Research Article

Solution of Space-Time-Fractional Problem by Shehu Variational Iteration Method

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In this study, we deal with the problem of constructing semianalytical solution of mathematical problems including space-time-fractional linear and nonlinear differential equations. The method, called Shehu Variational Iteration Method (SVIM), applied in this study is a combination of Shehu transform (ST) and variational iteration method (VIM). First, ST is utilized to reduce the time-fractional differential equation with fractional derivative in Liouville-Caputo sense into an integer-order differential equation. Later, VIM is implemented to construct the solution of reduced differential equation. The convergence analysis of this method and illustrated examples confirm that the proposed method is one of best procedures to tackle space-time-fractional differential equations.

1. Introduction

Last couple of decades, employing fractional differential equations in modelling of processes such as dynamical systems, biology, fluid flow, signal processing, electrical networks, reaction and diffusion procedure, and advection-diffusion-reaction process [1–4] has gained great importance since these models reflect the behaviour of the processes better than integer-order differential equations.

Consequently, a great deal of methods such as [3, 4] are established to construct analytical and numerical solutions of fractional differential equations. Moreover, their existence, uniqueness, and stability have been studied by many scientists.

One of the significant integral transformations is Shehu transformation proposed by Maitama and Zhao [5]. This linear transformation is a generalization of Laplace transformation. However, the Laplace transformation is obtained by substituting \( q = 1 \) in Shehu transformation. By this transformation, differential equations are reduced into simpler equations.

Various methods such as the homotopy perturbation method (HPM) and VIM are utilized to establish approximate solutions of differential equations of any kind [6, 7]. As a result, it is employed widely to deal with differential equations in various branches of science [8–11]. VIM has been modified by many researchers to improve this method. By modified VIM, the approximate solutions of initial value problems can be established by making use of an initial condition.

2. Preliminaries

In this section, preliminaries, notations, and features of the fractional calculus are given [12, 13]. Riemann-Liouville time-fractional integral of a real valued function \( u(x, t) \) is defined as

\[
I_t^\alpha u(x, t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(x, s) ds,
\]

where \( \alpha > 0 \) denotes the order of the integral.

The \( \alpha \)-order Liouville-Caputo time-fractional derivative operator of \( u(x, t) \) is defined as

\[
\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} = I_t^{m-\alpha} \left( \frac{\partial^m u(x, t)}{\partial y^m} \right)
= \begin{cases} 
\frac{1}{\Gamma(m-\alpha)} \int_0^t (t-y)^{m-\alpha-1} \frac{\partial^m u(x, y)}{\partial y^m} dy, & m - 1 < \alpha < m, \\
\frac{\partial^m u(x, t)}{\partial t^m}, & \alpha = m.
\end{cases}
\]
The function

\[ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak+\beta)}, \]

where \( \alpha, \beta \in \mathbb{C} \),

\[ \text{Re}(\alpha) > 0, \]

\[ z, \beta \in \mathbb{C}, \]

is called Mittag-Leffler function depending on two parameters \( \alpha \) and \( \beta \).

The following set of functions has Shehu transformation:

\[ \{ f(t) | \exists P, \tau_1, \tau_2 > 0, |f(t)| < Pe^{\beta/\tau_1}, \text{if} \ t \in (-1)^j \times [0, \infty) \}, \]

and it is defined as

\[ S[f(t)] = F(p, q) = \int_0^\infty e^{-p(t/t)^q} f(t) dt, \]

which has the following property:

\[ S[t^n] = \int_0^\infty e^{-p(t/t)^q} t^n dt = \Gamma(\alpha + 1) \left( \frac{q}{p} \right)^{\alpha+1}, \]

where \( \text{Re}(\alpha) > 0 \).

The inverse Shehu inverse transform of \( (q/p)^{na+1} \) is defined as

\[ S^{-1} \left[ \left( \frac{q}{p} \right)^{na+1} \right] = \frac{t^{na}}{\Gamma(na + 1)}, \]

where \( n > 0 \) [5].

For the \( \alpha \)-order of Liouville-Caputo time-fractional derivative of \( f(x, t) \), the Shehu transformation has the following form [14]:

\[ S \left[ \frac{\partial^\alpha f(x, t)}{\partial t^\alpha} \right] = \left( \frac{p}{q} \right)^{\alpha} S[f(x, t)] - \sum_{k=0}^{\alpha-1} \left[ \left( \frac{q}{p} \right)^{k} \frac{\partial f(x, 0)}{\partial t^k} \right]n-1 < \alpha \leq n, n \in \mathbb{N}. \]

3. Main Results

3.1. Fractional Shehu Variational Iteration Method. To reveal the fundamental notions of this method, let us take the following space-time-fractional initial value problem in the Liouville-Caputo fractional derivative:

\[ C^\alpha D^\alpha u(x, t) + R(u, C^\beta D^\beta u; x, t) + N(u, C^\beta D^\beta u; x, t) = g(x, t), \]

where \( N, R, \) and \( g(x, t) \) denote the nonlinear, linear part of the differential equation, and the source function, respectively.

Utilizing Shehu transformation for Equation (9), we have

\[ S[u(x, t)] = \sum_{k=0}^{m-1} \left[ \left( \frac{q}{p} \right)^{k+1} \frac{\partial^k u(x, 0)}{\partial t^k} \right] - \left( \frac{q}{p} \right)^{\alpha} S \left[ R(u, C^\beta D^\beta u; x, t) + N(u, C^\beta D^\beta u; x, t) \right] + \left( \frac{q}{p} \right)^{\alpha} S[g(x, t)]. \]

(11)

Employing the inverse Shehu transformation for Equation (11) leads to

\[ u(x, t) = k(x, t) - \sum_{k=0}^{m-1} \left[ \left( \frac{q}{p} \right)^{k+1} \frac{\partial^k u(x, 0)}{\partial t^k} \right] - \left( \frac{q}{p} \right)^{\alpha} S \left[ R(u, C^\beta D^\beta u; x, t) + N(u, C^\beta D^\beta u; x, t) \right] + \left( \frac{q}{p} \right)^{\alpha} S[g(x, t)]. \]

(12)

where \( k(x, t) = \sum_{k=0}^{m-1} \left[ \left( \frac{q}{p} \right)^{k+1} \frac{\partial^k u(x, 0)}{\partial t^k} \right] + \left( \frac{q}{p} \right)^{\alpha} S[g(x, t)]. \) and so

\[ \frac{\partial u(x, t)}{\partial t} + \frac{\partial}{\partial t} S^{-1} \left[ \left( \frac{q}{p} \right)^{\alpha} S \left[ R(u, C^\beta D^\beta u; x, t) + N(u, C^\beta D^\beta u; x, t) \right] \right] = \frac{\partial}{\partial t} k(x, t) = 0. \]

(13)

The following recurrence relation is established by VIM:

\[ u_{m+1}(x, t) = u_m(x, t) - \sum_{k=0}^{\alpha-1} \left[ \left( \frac{q}{p} \right)^k \frac{\partial^k u_{m+1}(x, 0)}{\partial t^k} \right]n-1 < \alpha \leq m, m \in \mathbb{N}. \]

Alternately,

\[ u_{m+1}(x, t) = k(x, t; \beta) - \sum_{k=0}^{\alpha-1} \left[ \left( \frac{q}{p} \right)^k \frac{\partial^k u_{m+1}(x, 0)}{\partial t^k} \right]n-1 < \alpha \leq m, m \in \mathbb{N}. \]

(15)

is called the \( (m+1)^{th} \)-order of truncated solution.
exists, then the analytical solution \( u \) such that
\[
\text{Theorem 1 \cite{16}.}
\]
exists under the condition that the series solution \( u(x,t) = \sum_{k=0}^{\infty} v_k \) defined in (18) converges.

### 3.2. Convergence Theorem.

Now, the convergence of VIM is investigated and required conditions and error estimate \cite{15} are established for Equation (9).

The operator \( V \) is introduced as
\[
V = -\int_{0}^{t} \left[ \frac{\partial u_m(x,t)}{\partial t} + \frac{d}{dt} S^{-1} \left[ \left( \frac{q}{p} \right)^a \left[ R \left( u_m, C D_x^\beta u_m ; x, t \right) \right] \right] \right] - \frac{\partial}{\partial t} k(x,t) \, dt,
\]
where \( v_k, k = 0, 1, 2, \ldots \), denote the components of the solution satisfying
\[
u(x,t) = \lim_{m \to \infty} u_m(x,t) = \sum_{k=0}^{\infty} v_k.
\]

### Theorem 2 \cite{16}.
The exact solution of nonlinear problem (9) exists under the condition that the series solution \( u(x,t) = \sum_{k=0}^{\infty} v_k \) defined in (18) converges.

### Theorem 3 \cite{16}.
Suppose that the series solution \( \sum_{k=0}^{\infty} v_k \) defined in (18) converges to the solution \( u(x,t) \). The maximum error \( E_j(x,t) \) for the approximate solution \( \sum_{k=0}^{j} v_k \) satisfies the following inequality:
\[
E_j(x,t) \leq \frac{1}{1 - p^j} \| v_0 \|.
\]

The series solution \( \sum_{k=0}^{\infty} v_k \) of problem (9) is convergent to an exact solution \( u(x,t) \), if the conditions
\[
0 < \chi_i \leq 1, \quad \forall \, i \in \mathbb{N} \cup \{0\},
\]
hold where the parameters \( \chi_i \) for \( i \in \mathbb{N} \cup \{0\} \) are introduced as
\[
\chi_i = \begin{cases} 
\| v_{i+1} \|, & \| v_i \| \neq 0, \\
0, & \| v_i \| = 0.
\end{cases}
\]
Furthermore, the maximum absolute truncation error satisfies the inequality
\[
\left\| u(x,t) - \sum_{k=0}^{\infty} v_k \right\| \leq \frac{1}{1 - \chi} \sum_{k=0}^{\infty} \| v_k \|,
\]
where \( \chi \) is defined as \( \chi = \max \{ \chi_i, i = 0, 1, 2, \ldots, j \} \).
4. Illustrative Examples

Example 1. Let us consider following space-time-fractional initial value problem

\[ C_{D_x}^\alpha u(x,t) = C_{D_t}^{2\beta} u(x,t), \quad 0 < \alpha \leq 1, 1 < 2\beta \leq 2, 0 \leq x \leq l, t > 0, \]

\[ u(x, 0) = E_{\beta,1} (x^\beta). \]  

**Step 1.** Implementing Shehu transform for (23), we have

\[ \mathcal{S}[u(x,t)] = \left( \frac{q}{p} \right) E_{\beta,1} (x^\beta) + \left( \frac{q}{p} \right)^\alpha \mathcal{S} \left[ C_{D_x}^{2\beta} u(x,t) \right]. \]  

**Step 2.** Taking the inverse Shehu transform of (25), we get

\[ u(x,t) = \mathcal{S}^{-1} \left[ \left( \frac{q}{p} \right) E_{\beta,1} (x^\beta) + \left( \frac{q}{p} \right)^\alpha \mathcal{S} \left[ C_{D_x}^{2\beta} u(x,t) \right] \right], \]

\[ u(x,t) = E_{\beta,1} (x^\beta) + \mathcal{S}^{-1} \left[ \left( \frac{q}{p} \right) \mathcal{S} \left[ C_{D_x}^{2\beta} u(x,t) \right] \right], \]

\[ \mathcal{S}[u_{m+1}(x,t)] = \left( \frac{q}{p} \right) E_{\beta,1} (x^\beta) + \frac{\partial}{\partial \tau} \mathcal{S}^{-1} \left[ \left( \frac{q}{p} \right)^\alpha \mathcal{S} \left[ C_{D_x}^{2\beta} u(x,t) \right] \right]. \]

\[ \mathcal{S}[u_0(x,t)] = \left( \frac{q}{p} \right) E_{\beta,1} (x^\beta), \]

\[ u_1(x,t) = E_{\beta,1} (x^\beta) + \frac{t^\alpha}{\Gamma(\alpha + 1)} \frac{\partial}{\partial \tau} \mathcal{S}^{-1} \left[ \left( \frac{q}{p} \right)^\alpha \mathcal{S} \left[ C_{D_x}^{2\beta} u(x,t) \right] \right], \]

\[ u_2(x,t) = E_{\beta,1} (x^\beta) \left[ 1 + \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha + 2)} \right], \]

\[ u_3(x,t) = E_{\beta,1} (x^\beta) \left[ 1 + \frac{t^\alpha}{\Gamma(\alpha + 1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha + 2)} + \frac{t^{3\alpha}}{\Gamma(3\alpha + 3)} \right]. \]

By using the recurrence relation, we obtained the \( m \)th approximate solution of (23) as follows:

\[ u_m(x,t) = E_{\beta,1} (x^\beta) \sum_{k=0}^{m-1} \frac{t^{ka}}{\Gamma(k\alpha + k + 1)}, \quad m = 0, 1, 2, \ldots \]
Table 2: Comparison of the exact solution with the truncated solutions by SVIM for various $\beta$ and $\alpha$ for Example 2.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$x$</th>
<th>$\alpha = 0.5$, $\beta = 0.5$</th>
<th>$\alpha = 0.75$, $\beta = 0.75$</th>
<th>$\alpha = 1$, $\beta = 1$</th>
<th>$u_{\text{exact}}$</th>
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<td></td>
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<td>3.00924465929121</td>
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</table>
As a result, the analytical solution of (23) is reached by taking the limit of (30):

$$u(x, t) = \lim_{m \to \infty} u_m(x, t) = E_{\alpha,1}(x^\beta) E_{\alpha,1}(t^\alpha).$$ (31)

where $E_{\alpha,1}(t^\alpha)$ is the two-parameter Mittag-Leffler function.

Notice from Table 1 and Figure 1 that the values of the solution for $\alpha = \beta = 1$ and exact solution are the same which implies that the method implemented in this study is one of the best one for the solution of space-time-fractional differential equations of any order. Moreover, it is clear from Figure 1 that as $\alpha$ and $\beta$ tend to 1, the corresponding solutions tend to exact solution. Three-dimensional graphs of exact solution and a truncated solution are given in Figure 2.

**Example 2.** Let us consider the space-time-fractional equation

$$\frac{CD_t^\alpha u(x,t)}{x} = \left(\frac{CD_x^\beta u(x,t)}{x}\right)^2 - u(x,t)\frac{CD_x^\beta u(x,t)}{x}, \quad 0 < \alpha, \beta, 1, 0 \leq x \leq l, \ t > 0,$$

with the condition at $t = 0$.

$$u(x, 0) = 3 + \frac{5}{2} E_{\beta,1}(x^\beta).$$ (33)

**Step 1.** Carrying out Shehu transform of (32), we have

$$\mathbb{S}[u(x, t)] = \left(\frac{q}{p}\right) u(x, 0) + \left(\frac{q}{p}\right)^a \mathbb{S} \left[ \left(\frac{CD_x^\beta u(x,t)}{x}\right)^2 - u(x,t)\frac{CD_x^\beta u(x,t)}{x} \right].$$ (34)

**Step 2.** Enforcing inverse Shehu transform of (34), we obtain

$$u(x, t) = u(x, 0) + \mathbb{S}^{-1} \left[ \left(\frac{q}{p}\right)^a \mathbb{S} \left[ \left(\frac{CD_x^\beta u(x,t)}{x}\right)^2 - u(x,t)\frac{CD_x^\beta u(x,t)}{x} \right] \right],$$ (35)

and so

$$\frac{\partial u(x, t)}{\partial t} - \frac{\partial}{\partial t} \mathbb{S}^{-1} \left[ \left(\frac{q}{p}\right)^a \mathbb{S} \left[ \left(\frac{CD_x^\beta u(x,t)}{x}\right)^2 - u(x,t)\frac{CD_x^\beta u(x,t)}{x} \right] \right] = 0.$$ (36)

**Step 3.** Utilizing the variational iteration method, we have

$$u_{m+1}(x, t) = u_m(x, t) - \int_0^t \left[ \frac{\partial u_m(x, \tau)}{\partial \tau} - \frac{\partial}{\partial \tau} \mathbb{S}^{-1} \left[ \left(\frac{q}{p}\right)^a \mathbb{S} \left[ \left(\frac{CD_x^\beta u_m(x, \tau)}{x}\right)^2 - u_m(x, \tau)\frac{CD_x^\beta u_m(x, \tau)}{x} \right] \right] \right] d\tau.$$ (37)

Based on the iteration formula (37), we have

$$u_0(x, t) = 3 + \frac{5}{2} E_{\beta,1}(x^\beta),$$

$$u_1(x, t) = 3 + \frac{5}{2} E_{\beta,1}(x^\beta) - \frac{15}{2} E_{\beta,1}(x^\beta) \frac{t^\alpha}{\Gamma(\alpha + 1)},$$

$$u_2(x, t) = 3 + \frac{5}{2} E_{\beta,1}(x^\beta) - \frac{15}{2} E_{\beta,1}(x^\beta) \frac{t^\alpha}{\Gamma(2\alpha + 1)} + \frac{45}{2} \frac{t^\alpha}{\Gamma(2\alpha + 1)} E_{\beta,1}(x^\beta) \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)},$$

$$u_3(x, t) = 3 + \frac{5}{2} E_{\beta,1}(x^\beta) - \frac{15}{2} E_{\beta,1}(x^\beta) \frac{t^\alpha}{\Gamma(2\alpha + 1)} + \frac{45}{2} \frac{t^\alpha}{\Gamma(2\alpha + 1)} E_{\beta,1}(x^\beta) \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} - \frac{135}{2} E_{\beta,1}(x^\beta) \frac{t^{4\alpha}}{\Gamma(4\alpha + 1)}.$$ (38)

By using the recurrence relation, the $m^{th}$ approximate solution of (32) is obtained as follows:
Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors’ Contributions

All authors contributed equally to this work.

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References


Data Availability

No data were used to support this study.

Figure 4: 6th order of truncated solutions for $\alpha = \beta = 2/3$ and exact solution for Example 2.

The authors declare that there is no conflict of interest regarding the publication of this paper.

Authors’ Contributions

All authors contributed equally to this work.

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