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Research Article

The Soliton Solutions and Long-Time Asymptotic Analysis for an Integrable Variable Coefficient Nonlocal Nonlinear Schrödinger Equation

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An integrable variable coefficient nonlocal nonlinear Schrödinger equation (NNLS) is studied; by employing the Hirota's bilinear method, the bilinear form is obtained, and the N-soliton solutions are constructed. In addition, some singular solutions and period solutions of the addressed equation with specific coefficients are shown. Finally, under certain conditions, the asymptotic behavior of the two-soliton solution is analyzed to prove that the collision of the two-soliton is elastic.

1. Introduction

In 1998, Bender and coworker first proposed the $\mathscr{P}\mathscr{T}$- (parity-time-) symmetry for non-Hermitian quantum mechanics [1]. Now, $\mathscr{P}\mathscr{T}$-symmetry has been extensively studied in diverse areas such as lasers [2], acoustics [3], nonlinear optics [4], Bose-Einstein condensation [5], and quantum mechanics [6, 7]. The nonlinear Schrödinger equation has been regarded as the basic model to describe the propagation of solitons in optical fiber, and its spatial solitons have become the research frontier of nonlinear science [8, 9]. In 2013, Ablowitz and Musslimani incorporated the $\mathscr{P}\mathscr{T}$-symmetry with nonlinear integrable systems and proposed the nonlocal or $\mathscr{P}\mathscr{T}$-symmetry nonlinear Schrödinger equation (NLS) [10],

$$iq_t(x,t) + q_{xx}(x,t) + 2q^2(x,t)q^*(-x,t) = 0, \quad (1)$$

where $*$ represents complex conjugation. Obviously, Equation (1) is invariant under the parity-time (PT) transformation, and its solution is evaluated at (x, t) and $(-x, t)$. Since Equation (1) was proposed, many researchers have carried out a lot of work on it. The integrability [10, 11], the Cauchy problem [12], the inverse scattering transform [13], and exact solutions, such as breathers, periodic, and rational solutions [14], general rogue waves [15], multiple bright soliton [16], higher order rational solutions [17], and N-soliton solutions [18] of (1) have been derived. Moreover, other nonlocal integrable systems have also been investigated like nonlocal modified Korteweg-de Vries equation [19, 20], nonlocal KP equation [21], nonlocal vector nonlinear nonlinear Schrödinger equation [22, 23], nonlocal discrete nonlinear Schrödinger equation [24–26], nonlocal Davey-Stewartson I equation [27], etc.

Although much advance has been made in nonlocal systems, there are very few studies on nonlocal equations with variable coefficients. From the realistic point of view, it is more accurate to describe the physical phenomena by using the variable coefficient equations in many physics situations [28]. So it is a meaningful work to study the exact solutions for nonlocal equations with variable coefficients. In [29], authors constructed the soliton solutions for the variable coefficient nonlocal NLS equation by using Darboux transformation. In [30], analytical matter wave solutions of a $(2+1)$-dimensional nonlocal Gross-Pitaevskii equation are investigated. In this paper, we consider the variable coefficient nonlocal NLS equation,

$$iq_t(x,t) - \delta(t)q_{xx}(x,t) - 2\delta(t)q(x,t)^2q^*(-x,t) + \alpha(t)q(x,t) = 0, \quad (2)$$

where the dispersion coefficient $\delta(t)$ and the gain/loss coefficient $\alpha(t)$ are arbitrary real continuous even functions of variable t. Obviously, Equation (2) keeps the parity-time transformation $x \longrightarrow -x$, $t \longrightarrow -t$, $q(x,t) \longrightarrow q^*(-x,-t)$ invariant, so it is $\mathscr{PT}$-symmetric. When $\delta(t) = -1$ and $\alpha(t) = 0$, Equation (2) reduces to the constant coefficient self-focusing nonlocal NLS equation (1). When $\alpha(t) = 0$, Equation (2) becomes variable coefficient nonlocal NLS equation which has been solved by Darboux transformations in [29]. The novelty of this paper is that the variable coefficient NLS equation is firstly solved by Hirota's bilinear method, the more general two-soliton solution and N-soliton solution are reported, and the collision of the two solitons is firstly discussed.

The paper is organized as follows: In Section 2, the bilinear form and the one-soliton, two-soliton, and N-soliton solutions of Equation (2) are obtained based on the Hirota's direct method. In Section 3, the asymptotic behavior is studied to prove that the two-soliton collision is elastic. Finally, conclusions are given in Section 4.

2. The Bilinear Form and Soliton Solutions

We implement the following dependent variable transformation to Equation (2)

$$q = e^{i\beta(t)} \frac{g}{f}, \tag{3}$$

where g and f are complex functions and $\beta(t)$ is a real function; then, the following bilinear equations of Equation (2) are obtained as follows:

$$\begin{gathered} \left(iD_t - \delta(t)D_x^2\right) g \cdot f = 0, \\ f^*(-x,t) D_x^2 f \cdot f = 2fgg^*(-x,t), \end{gathered} \tag{4}$$

where $\beta(t) = \int \alpha(t)dt$ and D is the bilinear operator [24]:

$$D_t^l D_x^m f(x,t) \cdot g(x,t) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^l \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m f(x,t) g\left(x',t'\right)\Big|_{x}^{\,t=x,t'=t}. \tag{5}$$

2.1. One-Soliton Solution. In order to construct the soliton solutions for Equation (2), we expend f and g as follows:

$$\begin{gathered} f = 1 + \varepsilon^2 f_2 + \varepsilon^4 f_4 + \varepsilon^6 f_6 + \cdots, \\ g = \varepsilon g_1 + \varepsilon^3 g_3 + \varepsilon^5 g_5 + \cdots, \end{gathered} \tag{6}$$

where ε is an arbitrary small parameter. Then, substituting Equation (6) into the bilinear equations (4) and collecting the same power coefficients in ε, we get the following equations:

$$\varepsilon^1 : \left(iD_t - \delta(t)D_x^2\right) g_1 \cdot 1 = 0, \tag{7}$$

$$\varepsilon^2 : D_x^2 (f_2 \cdot 1 + 1 \cdot f_2) = 2 g_1 g_1^*(-x,t), \tag{8}$$

$$\varepsilon^3 : \left(iD_t - \delta(t)D_x^2\right)(g_3 \cdot 1 + g_1 \cdot f_2) = 0, \tag{9}$$

$$\begin{aligned} \varepsilon^4 : {} & D_x^2(1 \cdot f_4 + f_4 \cdot 1 + f_2 \cdot f_2) + f_2^*(-x,t) D_x^2 (1 \cdot f_2 + f_2 \cdot 1) \\ & = 2\left(g_1 g_3^*(-x,t) + g_1^*(-x,t) g_3\right) + 2 f_2 g_1 g_1^*(-x,t), \end{aligned} \tag{10}$$

$$\varepsilon^5 : \left(iD_t - \delta(t)D_x^2\right)(g_5 \cdot 1 + g_3 \cdot f_2 + g_1 \cdot f_4) = 0, \tag{11}$$

$$\begin{aligned} \varepsilon^6 : {} & D_x^2(1 \cdot f_6 + f_6 \cdot 1 + f_2 \cdot f_4 + f_4 \cdot f_2) + f_2^*(-x,t) D_x^2 (1 \cdot f_4 \\ & + f_4 \cdot 1 + f_2 f_2) + f_4^*(-x,t) \times D_x^2 (1 \cdot f_2 + f_2 \cdot 1) \\ & = 2\left(g_5 g_1^*(-x,t) + g_3 g_3^*(-x,t) + g_1 g_5^*(-x,t)\right) \\ & \quad + 2 f_2 \left(g_1 g_3^*(-x,t) + g_3 g_1^*(-x,t)\right) + 2 f_4 g_1 g_1^*(-x,t). \end{aligned} \tag{12}$$

Now, we construct the one-soliton solution for Equation (2). Assuming $g_1 = e^{\eta}$ with $\eta = kx + w(t)$, $\eta^*(-x,t) = -k^* x + w^*(t)$, Equation (7) yields the dispersive relation with $w(t) = -ik^2 \int \delta(t)dt$. Then, substituting the obtained g_1 into Equation (8), we get $f_2 = Ae^{\eta + \eta^*(-x,t)}$ with $A = 1/(k-k^*)^2$. Hence, g_1 and f_2 can be expressed as

$$\begin{aligned} g_1 &= e^{kx - ik^2 \int \delta(t)dt}, \\ f_2 &= \frac{1}{(k-k^*)^2} e^{(k-k^*)x - i\left(k^2 - k^{*2}\right) \int \delta(t)dt}. \end{aligned} \tag{13}$$

Other left equations are satisfied if we set $g_3 = g_5 = \cdots = 0$ and $f_4 = f_6 = \cdots = 0$. Hence, we get the one-soliton solution for Equation (2) as

$$q = e^{i \int \alpha(t)dt} \left(\varepsilon e^{kx - ik^2 \int \delta(t)dt} / 1 + \varepsilon^2 \left(1/(k-k^*)^2\right) e^{(k-k^*)x - i\left(k^2 - k^{*2}\right) \int \delta(t)dt} \right). \tag{14}$$

Now, setting $\varepsilon = 1$ and $\alpha(t) = 0$, we get several special solutions for Equation (2):

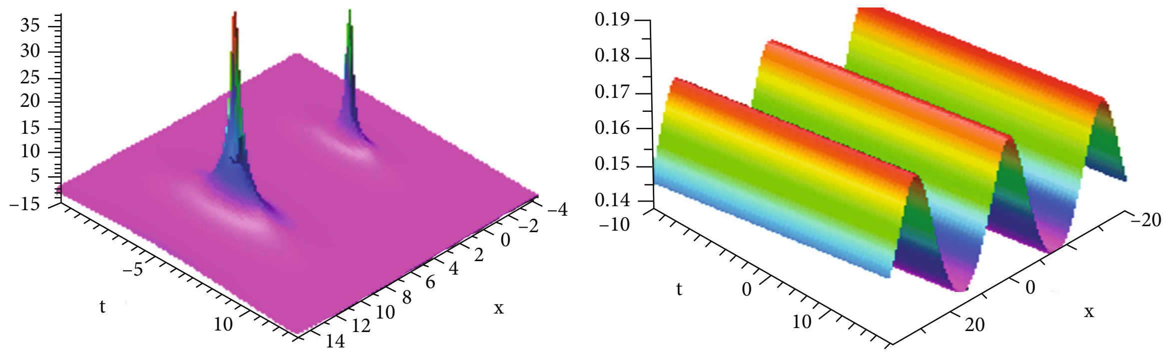
(i) If $k = -2\lambda_2 i$ and $-4\lambda_2 i = \gamma_2$, where λ_2 is a real number, Equation (14) turns into the following period solution which has been reported in [29],

$$q = -\frac{4i\lambda_2 \gamma_2 e^{4i\lambda_2^2 \int \delta(t)\mathrm{d}t}}{\gamma_2^2 e^{2i\lambda_2 x} + e^{-2i\lambda_2 x}}. \tag{15}$$

(ii) If $k = a + ib$ $(a, b \in \mathbf{R}$, and $ab \neq 0)$, Equation (14) becomes

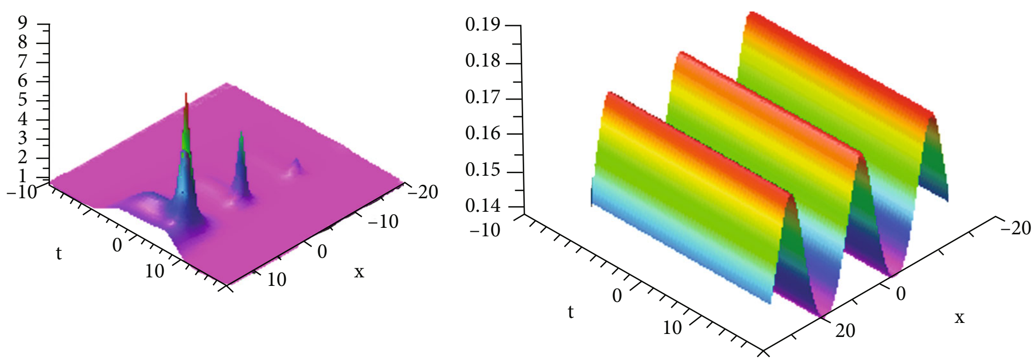
$$q = \frac{e^{ax + 2ab \int \delta(t)\mathrm{d}t} e^{i\left(bx + \int \left(\alpha(t) - \left(a^2 - b^2\right)\delta(t)\right)\mathrm{d}t\right)}}{1 - \left(1/4b^2\right) e^{4ab \int \delta(t)\mathrm{d}t} e^{2ibx}}. \tag{16}$$

Obviously, Equation (16) is the one-soliton solution with the singular point $(x_0, t_0) = (l\pi/b, t_0)$, where t_0 satisfies $\int \delta(t)dt = -\ln 4b^2/4ab$, and $l \in \mathbf{Z}$.



(a) Singular solution ($\delta(t) = -1$) (b) Period solution ($\delta(t) = -1$)

FIGURE 1: (a) Soliton solution with singularity with parameters $a = 0.12$ and $b = -0.35$. (b) Spatial period soliton solution with parameters $a = 0$ and $b = 0.2$, period $M = 5\pi$.



(a) Singular solution ($\delta(t) = t^2$) (b) Period solution ($\delta(t) = t^2$)

FIGURE 2: (a) Soliton solution with singularity when $a = 0.1$ and $b = -0.3$. (b) Spatial period soliton solution with parameter $a = 0$ and $b = 0.2$, period $M = 5\pi$.

(iii) If $k = ib$, $b \in \mathbf{R}$, and $b \neq 0$, we get the spatial period solution

$$|q| = \frac{4b^2}{\sqrt{16b^4 - 8b^2 \cos 2bx + 1}}, \tag{17}$$

where the period $M = \pi/b$.

To show the characteristics of the one-soliton solution, we illustrate the singular solution (16) and the period solution (17) in Figure 1 (when $\delta(t) = -1$) and Figure 2 (when $\delta(t) = t^2$).

2.2. Two-Soliton Solution. To get the two-soliton solution, we let $g_1 = e^{\eta_1} + e^{\eta_2}$ with $\eta_j = k_j x + w_j(t)$, $\eta_j^*(-x,t) = -k_j^* x + w_j^*(t)$, $j = 1, 2$. From Equation (7), we have $w_j(t) = -ik^2 \int \delta(t)dt$, $j = 1, 2$. Plugging the obtained g_1 into Equation (8) leads to

$$\begin{aligned} f_2 = {} & a(1,1^*)e^{\eta_1+\eta_1^*(-x,t)} + a(1,2^*)e^{\eta_1+\eta_2^*(-x,t)} \\ & + a(2,1^*)e^{\eta_2+\eta_1^*(-x,t)} + a(2,2^*)e^{\eta_2+\eta_2^*(-x,t)}, \end{aligned} \tag{18}$$

where $a(l, m^*) = 1/(k_l - k_m^*)^2$, $l, m = 1, 2$.

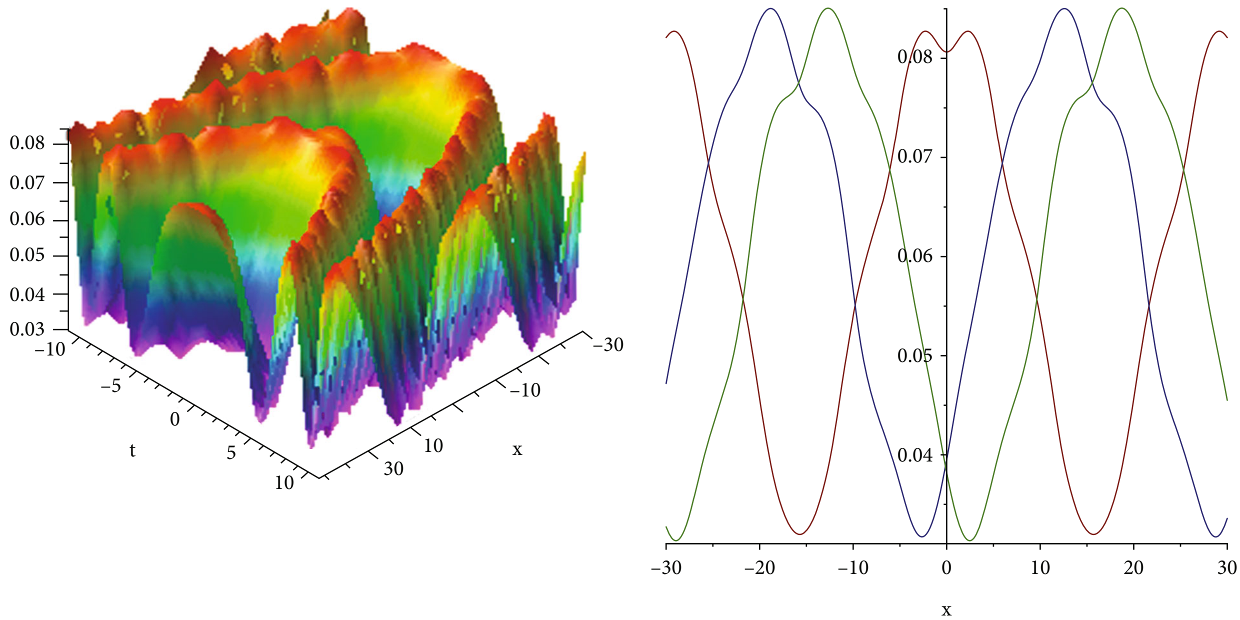
Then, plugging the known g_1 and f_2 into Equation (9) and Equation (10), we derive g_3 and f_4 as

$$\begin{aligned} g_3 &= a(1,2,1^*)e^{\eta_1+\eta_2+\eta_1^*(-x,t)} + a(1,2,2^*)e^{\eta_1+\eta_2+\eta_2^*(-x,t)}, \\ f_4 &= a(1,2,1^*,2^*)e^{\eta_1+\eta_2+\eta_1^*(-x,t)+\eta_2^*(-x,t)}. \end{aligned} \tag{19}$$

where

$$\begin{gathered} a(l,m) = \frac{1}{(k_l - k_m)^2}, a(l, m*) = \frac{1}{(k_l - k_m^*)^2}, l, m = 1, 2, \\ a(1,2,1^*) = a(1,2)a(1,1^*)a(2,1^*), \\ a(1,2,2^*) = a(1,2)a(1,2^*)a(2,2^*), \\ a(1,2,1^*,2^*) = a(1,2)a(1,1^*)a(1,2^*)a(2,1^*)a(2,2^*)a(1^*,2^*). \end{gathered} \tag{20}$$

Other equations are satisfied if we let $f_6 = f_8 = \cdots = 0$ and $g_5 = g_7 = \cdots = 0$. Therefore, for $\varepsilon = 1$, we get the two-soliton solution as



(a) Two-soliton solution ($\delta(t)=t^2$)

(b) Cross-sectional display diagrams ($\delta(t)=t^2$)

FIGURE 3: (a) Double spatial-period soliton solution with parameters: $k_1=0.4i$, $k_2=0.6i$. (b) Cross-sectional shots of solution (a) at $t=0$ (red), $t=5$ (blue), and $t=10$ (green).

$$q=e^{i\int\alpha(t)dt}\frac{e^{\eta_1}+e^{\eta_2}+\gamma_1 e^{\eta_1+\eta_2+\eta_1^*(-x,t)}+\gamma_2 e^{\eta_1+\eta_2+\eta_2^*(-x,t)}}{1+\rho_1 e^{\eta_1+\eta_1^*}+\rho_2 e^{\eta_1+\eta_2^*}+\rho_3 e^{\eta_2+\eta_1^*}+\rho_4 e^{\eta_2+\eta_2^*}+\theta e^{\eta_1+\eta_2+\eta_1^*+\eta_2^*}}. \tag{21}$$

where $\gamma_1=a(1,2,1^*)$, $\gamma_2=a(1,2,2^*)$, $\rho_1=a(1,1^*)$, $\rho_2=a(1,2^*)$, $\rho_3=a(2,1^*)$, $\rho_4=a(2,2^*)$, $\rho_5=a(1,2)$, and $\theta=a(1,2,1^*,2^*)$. Specially, if $k_1=b_1 i$ and $k_2=b_2 i$, the solution Equation (21) becomes a double spatial-period solution which is illustrated in Figure 3 (when $\delta(t)=t^2$).

2.3. N-Soliton Solution. The N-soliton solution for Equation (2) can be shown as follows:

$$q=e^{i\int\alpha(t)dt}\frac{g}{f}, \tag{22}$$

where

$$\begin{aligned}
f&=\sum_{\mu=0,1}^{(e)}\exp\left(\sum_{l=1}^{2N}\mu_l\eta_l+\sum_{l<m}^{2N}\mu_l\mu_m A_{lm}\right),\\
g&=\sum_{\mu=0,1}^{(o)}\exp\left(\sum_{l=1}^{2N}\mu_l\eta_l+\sum_{l<m}^{2N}\mu_l\mu_m A_{lm}\right),\\
g^*(-x,t)&=\sum_{\mu=0,1}^{(c)}\exp\left(\sum_{l=1}^{2N}\mu_l\eta_l+\sum_{l<m}^{2N}\mu_l\mu_m A_{lm}\right),
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
&\eta_l=k_l x+\omega_l(t),\ \omega_l(t)=-ik^2\int\delta(t)dt,\\
&\eta_{l+N}=\eta_l^*(-x,t),\ k_{l+N}=k_l^*\,(l=1,2,\cdots,N),\\
&A_{lm}=\ln\frac{1}{(k_l-k_m)^2}\,(l=1,2,\cdots,N,\ m=N+1,\cdots,2N),\\
&A_{lm}=\ln\,(k_l-k_m)^2\,(l,m=1,2,\cdots,N,\ orl,m=N+1,\cdots,2N),
\end{aligned} \tag{24}$$

and for $\mu_l=0$ or 1 $(l=1,2,\cdots N)$, $\sum_{\mu=0,1}^{(e)}$, $\sum_{\mu=0,1}^{(o)}$, and $\sum_{\mu=0,1}^{(c)}$ satisfy the following conditions, respectively,

$$\sum_{l=1}^{N}\mu_l=\sum_{l=1}^{N}\mu_{l+N},\ \sum_{l=1}^{N}\mu_l=1+\sum_{l=1}^{N}\mu_{l+N},\ 1+\sum_{l=1}^{N}\mu_l=\sum_{l=1}^{N}\mu_{l+N}. \tag{25}$$

3. Asymptotic Analysis on Two-Soliton Solution

The asymptotic behavior of the two-soliton solution is dependent on $\delta(t)$. In this section, under certain assumption that $\lim_{t\longrightarrow+\infty}\int\delta(t)dt=+\infty$, we investigate the asymptotic behavior of the two-soliton solution. Since $\delta(t)$ is an even real function, we have $\lim_{t\longrightarrow-\infty}\int\delta(t)dt=-\infty$. For simplicity, we denote $-ik_j^2$ by ω_j, $j=1,2$, then $\eta_j=k_j x+\omega_j\int\delta(t)dt$, $j=1,2$.

For fixed η_1, we get

$$\begin{aligned}\eta_2 &= \frac{k_2}{k_1}\eta_1 + \left(w_2 - \frac{k_2}{k_1}w_1\right)\int \delta(t)dt,\\ \eta_2^*(-x,t) &= \frac{k_2^*}{k_1^*}\eta_1^*(-x,t) + \left(w_2^* - \frac{k_2^*}{k_1^*}w_1^*\right)\int \delta(t)dt,\\ \eta_2 + \eta_2^*(-x,t) &= 2\,\mathrm{Re}\left(\frac{k_2}{k_1}\xi_1\right) + 2\,\mathrm{Re}\left(w_2 - \frac{k_2}{k_1}w_1\right)\int \delta(t)dt.\end{aligned} \tag{26}$$

where $w_2 - (k_2/k_1)w_1 = i(-k_2^2 + k_2k_1)$.

Suppose that $\mathrm{Re}\,(w_2 - (k_2/k_1)w_1) > 0$, that is, $\mathrm{Im}\,(k_2^2 - k_2k_1) < 0$. The two-soliton solution asymptotically tends to be one-soliton solution as follows:

$$q \sim \frac{1}{2}e^{(\eta_1 - \eta_1^*(-x,t) - \ln \rho_1/2) + i\int \delta(t)dt}\ \sec h\frac{\eta_1 + \eta_1^*(-x,t) + \ln \rho_1}{2},\ t \longrightarrow -\infty, \tag{27}$$

$$q \sim \frac{\rho_2\rho_5}{2}e^{(\eta_1 - \eta_1^*(-x,t) - \ln(\theta/\rho_4)/2) + i\int \delta(t)dt}\ \sec h\frac{\xi_1 + \xi_1^*(-x,t) + \ln(\theta/\rho_4)}{2},\ t \longrightarrow +\infty. \tag{28}$$

For fixed η_2, suppose that $\mathrm{Re}\,(w_2 - (k_2/k_1)w_1) > 0$, in a similar way, we get the asymptotic expressions of Equation (21):

$$q \sim \frac{1}{2}e^{(\eta_2 + \eta_2^*(-x,t) - \ln \rho_4/2) + i\int \delta(t)dt}\ \sec h\frac{\eta_2 + \eta_2^*(-x,t) + \ln \rho_4}{2},\ t \longrightarrow -\infty, \tag{29}$$

$$q \sim \frac{\rho_3\rho_5}{2}e^{(\eta_2 + \eta_2^*(-x,t) - \ln(\theta/\rho_1)/2) + i\int \delta(t)dt}\ \sec h\frac{\eta_2 + \eta_2^*(-x,t) + \ln(\theta/\rho_1)}{2},\ t \longrightarrow +\infty. \tag{30}$$

We can see that the asymptotic solutions Equation (27) and Equation (28), Equation (29) and Equation (30) have the same form, which implies that the two-soliton collision is elastic. But the two-soliton solution is not a travelling wave. If we suppose that $\lim_{t\longrightarrow+\infty}\int \delta(t)dt = -\infty$, the same conclusion can be drawn.

4. Conclusion and Remarks

In the current paper, we studied an integrable variable coefficient nonlocal nonlinear Schrödinger equation via the Hirota's bilinear method. We first constructed the bilinear form and then the N-soliton solution. Furthermore, under certain conditions, we analyzed the asymptotic behavior of the two-soliton solution and proved that the collision of the two soliton is elastic. Also, we demonstrated that by choosing different special parameters, the obtained soliton solutions can reduce to spatial period solution or singular solution. We know that sometimes the higher-dimensional nonlinear systems are more suitable to model the physical phenomena such as ultrafast nonlinear optics, so we hope to investigate the $(2+1)$-dimensional variable coefficient nonlocal partial equations in the future.

Data Availability

All data used to support the findings of this study is included in the submitted paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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