}}, \\
b_{11} &= 0, \\
b_{13} &= \frac{a_{13}b_{12}}{a_{12}}, \\
b_{14} &= 0, \quad a_{12}a_{21} \neq 0
\end{aligned} \tag{41}$$

Case 2.

$$\begin{aligned}
m_2 &= 0, \\
a_{11} &= 0, \\
a_{13} &= \frac{2a_{12}a_{34}}{3a_{31}}, \\
a_{14} &= 0, \\
a_{21} &= 0, \\
a_{22} &= \frac{3a_{23}a_{31}}{2a_{34}}, \\
a_{24} &= 0, \\
a_{32} &= 0, \\
a_{33} &= 0, \\
b_{11} &= 0, \\
b_{13} &= \frac{2a_{34}b_{12}}{3a_{31}}, \\
b_{14} &= 0, \quad a_{31}a_{34} \neq 0
\end{aligned} \tag{42}$$

Case 3.

$$\begin{aligned}
m_2 &= 0, \\
a_{11} &= 0, \\
a_{13} &= \frac{2a_{12}a_{24}}{3a_{21}}, \\
a_{14} &= 0, \\
a_{22} &= 0, \\
a_{23} &= 0, \\
a_{31} &= 0, \\
a_{33} &= \frac{2a_{24}a_{32}}{3a_{21}}, \\
a_{34} &= 0, \\
b_{12} &= 0, \\
b_{13} &= 0, \\
b_{14} &= \frac{a_{24}b_{11}}{a_{21}}, \quad a_{21} \neq 0
\end{aligned} \tag{43}$$

Case 4.

$$\begin{aligned}
m_2 &= 0, \\
a_{11} &= 0, \\
a_{14} &= 0, \\
a_{21} &= \frac{2a_{12}a_{24}}{3a_{13}},
\end{aligned}$$

$$\begin{aligned}
a_{22} &= 0, \\
a_{23} &= 0, \\
a_{31} &= \frac{2a_{12}a_{34}}{3a_{13}}, \\
a_{32} &= 0, \\
a_{33} &= 0, \\
b_{12} &= 0, \\
b_{13} &= 0, \\
b_{14} &= \frac{3a_{13}b_{11}}{2a_{12}}, \quad a_{12}a_{13} \neq 0
\end{aligned} \tag{44}$$

Case 5.

$$\begin{aligned}
m_2 &= 0, \\
a_{11} &= 0, \\
a_{12} &= \frac{3a_{13}a_{31}}{2a_{34}}, \\
a_{14} &= 0, \\
a_{21} &= 0, \\
a_{22} &= 0, \\
a_{23} &= 0, \\
a_{24} &= 0, \\
a_{32} &= 0, \\
a_{33} &= 0, \\
b_{12} &= 0, \\
b_{13} &= 0, \\
b_{14} &= \frac{a_{34}b_{11}}{a_{31}}, \quad a_{31}a_{34} \neq 0
\end{aligned} \tag{45}$$

Case 6.

$$\begin{aligned}
m_2 &= 0, \\
a_{11} &= 0, \\
a_{14} &= 0, \\
a_{21} &= 0, \\
a_{22} &= 0, \\
a_{23} &= 0,
\end{aligned}$$

$$\begin{aligned}
a_{24} &= 0, \\
a_{31} &= 0, \\
a_{33} &= \frac{a_{13}a_{32}}{a_{12}}, \\
a_{34} &= 0, \\
b_{12} &= 0, \\
b_{13} &= 0, \\
b_{14} &= \frac{3a_{13}b_{11}}{2a_{12}}, \quad a_{12} \neq 0
\end{aligned} \tag{46}$$

where other parameters in Cases 1–6 are arbitrary real constants.

Remark 7. When $m_1 = 0, \eta_2 = \eta_1$, we can obtain the same solutions as Cases 1–6.

4.3. Between Lump and Periodic Solitary Wave Solutions. When $\eta_3 = \eta_2 = \eta_1, m_2 = m_3 = m_4 = m_1 \neq 0$ in (33), the solution (33) represents the interaction solutions between the high-order lump-type solution and periodic solitary wave solutions $f = a_0 + \xi_1^4 + \xi_2^2 + \xi_3^2 + m_1 e^{\eta_1} + m_1 e^{-\eta_1} + m_1 \tan \eta_1 + m_1 \tanh \eta_4$,

Case 1.

$$\begin{aligned}
a_{11} &= 0, \\
a_{13} &= \frac{2a_{12}a_{34}}{3a_{31}}, \\
a_{14} &= 0, \\
a_{21} &= 0, \\
a_{22} &= \frac{3a_{23}a_{31}}{2a_{34}}, \\
a_{24} &= 0, \\
a_{32} &= 0, \\
a_{33} &= 0, \\
b_{11} &= 0, \\
b_{13} &= \frac{2a_{34}b_{12}}{3a_{31}}, \\
b_{14} &= 0, \\
b_{41} &= 0, \\
b_{43} &= \frac{2a_{34}b_{42}}{3a_{31}}, \\
b_{44} &= 0, \quad a_{31}a_{34} \neq 0
\end{aligned} \tag{47}$$

Case 2.

$$\begin{aligned}
a_{11} &= 0, \\
a_{14} &= 0, \\
a_{22} &= 0, \\
a_{23} &= 0, \\
a_{24} &= \frac{3a_{13}a_{21}}{2a_{12}}, \\
a_{31} &= \frac{2a_{12}a_{34}}{3a_{13}}, \\
a_{32} &= 0, \\
a_{33} &= 0, \\
b_{12} &= 0, \\
b_{13} &= 0, \\
b_{14} &= \frac{3a_{13}b_{11}}{2a_{12}}, \\
b_{42} &= 0, \\
b_{43} &= 0, \\
b_{44} &= \frac{3a_{13}b_{41}}{2a_{12}}, \quad a_{12}a_{13} \neq 0
\end{aligned} \tag{48}$$

Case 3.

$$\begin{aligned}
a_{11} &= 0, \\
a_{14} &= 0, \\
a_{22} &= -\frac{a_{31}a_{32}}{a_{21}}, \\
a_{23} &= -\frac{a_{13}a_{31}a_{32}}{a_{12}a_{21}}, \\
a_{24} &= \frac{3a_{13}a_{21}}{2a_{12}}, \\
a_{33} &= \frac{a_{13}a_{32}}{a_{12}}, \\
a_{34} &= \frac{3a_{13}a_{31}}{2a_{12}}, \\
b_{12} &= 0, \\
b_{13} &= 0, \\
b_{14} &= \frac{3a_{13}b_{11}}{2a_{12}}, \\
b_{42} &= 0, \\
b_{43} &= 0, \\
b_{44} &= \frac{3a_{13}b_{41}}{2a_{12}}, \quad a_{12}a_{21} \neq 0
\end{aligned} \tag{49}$$

Case 4.

$$\begin{aligned}
 a_{11} &= 0, \\
 a_{13} &= \frac{2a_{12}a_{34}}{3a_{31}}, \\
 a_{14} &= 0, \\
 a_{21} &= 0, \\
 a_{23} &= \frac{2a_{22}a_{34}}{3a_{31}}, \\
 a_{24} &= 0, \\
 a_{32} &= 0, \\
 a_{33} &= 0, \\
 b_{12} &= 0, \\
 b_{13} &= 0, \\
 b_{14} &= \frac{a_{34}b_{11}}{a_{31}}, \\
 b_{41} &= b_{11}, \\
 b_{42} &= 0, \\
 b_{43} &= 0, \\
 b_{44} &= \frac{a_{34}b_{41}}{a_{31}}, \quad a_{31} \neq 0
 \end{aligned} \tag{50}$$

Case 5.

$$\begin{aligned}
 a_{11} &= 0, \\
 a_{13} &= \frac{a_{12}a_{33}}{a_{32}}, \\
 a_{14} &= 0, \\
 a_{21} &= 0, \\
 a_{23} &= \frac{a_{22}a_{33}}{a_{32}}, \\
 a_{24} &= 0, \\
 a_{31} &= 0, \\
 a_{34} &= 0, \\
 b_{12} &= 0, \\
 b_{13} &= 0, \\
 b_{14} &= \frac{3a_{33}b_{11}}{2a_{32}}, \\
 b_{41} &= b_{11}, \\
 b_{42} &= 0, \\
 b_{43} &= 0, \\
 b_{44} &= \frac{3a_{33}b_{41}}{2a_{32}}, \quad a_{32} \neq 0
 \end{aligned} \tag{51}$$

Case 6.

$$\begin{aligned}
 a_{11} &= 0, \\
 a_{14} &= 0, \\
 a_{21} &= 0, \\
 a_{23} &= \frac{a_{13}a_{22}}{a_{12}}, \\
 a_{24} &= 0, \\
 a_{31} &= 0, \\
 a_{32} &= \frac{a_{12}a_{33}}{a_{13}}, \\
 a_{34} &= 0, \\
 b_{12} &= 0, \\
 b_{13} &= 0, \\
 b_{14} &= \frac{3a_{13}b_{11}}{2a_{12}}, \\
 b_{41} &= b_{11}, \\
 b_{42} &= 0, \\
 b_{43} &= 0, \\
 b_{44} &= \frac{3a_{13}b_{41}}{2a_{12}}, \quad a_{12}a_{13} \neq 0,
 \end{aligned} \tag{52}$$

where other parameters are arbitrary real constants.

4.4. *Between Lump and Solitary Wave Solutions.* When $\eta_2 = \eta_1, m_2 = m_1 \neq 0, m_3 = 0$ in (33), the solution (33) represents the interaction solutions between the high-order lump-type solution and solitary wave solutions:

$$f = a_0 + \xi_1^4 + \xi_2^2 + \xi_3^2 + m_1 e^{\eta_1} + m_1 e^{-\eta_1} + m_4 \tanh \eta_4 \tag{53}$$

Case 1.

$$\begin{aligned}
 a_{11} &= 0, \\
 a_{13} &= \frac{2a_{12}a_{34}}{3a_{31}}, \\
 a_{14} &= 0, \\
 a_{21} &= 0, \\
 a_{22} &= \frac{3a_{23}a_{31}}{2a_{34}}, \\
 a_{24} &= 0, \\
 a_{32} &= 0, \\
 a_{33} &= 0, \\
 b_{11} &= 0, \\
 b_{13} &= \frac{2a_{34}b_{12}}{3a_{31}}, \\
 b_{14} &= 0,
 \end{aligned}$$

$$\begin{aligned}
b_{41} &= 0, \\
b_{42} &= b_{12}, \\
b_{43} &= \frac{2a_{34}b_{42}}{3a_{31}}, \\
b_{44} &= 0, \quad a_{31}a_{34} \neq 0,
\end{aligned} \tag{54}$$

where other parameters are arbitrary real constants.

Remark 8. In addition to the above result Case 1, we can also get the same solutions as Cases 1–4 and 6 in Section 4.3 when $m_4 = m_1$ and the special result of Case 5 in Section 4.3 when $m_4 = m_1, a_{22} = 0$, respectively.

4.5. Between Lump and Tan Function Solutions. When $\eta_2 = \eta_1, m_2 = m_1 \neq 0, m_4 = 0$ in (33), the solution (33) represents the interaction solutions between the high-order lump-type solution and tan function solutions $f = a_0 + \xi_1^4 + \xi_2^2 + \xi_3^2 + m_1 e^{\eta_1} + m_1 e^{-\eta_1} + m_3 \tan \eta_3$,

Case 1.

$$\begin{aligned}
a_{11} &= 0, \\
a_{13} &= \frac{2a_{12}a_{34}}{3a_{31}}, \\
a_{14} &= 0, \\
a_{21} &= 0, \\
a_{22} &= \frac{3a_{23}a_{31}}{2a_{34}}, \\
a_{24} &= 0, \\
a_{32} &= 0, \\
a_{33} &= 0, \\
b_{11} &= 0, \\
b_{13} &= \frac{2a_{34}b_{12}}{3a_{31}}, \\
b_{14} &= 0, \\
b_{31} &= 0, \\
b_{32} &= b_{12}, \\
b_{33} &= \frac{2a_{34}b_{32}}{3a_{31}}, \\
b_{34} &= 0, \quad a_{31}a_{34} \neq 0,
\end{aligned} \tag{55}$$

where other parameters are arbitrary real constants.

Remark 9. In addition to the above result Case 1, we can also get the same solutions as Cases 1–6 in Section 4.3 when $m_3 = m_1, \eta_3 = \eta_4$, respectively.

4.6. Between Lump and Tan-Tanh Wave Solutions. When $m_1 = m_2 = 0, m_4 = m_3 \neq 0$ in (33), the solution (33) represents the interaction solutions between the high-order lump-type solution and tan-tanh wave solutions $f = a_0 + \xi_1^4 + \xi_2^2 + \xi_3^2 + m_3 \tan \eta_3 + m_3 \tanh \eta_4$,

Case 1.

$$\begin{aligned}
a_{11} &= 0, \\
a_{13} &= \frac{2a_{12}a_{34}}{3a_{31}}, \\
a_{14} &= 0, \\
a_{21} &= 0, \\
a_{23} &= \frac{2a_{22}a_{34}}{3a_{31}}, \\
a_{24} &= 0, \\
a_{32} &= 0, \\
a_{33} &= 0, \\
b_{32} &= 0, \\
b_{33} &= 0, \\
b_{34} &= \frac{a_{34}b_{31}}{a_{31}}, \\
b_{42} &= 0, \\
b_{43} &= 0, \\
b_{44} &= \frac{a_{34}b_{41}}{a_{31}}, \quad a_{31} \neq 0
\end{aligned} \tag{56}$$

Case 2.

$$\begin{aligned}
a_{11} &= 0, \\
a_{14} &= 0, \\
a_{21} &= 0, \\
a_{23} &= \frac{a_{13}a_{22}}{a_{12}}, \\
a_{24} &= 0, \\
a_{31} &= 0, \\
a_{33} &= \frac{a_{13}a_{32}}{a_{12}}, \\
a_{34} &= 0, \\
b_{32} &= 0, \\
b_{33} &= 0, \\
b_{34} &= \frac{3a_{13}b_{31}}{2a_{12}}, \\
b_{42} &= 0, \\
b_{43} &= 0, \\
b_{44} &= \frac{3a_{13}b_{41}}{2a_{12}}, \quad a_{12} \neq 0
\end{aligned} \tag{57}$$

Case 3.

$$\begin{aligned}
a_{10} &= 0, \\
a_{11} &= 0, \\
a_{13} &= \frac{2a_{12}a_{34}}{3a_{31}}, \\
a_{14} &= 0, \\
a_{21} &= -\frac{a_{31}a_{32}}{a_{22}}, \\
a_{23} &= \frac{2a_{22}a_{34}}{3a_{31}}, \\
a_{24} &= -\frac{a_{32}a_{34}}{a_{22}}, \\
a_{33} &= \frac{2a_{32}a_{34}}{3a_{31}}, \\
b_{32} &= 0, \\
b_{33} &= 0, \\
b_{34} &= \frac{a_{34}b_{31}}{a_{31}}, \\
b_{42} &= 0, \\
b_{43} &= 0, \\
b_{44} &= \frac{a_{34}b_{41}}{a_{31}}, \quad a_{22}a_{31} \neq 0
\end{aligned} \tag{58}$$

Case 4.

$$\begin{aligned}
a_{10} &= 0, \\
a_{11} &= 0, \\
a_{12} &= \frac{3a_{13}a_{31}}{2a_{34}}, \\
a_{14} &= 0, \\
a_{21} &= -\frac{a_{31}a_{32}}{a_{22}}, \\
a_{23} &= \frac{2a_{22}a_{34}}{3a_{31}}, \\
a_{24} &= -\frac{a_{32}a_{34}}{a_{22}}, \\
a_{30} &= \frac{a_{20}a_{32}}{a_{22}}, \\
a_{33} &= \frac{2a_{32}a_{34}}{3a_{31}}, \\
b_{32} &= 0, \\
b_{33} &= 0, \\
b_{34} &= \frac{a_{34}b_{31}}{a_{31}}, \\
b_{42} &= 0, \\
b_{43} &= 0, \\
b_{44} &= \frac{a_{34}b_{41}}{a_{31}}, \quad a_{22}a_{31}a_{34} \neq 0
\end{aligned} \tag{59}$$

Case 5.

$$\begin{aligned}
a_{10} &= 0, \\
a_{11} &= 0, \\
a_{14} &= 0, \\
a_{21} &= \frac{2a_{12}a_{24}}{3a_{13}}, \\
a_{22} &= 0, \\
a_{23} &= 0, \\
a_{31} &= \frac{2a_{12}a_{34}}{3a_{13}}, \\
a_{32} &= 0, \\
a_{33} &= 0, \\
b_{32} &= 0, \\
b_{33} &= 0, \\
b_{34} &= \frac{3a_{13}b_{31}}{2a_{12}}, \\
b_{42} &= 0, \\
b_{43} &= 0, \\
b_{44} &= \frac{3a_{13}b_{41}}{2a_{12}}, \quad a_{12}a_{13} \neq 0
\end{aligned} \tag{60}$$

Case 6.

$$\begin{aligned}
a_{10} &= 0, \\
a_{11} &= 0, \\
a_{13} &= \frac{a_{12}a_{33}}{a_{32}}, \\
a_{14} &= 0, \\
a_{21} &= \frac{2a_{24}a_{32}}{3a_{33}}, \\
a_{22} &= 0, \\
a_{23} &= 0, \\
a_{31} &= 0, \\
a_{34} &= 0, \\
b_{32} &= 0, \\
b_{33} &= 0, \\
b_{34} &= \frac{3a_{33}b_{31}}{2a_{32}}, \\
b_{42} &= 0, \\
b_{43} &= 0, \\
b_{44} &= \frac{3a_{33}b_{41}}{2a_{32}}, \quad a_{32}a_{33} \neq 0,
\end{aligned} \tag{61}$$

where other parameters are arbitrary real constants.

Remark 10. In addition to the above results Cases 1–6, we can also get the same solutions as Cases 1–3 in Section 4.3 when $\eta_3 = \eta_1$, respectively.

Step 4. By substituting the parameters a_0, a_{ij}, b_{jk}, m_j in the Sections 4.1–4.6 into the solution (33) and using transformation (9), we can obtain abundant interaction solutions of Jimbo-Miwa equation (1).

These sets of solutions for the parameters generate 42 classes of combination solutions $f_i, 1 \leq i \leq 42$ to the generalized bilinear Jimbo-Miwa equation (13), and then, the resulting combination solutions present 42 classes of interaction solutions $u_i, 1 \leq i \leq 42$ to Equation (8) under transformation (9). Therefore, various kinds of interaction solutions could be constructed explicitly this way.

As the example, substituting (38) into (33), we can get f as follows:

$$\begin{aligned} f(x, y, z, t) = & a_0 + \left(a_{10} + a_{12}y + \frac{2a_{12}a_{34}}{3a_{31}}z \right)^4 \\ & + \left(a_{20} - \frac{a_{31}a_{32}}{a_{22}}x + a_{22}y + \frac{2a_{22}a_{34}}{3a_{31}}z - \frac{a_{32}a_{34}}{a_{22}}t \right)^2 \\ & + \left(a_{30} + a_{31}x + a_{32}y + \frac{2a_{32}a_{34}}{3a_{31}}z + a_{34}t \right)^2 \\ & + m_1 e^{\eta_1} + m_1 e^{-\eta_2}, \end{aligned} \quad (62)$$

where

$$\eta_1 = b_{10} + b_{11}x + \frac{b_{11}a_{34}}{a_{31}}t, \eta_2 = b_{10} + b_{11}x + \frac{b_{11}a_{34}}{a_{31}}t. \quad (63)$$

By using transformation (9), we get the interaction solution between the high-order lump-type solutions and a pair of line-soliton solution of (3 + 1)-dimensional Jimbo-Miwa equation (8)

$$u(x, y, z, t) = \frac{2f_x(x, y, z, t)}{f(x, y, z, t)}, \quad (64)$$

where $f(x, y, z, t)$ is given in (62).

In order to exhibit the dynamical characteristics of these waves, we plot various three-dimensional, contour, and density plots as follows. We choose the following parameters to illustrate interaction solution (64),

$$\begin{aligned} a_0 &= 0.5, \\ a_{10} &= 1, \\ a_{12} &= 1, \\ a_{20} &= -1, \\ a_{22} &= -1, \\ a_{30} &= 1, \end{aligned}$$

$$\begin{aligned} a_{31} &= -2, \\ a_{32} &= -2, \\ a_{34} &= -10, \\ b_{10} &= -1, \\ b_{11} &= 1.3, \\ m_1 &= 1, z = y. \end{aligned} \quad (65)$$

The physical properties and structures for interaction solution (64) are shown in Figure 2. Figure 2 shows the three-dimensional dynamic graphs A_1, B_1, C_1 , corresponding contour maps A_2, B_2, C_2 , and density plots A_3, B_3, C_3 in the (x, y) -plane when $t = -1, 0$, and 1 , respectively. The three-dimensional graphs reflect the localized structures, and the density plots show the energy distribution. We can see that the high-order lump-type wave and the exponential function wave react with each other.

When we choose the following parameters and $t = -2$, we illustrate interaction solution (64) of (3 + 1)-dimensional Jimbo-Miwa equation (8),

$$\begin{aligned} a_0 &= 5, \\ a_{10} &= 6, \\ a_{12} &= 2, \\ a_{20} &= 1, \\ a_{22} &= -1, \\ a_{30} &= 1, \\ a_{31} &= 5, \\ a_{32} &= 2, \\ a_{34} &= -1, \\ b_{10} &= -1, \\ b_{11} &= 1.5, \\ m_1 &= -1, z = y. \end{aligned} \quad (66)$$

The physical properties and structures for interaction solution (64) are shown in Figure 3. Figure 3 shows the three-dimensional dynamic graph D_1 , corresponding contour map D_2 , and density plot D_3 in the (x, y) -plane, respectively.

5. Conclusion

In this paper, we gave the form of rational solution and their interaction solution to NLEE. The rational solution contained lump solution, general lump solution, high-order lump solution, lump-type solution, etc. The general interaction solution contain the classical interaction solution, such as the lump-kink solution and the lump-soliton solution. As the example, by using the generalized bilinear method and symbolic computation Maple, we successfully constructed

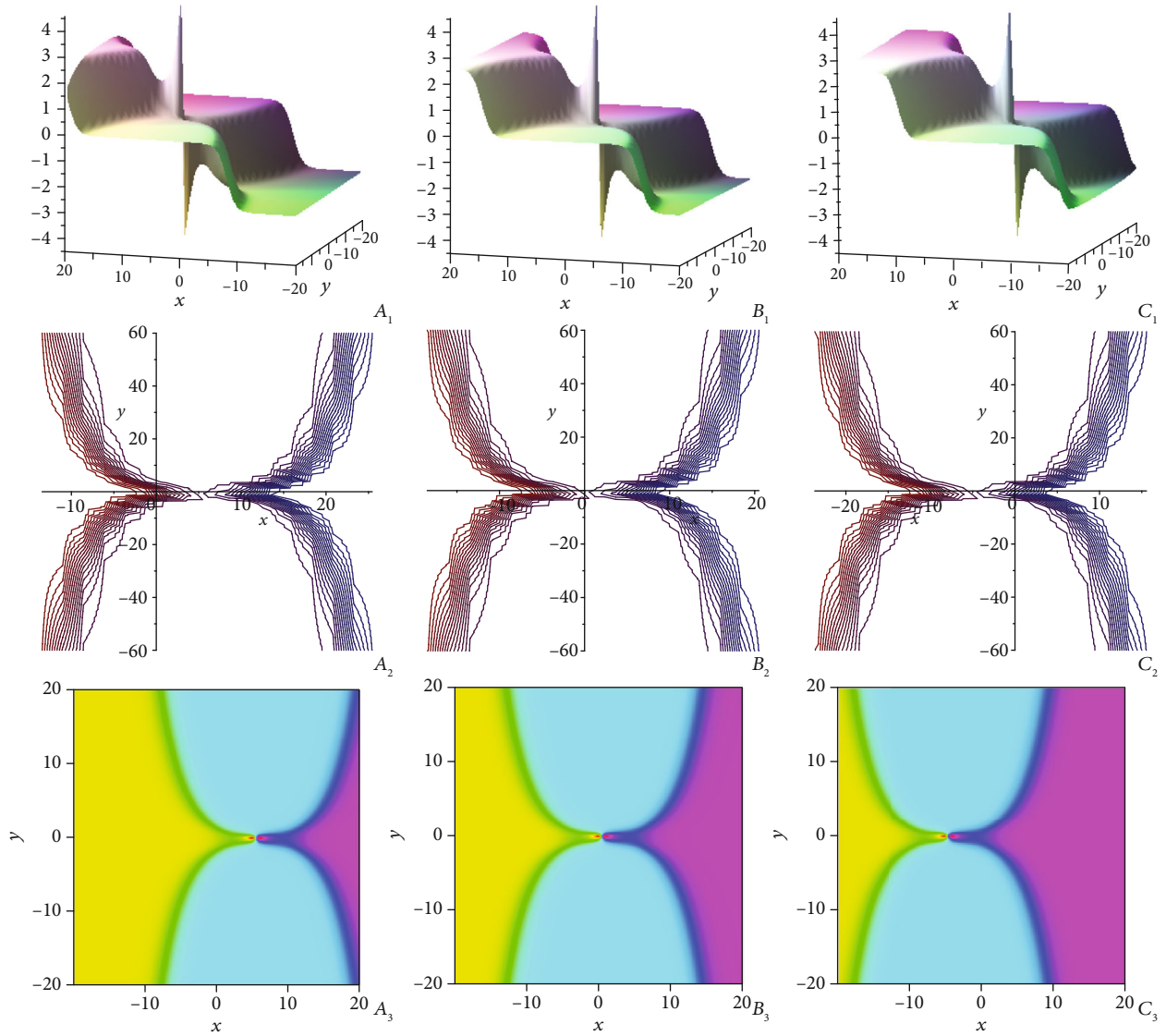


FIGURE 2: Three-dimensional plots, contour plots, and density plots of the wave with parameters (65) at times $t = -1$ (A_1, A_2, A_3), $t = 0$ (B_1, B_2, B_3), and $t = 1$ (C_1, C_2, C_3).

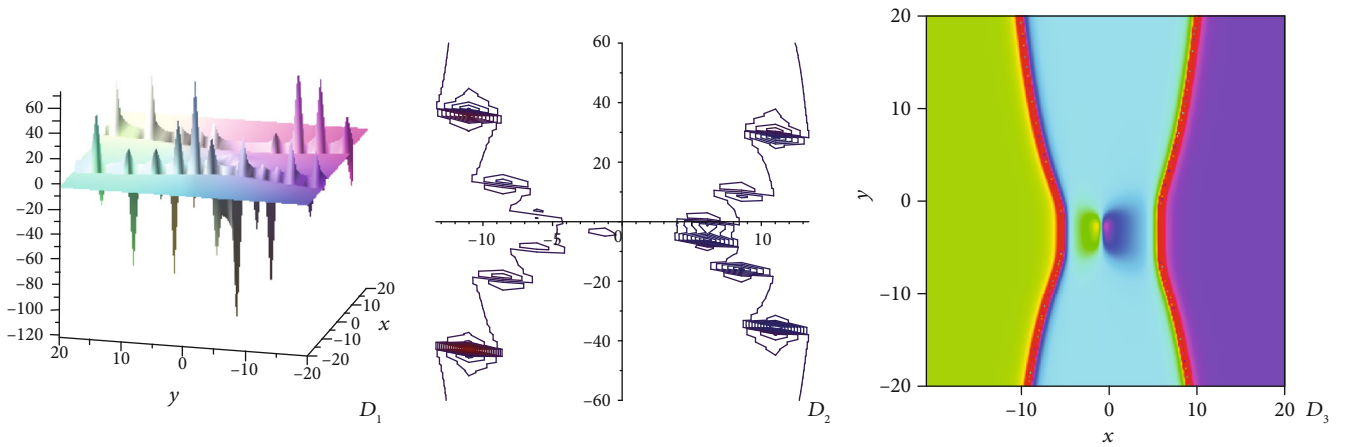


FIGURE 3: Three-dimensional plot, contour plot, and density plot of the wave with parameters (66) at time $t = -2$.

the high-order lump-type solutions and their interaction solutions between lumps and other function solutions under certain constraints of $(3 + 1)$ -dimensional Jimbo-Miwa equation. Three-dimensional plots, contour plots, and density plots of these waves are observed in Figures 1–3, respectively. We can find the physical structure and characteristics of the interactions between the high-order lump-type solutions and the exponential function wave.

Many researchers have studied the exact solutions of $(3 + 1)$ -dimensional Jimbo-Miwa equation, such as the generalized solitary solutions [6]; the various travelling wave solutions [7]; and one-soliton, two-soliton and dromion solutions [11], and multiple-soliton solutions [12, 13]; periodic solitary wave solutions [14]; several interaction solutions (the interaction phenomenon between the exponential function, the cosine function, and the hyperbolic cosine function; the interaction phenomenon between the exponential function, the sine function, and the hyperbolic sine function) [20, 21]; lump-type solutions ($N = 2, n_1 = n_2 = 1$); lump-kink solution ($N = 2, n_1 = n_2 = 1, M = 1, g_1(\eta_1) = e^{\eta_1}$); and lump-soliton solution [36–41]

$$N = 2, n_1 = n_2 = 1, M = 1, g_1(\eta_1) = \cosh \eta_1. \quad (67)$$

The obtained new high-order lump-type solutions (16) and their interaction solutions (33) in this paper are different from the lump-type solutions and the interaction solutions in [36–41]. These solutions will greatly expand the exact solutions of $(3 + 1)$ -dimensional Jimbo-Miwa equation on the existing literature [6, 7, 11–14, 20, 21, 36–41, 58]. These results are significant to understand the propagation processes for nonlinear waves in fluid mechanics and the explanation of some special physical problems.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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