Research Article

M-Breather, Lumps, and Soliton Molecules for the (2 + 1)-Dimensional Elliptic Toda Equation

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The (2 + 1)-dimensional elliptic Toda equation is a higher dimensional generalization of the Toda lattice and also a discrete version of the Kadomtsev-Petviashvili-1 (KP1) equation. In this paper, we derive the M-breather solution in the determinant form for the (2 + 1)-dimensional elliptic Toda equation via Bäcklund transformation and nonlinear superposition formulae. The lump solutions of the (2 + 1)-dimensional elliptic Toda equation are derived from the breather solutions through the degeneration process. Hybrid solutions composed of two line solitons and one breather/lump are constructed. By introducing the velocity resonance to the N-soliton solution, it is found that the (2 + 1)-dimensional elliptic Toda equation possesses line soliton molecules, breather-soliton molecules, and breather molecules. Based on the N-soliton solution, we also demonstrate the interactions between a soliton/breather-soliton molecule and a lump and the interaction between a soliton molecule and a breather. It is interesting to find that the KP1 equation does not possess a line soliton molecule, but its discrete version—the (2 + 1)-dimensional elliptic Toda equation—exhibits line soliton molecules.

1. Introduction

The Toda lattice is an integrable one-dimensional lattice model which originally describes the motion of a chain of particles due to nearest neighbor interaction through an exponential potential function [1]. The Toda equation takes the form

\[ v_{tt}(n) = e^{v(n-1) - v(n)} - e^{v(n) - v(n+1)}, \quad n \in \mathbb{Z}, \]

(1)

which is the equation of motion for the nth particle. Here, we denote \( v(n, t) \) as \( v(n) \) for simplicity. This equation also describes nonlinear wave propagation in many areas of physics such as ladder circuits [2], biophysics [3], and elementary particle physics [4]. The (2 + 1)-dimensional elliptic Toda lattice which is a natural dimensional generalization of the Toda lattice (1) reads

\[ \Delta v(n) = e^{v(n-1) - v(n)} - e^{v(n) - v(n+1)}, \quad n \in \mathbb{Z}, \]

(2)

where \( \Delta = \partial_{xx} + \partial_{yy} \) is a two-dimensional Laplacian operator. It first appears in connection with Laplace-Darboux transformation for general second-order partial differential equations in the work of Darboux in 1887 [5]. In 1979, the integrability of the (2 + 1)-dimensional Toda lattice was established through the inverse scattering method [6, 7] and Lie group theory [8]. The (2 + 1)-dimensional Toda lattice and its relatives have important applications in 2D gravity [9, 10], string theory [11, 12], differential geometry [13], and random matrices and orthogonal polynomials [14, 15]. The Bessel-type solutions for the (2 + 1)-dimensional elliptic Toda lattice (1) were derived in [16], and its various classes of special solutions such as lump-type solutions, periodic solutions, and line solitons were investigated via the inverse scattering transform in [17]. In [18], the rational solution and breather solution for (2) were studied applying the Hirota bilinear method. Nakamura derived exact solutions for the (2 + 1)-dimensional cylindrical Toda equation and the (3 + 1)-dimensional elliptic Toda equation in [19, 20]. In [21], three classes of lump solutions for (2) were constructed through symbolic computation.

The study of the nonlinear localized waves such as solitons, breathers, lumps, and rough waves has attracted great attention due to their important applications in nonlinear
physical areas such as nonlinear optics, biophysics, oceanography, Bose-Einstein condensates, and plasma [22–27]. A breather is a special localized solitary wave that is periodic in space or time. Breathers have important applications in many physical areas such as optics, hydrodynamics, and quantized superfluid [28–30]. A lump solution is a kind of two-dimensional localized wave that decays algebraically in all directions [31]. Bäcklund transformation, which owes its origin to classical differential geometry in the 19th century, is an important tool in studying nonlinear integrable equations [32, 33]. The Bäcklund transformations and their associated nonlinear superposition formulae allow the generation of the various solutions of the nonlinear equations by purely algebraic procedures. In Hirota bilinear formalism, the original bilinear equation is bilinear in the dependent variables, whereas its bilinear Bäcklund transformations are linear in both the old and new dependent variables; therefore, one only needs to solve a set of linear partial differential equations to obtain new solutions from old ones [34]. Combining the bilinear Bäcklund transformation and associated nonlinear superposition formulae, we may derive an infinite sequence of solutions for nonlinear equations. In this paper, we derive an $M$-breather solution in the determinant form for the $(2 + 1)$-dimensional elliptic Toda equation (2) by applying the bilinear Bäcklund transformation and associated nonlinear superposition formulae. We also obtain its lump solutions by taking the infinite period of the breathers. Some bound states of solitons such as soliton molecules, breather molecules, and breather-soliton molecules have been theoretically and experimentally found in optics [35, 36] and Bose-Einstein condensation [37]. They are of great interest for applications in optical technologies because they would provide a doubling of the data-carrying capacity of the fiber [38, 39]. The velocity resonance mechanism has been proposed in [40] to study the soliton molecule. Many novel soliton molecules such as dark soliton molecules, dro-mion molecules, breather molecules, and breather-soliton molecules for continuous nonlinear wave equations have been found by utilizing this method [41–43]. However, the soliton molecules for the discrete nonlinear wave equations have not been reported yet. In this paper, we discuss the resonant structures for the solitons such as line soliton molecules, breather-soliton molecules, and breather molecules for the $(2 + 1)$-dimensional elliptic Toda equation via the velocity resonance.

The paper is organized as follows. In Section 2, the $M$-breather solution and hybrid solution composed of line solitons and breathers for the $(2 + 1)$-dimensional elliptic Toda equation are derived via the Bäcklund transformation and nonlinear superposition formulae. In addition, we analyze the dynamical properties of 1-breather and 2-breather. In Section 3, we derive the lump solutions for the $(2 + 1)$-dimensional elliptic Toda equation by taking the infinite period of the breathers. Furthermore, we construct hybrid solutions consisting of line solitons, breathers, and lumps. In Section 4, line soliton molecules, breather-soliton molecules, and breather molecules for the $(2 + 1)$-dimensional elliptic Toda equation are investigated through the velocity resonance mechanism, and interactions between soliton molecules and breathers/lumps are illustrated. A summary and discussion are given in Section 5.

2. $M$-Breather of the $(2 + 1)$-Dimensional Elliptic Toda Equation

By introducing $u(n) = e^{i(n-1)-\eta(n)}$, the $(2 + 1)$-dimensional elliptic Toda equation (2) can be written as

$$
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \ln u(n) = u(n + 1) - 2u(n) + u(n - 1),
$$

(3)

where we denote $u(n, x, y)$ as $u(n)$ for simplicity. Through the dependent variable transformation $u(n) = (f(n + 1)f(n - 1))/f(n)^2$, equation (3) can be transformed into the bilinear form

$$
(D^2_x + D^2_y)f(n) \cdot f(n) = (2e^{Dn} - 2)f(n) \cdot f(n).
$$

(4)

The bilinear equation (4) admits the following Bäcklund transformation:

$$
(D_x + iD_y + \lambda^{-1}e^{-D_n} + \mu)f(n) \cdot g(n) = 0,
$$

$$
((D_x - iD_y)e^{-1/2D_n} - \lambda e^{1/2D_n} + ye^{-1/2D_n})f(n) \cdot g(n) = 0.
$$

(5)

Here, the bilinear operators $D^m_xD^n_y$ and $e^{Dn}$ are defined by [44]

$$
D^m_xD^n_yf \cdot g = \left. \frac{\partial^m}{\partial y^m} \frac{\partial^n}{\partial x^n} f(x + y, t + s)g(x - y, t - s) \right|_{y=0, s=0},
$$

$$
e^{Dn}f(n) \cdot g(n) = f(n + 1) \cdot g(n - 1),
$$

$$
e^{-Dn}f(n) \cdot g(n) = f(n - 1) \cdot g(n + 1).
$$

(6)

If we take $f(n) = 1$, $\mu = -\lambda^{-1}$, and $y = \lambda$ in equation (5), then we obtain $g(n) = 1 + e^{\phi}$, $n = px + qy + rn + s\beta$ and $\lambda = (p - iq)/(1 - e^{r})$ and the dispersion relation for the $(2 + 1)$-dimensional elliptic Toda equation (3):

$$
p^2 + q^2 = 4 \sinh^2 \frac{r}{4}.
$$

(7)

We apply the following nonlinear superposition formula presented in [45] to derive the 1-breather solution for equation (3).

**Proposition 1.** Let $f_0(n)$ be a nonzero solution of equation (5) and suppose that $f_i(n)$ and $f_j(n)$ are solutions of (5) such that $f_0(n) \rightarrow f_i(n)(i = 1, 2)$; then, there exists the following nonlinear superposition formula:
Figure 1: (a) 1-breather. (b) Density picture of 1-breather.

Figure 2: Propagation of 2-breather for $L_1, L_2 > 0$ at different times: (a) $n = -30$, (b) $n = -8$, and (c) $n = 35$. 
Consequently, equation (9), we get

\[ e^{-i/2D}f_0(n) \cdot f_{12}(n) = c\left(\lambda_1 e^{-i/2D} - \lambda_2 e^{i/2D}\right)f_1(n) \cdot f_2(n), \]

(8)

where \( f_{12}(n) \) is a new solution of (4) related to \( f_1(n) \) and \( f_2(n) \) with parameters \( \lambda_2 \) and \( \lambda_1 \), respectively. Here, \( c \) is a nonzero constant.

By taking \( c = 1/(\lambda_1 - \lambda_2) \), \( f_0(n) = 1 \), and \( f_1(n) = 1 + \delta_i \) (\( i = 1, 2 \)) in nonlinear superposition formula (8), we derive

\[ f_{12}(n) = 1 + \delta_i + \eta_i + A_{12}e^{\eta_i}, \]

(9)

where \( \lambda_i = (p_i - iq_i)/(1-e^{-r_i}) \) and \( \eta_i = p_i x + q_i y + r_i n + \beta_i^0 \) (\( i = 1, 2 \)). According to dispersion relation (7), we may take \( p_i = 2 \sinh k_i \), \( q_i = 2 \sinh k_i \sin l_i \), \( r_i = 2k_i \) (\( i = 1, 2 \)). Consequently,

\[ A_{12} = \cos (l_i - l_2) - \cosh (k_i - k_2) \cos (l_i - l_2) - \cosh (k_i + k_2). \]

(10)

Furthermore, if we take \( k_1 = k_1^* = \alpha_i + \beta_i^i, l_1 = l_2 = \gamma_1 + \delta_i, \beta_0^i = \beta_0^* = \ln (a/2) + i\theta\), and \( e^{\theta_i} = -(1/b^i)e^{r_i, \delta_i^i} \) in equation (9), we get

\[ f^{ib}(n) = 1 - \frac{a^2}{b^2}e^{\beta_i} \cos I_1 + \frac{A_{12}a^2}{4b^4}e^{2R_i}, \]

(11)

where

\[ R_1 = 2\left(\alpha_i x + \beta_i y + \alpha_i n \right) + \sigma_i, \]

\[ I_1 = 2\left(\alpha_i x + \beta_i y + \beta_i n \right) + \theta_i, \]

\[ \tilde{a}_i = (P_1 \cos \gamma_i + Q_1 \sin \gamma_i), \]

\[ \tilde{b}_i = (P_1 \sin \gamma_i - Q_1 \cos \gamma_i), \]

\[ \tilde{c}_i = (\xi_i \cos \gamma_i - \zeta_i \sin \gamma_i), \]

\[ \tilde{d}_i = (\xi_i \sin \gamma_i + \zeta_i \cos \gamma_i), \]

\[ P_i = \sinh \alpha_i \cosh \beta_i, \]

\[ Q_i = \cosh \alpha_i \sinh \beta_i, \]

\[ \xi_i = \cosh \alpha_i \cosh \beta_i, \]

\[ \zeta_i = \sinh \alpha_i \sinh \beta_i, \]

and \( \alpha_i, \beta_i, \gamma_i, \delta_i, \sigma_i, \) and \( \theta_i \) are arbitrary real-valued.
constants. Since $k_j = k_2 = \alpha_j + \beta_j i$ and $l_j = l_2 = \gamma_j + \delta_j i$, we can get

$$A_{12} = \frac{\cosh 2\delta_j - \cos 2\beta_j}{\cosh 2\delta_j - \cos 2\alpha_j}. \quad (13)$$

By substituting equation (11) into $u(n) = (f(n + 1)f(n - 1))/f(n)^2$, we derive the 1-breather solution for equation (3) as follows:

$$u(n)_{16} = \frac{A_{12}(\cosh^2(R_j + \epsilon) + \sinh^2 2\alpha_j) + \cosh I_j - \sin I_j - 2\sqrt{A_{12}^2}}{(\sqrt{A_{12}^2 \cosh (R_j + \epsilon) - \cos I_j)}^2}, \quad (14)$$

where $\kappa = \cos 2\beta_j \cosh 2\alpha_j \cos I_j \cosh (R_j + \epsilon) + \sin 2\beta_j \sinh 2\alpha_j \sin I_j \sinh (R_j + \epsilon)$ and $\epsilon = \ln (a\sqrt{A_{12}^2} 2^\delta).$

To obtain the nonsingular solution, we impose the condition $A_{12} > 1$. Figure 1 shows the 1-breather (14) with $\alpha_j = 0.5$, $\beta_j = 0.5$, $\gamma_j = 5$, $\delta_j = 1$, $n = 0$, $a = 1$, $b = 1$, $\sigma = 0$, and $\theta = 0$. Its top trace is a line $l_1$ on the $(x,y)$-plane for a given time $n$ (see Figure 1(b)), which is defined by $l_1 : 2(\tilde{a}_1 x + \tilde{b}_1 y + \alpha_j x + \gamma_j y + 1) = 0$. The period of 1-breather (14) is $T_{[a]} = (Q_1 \cos \gamma_1 - P_1 \sin \gamma_1)\pi/W$ along the $x$-direction and $T_{[y]} = (P_1 \cos \gamma_1 + Q_1 \sin \gamma_1)\pi/W$ for the $y$-direction, where $W = \sinh \delta_1 \cosh \delta_1 (\sinh^2 \alpha_i\cos^2 \beta_j + \cosh^2 \alpha_i \sin^2 \beta_j)$. Then, the distance of two neighboring peaks in 1-breather (14) is

$$T = \frac{\pi \sqrt{P_1^2 + Q_1^2}}{W}. \quad (15)$$

Note that $T$ is the period of 1-breather $u(n)_{16}$.

**Proposition 2.** The elliptic Toda equation admits the general nonlinear superposition formula [45]

$$e^{i/2D_x} f_{N-1}(n) \cdot \hat{f}_N(n) = c_N \left( \lambda_N e^{i/2D_x} - \hat{\lambda}_N e^{i/2D_x} \right) \hat{f}_N(n) \cdot \hat{f}_N(n), \quad (16)$$

Figure 4: The degeneration of 1-breather (14) with $y = 0.1$ and $\delta = 0.7$: (a) $\alpha = \beta = 1/4$, (b) $\alpha = \beta = 1/10$, (c) $\alpha = \beta = 1/60$, and (d) 1-lump.
where

$$f_N = c_N |I(n), \cdots, N(n)|,$$

$$\tilde{f}_N(n) = |I(n), \cdots, N - 1(n), N + 1(n)|,$$

$$j(n) = \left(\varphi_j(n), \cdots, (-\lambda_j)^{N-1} \varphi_j(n - N + 1)\right)^T.$$

If we take $$c_N = \prod_{1 \leq i < j \leq N} 1/(\lambda_j - \lambda_i), \varphi_j(n) = 1 + e^{\theta_j}, \lambda_j = (p_j - iq_j)/(1 - e^{\gamma_j}),$$ and $$\eta_j = p_j x + q_j y + r_j n + \beta_j = 2 \sinh l_j x + 2 \sinh k_j y + 2k_j n + \beta_j^0 (j = 1, 2, \cdots, N)$$ in non-linear superposition formula (17), we derive N-soliton as

$$f = \sum_{\mu = 0 \leq l}^N \exp \left(\sum_{i<j}^N \mu_i B_{ij} + \sum_{j=1}^N \mu_j H_j\right).$$

where $$e^{\lambda_i j} \triangleq A_{ij} = (\cos (l_i - l_j) - \cosh (k_i - k_j))/(\cos (l_i - l_j) - \cosh (k_i + k_j)), \sum_{\mu = 0 \leq l}^N$$ implies the summation over all possible combinations of $$\mu_1 = 0.1, \mu_2 = 0.1, \cdots, \mu_N = 0.1,$$ and $$\sum_{j=1}^N$$ indicates the summation over all possible pairs chosen from N elements.

By taking $$N = 2M, c_N = \prod_{1 \leq i < j \leq N} 1/(\lambda_j - \lambda_i), \varphi_j(n) = 1 + e^{\theta_j}, k_{j+M} = k_j, l_{j+M} = l_j,$$ and $$\beta_j^0 = \beta_j^0 (j = 1, 2, \cdots, M)$$ in equation (17), we derive the determinant form of M-breather for the (2 + 1)-dimensional elliptic Toda equation (3) under certain nonsingular conditions. When $$M = 2,$$ we derive the following 2-breather for equation (3):

$$f = 1 - \frac{e^{\theta_1}}{b_1^2} \cos I_1 - \frac{e^{\theta_2}}{b_2^2} \cos I_2 + \frac{K_1}{4b_1^2} e^{2R_1} + \frac{K_2}{4b_2^2} e^{2R_2} + \frac{K_1 K_2}{16b_1^2 b_2^2} e^{2R_1 + 2R_2} + \frac{e^{R_1 + R_2}}{2b_1^2 b_2^2} (I_1 \cos (I_1 + I_2) - M_1 \sin (I_1 + I_2)) + \frac{e^{R_1 + R_2}}{2b_1^2 b_2^2} (I_2 \cos (I_1 - I_2) - M_2 \sin (I_1 - I_2))$$

Figure 5: The degeneration of 2-breather (23) with $$\beta_1 = \beta_2 = 0, \gamma_1 = \gamma_2 = 0, \delta_1 = 1.7/\sqrt{3},$$ and $$\delta_2 = 1.6/\sqrt{3};$$ (a) $$\alpha_1 = \alpha_2 = 0.45,$$ (b) $$\alpha_1 = \alpha_2 = 0.1,$$ (c) $$\alpha_1 = \alpha_2 = 0.05,$$ and (d) 2-lump.
\[
K_1 e^{2R_i + R_2} \frac{R_i + R_2}{4b_1^2 b_2^2} \left( (L_1 L_2 + M_1 M_2) \cos I_2 - (L_2 M_1 - L_1 M_2) \sin I_2 \right)
\]
\[
K_2 e^{2R_i + 2R_2} \frac{R_i + 2R_2}{4b_1^2 b_2^2} \left( (L_1 L_2 - M_1 M_2) \cos I_1 - (L_2 M_1 + L_1 M_2) \sin I_1 \right),
\]
\(\quad (21)\)

where

\[
\begin{align*}
R_i &= 2 \left( \tilde{a}_i x + \tilde{b}_i y + \alpha_i n \right) + \sigma_i, \\
I_i &= 2 \left( \tilde{c}_i x + \tilde{d}_i y + \beta_i n \right) + \theta_i, \\
\tilde{a}_i &= (P_i \cos \gamma_i + Q_i \sin \gamma_i), \\
\tilde{b}_i &= (P_i \sin \gamma_i - Q_i \cos \gamma_i), \\
\tilde{c}_i &= (\xi_i \cos \gamma_i - \zeta_i \sin \gamma_i), \\
\tilde{d}_i &= (\xi_i \sin \gamma_i + \zeta_i \cos \gamma_i), \\
P_i &= \sinh \alpha_i \cosh \delta_i \cos \beta_i, \\
Q_i &= \cosh \alpha_i \sinh \delta_i \sin \beta_i,
\end{align*}
\]

\(\xi_i = \cosh \alpha_i \cosh \delta_i \sin \beta_i,\)

\(\zeta_i = \sinh \alpha_i \sinh \delta_i \cos \beta_i \quad (i = 1, 2),\)

\(L_1 = \text{Re} (A_{12}),\)

\(L_2 = \text{Re} (A_{14}),\)

\(K_1 = A_{13},\)

\(K_2 = A_{24},\)

\(M_1 = \text{Im} (A_{12}),\)

\(M_2 = \text{Im} (A_{14}),\)

and \(\alpha_i, \beta_i, \gamma_i, \delta_i, \sigma_i, \text{ and } \theta_i (i = 1, 2)\) are arbitrary real-valued constants.

Now, we show that the interaction of two breathers is elastic and calculate their phase shift before and after the interaction. We consider \(y\)-periodic 2-breather-soliton (i.e.,
\( \beta_i = \gamma_i = 0 \ (i = 1, 2) \) in equation (21):

\[
\begin{align*}
    f &= 1 + \frac{K_1}{b_1^4} e^{2R_1} + \frac{K_2}{b_2^4} e^{2R_2} + \frac{K_1 K_2 L_1^2 L_2^2}{16 b_1^4 b_2^4} e^{2R_1 + 2R_2} \\
    &- \frac{e^{R_1}}{b_1^4} \cos I_1 \left( \frac{L_1 L_2 K_2}{4 b_2^4} e^{2R_2} + 1 \right) \\
    &- \frac{e^{R_2}}{b_2^4} \cos I_2 \left( \frac{L_1 L_2 K_1}{4 b_1^4} e^{2R_1} + 1 \right) \\
    &+ \frac{e^{R_1 + R_2}}{2 b_1^4 b_2^4} \left( L_1 \cos (I_1 + I_2) + L_2 \cos (I_1 - I_2) \right),
\end{align*}
\]

where

\[
\begin{align*}
    L_1 &= \frac{\cosh (\delta_1 - \delta_2) - \cosh (\alpha_1 - \alpha_2)}{\cosh (\delta_1 - \delta_2) - \cosh (\alpha_1 + \alpha_2)}, \\
    L_2 &= \frac{\cosh (\delta_1 + \delta_2) - \cosh (\alpha_1 - \alpha_2)}{\cosh (\delta_1 + \delta_2) - \cosh (\alpha_1 + \alpha_2)}, \\
    K_1 &= \frac{\cosh 2 \delta_1 - 1}{\cosh 2 \delta_1 - \cosh 2 \alpha_1}, \\
    K_2 &= \frac{\cosh 2 \delta_2 - 1}{\cosh 2 \delta_2 - \cosh 2 \alpha_2},
\end{align*}
\]

\[
\begin{align*}
    u_{pj} &= 2b_j^2 K_j \frac{1 - \left( 1/\sqrt{K_j} \right) \cosh R_j \cos I_j}{\sqrt{K_j} \cosh R_j - \cos I_j} \quad (j = 1, 2). \quad (24)
\end{align*}
\]

Assuming that \( \alpha_1 > 0, \alpha_2 > 0, \) and \( \alpha_1/\sinh \alpha_1 \cosh \delta_1 > \alpha_2/\sinh \alpha_2 \cosh \delta_2, \) we show that the interaction of two breathers is elastic and obtain the phase shift between two breathers after the interaction:

(1) Before interaction (\( n \to -\infty \))

Breather 1 (\( R_1 \) is finite, and \( R_2 \to -\infty \)):

\[
\begin{align*}
    f_1 (R_1, I_1) &= 1 + \frac{K_1}{4 b_1^4} e^{2R_1} - \frac{1}{b_1^4} e^{R_1} \cos I_1. \quad (25)
\end{align*}
\]

Figure 7: Propagation of hybrid solution composed of a breather and a lump at different times: (a) \( n = -20 \), (b) \( n = 0 \), and (c) \( n = 20 \).
Breather 2 ($R_1 \to -\infty$, and $R_2$ is finite):

$$f_2(R_2, I_2) = \frac{K_1}{4b_1^2} e^{2R_i} \left(1 + \frac{K_2L_2^2L_1^2}{4b_2^4} e^{2R_i} - \frac{L_1L_2^2 \cos I_2}{b_2^2} e^{R_i} \right).$$

(26)

After interaction ($n \to \infty$)

Breather 1 ($R_1$ is finite, and $R_2 \to \infty$):

$$f_1(R_2, I_1) = \frac{K_2}{4b_2^2} e^{2R_i} \left(1 + \frac{K_1L_1^2L_2^2}{4b_1^4} e^{2R_i} - \frac{L_1L_2^2 \cos I_1}{b_1^2} e^{R_i} \right).$$

(27)

Breather 2 ($R_1 \to \infty$, and $R_2$ is finite):

$$f_2(R_2, I_2) = 1 + \frac{K_2}{4b_2^2} e^{2R_i} - \frac{1}{b_2^2} e^{R_i} \cos I_2.$$  

(28)

Taking into account $u_j(n) = (f_j(n + 1)f_j(n - 1))/f_j(n)^2$ ($j = 1, 2$), we find that the two separated breathers before and after the interaction are of the form

$$[u_1(R_1, I_1), u_2(R_2 + \ln |L_1L_2|, I_2)]$$
$$\longrightarrow [u_1(R_1 + \ln |L_1L_2|, I_1), u_2(R_2, I_2)],$$  

$L_1L_2 > 0,$
$$\longrightarrow [u_1(R_1 + \ln |L_1L_2|, I_1 + \pi), u_2(R_2, I_2 - \pi)],$$  

$L_1L_2 < 0.$

(29)

From the above expression, we conclude that whether $L_1L_2 > 0$ or $L_1L_2 < 0$, the interaction of two breathers in 2-breather (23) is elastic. The interaction of 2-breather solution (23) for $L_1L_2 > 0$ is depicted in Figure 2 by taking $\alpha_1 = 0.2$, $\alpha_2 = 0.15$, $\beta_1 = 0$, $\beta_2 = 0$, $\gamma_1 = 0$, $\gamma_2 = 0$, $\delta_1 = 0.75$, $\delta_2 = 1.5$, $\sigma_1 = 0$, $\sigma_2 = 0$, $\theta_1 = 0$, and $\theta_2 = 0$. The sequence of snapshots of Figure 2 shows the interaction between two $y$-periodic breathers under the case $\beta_1 = \gamma_1 = 0$ or $\beta_2 = \gamma_2 = 0$, while the two $y$-periodic breathers propagate in the negative direction of the $x$-coordinate. The humps of the second breather pass through between the humps of the first breather as shown in Figure 2(b). After that, they begin to separate and recover the initial shapes and velocities in Figure 2(c).

Now, we construct a hybrid solution composed of two line solitons and one breather. By taking $N = 4$, $k_3^* = k_4^*$, $\lambda_3^* = \lambda_4^*$, and $\beta_3^* = \beta_4^*$ in $N$-soliton (20), we obtain the hybrid

**Figure 8:** Propagation of hybrid solution composed of two breathers and a lump at different times: (a) $n = -180$, (b) $n = -50$, and (c) $n = -10$. 

(a) 

(b) 

(c)
solution of two line solitons and one breather which is shown in Figure 3 with \( k_1 = 0.2, k_2 = 0.3, k_3 = k_3^* = 0.2 + 0.2i, l_1 = 0.5, l_2 = -0.5, l_3 = l_3^* = 4.5 + 0.4i, \beta_1^0 = -5, \beta_2^0 = 15, \beta_3^0 = 0, \) and \( \beta_3^* = 0. \) In addition, we find that two line solitons propagate along the negative direction of the \( x \)-coordinate. In Figure 3(a) at \( n = -40, \) two line solitons are in front of the 1-breather. Then, the 1-breather overtakes and collides with two line solitons in Figure 3(b) at \( n = 0. \) After that, they become farther and farther without changing their shapes and directions of movement in Figure 3(c) at \( n = 40. \)

3. Degeneration of Breathers

In this section, we derive lump solutions for the \((2+1)\) -dimensional elliptic Toda equation (3) by taking the limit of an infinitely large period of breathers derived in the previous section. We also construct the hybrid solution composed of two line solitons and one lump and the hybrid solution composed of one breather and one lump.

By taking the limits \( \alpha \to 0 \) and \( \beta \to 0 \) in the 1-breather (14), the period (15) of 1-breather tends to infinity. Under these limits, 1-breather (14) has degenerated into 1-lump which is given by

\[
\begin{align*}
\eta_i^{[n]} &= j_{[i]}^{[n]}(n+1) \cdot j_{[i]}^{[n]}(n-1) \\
&= \left( \frac{1}{(1/(\cosh 2\delta_1 - 1)) + 4\omega^2 + v^2 + 4} - 64\omega^2 \right),
\end{align*}
\]

where \( j_{[i]}^{[n]}(n) = (1/(\cosh 2\delta_1 - 1)) + 4\omega^2 + 4v^2, \) \( \omega = \cosh \delta_1 (\cos \gamma x + \sin \gamma y) + n \) and \( v = \sinh \delta_1 (\sin \gamma x - \cos \gamma y). \)

Figures 4(a)–4(c) show the degeneration process of 1-breather (14) along the line \( l_1 : 2(\alpha x + \beta y + \alpha n) + \sigma_1 + \epsilon = 0 \) at \( n = 0. \) Figure 4 shows the density pictures of the degeneration process of 1-breather (14) by taking the parameters \( \gamma = 0.1, \delta = 0.7, a = 1, b = 1, \sigma = \theta = 0, \) and \( n = 0. \) The degeneration of the 1-breather given in Figure 4(c) is very similar to that of the 1-lump (30) depicted in Figure 4(d) with the parameter selection of \( \gamma = 0.1, \delta = 0.7, \) and \( n = 0, \) and thus, the former is a good approximation of the latter.
where

\[ \omega_j = \cosh \delta_j \left( \cos \gamma_j x + \sin \gamma_j y \right) + n, \]

\[ v_j = \sinh \delta_j \left( \sin \gamma_j x - \cos \gamma_j y \right) \quad (j = 1, 2), \]

\[ Q_1 = \cos 2(\gamma_1 - \gamma_2) + \cosh 2(\delta_1 - \delta_2) \]

\[ - 4 \cos (\gamma_1 - \gamma_2) \cosh (\delta_1 - \delta_2) + 2, \]

\[ Q_2 = \cos 2(\gamma_1 - \gamma_2) + \cosh 2(\delta_1 + \delta_2) \]

\[ - 4 \cos (\gamma_1 - \gamma_2) \cosh (\delta_1 + \delta_2) + 2, \]

\[ h = 2((x \cos \gamma_1 + y \sin \gamma_1)(x \cos \gamma_2 + y \sin \gamma_2) \cosh \delta_2 \]

\[ + (x \cos \gamma_2 + y \sin \gamma_2)n \cosh \delta_2 \]

\[ - \sinh \delta_1 \sinh \delta_2 (x \sin \gamma_2 - y \cos \gamma_2) \cos \gamma_1 \]

\[ + \sin \gamma_1 \sin \gamma_2 \sinh \delta_1 \sinh \delta_2 x^2 \]

\[ - \sin \gamma_1 \cos \gamma_2 \sinh \delta_1 \sinh \delta_2 xy + n^2 \cosh (\delta_1 + \delta_2) \]

\[ \cdot \cos (\gamma_1 - \gamma_2) \cosh (\delta_1 + \delta_2)(-x \sin \delta_2 \sin \gamma_1 \]

\[ - 2 \sin \gamma_1 \sin \gamma_2 \cosh (\delta_1 - \delta_2) \cosh \delta_2 \]

\[ + n(x \sin \gamma_1 \sin \delta_1 - y(\sinh \delta_1 \cos \gamma_1 - \sin \delta_2 \cos \gamma_2) \]

\[ - \sin \gamma_1 \sin \delta_1) \sin (\gamma_1 - \gamma_2) - 2(y \sin \gamma_1 + x \cos \gamma_1) \]

\[ \cdot ((y \sin \gamma_2 + x \cos \gamma_2) \cosh \delta_2 + n) \cosh \delta_1 \]

\[ - 2n(x \cos \gamma_2 + y \sin \gamma_2) \cosh \delta_2 \]

\[ + 2y \sin \delta_1 \sinh \gamma_1 (x \sin \gamma_2 - y \cos \gamma_2) \cos \gamma_1 \]

\[ - 2 \sin \gamma_1 \sin \gamma_2 \sinh \delta_1 \cosh \delta_2 x^2 \]

\[ + 2 \sin \gamma_1 \cos \gamma_2 \sinh \delta_1 \cosh \delta_2 xy - 2n^2, \]

Similarly, if we let \( \alpha_1, \alpha_2 \rightarrow 0 \) and \( \beta_1, \beta_2 \rightarrow 0 \) in 2-breather (21), we obtain the following 2-lump solution for (3):

\[
\begin{align*}
 f &= \left( \omega_1^2 + v_1^2 \right) \left( \omega_2^2 + v_2^2 \right) + \frac{\omega_1^2 + v_1^2}{2 \cosh \delta_1 - 2} + \frac{\omega_2^2 + v_2^2}{2 \cosh \delta_2 - 2} \\
 &+ \frac{1 + 2H}{2Q_1} + \frac{1 + 2h}{2Q_2} + \frac{1}{(2 \cosh \delta_1 - 2)(2 \cosh \delta_2 - 2)},
\end{align*}
\]

\[ (31) \]

Figure 10: Propagation of hybrid solution composed of a soliton molecule and a breather at different times: (a) \( n = -30 \), (b) \( n = 0 \), and (c) \( n = 30 \).
Here, we take the parameters \( \beta_1 = \beta_2 = 0, \gamma_1 = \gamma_2 = 0, \delta_1 = 1.7/\sqrt{3}, \delta_2 = 1.6/\sqrt{3}, \theta_1 = 0, \theta_2 = \pi, \) and \( n = 0 \) in y-periodic 2-breather (23) and show the degeneration process of the 2-breather in Figures 5(a)–5(c). The degeneration of the 2-breather given in Figure 5(c) is very similar to that of the 2-lump (31) depicted in Figure 5(d) with the parameter selection of \( \gamma_1 = 0, \gamma_2 = 0, \delta_1 = 1.7/\sqrt{3}, \delta_2 = 1.6/\sqrt{3}, \) and \( n = 0, \) and thus, the former is a good approximation of the latter. Therefore, we find that an \( M \)-lump can be degenerated from an \( M \)-breather in the same way.

By taking \( \alpha_j \to 0 \) and \( \beta_i \to 0 (i = 1, 2) \) in the 4-soliton solution \( (N = 4 \) for (20)), we derive the hybrid solution composed of two line solitons and one lump:

\[
f = \left( \frac{B_{12}}{2} + \theta_1 \theta_2 \right) (1 + e^{\eta_1} + e^{\eta_2} + A_{34}e^{\eta_1+\eta_2}) + (M_{33} + N_{23} \theta_1 + N_{13} \theta_2) e^{\eta_1} + (G_{34} + N_{23} \theta_1 + N_{13} \theta_2) A_{34}e^{\eta_1+\eta_2},
\]

where \( B_{12} = 1/(\cos (l_1 - l_2) - 1), \) \( B_{13} = 1/(\cos (l_1 - l_3) - \cosh (k_3)), \) \( B_{14} = 1/(\cos (l_1 - l_4) - \cosh (k_4)), \) \( B_{23} = 1/(\cos (l_2 - l_3) - \cosh (k_3)), \) \( B_{24} = 1/(\cos (l_2 - l_4) - \cosh (k_4)), \) \( \theta_m = \sin (l_m)x + \cos (l_m)y + n, \) \( M_{ij} = \sin (k_j) \sin (k_i) B_{ij}, \) \( N_{mi} = \sin (k_i) B_{mi}, \) \( N_{mij} = \sin (k_j) B_{mi} + \sin (k_i) B_{mj}, \) and \( G_{34} = \)
of movement and shapes at equation (33) with the conjugation of two solitons and changing their shapes and directions of movement. After that, they keep moving forward without keeping moving forward without changing their shapes and directions of movement.

The hybrid solution of one breather and one lump was constructed by equation (33) with the conjugation of two solitons by taking $k_3 = k_4^* = 0.5, l_3 = l_4 = 0.8$, and $\beta_3 = \beta_4^* = 0$. Figure 7 depicts the hybrid solution of one breather and one lump from equation (33) with the parameter selection of $l_1 = l_2^* = 3.5 + i, k_3 = 0.5, k_4 = 0.8, l_3 = 0.5, l_4 = 0.75$, and $\beta_3 = \beta_4^* = 0$ in (33). At first, they move toward each other at $n = -10$ in Figure 6(a); then, the lump is collided and swallowed by two line solitons at $n = 0$ in Figure 6(b). After that, they keep moving forward without changing their shapes and directions of movement.

In the same way of obtaining equation (33), we construct the hybrid solution of three line solitons and one lump:

$$f = \left( \frac{B_{12}}{2} + \theta_1 \theta_2 \right) \left( \sum_{j=0,1} \exp \left( \sum_{k=3,4} A_{jk} \mu_j \mu_k + \sum_{j=1,2} \mu_j \eta_j \right) \right)$$

and we also construct the hybrid solution of four line solitons and one lump:

$$f = \left( \frac{B_{12}}{2} + \theta_1 \theta_2 \right) \left( \sum_{j=0,1} \exp \left( \sum_{k=3,4} A_{jk} \mu_j \mu_k + \sum_{j=1,2} \mu_j \eta_j \right) \right)$$

$$+ \sum_{j=3,4} (G_{j1} + N_{j,1} \theta_1 + N_{j,2} \theta_2) \sum_{i=1,2} (G_{ij} + N_{ij,1} \theta_1 + N_{ij,2} \theta_2) \prod_{j=1}^{5} A_{ij} e^{\mu_1 \eta_1 + \eta_1 + \mu_3 \eta_3 + \mu_4 \eta_4 + \eta_4}$$

$$(34)$$

$$(35)$$

$M_{33} + M_{34} + M_{44}(i, j = 3, 4, m = 1, 2)$. As shown in Figure 6, a 4-soliton solution exhibits the interaction between two line solitons and one lump under the longwave limits with the parameter selection of $l_1 = l_2^* = 3.5 + i, k_3 = 0.5, k_4 = 0.8, l_3 = 0.5, l_4 = 0.75$, and $\beta_3 = \beta_4^* = 0$ in (33). At first, they move toward each other at $n = -10$ in Figure 6(a); then, the lump is collided and swallowed by two line solitons at $n = 0$ in Figure 6(b). After that, they keep moving forward without changing their shapes and directions of movement.
where
\[ \theta_m = \sin (l_m) x + \cos (l_m) y + n, \]
\[ M_{ij} = \sinh (k_i) \sinh (k_j) B_{1i} B_{2j}, \]
\[ G_{ij} = \sum_{n=1}^{2} \sum_{n=1}^{2} M_{mn}, \]
\[ N_m = \sinh (k_i) B_{mi}, \]
\[ N_{mj} = N_{mi} + N_{mj}, \]
\[ G_{ij} = \sum_{n=1}^{2} \sum_{n=1}^{2} M_{mn}, \]
\[ N_{mij} = N_{mi} + N_{mj} + N_{ml}, \]
\[ G_{3456} = \sum_{n=1}^{2} \sum_{n=1}^{2} M_{mn} \]
\[ B_{mn} = \frac{1}{\cos (l_n - l_m) - \cosh (k_i + k_n)} \]
\[ (s, n = 1, 2, 3, 4, 5, 6, m = 1, 2). \]

(36)

Figure 8 describes the interaction between two breathers and one lump in (35) by taking the parameters \( k_3 = k_4^* = 0.828 + 0.2i, k_5 = k_6^* = 0.134 + 5i, l_1 = l_5^* = 0.1 + 0.7i, l_2 = 1 = l_6^* = 1 + 0.750i, l_3 = l_6^* = 8 + 0.275i, \beta_1^0 = -15, \beta_2^0 = -15, \beta_3^0 = 15, \beta_4^0 = 15, \beta_5^0 = 15, \) and \( \beta_6^0 = 15. \) It can be observed in Figure 8 that the two breathers move in opposite directions on the x-coordinate and the lump propagates in the negative direction of the x-coordinate; then, the direction of two breathers becomes smaller. Figure 8(a) depicts the lump gradually approaching the two breathers at \( n = -180; \) then, they collide at \( n = -50 \) in Figure 8(b), and Figure 8(c) shows their separation with initial structures at \( n = -10. \)

### 4. Soliton Molecules and Breather Molecules

In this section, we investigate the soliton molecules, the breather-soliton molecules, the breather molecules, the interaction between soliton/breather-soliton molecules and lumps, and the interaction between soliton molecules and breathers for the \((2 + 1)\)-dimensional elliptic Toda equation (3) via the velocity resonant method and degeneration of breathers.

**Case 1. Soliton molecule.** The soliton molecule is constructed by imposing the velocity resonance condition on a 2-soliton solution \((N = 2)\) in (20)). The velocity resonance condition for 2-soliton is

\[ \frac{r_1}{p_1} = \frac{r_2}{p_2}, \]
\[ \frac{r_1}{q_1} = \frac{r_2}{q_2}, \]
\[ \text{Re} \left( \frac{r_i}{p_i} \right) = \frac{r_3}{p_3}, \]
\[ \text{Re} \left( \frac{r_i}{q_i} \right) = \frac{r_3}{q_3}, \]

(37)

where \( p_i = 2 \sinh k_i \cos l_i, \ q_i = 2 \sinh k_i \sin l_i, \) and \( r_i = 2k_i \) \((i = 1, 2, 3)\).

The 2-soliton solution \((N = 2)\) exhibits one soliton molecule shape under the velocity resonance condition (37) in Figure 9 with the parameter selection of \( k_1 = 0.1, k_2 = 0.2, l_1 = \arcsin (k_1 \sinh (k_1)) - \pi/4, l_2 = \arcsin (k_2 \sinh (k_2)) - \pi/4, \) and \( n = 0. \) It can be observed in Figures (9a)–(9c) that the sizes of the soliton molecule depend on the parameters \( \beta_1^0 \) and \( \beta_2^0. \)

In addition, the two line solitons in the soliton molecule are different because of \( k_1 \neq k_2 \) and \( l_1 \neq l_3, \) although the velocities of the two solitons are the same. Comparing Figures (9a)–9(c), we can find that an asymmetric soliton can be obtained by changing the size decided by parameters \( \beta_1^0 \) and \( \beta_2^0 \) in the molecule. The height of the asymmetric soliton (see in Figure 9(b)) is between the heights of the two solitons, and the wave width of the asymmetric soliton is widened.

Now, we describe the interaction between a soliton molecule and a breather under the velocity resonance condition (37) and the conjugation of two solitons \((k_3 = k_4^* = l_1 = l_5^*, \) and \( \beta_3^0 = \beta_4^0 \)) in 4-soliton \((N = 4)\) in (20). Figure 10 depicts the hybrid solution composed of a soliton molecule and a breather with \( k_1 = 0.5, k_2 = 0.4, k_3 = k_4^* = 0.1 + 0.2i, l_1 = \arcsin (k_1 \sinh (k_2)) - \pi/4, l_2 = \arcsin (k_1 \sinh (k_2)) - \pi/4, l_3 = l_4^* = 4.5 + 0.4i, \beta_1^0 = -15, \beta_2^0 = 15, \) and \( \beta_3^0 = \beta_4^0 = 0 \) in 4-soliton.

As the breather is approaching the soliton molecule in Figure 10(b), we can see that the four humps of the breather collide with the soliton molecule in Figure 10(b); then, the breather and the soliton molecule still propagate in their original directions and keep their original shapes after they separated in Figure 10(c).

If two solitons satisfy the velocity resonance condition (37) in equation (33), we can get a hybrid solution composed of a soliton molecule and a lump in Figure 11 with \( k_3 = k_4^* = 0.8, k_5 = 0.5, l_1 = l_5^* = 0.25 + i, l_2 = \arcsin (k_2 \sinh (k_2)) - \pi/4, l_4 = \arcsin (k_2 \sinh (k_2)) - \pi/4, l_4 = 2 = \arcsin (k_2 \sinh (k_2)) - \pi/4, l_5 = 2 = \arcsin (k_2 \sinh (k_2)) - \pi/4, l_6 = 2 = \arcsin (k_2 \sinh (k_2)) - \pi/4, l_3 = 2 = \arcsin (k_2 \sinh (k_2)) - \pi/4, l_3 = 2 = \arcsin (k_2 \sinh (k_2)) - \pi/4, l_3 = 2 = \arcsin (k_2 \sinh (k_2)) - \pi/4, \)

(38)

where \( p_i = 2 \sinh k_i \cos l_i, q_i = 2 \sinh k_i \sin l_i, \) and \( r_i = 2k_i \) \((i = 1, 2, 3)\). To derive the breather-soliton molecule in Figure 12, we need to take \( N = 3 \) in (20) and choose the parameters \( k_1 = k_2 = 0.1 + 0.25i, l_1 = l_2^* = 1.1 + i, k_3 = 2.117, l_3 = \Re (\sinh (k_1) \sin (1)) / \Re (\sinh (k_1) \cos (1)), \) and \( n = 0. \) Similarly, it can be observed that the size of the breather-
soliton molecule is controlled by the parameters $\beta_0^1$, $\beta_0^2$, and $\beta_0^3$. Figures 12(a) and 12(c) are breather-soliton molecules of different sizes, and Figure 12(b) can be regarded as a collision between a soliton and a breather.

The hybrid solution of a breather-soliton molecule and a lump can be constructed by the condition of velocity resonance (38) in equation (34). The interaction between a breather-soliton molecule and a lump can be obtained, which is depicted in Figure 13 with $k_3 = k_4 = 0.2 + 0.4i$, $k_2 = 0.2$, $l_1 = l_2 = 1.5 + 0.7i$, $l_3 = l_4 = 0.5 + 0.35i$, $l_5 = \text{Re}(\sinh(k_3) \sin(1_3)) / \text{Re}(\sin(k_3) \cos(1_3))$, $\beta_3^0 = -10$, $\beta_4^0 = -10$, and $\beta_5^0 = 10$. As time $t$ goes on, the breather-soliton molecule and the lump move closer both along the negative direction of the $x$-coordinate; then, they interact with each other at $t = 0$ in Figure 13(b). Finally, they move apart, keeping their shapes and directions of movement, but the size of the breather-soliton molecule becomes larger.

Case 3. Breather-breather molecule. The breather-breather molecule is constructed by imposing the velocity resonance condition on a 4-soliton solution ($N = 4$ in (20)). The velocity resonance condition for 4-soliton is

$$\frac{\text{Re}(k_1)}{\text{Re}(\sinh(k_1) \cos(l_1))} = \frac{\text{Re}(k_2)}{\text{Re}(\sinh(k_2) \cos(l_2))},$$

$$\frac{\text{Re}(k_3)}{\text{Re}(\sinh(k_3) \sin(l_3))} = \frac{\text{Re}(k_4)}{\text{Re}(\sinh(k_4) \sin(l_4))},$$

where $p_i = 2 \sinh(k_i) \cos(l_i)$, $q_i = 2 \sinh(k_i) \sin(l_i)$, and $r_i = 2k_i$ ($i = 1, 2, 3, 4$). Figure 14 illustrates a breather-breather molecule with the parameter selection of $k_1 = k_4 = -0.326551718 + 0.2i$, $k_2 = k_3 = 0.1340192245 + 5i$, $l_1 = l_2 = 1.8 + 0.848416875i$, $l_3 = l_4 = 8 + 0.2746749837i$, and $n = 0$ in a 4-soliton solution. Similarly, Figures 14(a)–14(c), respectively, show the breather-breather molecules with different sizes depending on $\beta_1^0$, $\beta_2^0$, $\beta_3^0$, and $\beta_4^0$. In Figure 14(b), two breathers of the breather-breather molecule become in line and also can be regarded as a collision between a breather and a breather.

Figure 13: Propagation of hybrid solution composed of a breather-soliton molecule and a lump at different times: (a) $n = -60$, (b) $n = 0$, and (c) $n = 65$. 
5. Conclusion and Discussion

In this paper, we have considered a variety of solutions for the $(2+1)$-dimensional elliptic Toda equation. This equation is investigated to search for $M$-breather, lumps, possible molecules constructed by solitons and breathers, and hybrid solutions of them. It is interesting to find that the $(2+1)$-dimensional elliptic Toda equation possesses the line soliton molecules, but its continuous version KP1 does not exhibit soliton molecule structures. This shows that the discrete nonlinear wave equations have more diverse molecule structures than their continuous counterparts. We expect that the method of bilinear Bäcklund transformation and nonlinear superposition formulae can be applied to more continuous and discrete nonlinear wave equations to investigate various nonlinear wave phenomena.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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