

Research Article

A Numerical Algorithm Applied to Free Convection Flows of the Casson Fluid along with Heat and Mass Transfer Described by the Caputo Derivative

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Received 3 May 2021; Revised 3 June 2021; Accepted 12 June 2021; Published 23 June 2021

Academic Editor: Mustafa Inc

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In this paper, we present a class of numerical schemes and apply it to the diffusion equations. The objective is to obtain numerical solutions of the constructive equations of a type of Casson fluid model. We investigate the solutions of the free convection flow of the Casson fluid along with heat and mass transfer in the context of modeling with the fractional operators. The numerical scheme presented in this paper is called the fractional version of the Adams Basford numerical procedure. The advantage of this numerical technique is that it combines the Laplace transforms and the classical Adams Basford numerical procedure. Note that the usage of the Laplace transforms makes possible the applicability of the numerical approach to diffusion equations in general. The Caputo derivative will be used in the investigations. The influence of the considered Casson fluid model parameters as the Prandtl number Pr , the Schmidt number Sc , the material parameter of the Casson fluid β , and the order of the Caputo fractional derivative on the dynamics of the temperature, concentration, and velocity profiles has been presented analyzed. Graphical representations have supported the results of the paper.

1. Introduction

Fractional calculus has received much attraction over this last decade and grew many papers with many applications in sciences and engineering [1, 2], in mathematical physics [3, 4], in physics [3, 5], biology [6], and many other fields of sciences [7]. There exist, in our opinion, many interesting works on the application of fractional calculus in the literature; the authors of the following investigations [8, 9] can be considered references in fractional calculus. The importance of fractional calculus in real-world problems exists in many types of operators, those with singular kernels and those without singular kernels. For the operators with singular kernels, we can cite the Riemann-Liouville derivative [8, 9] and the Caputo derivative. The most used fractional operator with the singular kernel is the Caputo derivative [8, 9] because it has no problem in considering initial conditions and the fact that the derivative of the constant function gives a zero function, contrary to the Riemann-Liouville derivative,

which considers the unphysical initial conditions and the derivative of a constant function is not zero. We also have novel fractional operators called fractional operators with the exponential kernel as the Caputo-Fabrizio derivative [10] and the fractional derivative with Mittag-Leffler kernel as the Atangana-Baleanu fractional derivative [11]. One advantage of these two recent derivatives is that they remove the singularity existing in the definitions of the old fractional operators. Many other fractional operators and generalizations of fractional operators exist in the literature (see in [12–14]). The advantages of the Caputo derivative, the Atangana-Baleanu derivative, and the Caputo-Fabrizio derivative in real-world modeling problems are that they consider the memory effect and generalize the integer-order derivative. New diffusion processes appear with the fractional operators as a subdiffusion process, superdiffusion process, and ballistic diffusion. For further applications of fractional derivatives and Laplace transforms in [15], Riaz et al. have presented the analytical solution of Oldroyd-B fluid in a circular duct that

applies a constant couple described by the fractional derivatives using the Laplace transform; in [16], Imran et al. have used the Caputo derivative to model natural convection flow subject to arbitrary velocity and Newtonian heating; the authors have proposed the exact solutions of the constructive equations of the considered model; in [17], Imran et al. have also used the Laplace transforms to get the exact solution of a class of Casson fluid models; in [18], Fetecau et al. study the Stokes problems for fluids of Brinkman type using Fourier sine transform. We can also draw recent investigations on fractional calculus; in [19], the authors have proposed the analytical solutions of the governing equations of gravity-driven thermal convection flow, which is based on a new method of derivations of the velocity field and temperature distribution. In [20], Abro has investigated the thermo-diffusion process on free convection flow in the context of modeling with the fractional operator. In [21], Qureshi et al. have focussed on the blood ethanol concentration system described by the fractional operator, and they have used real data application. In [22], the authors have used a fractional operator to propose a new iterative method for solving the fishery model. In [23], the authors have modeled the varicella-zoster virus using integer derivative, Caputo-Fabrizio derivative, and Atangana-Baleanu derivative. In [24], Jajarmi et al. have modeled the epidemic model with the nonsingular fractional operator and propose optimal control to control the disease.

Many papers in the literature address the analytical and semianalytical solutions of fluid and nanofluid models in general. In [25], Sheikh et al. propose a comparative study of the convective flow of a generalized Casson fluid described by the Atangana-Baleanu fractional derivative and the convective flow of a generalized Casson fluid described by the Caputo-Fabrizio fractional derivative. In [26], Hussanan et al. proposed an analytical solution via Laplace transform of the micropolar fluid flow model. In [27], the authors consider a generalized Casson fluid model with heat generation and chemical reaction and compare the model with Atangana-Baleanu derivative and the model with the derivative with the exponential kernel. In [28], Ali et al. propose applying a new fractional operator, namely, Caputo-Fabrizio derivative for MHD-free convection flow of the generalized Walters'-B fluid model. In [29], Khalid et al. address the unsteady MHD-free flow of a Casson fluid past an oscillating vertical plate with a constant wall temperature. For investigations of the nanofluids model, see the following investigations [30, 31]. In [32], the authors propose solutions to the Rayleigh-Stokes problem for a heated generalized second-grade fluid with Caputo fractional derivative model, the Laplace transforms, and the Fourier transform have been used for obtaining the analytical solutions. About works on Casson fluid model in fractional operators and integer-order derivative, in [33], Khan et al. propose fractional constitutive model of a generalized Casson fluid moving over an infinite, oscillating flat plate; they remarkably offer the exact solution using the exact solution using the Laplace transform method; in [34], Ali et al. investigate on a Casson fluid, along with magnetic particles in a horizontal cylinder; the exact solution is also proposed with the same Laplace transform, and the Caputo derivative has been used in the

modeling of the constructive equation; in [29], Khalid et al. studied the unsteady MHD free flow of a Casson fluid past an oscillating vertical plate with constant wall temperature; in [35], the author focused on the boundary layer flow of a non-Newtonian fluid accompanied by heat transfer toward an exponentially stretching surface in presence of suction or blowing at the surface and used Casson fluid to arrive at his objective.

Modeling fluid models using fractional-order derivative is also addressed in the literature (see in [25, 27]). Many papers in the literature use the homotopy procedure or the application of Fourier Laplace transforms. The Homotopy method application is limited due to the stability and convergence of the method. For example, with the homotopy method, the number of steps to consider making the scheme stable and convergent is the main inconvenience to the credibility of the obtained solution. The combination between the Laplace transforms and the Fourier-Laplace transform has many advantages, but the inconvenience is this method does not apply to some fluid models. Furthermore, we notice in the paper [25], the authors have studied the problem addressed in this paper by considering the Laplace transform. In many other investigations in the literature of Casson fluid models, analytical and semianalytical methods as in [29, 33–35] are proposed; the novelty of this paper in difference from the previous works is that we proposed a new numerical method to provide the solutions of the constructive equation of the model considered in this paper. The numerical scheme proposed in this paper combines the Adams Bashford numerical scheme and the Laplace transform. This paper novelty presents the solution of the free convection flow of the Casson fluid along with heat and mass transfer described by Caputo derivative using the previously cited fractional numerical procedure. The impact of the order of the Caputo derivative will be analyzed and illustrated as well. The effect on the dynamics of the free convection flow of the Casson fluid along with heat and mass transfer in the termal Grashof number, Prandtl number, mass Grashof number, Schmidt number, and material parameters of the Casson fluid will be analyzed.

2. Fractional Calculus Tool

We recall the fractional operators necessary for the rest of our investigations. We mean the Riemann-Liouville integral and the Caputo fractional derivative. These two operators are classical fractional calculus operators and are known to be with singularities. In the literature, there exist many other operators which can be used for the present investigations and are without singular kernels, like the derivative with Mittag-Leffler kernel and the derivative with the exponential kernel.

We describe the Riemann-Liouville fractional integral [8, 9] of the function $x : [0, +\infty[\rightarrow \mathbb{R}$ in the following form

$$(I^\alpha x)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds, \quad (1)$$

where $\Gamma(\dots)$ is known as the Gamma function and with order α obeying the assumption that $\alpha > 0$.

Its associated derivative can also be represented; therefore, we describe the Riemann-Liouville fractional derivative [8, 9] of the function $x : [0, +\infty[\rightarrow \mathbb{R}$, of order α in the form

$$D^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t x(s)(t-s)^{-\alpha} ds, \quad (2)$$

where $t > 0$, the order of the operator obeys to the condition $\alpha \in (0, 1)$, and $\Gamma(\dots)$ denotes the Gamma Euler function.

The derivative which can be used in the modeling is called Caputo derivative [8, 9], and we represent it with the function $x : [0, +\infty[\rightarrow \mathbb{R}$, of order α as the following term:

$$D^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{dx}{ds} (t-s)^{-\alpha} ds, \quad (3)$$

where $t > 0$, the order of the fractional operator obeys to $\alpha \in (0, 1)$, and $\Gamma(\dots)$ is the so-called gamma Euler function.

The Laplace transform, which helps in general to solve the fractional differential equation, will also be used in this paper, and we try recalling it in this section. The Laplace transform of the Caputo derivative [8, 9] is represented as the following form:

$$\mathcal{L}\{(D_t^\alpha x)(t)\} = s^\alpha \mathcal{L}\{x(t)\} - s^{\alpha-1} x(0). \quad (4)$$

where the order α satisfies the relationship $\alpha \in (0, 1)$.

In the rest of this paper, the Laplace transform in Equation (4) will be combined with the numerical discretization of the Riemann-Liouville integral in Equation (1) to give a numerical approximation of the Caputo derivative represented in Equation (2).

3. Constructive Equations

In this section, we consider the fluid model described by fractional diffusion equations. The model investigated in this paper can be found in [25]. We particularly consider the governing equations of the Casson fluid's free convection flow along with the heat and mass transfer described by the Caputo fractional derivative. In this section, we set the fractional differential velocity equation described by the following form:

$$D_t^\alpha u = \left[1 + \frac{1}{\beta} \right] \frac{\partial^2 u}{\partial x^2} + Grw + Gmv, \quad (5)$$

combined with the fractional differential temperature equation

$$D_t^\alpha w = \frac{1}{Pr} \frac{\partial^2 w}{\partial x^2}, \quad (6)$$

and we add the fractional differential concentration equation

$$D_t^\alpha v = \frac{1}{Sc} \frac{\partial^2 v}{\partial x^2}, \quad (7)$$

with the initial conditions given by

$$u(x, 0) = v(x, 0) = w(x, 0) = 0, \quad (8)$$

and the functions u and v have as boundary conditions the following relation

$$u(0, t) = 1 \text{ and } v(0, t) = 1 \text{ and } w(0, t) = 1. \quad (9)$$

In the above equation, Gr represents the thermal Grashof number, Pr denotes the Prandtl number, Gm denotes the mass Grashof number, Sc denotes the Schmidt number, and β denotes the material parameter of the Casson fluid. Furthermore, u represents the velocity of the considered fluid, w represents the temperature, and v denotes the considered fluid concentration. As we notice, Equations (6) and (7) are fractional diffusion equations and represent the input of the fractional differential Equation (5). Equation (5) is specially called the fractional diffusion equation with a reaction term. Equation (5) describes the velocity of the model. Equation (7) represents the temperature distribution, and Equation (6) describes the fluid's concentration. Equations (5)–(7) can be solved using integral balance methods, double integral method and homotopy analysis method, and homotopy perturbation method. In this paper, Equations (5)–(7) will be subject to discretization. Note that it is important to mention that when the order $\alpha = 1$ in the Caputo derivative mentioned in Equation (2), we recover the classic fluid model described by the integer-order derivative. Then, the fractional context of the fluid model (5)–(7) is obtained when the order of the Caputo derivative is in $\alpha \in (0, 1)$. Furthermore, the behavior of the solution of model (5)–(7) in the interval $\alpha \in (1, 2)$ for example can be reported in the interval $(0, 1)$ by considering the change variable given as the form $\alpha = 1 + \beta$ where $\beta \in (0, 1)$. In conclusion, considering the study with the order of the Caputo derivative in $(0, 1)$ is sufficient in this investigation.

4. Numerical Procedure

In this section, the numerical scheme will be applied to obtain the solutions of the model presented in Equations (5)–(7). We adopt the Adams Bashford numerical scheme [36, 37] presented in the next paragraph. Let the fractional differential equations described by the following differential equation:

$$D_t^\alpha y = f(t, x), \quad (10)$$

where the function f is assumed to be Lipchitz continuous. This condition will guarantee the stability of the scheme described in this section. Here, we use the Caputo derivative, then the analytical solution of Equation (10) can be represented as the following form

$$x(t) = x(0) + I^\alpha f(t, x). \quad (11)$$

We now begin the discretization of the Equation (11) at the points t_n and t_{n+1} ; thus, we have the following representation

$$x(t_{n+1}) = x(0) + I^\alpha f(t_{n+1}, x_{n+1}), \quad (12)$$

$$x(t_n) = x(0) + I^\alpha f(t_n, x_n). \quad (13)$$

The difference between Equation (11) and Equation (12) gives the following relationship:

$$x(t_{n+1}) = x(t_n) + I^\alpha f(t_{n+1}, x_{n+1}) - I^\alpha f(t_n, x_n). \quad (14)$$

We now give the discretization of the fractional integral part of the previous equation; we have the following formula:

$$I^\alpha f(t_{n+1}, x_{n+1}) = A_{\alpha,1} = \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} (t_{n+1} - s)^{\alpha-1} f(s, x(s)) ds, \quad (15)$$

with the aid of the first-order interpolant polynomial of the function $f(\tau, x(\tau))$ described by the following representation:

$$p(\tau) = f(t_{j+1}, x_{j+1}) + \frac{\tau - t_{j+1}}{h} [f(t_{j+1}, x_{j+1}) - f(t_j, x_j)], \quad (16)$$

Using Equation (16) into Equation (15), we have the following relation:

$$A_{\alpha,1} = \frac{f(t_n, x_n)}{\Gamma(\alpha)h} \left[\frac{2ht_{n+1}^\alpha}{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} \right] - \frac{f(t_{n-1}, x_{n-1})}{\Gamma(\alpha)h} \left[\frac{ht_{n+1}^\alpha}{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} \right], \quad (17)$$

Similarly by the discretization of the integral form $A_{\alpha,2} = I^\alpha f(t_n, y_n)$, after calculation, we obtain the formula given by

$$A_{\alpha,2} = \frac{f(t_n, x_n)}{\Gamma(\alpha)h} \left[\frac{ht_n^\alpha}{\alpha} - \frac{t_n^{\alpha+1}}{\alpha+1} \right] - \frac{f(t_{n-1}, x_{n-1})t_n^{\alpha+1}}{\Gamma(\alpha)h}, \quad (18)$$

Combining Equation (17) and Equation (18) into Equation (14), we arrive at a numerical scheme with Adams Bashford procedure given as the following form:

$$x_{n+1} = x_n + A_{\alpha,1} - A_{\alpha,2}. \quad (19)$$

We set $t_n = nh$, and then, $t_{n+1} = (n+1)h$, replacing into Equation (17) the discretized form takes the form described by

$$A_{\alpha,1} = \frac{h^\alpha(n+1)^\alpha f(t_n, x_n)}{\Gamma(\alpha)} \left[\frac{1}{\alpha} - \frac{n+1}{\alpha+1} \right] - \frac{h^\alpha(n+1)^\alpha f(t_{n-1}, x_{n-1})}{\Gamma(\alpha)\alpha(\alpha+1)}, \quad (20)$$

Similarly, with the assumption $t_n = nh$, then $t_{n+1} = (n+1)h$, we replace in Equation (18) the discretized form takes the form described by

$$A_{\alpha,2} = \frac{h^\alpha(n)^\alpha f(t_n, x_n)}{\Gamma(\alpha)} \left[\frac{1}{\alpha} - \frac{n}{\alpha+1} \right] - \frac{h^\alpha(n)^{\alpha+1} f(t_{n-1}, x_{n-1})}{\Gamma(\alpha)}, \quad (21)$$

We can observe when the order α of the Caputo derivative converges to 1; we recover the original scheme

proposed by Adams Bashforth two-step method described as the form

$$x_{n+1} = x_n + \frac{3}{2}f(t_n, x_n) - \frac{1}{2}f(t_{n-1}, x_{n-1}). \quad (22)$$

Before closing this section, we focus on the stability of the proposed method. The necessary and sufficient condition will be to prove the function defined by $A_{\alpha,1} - A_{\alpha,2}$ is Lipchitz continuous. We can observe that the function $A_{\alpha,1} - A_{\alpha,2}$ depends on the function f . Thus, the Lipchitz continues of the function f implies the Lipchitz continuous of the function $A_{\alpha,1} - A_{\alpha,2}$.

This subsection will apply the numerical scheme previously described to obtain solutions of the diffusion equations presented in this section. The main idea of the application is for the first time to use the Laplace transforms. The obtained equation after applying the Laplace transform will be of one variable, and in this case, the numerical procedure can be used.

We begin the application of Equation (7). The first step will be to apply the Laplace transform to both sides of Equation (7). Then, we have the following relationship:

$$D_t^\alpha \tilde{v}(p, t) = \frac{1}{Sc} [p^2 \tilde{v}(p, t) - pv(0, t) - v(0, t)] - Sc\eta \tilde{v}(p, t). \quad (23)$$

For simplification in the style of writing, we suppose $\tilde{v} = \tilde{v}(p, t)$ and $f(t, \tilde{v}) = (1/Sc)[p^2 \tilde{v}(p, t) - pv(0, t) - v(0, t)] - Sc\eta \tilde{v}(p, t)$. Thus, Equation (23) can be rewritten as the following form:

$$D_t^\alpha \tilde{v} = f(t, \tilde{v}). \quad (24)$$

The numerical discretization of the previous Equation (24) is obtained from Equation (19) and is represented by the following form:

$$\tilde{v}_{n+1} = \tilde{v}_n + A_{\alpha,1} - A_{\alpha,2}, \quad (25)$$

where the intermediary discretizations are given by the forms

$$A_{\alpha,1} = \frac{h^\alpha(n+1)^\alpha f(t_n, \tilde{v}_n)}{\Gamma(\alpha)} \left[\frac{1}{\alpha} - \frac{n+1}{\alpha+1} \right] - \frac{h^\alpha(n+1)^\alpha f(t_{n-1}, \tilde{v}_{n-1})}{\Gamma(\alpha)\alpha(\alpha+1)}, \quad (26)$$

$$A_{\alpha,2} = \frac{h^\alpha(n)^\alpha f(t_n, \tilde{v}_n)}{\Gamma(\alpha)} \left[\frac{1}{\alpha} - \frac{n}{\alpha+1} \right] - \frac{h^\alpha(n)^{\alpha+1} f(t_{n-1}, \tilde{v}_{n-1})}{\Gamma(\alpha)}. \quad (27)$$

Applying the inverse of the Laplace transform to both sides of Equation (25), we obtain the following solution for our model:

$$v_{n+1} = v_n + A_{\alpha,1} - A_{\alpha,2}, \quad (28)$$

where the inverse of the Laplace transforms of the

transformations in Equations (26) and (27) are given by the following form:

$$A_{\alpha,1} = \frac{h^\alpha(n+1)^\alpha f(t_n, v_n)}{\Gamma(\alpha)} \left[\frac{1}{\alpha} - \frac{n+1}{\alpha+1} \right] - \frac{h^\alpha(n+1)^\alpha f(t_{n-1}, v_{n-1})}{\Gamma(\alpha)\alpha(\alpha+1)}, \quad (29)$$

$$A_{\alpha,2} = \frac{h^\alpha(n)^\alpha f(t_n, v_n)}{\Gamma(\alpha)} \left[\frac{1}{\alpha} - \frac{n}{\alpha+1} \right] - \frac{h^\alpha(n)^{\alpha+1} f(t_{n-1}, v_{n-1})}{\Gamma(\alpha)}, \quad (30)$$

and furthermore, the discretizations of the functions are given by the following forms:

$$f(t_n, v_n) = \frac{1}{\text{Sc}} \left[\frac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{\Delta^2 x} \right], \quad (31)$$

and that

$$f(t_{n-1}, v_{n-1}) = \frac{1}{\text{Sc}} \left[\frac{v_{i+1}^{n-1} - 2v_i^{n-1} + v_{i-1}^{n-1}}{\Delta^2 x} \right]. \quad (32)$$

We continue the application. We consider Equation (6). We repeat the same procedure by applying the Laplace transform to both sides of Equation (6); we have that

$$D_t^\alpha \tilde{w}(p, t) = \frac{1}{\text{Pr}} [p^2 \tilde{v}(p, t) - p w(0, t) - v(0, t)] + \kappa \tilde{w}(p, t). \quad (33)$$

We consider that the function $\tilde{w} = \tilde{w}(p, t)$ and furthermore the function $f(t, \tilde{w}) = (1/\text{Pr})[p^2 \tilde{w}(p, t) - p w(0, t) - w(0, t)] + \kappa \tilde{w}(p, t)$. Equation (33) becomes the following representation

$$D_t^\alpha \tilde{w} = f(t, \tilde{w}). \quad (34)$$

The numerical scheme of Equation (34) can be obtained from Equation (19) and is described by the following equation:

$$\tilde{w}_{n+1} = \tilde{w}_n + A_{\alpha,1} - A_{\alpha,2}, \quad (35)$$

where the intermediary schemes are given by the forms

$$A_{\alpha,1} = \frac{h^\alpha(n+1)^\alpha f(t_n, \tilde{w}_n)}{\Gamma(\alpha)} \left[\frac{1}{\alpha} - \frac{n+1}{\alpha+1} \right] - \frac{h^\alpha(n+1)^\alpha f(t_{n-1}, \tilde{w}_{n-1})}{\Gamma(\alpha)\alpha(\alpha+1)}, \quad (36)$$

$$A_{\alpha,2} = \frac{h^\alpha(n)^\alpha f(t_n, \tilde{w}_n)}{\Gamma(\alpha)} \left[\frac{1}{\alpha} - \frac{n}{\alpha+1} \right] - \frac{h^\alpha(n)^{\alpha+1} f(t_{n-1}, \tilde{w}_{n-1})}{\Gamma(\alpha)}. \quad (37)$$

Applying the inverse of the Laplace transform to both sides of Equation (35), we have the following solution for our fluid model:

$$w_{n+1} = w_n + A_{\alpha,1} - A_{\alpha,2}, \quad (38)$$

where the inverse of the Laplace transform of Equations (36) and (37) are represented in the following equations:

$$A_{\alpha,1} = \frac{h^\alpha(n+1)^\alpha f(t_n, w_n)}{\Gamma(\alpha)} \left[\frac{1}{\alpha} - \frac{n+1}{\alpha+1} \right] - \frac{h^\alpha(n+1)^\alpha f(t_{n-1}, w_{n-1})}{\Gamma(\alpha)\alpha(\alpha+1)}, \quad (39)$$

$$A_{\alpha,2} = \frac{h^\alpha(n)^\alpha f(t_n, w_n)}{\Gamma(\alpha)} \left[\frac{1}{\alpha} - \frac{n}{\alpha+1} \right] - \frac{h^\alpha(n)^{\alpha+1} f(t_{n-1}, w_{n-1})}{\Gamma(\alpha)}, \quad (40)$$

and Laplace transform furthermore; the numerical schemes of the functions coming from Equation (6) are given by the following equations:

$$f(t_n, w_n) = \frac{1}{\text{Pr}} \left[\frac{w_{i+1}^n - 2w_i^n + w_{i-1}^n}{\Delta^2 x} \right], \quad (41)$$

and that

$$f(t_{n-1}, w_{n-1}) = \frac{1}{\text{Pr}} \left[\frac{w_{i+1}^{n-1} - 2w_i^{n-1} + w_{i-1}^{n-1}}{\Delta^2 x} \right]. \quad (42)$$

We finish this section by giving the numerical scheme of the fluid model Equation (5). We apply the Laplace transform to Equation (5); we have the following:

$$D_t^\alpha \tilde{u} = \left[1 + \frac{1}{\beta} \right] [p^2 \tilde{u} - p u(0, t) - u(0, t)] + \text{Gr} \tilde{w} + \text{Gm} \tilde{v}. \quad (43)$$

We suppose the following functions which will help us in understanding the numerical scheme, $\tilde{u} = \tilde{u}(p, t)$ and $f(t, \tilde{u}) = [1 + (1/\beta)][p^2 \tilde{u} - p u(0, t) - u(0, t)] + \text{Gr} \tilde{w} + \text{Gm} \tilde{v}$. Thus, Equation (43) can be rewritten as the form

$$D_t^\alpha \tilde{u} = f(t, \tilde{u}). \quad (44)$$

The numerical procedure of the previous Equation (44) is obtained from Equation (19) and can be symbolized by the following representation

$$\tilde{u}_{n+1} = \tilde{u}_n + A_{\alpha,1} - A_{\alpha,2}, \quad (45)$$

with the intermediary discretization given by the equations

$$A_{\alpha,1} = \frac{h^\alpha(n+1)^\alpha f(t_n, \tilde{u}_n)}{\Gamma(\alpha)} \left[\frac{1}{\alpha} - \frac{n+1}{\alpha+1} \right] - \frac{h^\alpha(n+1)^\alpha f(t_{n-1}, \tilde{u}_{n-1})}{\Gamma(\alpha)\alpha(\alpha+1)}, \quad (46)$$

$$A_{\alpha,2} = \frac{h^\alpha(n)^\alpha f(t_n, \tilde{u}_n)}{\Gamma(\alpha)} \left[\frac{1}{\alpha} - \frac{n}{\alpha+1} \right] - \frac{h^\alpha(n)^{\alpha+1} f(t_{n-1}, \tilde{u}_{n-1})}{\Gamma(\alpha)}. \quad (47)$$

Utilizing the inverse of the Laplace transform on both sides of Equation (45), we obtain the following solution for our fluid model:

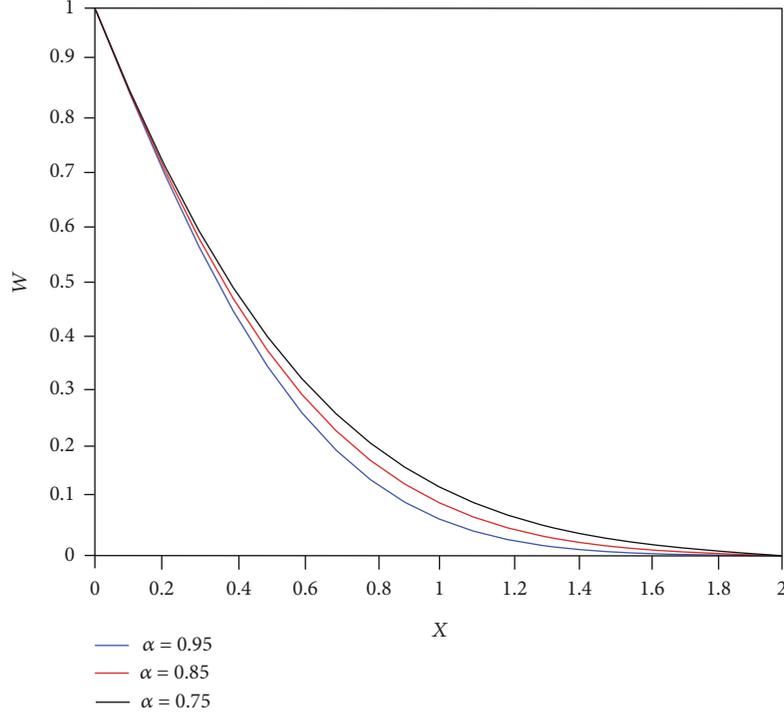


FIGURE 1: Dynamics of concentration in the fractional diffusion equation (Equation (6)).

$$u_{n+1} = u_n + A_{\alpha,1} - A_{\alpha,2}, \quad (48)$$

with the inverse of the Laplace transform of the transformations in Equations (49) and (50) are described by the following equations:

$$A_{\alpha,1} = \frac{h^\alpha (n+1)^\alpha f(t_n, u_n)}{\Gamma(\alpha)} \left[\frac{1}{\alpha} - \frac{n+1}{\alpha+1} \right] - \frac{h^\alpha (n+1)^\alpha f(t_{n-1}, u_{n-1})}{\Gamma(\alpha)\alpha(\alpha+1)}, \quad (49)$$

$$A_{\alpha,2} = \frac{h^\alpha (n)^\alpha f(t_n, u_n)}{\Gamma(\alpha)} \left[\frac{1}{\alpha} - \frac{n}{\alpha+1} \right] - \frac{h^\alpha (n)^\alpha f(t_{n-1}, u_{n-1})}{\Gamma(\alpha)}, \quad (50)$$

and in addition, the discretizations of the functions are given by the following equations:

$$f(t_n, u_n) = \left[1 + \frac{1}{\beta} \right] \left[\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta^2 x} \right] + Grw_i^n + Gmv_i^n, \quad (51)$$

and that

$$f(t_{n-1}, v_{n-1}) = \left[1 + \frac{1}{\beta} \right] \left[\frac{u_{i+1}^{n-1} - 2u_i^{n-1} + u_{i-1}^{n-1}}{\Delta^2 x} \right] + Grw_i^n + Gmv_i^n, \quad (52)$$

This section presents the numerical procedure to obtain the solutions of the fluid model shown in section 3. The main advantage is using the Laplace transform to make possible the

application of the numerical scheme presented in this section else; the applicability of the Adams Bashforth method is not trivial in many cases.

5. Discussion and Interpretation

In this section, we apply the Adams Bashford numerical scheme presented in the previous quarter. The numerical technique will be used to obtain the graphical representations. Additionally, the profile of the solution will be analyzed according to the variation of the model's parameters.

In our analysis, we begin with the constructive equation described in Equation (7). The first objective will be to analyze the impact of the Caputo derivative's order, the impact of the thermal Grashof number, the Prandtl number, the mass Grashof number, the Schmidt number, and the material parameter of the Casson fluid. We fix the time, and we depict the dynamics of the model according to the variation of the order of the Caputo derivative α . In Figures 1 and 2, we give the profile of the concentration and temperature with $Sc = 20$ and $Pr = 30$.

In Figures 1 and 2, we have considered the following orders $\alpha = 0.99$, $\alpha = 0.85$, and $\alpha = 0.75$. We notice from the statement that when the fractional operator's order increases, the temperature and concentration of the fluid decrease. This statement justifies the impact of the fractional-order derivative on the considered model. The main conclusion is the order of the Caputo derivative has an acceleration effect in the diffusion processes. The order of the fractional operator has the same influence on the temperature and the concentration distribution; it is normal because Equations (6) and (7) are diffusion equations, the small difference is, in Equation (6), we have

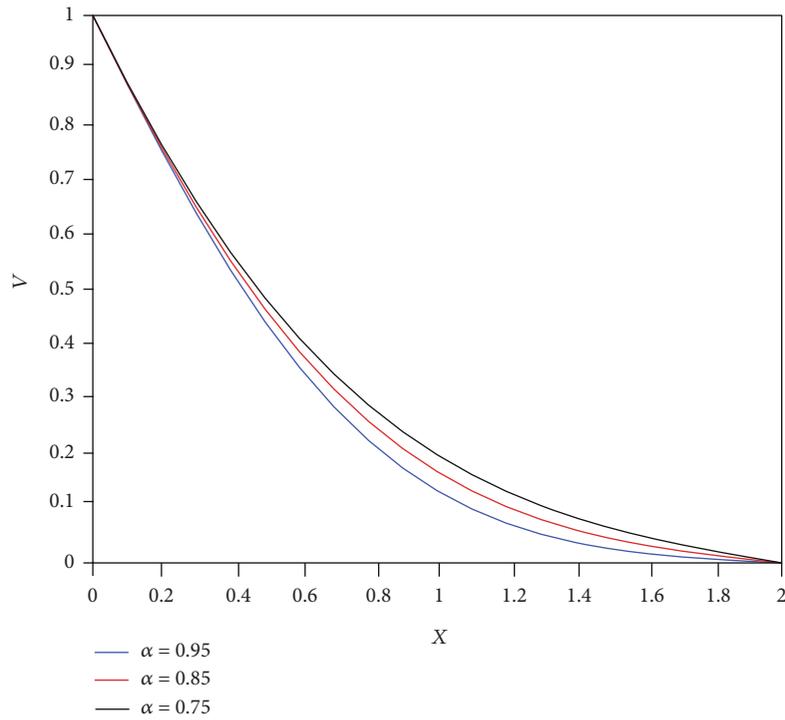


FIGURE 2: Dynamics of temperature in the fractional diffusion equation (Equation (7)).

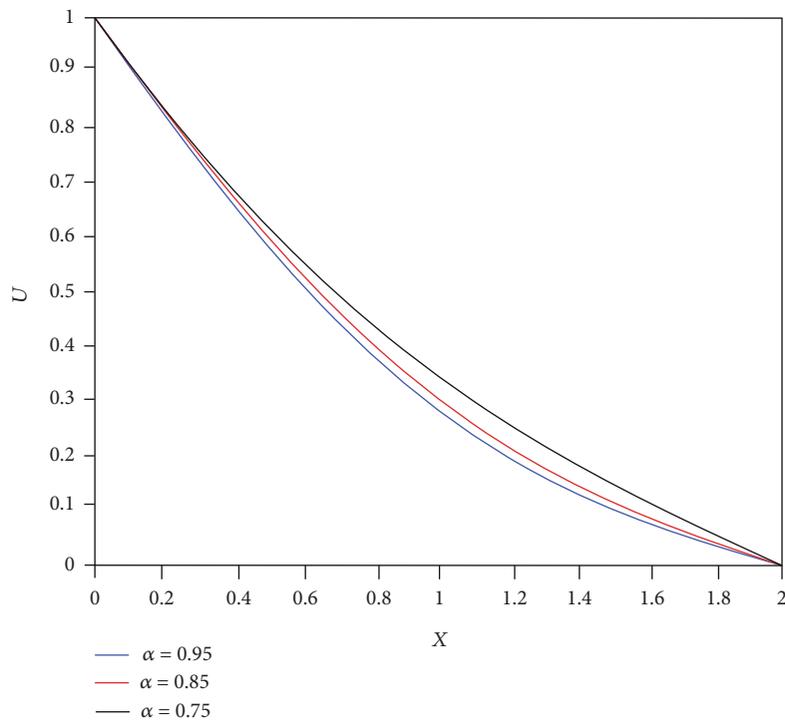


FIGURE 3: Dynamics of velocity using the fractional diffusion equation (Equation (5)).

$Pr = 30$, and in Equation (7), we have $Sc = 20$. In the studies in the paper [25], we can observe the behaviors of the profiles, the temperature, and the concentration distribution of the model (5)–(7) in [25] are the same as in Figures 1 and

2, which in particular validate our present investigations. In Figure 3, we consider the profile of the velocity with $\beta = 1$, $Gm = 15$, and $Gr = 15$. and $Pr = 30$, $Sc = 20$. The considered orders are $\alpha = 0.99$, $\alpha = 0.95$, and $\alpha = 0.90$.

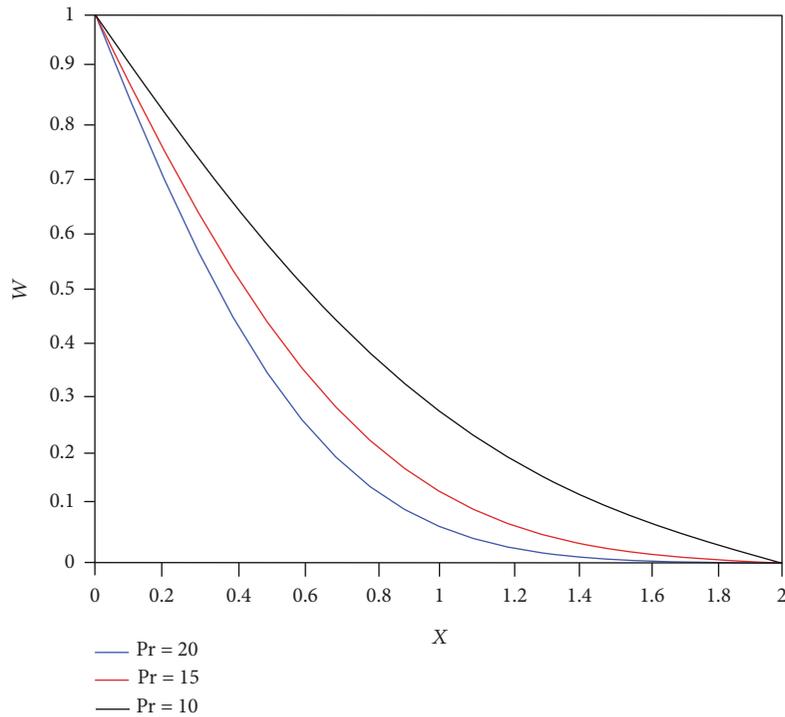


FIGURE 4: Dynamics of the temperature using the fractional diffusion equation (Equation (6)).

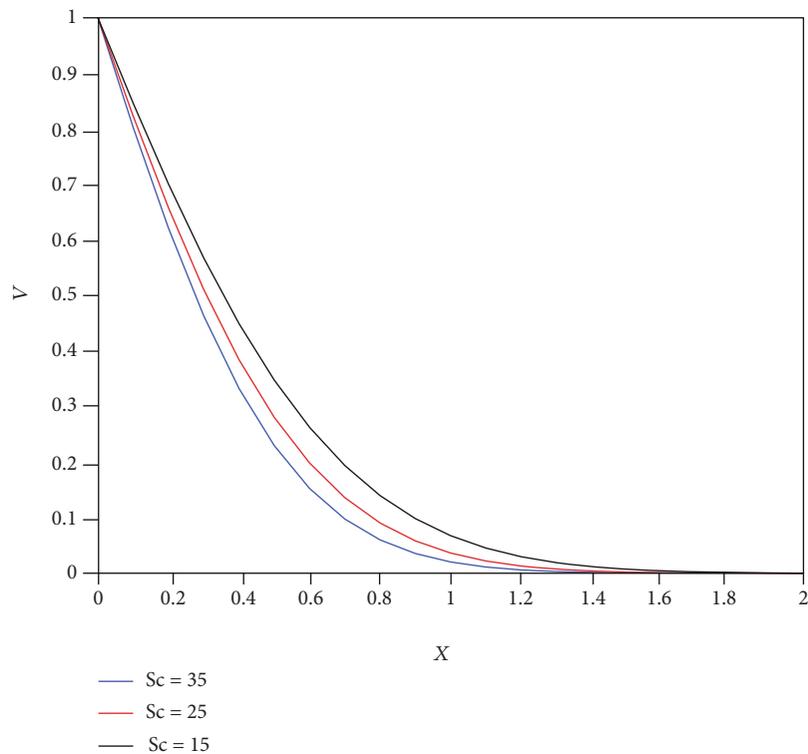


FIGURE 5: Dynamics of the concentration using the fractional diffusion equation (Equation (7)).

In Figure 3, we notice that the fractional-order derivative has an acceleration effect because when the order increases, the velocity of the temperature decreases as well. We now

analyze the impact of the Prandtl number on the temperature of the fluid model. Therefore, we fix into Equation (6) the order $\alpha = 0.95$. We consider the following values $Pr = 10$, Pr

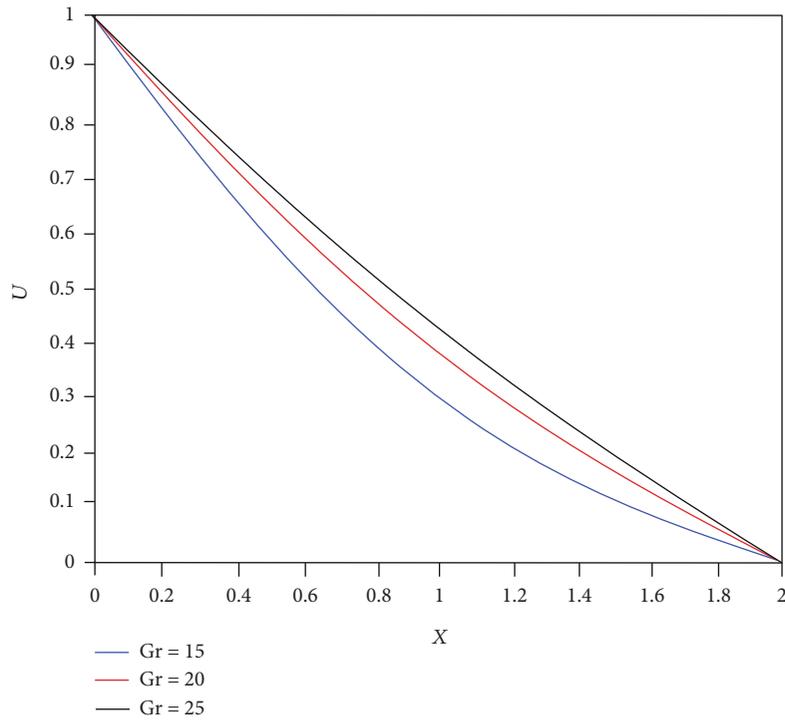


FIGURE 6: Dynamics of velocity of the fractional diffusion equation (Equation (5)).

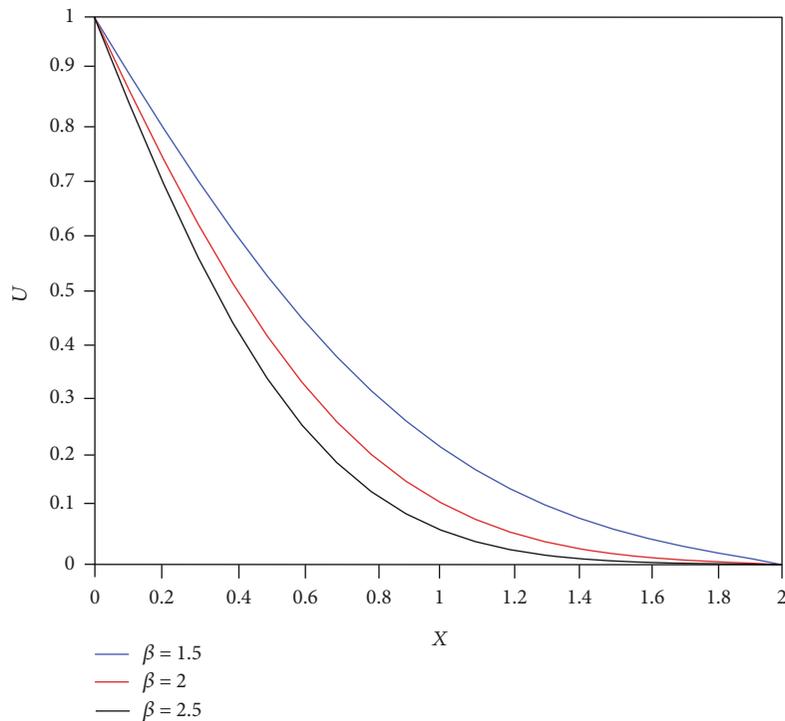


FIGURE 7: Dynamics of velocity using the fractional diffusion equation (Equation (5)).

= 20, and Pr = 30 in the following Figure 4. Our main observation in this figure is that when the Prandtl number increases, then the temperature of the fluid decreases considerably. Therefore, the Prandtl number plays a regulator role in the concentration model.

We now repeat the same analysis with the Schmidt number of the dynamics of the considered fluid concentration. We fix the order to $\alpha = 0.99$. In Figure 5, we set Sc = 15, Sc = 25, and Sc = 35, and we represent graphically the concentration profile.

We also notice that the Schmidt number has an acceleration effect in the diffusion process. We notice that when the order is fixed and when the Schmidt number increases, then the concentration profile decreases. Referring to Figures 4 and 5, we conclude that the Schmidt number and the Prandtl number have the same effects. We finish our analysis by analyzing the impact of the thermal Grashof number, mass Grashof number, and the material parameter of the Casson fluid on the dynamics of the velocity of the considered fluid. We begin by fixing $\beta = 0.5$, we set $Gm = 10$ and fix the order to $\alpha = 0.99$. We consider in this section the thermal Grashof number varies. In Figure 6 are represented the profiles of the velocities of the fluid at different values of the thermal Grashof number.

We remark that from Figure 6, we notice that when the thermal Grashof number increases, the velocity of the fluid increases with him. It is also not hard to see the mass Grashof number has the same influence on the velocity dynamics. We finish this section by analyzing the impact of the material parameters of the Casson fluid. We fix $Gm = 10$, $Pr = 10$, $Sc = 15$, and $Gr = 15$, and the order is considered as $\alpha = 0.99$. In Figure 7 are represented the velocity according to different values of the material parameters of the Casson fluid value.

In Figure 7, we notice that when the values of the Casson fluid's material parameter increase, then the velocity decreases as well.

The main findings of this section are summarized as follows. The increase in the mass Grashof number generates an increase in the dynamics of the Casson fluid's velocity. The increase in the Casson fluid's material parameters decreases the dynamics of the Casson fluid's velocity. The increase in the Prandtl number and Schmidt number generates a reduction in the temperature distribution and the concentration distribution.

6. Conclusion

We have proposed a numerical scheme to obtain the profile of the velocity, temperature, and concentration of the considered fluid. The method proposed in this paper is novel for this fluid model in the context of fractional-order derivative. We have found the Caputo derivative has an acceleration effect in the diffusion processes. Furthermore, the parameters used in our model play an important role in increasing or decreasing the Casson fluid dynamics considered in this paper. In this paper, we find that the Prandtl number Pr , the mass Grashof number Gr , the Schmidt number Sc , the material parameter of the Casson fluid β , and the order of the Caputo fractional derivative have an acceleration effect on the dynamics of the temperature, concentration, and velocity profiles of the considered Casson fluid model. In particular, the increase in the material parameter of the Casson fluid β generates the decrease of the velocity of the fluid. Contrary to the influence of the mass Grashof number Gr , which generates an increase in the velocity when his values increase. The Schmidt number Sc generates a decrease in the concentration when its values increase. The Prandtl number Pr also has the same influence; it generates a decrease in the temperature when its values increase. The method

applied in this paper can be experienced by the Stokes first problem described by the Caputo derivative and other second-grade fluids described by the fractional operators.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflict of interest.

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