

Research Article

Statistical Study of the Entanglement for Qubit Interacting with an Electromagnetic Field

Neveen Sayed-Ahmed ¹, M. M. Amein,¹ Taghreed M. Jawa,¹ Tahani A. Aloafi,¹
F. S. Bayones,¹ Azhari A. Elhag,¹ and J. Bouslimi²

¹Department of Mathematics, College of Science, Taif University, P.O. Box 11099 Taif 21944, Saudi Arabia

²Department of Physics, College of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia

Correspondence should be addressed to Neveen Sayed-Ahmed; nevensayd@yahoo.com

Received 12 February 2021; Accepted 20 September 2021; Published 11 October 2021

Academic Editor: Mustafa Inc

Copyright © 2021 Neveen Sayed-Ahmed et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

A statistical method is applied to predict the behaviour of a quantum model consisting of a qubit interacting with a single-mode cavity field. The qubit is prepared in excited state while the field starts from the binomial distribution state. The wave function of the proposed model is obtained. A von Neumann entropy is used to investigate the behaviour of the entanglement between the field and the qubits. Moreover, the atomic Q and Wigner functions are used to identify the behaviour of the distribution in a phase space. The simulation method is used to estimate the parameters of the proposed model to reach the best results. A numerical study is performed to estimate the specific dependency of the binomial distribution state. The results of entanglement were compared with the atomic Q and Wigner functions. The results showed that there are many maximum values of entanglement periodically. The results also confirmed a correlation between von Neumann entropy, the atomic Q , and Wigner functions.

1. Introduction

The problem of field interaction with an atom or atoms has gained the attention of many researchers in the field of optics and quantum information. Through the previous literature, it becomes clear that the interaction model of a qubit with two levels with a single-mode of the optical field is the most easiest model for estimating properties that have physical applications [1]. Therefore, researchers have been interested in studying this model on a large scale in the direction of information and quantum optics [2]. Moreover, many generalizations of this model were appeared, especially in field formulas; therefore, the combinations of the field modes such as bimode as either a converter or amplifier were discussed [3, 4]. Continuing the development of studies in the interaction of the field with the qubit, the effect of the interaction of two fields together on the interaction of the atom with the field was studied. The results showed nonclassical features in the phenomenon of collapse, revival, and

entropy of squeezing [3, 4]. On the other hand, a new type of generalization emerged from the Jaynes-Cummings model, which is the dependence of the coupling between the qubit and the field on time, which led to the results of increasing the correlation between the parts of the quantum system [5]. The theoretical efforts have been dealt with extensively by the experimental cavity of QED. Progress has been made in the quantum computation model, in which the quantum system is studied as a density operator or a matrix [6]. The study of the interaction of qubits with the single-mode field has received great interest from researchers. For example, the effect of stark displacement on the entanglement between quantum systems was investigated [7]. The analytical solution of the fractional Schrödinger equation was also used to describe the interaction between an atom with two levels and the placement of a single electromagnetic field inside a cavity [8]. Moreover, the effect of the interaction between the time-dependent field and a two-level atom with a single electromagnetic field

TABLE 1: ($M = 5$).

| | Max | Min | | Max | Min |
|-----------|-----------|-------------------------|-----------|------------|-------------|
| S | 0.693013 | 0.0000226021 | ϕ | 136.362 | 106.99 |
| $\tau(S)$ | 7.02325 | 31.4164 | W | 0.00553529 | -0.00140655 |
| $\eta(S)$ | 0.0203785 | 0.620389 | $\tau(W)$ | 2.87961 | 11.0223 |
| Q | 0.0027748 | 5.2415×10^{-6} | $\eta(W)$ | 0.985884 | 0.99747 |
| $\tau(Q)$ | 0.0100093 | 31.3941 | x | 0.28476 | -0.0930154 |
| $\eta(Q)$ | 0.217574 | 0.00494274 | y | 0.79185 | -0.738397 |
| θ | 113.035 | 128.731 | | | |

TABLE 2: ($M = 20$).

| | Max | Min | | Max | Min |
|-----------|-----------|--------------------------|-----------|------------|------------|
| S | 0.692933 | 1.49224×10^{-7} | ϕ | 349.464 | 40.5101 |
| $\tau(S)$ | 9.02077 | 47.1239 | W | 0.00552633 | -0.0006944 |
| $\eta(S)$ | 0.811648 | 0.664846 | $\tau(W)$ | 19.3756 | 32.0177 |
| Q | 0.0027569 | 5.47404×10^{-6} | $\eta(W)$ | 0.999427 | 0.996774 |
| $\tau(Q)$ | 37.6882 | 34.555 | x | -0.2998 | -1.01915 |
| $\eta(Q)$ | 0.352551 | 0.184801 | y | 0.853462 | 0.785669 |
| θ | 75.4497 | 65.8915 | | | |

TABLE 3: ($M = 50$).

| | Max | Min | | Max | Min |
|-----------|------------|--------------|-----------|--------------------------|--------------------------|
| S | 0.693125 | 0.0000714152 | ϕ | 197.615 | 64.7959 |
| $\tau(S)$ | 37.9639 | 34.5578 | W | 0.00550693 | 2.45124×10^{-6} |
| $\eta(S)$ | 0.745945 | 0.548057 | $\tau(W)$ | 24.0254 | 34.5578 |
| Q | 0.00276372 | 0.0000900447 | $\eta(W)$ | 3.41675×10^{-6} | 0.548057 |
| $\tau(Q)$ | 47.1265 | 6.23383 | x | 0.104398 | -1.05545 |
| $\eta(Q)$ | 0.156082 | 0.0424479 | y | 0.105826 | 0.345778 |
| θ | 131.806 | 159.928 | | | |

TABLE 4: ($M = 100$).

| | Max | Min | | Max | Min |
|-----------|------------|--------------|-----------|-----------|---------------------------|
| S | 0.693061 | 0.0000714152 | ϕ | 205.49 | 64.7959 |
| $\tau(S)$ | 28.082 | 34.5578 | W | 0.0052686 | -5.89211×10^{-8} |
| $\eta(S)$ | 0.360209 | 0.548057 | $\tau(W)$ | 40.7896 | 9.4861 |
| Q | 0.00276632 | 0.0000900447 | $\eta(W)$ | 0.295508 | 0.646471 |
| $\tau(Q)$ | 12.6124 | 6.23383 | x | 0.113395 | 0.426285 |
| $\eta(Q)$ | 0.00199163 | 0.0424479 | y | -0.374566 | 0.37842 |
| θ | 113.02 | 159.928 | | | |

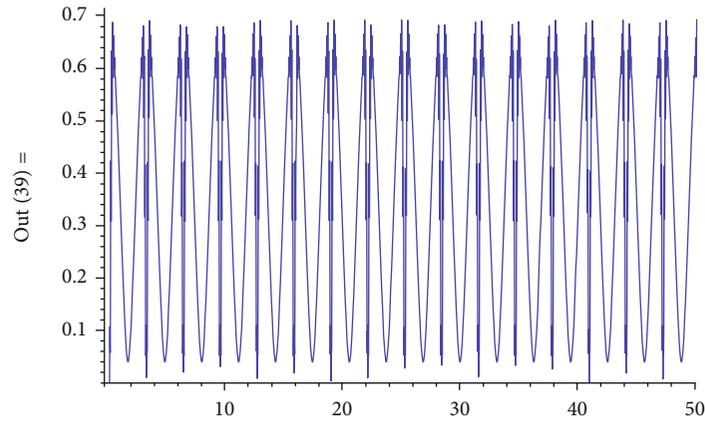


FIGURE 1: Maximum points for the S -function, $M = 50$, $0 \leq \tau \leq 50$.

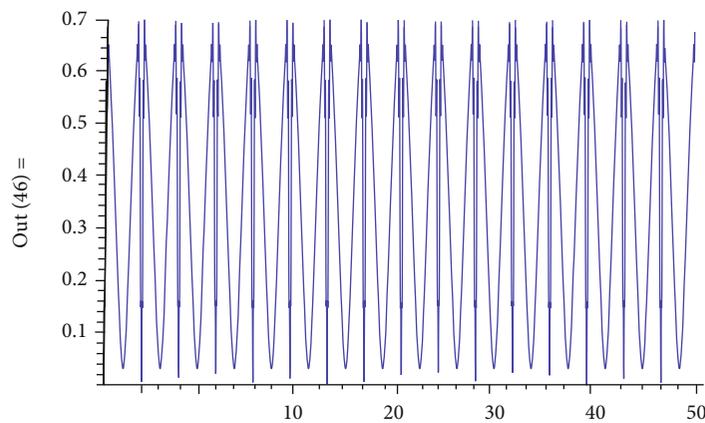


FIGURE 2: Minimum points for the S -function, $M = 50$, $0 \leq \tau \leq 50$.

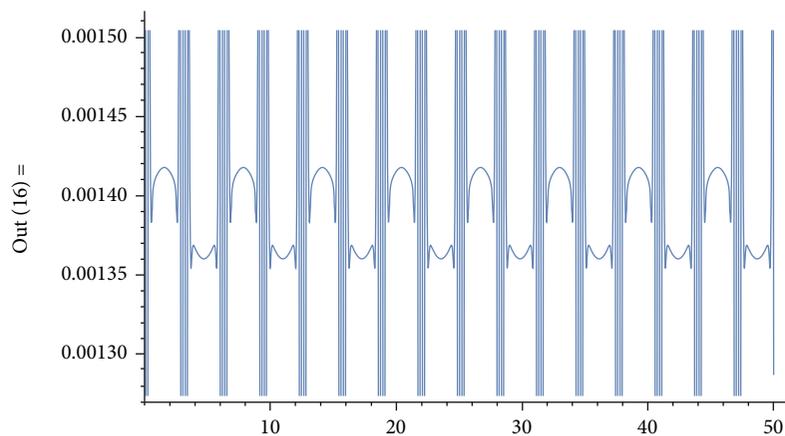


FIGURE 3: Maximum points for the Q -function, $M = 50$, $0 \leq \tau \leq 50$.

was studied [9]. The effect of the decay on the interaction of a three-level atom with a multiphoton field in the presence of Kerr like medium was also studied [10, 11]. In fact, the initial values of the parameters describing the quantum system are taken at random, while in this work, a statistical method is used to estimate the initial values that improve the entanglement between the parts of the system.

Undoubtedly, the entanglement of parts of quantum systems is an important measure for discovering the strength of the interaction between these systems. Therefore, attention has been given to many types of these measures to measure the degree of quantum entanglement between two systems. For example, the von Neumann entropy or linear entropy was used in the case of closed systems that start the

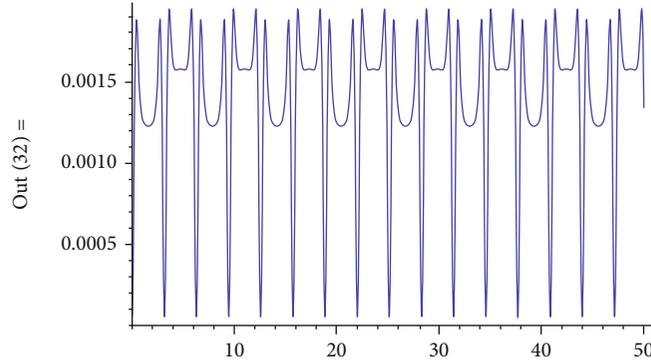


FIGURE 4: Minimum points for the Q -function, $M = 50$, $0 \leq \tau \leq 50$.

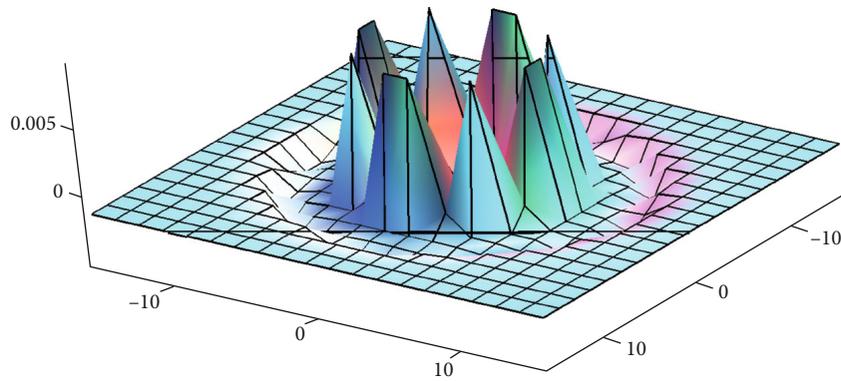


FIGURE 5: Minimum points for the w -function, $M = 50$.

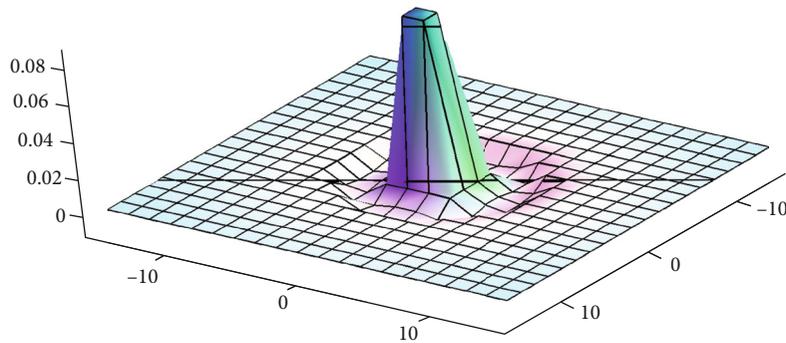


FIGURE 6: Maximum points for the w -function, $M = 50$.

interaction from a separate state [12]. Moreover, the qubit and the electromagnetic field were prepared in a separate or superposition states. They were followed by a partially entangled state during the interaction of the quantum system. Therefore, von Neumann entropy or linear entropy is the two main measures that mean the local classical entanglement process.

The process of controlling the degree of entanglement between two atoms with two thermal fields through the amount of decomposition between the atomic transition and the gaps is one of the most important problems in quantum optics [13]. Open system correlation cannot be measured with a standard scale such as entropy or von

Neumann. Therefore, various attempts have been made to discover a new quantum estimator of correlation between parts of quantum systems [14, 15]. It is worth noting that the initial state is strongly influencing the amount of entanglement between parts of a quantum system. The effect of the nondeformed pair coherent states on a quantum system was investigated [16]. The effect of the nondeformed pair coherent state on the field cavity was studied, where strong entanglement was established in most of the interaction periods between quantum systems, while the deformed state generated weak entanglement [17].

The main goal of this work is to predict the degree of entanglement between quantum systems containing a qubit

interaction with a single-mode cavity field. Statistical simulation is used to obtain the regions of entanglement strength and to obtain nonclassical behaviour by studying the Q and Wigner functions.

The paper is arranged as follows: Section 2 contains the solution of the differential equations to obtain the general solution of the quantum system. Section 3 is devoted to discover the regions of entanglement between the parts of the system. Moreover, von Neumann entropy, the atomic Q-function, and W-function are examined. Some statistical characteristics are presented in Section 4. Finally, conclusion is drawn in Section 5.

2. Quantum System

Now, we assume that a two-level atom is injected into a cavity containing a medium described by Hamiltonian containing single-mode of cavity field and the two-level atom.

$$\frac{\hat{H}}{\hbar} = \Omega c\Lambda^\dagger \hat{c} + \frac{\omega_0}{2} \hat{\sigma}_z + \lambda (c\Lambda^\dagger \hat{\sigma}_- + \hat{c} \hat{\sigma}_+), \quad (1)$$

where $\hat{\sigma}_+$, $\hat{\sigma}_-$, and $\hat{\sigma}_z$ are su(2) representation, and \hat{c} and $c\Lambda^\dagger$ are the annihilation and creation operator, respectively. The frequency ω_0 is the energy difference between the upper and the lower levels of the atom. The $\hat{\sigma}_\pm$ is connected with the inversion operator $\hat{\sigma}_z$ by the commutation relations

$$[\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_z, [\hat{\sigma}_z, \hat{\sigma}_\pm] = \pm 2\hat{\sigma}_\pm. \quad (2)$$

The Heisenberg equation of motion is used to obtain the constants of motion. Through it, the general solution for the proposed quantum system is obtained. The equation of motion for any dynamical operator \hat{Y} is given by

$$\frac{d}{dt} \hat{Y} = \frac{1}{i\hbar} [\hat{Y}, \hat{H}] + \frac{\partial \hat{Y}}{\partial t}. \quad (3)$$

Hence, the equation of motion for $\hat{\sigma}_z$ and $c\Lambda^\dagger \hat{c}$ can be given as follows:

$$\frac{d\hat{\sigma}_z}{dT} = 2i\lambda (c\Lambda^\dagger \hat{\sigma}_- - \hat{c} \hat{\sigma}_+), \quad (4)$$

$$\frac{dc\Lambda^\dagger \hat{c}}{dT} = -i\lambda (c\Lambda^\dagger \hat{\sigma}_- - \hat{c} \hat{\sigma}_+), \quad (5)$$

Thus, we deduce that the operator $2\hat{N} = \hat{\sigma}_z + 2c\Lambda^\dagger \hat{c}$ is a constant of motion. Therefore, the Hamiltonian (1) becomes

$$\frac{\hat{H}}{\hbar} = \omega \hat{N} - \Delta \hat{\sigma}_z + \lambda_1 (\hat{c} \hat{\sigma}_+ + c\Lambda^\dagger \hat{\sigma}_-), \quad (6)$$

where $\Delta = 1/2(\omega - \omega_0)$.

The time evolution operator is given by

$$\hat{U}(t) = \exp(-i\hat{H}t) = \begin{bmatrix} \hat{F}_1(\hat{n}, t) \exp\{-i\omega \hat{n}t\} & -i \exp\{-i\omega \hat{n}t\} \lambda \hat{E}_1(\hat{n}, t) \hat{c} \\ -i \exp\{-i\omega(\hat{n}-1)t\} \lambda \hat{E}_2(\hat{n}, t) c\Lambda^\dagger & \exp\{-i\omega(\hat{n}-1)t\} \hat{F}_2(\hat{n}, t) \end{bmatrix}, \quad (7)$$

$$\hat{F}_j(\hat{n}, t) = \cos \hat{\mu}_j(\hat{n})t - \frac{i\Delta}{\hat{\mu}_j(\hat{n})} \sin \hat{\mu}_j(\hat{n})t, \hat{E}_j(\hat{n}, t) = \frac{\sin \hat{\mu}_j(\hat{n})t}{\hat{\mu}_j(\hat{n})}, j = 1, 2, \quad (8)$$

$$\hat{\mu}_j^2(\hat{n}) = \Delta^2 + \nu_j(\hat{n}), j = 1, 2, \quad (9)$$

$$\hat{\nu}_1(\hat{n}) = \lambda^2(\hat{n}+1), \hat{\nu}_2 = \hat{\nu}_1(\hat{n}-1). \quad (10)$$

Assume that the atom is initially in the excited state $|+\rangle$, and the field in a Binomial state $|\eta, M\rangle$ [18], where

$$|\eta, M\rangle = \sum_{n=0}^M B(\eta, M, n) |n\rangle, \quad (11)$$

$$B(\eta, M, n) = \sqrt{\binom{M}{n} \eta^n (1-\eta)^{M-n}}, \quad (12)$$

where η is the characteristic probability of having each photon occurring, and M is the maximum photon number present in the field.

Therefore, the time-dependent wave function describing the proposed quantum system is given from the following relation,

$$|\Psi_{AF}(t)\rangle = |a\rangle|+\rangle + |b\rangle|-\rangle, \quad (13)$$

where

$$|a\rangle = \sum_{n=0}^M B(\eta, M, n) F_1(n, t) |n\rangle, \quad (14)$$

$$|b\rangle = -i \sum_{n=0}^M B(\eta, M, n) \exp\{-i\omega nt\} \sqrt{v_1(n)} E_1(n, t) |n+1\rangle. \quad (15)$$

3. Nonclassical Properties

In this part, the values of the parameters of this system are estimated. This leads to the improvement of the values of the entanglement between the proposed system. There is no doubt that the use of a statistical method in estimating the parameters leads to better results than the use of random values for these parameters. In quantum information, systems of qubits that interact remarkably with photons (quantum electrodynamics) are a suitable environment for studying open quantum systems [19]. Aside from being a major candidate for quantum information manipulation, such quantum optical systems are of fundamental theoretical importance [20]. In particular, the experimental and theoretical study of quantum optical systems may yield insight into the links between mixedness and entanglement. The mixedness which is associated with the impurity of the quantum state is always measured by atomic entropy [21]. The reduced density operator is given by

$$\rho_A(t) = \text{Tr}_F(|\psi_{AF}(t)\rangle\langle\psi_{AF}(t)|) = \begin{bmatrix} \langle a|a\rangle & \langle b|a\rangle \\ \langle a|b\rangle & \langle b|b\rangle \end{bmatrix}, \quad (16)$$

Here, the von Neumann formula is used to measure the entanglement between parts of a quantum system. Therefore, the von Neumann entropy is given by the following relation,

$$S_A(T) = -\text{Tr}_A(\rho_A(t) \ln(\rho_A(t))) = -\sum_{j=1}^2 e_j \ln(e_j), \quad (17)$$

where e_1, e_2 are the eigenvalues of the atomic reduced density matrix, which are defined as follows,

$$e_1(t) = \frac{1 + \sqrt{1 - 4(\langle a|a\rangle\langle b|b\rangle - |\langle a|b\rangle|^2)}}{2}, \quad (18)$$

$$e_2(t) = \frac{1 - \sqrt{1 - 4(\langle a|a\rangle\langle b|b\rangle - |\langle a|b\rangle|^2)}}{2}. \quad (19)$$

Recently, some measures have emerged that lead to measuring the entanglement between parts of quantum systems. One of the most well-known of these measures is Wehrl entropy [22], which is formulated on atomic Q-function. Therefore, the atomic Q-function is defined as follows,

$$Q_A(t) = \frac{1}{2\pi} \langle \theta, \phi | \rho_A(t) | \theta, \phi \rangle, \quad (20)$$

where θ, ϕ are the atomic phase space parameters and $|\theta, \phi\rangle$

is the atomic coherent state, which is defined as

$$|\theta, \phi\rangle = \cos\left(\frac{\theta}{2}\right) |+\rangle + \sin\left(\frac{\theta}{2}\right) \exp(-i\phi) |-\rangle. \quad (21)$$

The expansion in terms of θ and ϕ can be deciphered as follows:

$$Q_A(t) = \frac{1}{2\pi} \left(\begin{array}{c} 1 + (\langle a|a\rangle - \langle b|b\rangle) \cos(\theta) + \\ 2 \text{Re}(\langle a|b\rangle) \sin(\theta) \cos(\phi) + 2 \text{Im}(\langle a|b\rangle) \sin(\theta) \sin(\phi) \end{array} \right), \quad (22)$$

$$0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi. \quad (23)$$

Another measure of classical correlation for quantum systems in phase space is the Wigner function [23, 24], which is given by

$$W(x, y) = \frac{1}{\pi} \sum_{k=0}^{\infty} \sum_{n,m=0}^{\infty} (-1)^k (A_n(t) A_m^*(t) G_{k,n}(\Gamma) G_{m,k}(\Gamma) + B_n(t) B_m^*(t) G_{k,n+1}(\Gamma) G_{m+1,k}(\Gamma)), \quad (24)$$

$$A_n(t) = B(\eta, M, n) F(n, t), B_n(t) = -iB(\eta, M, n) \exp\{-i\omega nt\} \sqrt{v_1(n)} E_1(n, t), \quad (25)$$

$$G_{k,n}(\Gamma) = \exp\left(-\frac{|\Gamma|^2}{2}\right) \sum_{j=0}^{\min(k,n)} \frac{(-\Gamma)^{k-j} (\Gamma^*)^{n-j} \sqrt{k!n!}}{(k-j)!(n-j)!j!}, \quad (26)$$

$$\Gamma = x + iy. \quad (27)$$

4. Statistical Study

4.1. Statistical Study and Discussions. It is worth to investigate the relationship between S, Q , and W (i.e., when S is a maximum, is that would lead to Q to be maximum or minimum? Then what is the effect on the related parameters η, τ , and else). Some questions about the behaviour of equations (17), (23), and (24) and the related parameters in many cases have not a precise answer. Therefore, statistical studies are given in this subsection to give suitable answers in this point.

Statistical studies are used frequently to address and solve many of research questions. Simulation studies are creating data with numerical techniques like Monte-Carlo simulation by pseudorandom sampling based on computer packages for conducting experiments on the computer [25–30]. A Monte-Carlo simulation is defined as using random sampling from probability distributions in the computer experiments. Many researchers have used the simulation studies in their research, among them [16, 17]. The aim of simulation studies of this research is the ability to understand the behaviour of certain functions. This allows to consider some properties of the required equations (17), (23), and (24). The Monte-Carlo simulations are employed to investigate the behaviour of S, Q , and W functions and the related parameters by assuming that the parameters of

the S , Q , and W are random variables. Mathematica 10 program with 4 GB RAM and processor Core i7 is employed to generate samples for $\tau, \eta, \theta, \phi, x, y$, and M and then finding the corresponding dependent variables S, Q , and W . Depending on the interval of each variable, we assumed that

- (1) The parameter η of the binomial distribution is a random variable follows the standard uniform distribution in the form $f(\eta) = 1, 0 \leq \eta \leq 1$
- (2) The parameter τ is a random variable follows a uniform distribution in the form $f(\tau) = 1/50, 1 < \tau < 50, 0 \leq \tau \leq 50$
- (3) X has a standard normal distribution in the form $f(x) = 1/\sqrt{2\pi}e^{-0.5x^2}, -\infty < x < \infty$
- (4) Y has a standard normal distribution in the form $f(y) = 1/\sqrt{2\pi}e^{-0.5y^2}, -\infty < y < \infty$
- (5) The number of values which simulated was 5000 runs and $M = 5, 20, 50$ and 100. Then, finding the maximum and minimum for $S, Q, W, \tau(d), \eta(d), \theta, \phi, x$, and y for each value of M , where $d = S, Q$, and W . For instance, $\max \eta(S) = 0.0203785$ when $M = 5$ means that the value of $\eta = 0.0203785$ when S was maximum and similarly for Q and W . Some graphs are obtained to see the behaviour of S, Q , and W graphically at the maximum and minimum points which are given at Tables 1–4

Here, the results were obtained from the simulation method which are used to discuss the entanglement in the case of $M = 50$. For the data that estimated the maximum values of $S_A(T)$, the entanglement between the cavity field and the qubit returns between maximum and minimum values $(0, \ln(2))$; moreover, it repeats periodically. The function $S_A(T)$ reaches zero (pure state) only in two points, as evident from Figure 1. Whereas in the case of the minimum function $S_A(T)$, the entanglement between the field and the qubits reaches a state of separation periodically, as seen in Figure 2.

For the data that estimated the maximum values of $Q_A(t)$, in the case of $M = 50$, the atomic Q -function fluctuates chaotically, and the negative values are not shown completely, see Figure 3. Moreover, the function $Q_A(t)$ is improved and oscillations appeared regularly as observed in Figure 3. For the data that estimated the minimum values of $Q_A(t)$, in the case of $M = 50$, the oscillations have the chaotic behaviour. The intensity of the oscillations increases when $M = 50$ is taken into account, as shown in Figure 4.

For the data that estimated the maximum values of $W(x, y)$, in the case $M = 50$, for the maximum data, the quasi-distribution $W(x, y)$ has one-peak centered in the center $(0, 0)$. The nonclassical behaviour appears around the peak. The maximum values of the function $W(x, y)$ which decrease after the minimum values data are included. The previous peak was divided into a crater with multiple peaks at the edges. In addition, the negative values appear significantly as shown in Figure 5.

4.2. Concluding Remarks. The aim of this study is to discover the behaviour of the three main functions S, Q , and W between each others and studying the impact of the related variables. A standard uniform distribution has used to generate values for η, τ, θ, ϕ , and the standard normal distribution has used to generate values for x and y . A random samples of $0 < \tau < 50, 0 < \eta \leq 1, 0 < \theta < \pi, 0 < \phi < 2\pi, -\infty < x < \infty$ and $-\infty < y < \infty$ are generated and then S, Q , and W are estimated by using Mathematica 10 program to make a numerical study. Depending on the statistical study presented on the previous Section 4, the values of S and Q are always positive but W may take negative values. There is no affection of $\tau, \eta, \theta, \phi, x, y$, and M on the behaviour of S, Q , and W functions. In addition, all cases of $M, W < Q < S$ as seen in Tables 1–4 and Figures 1–6, the remarks on the Tables can be written as follows.

4.3. Remarks

- (1) The S function always positive and less than $\ln 2$ and may take 0 by rounding, while the W function may take negative values
- (2) The behaviour of the functions S, Q , and W does not depend on the values of τ and η . A direct or an indirect relation between the specific functions and their parameters cannot be discovered
- (3) A similar behaviour of the S, Q , and W does not observe. There is no remark recorded that the three functions have a maximum or minimum at the same time and at the same values
- (4) The impact of increasing M is too low on S, Q , and W
- (5) The upper limit of Q function is $\ln 1.002779 \approx 0.0027748$ and does not less than 0
- (6) The upper limit of W function is $\ln 1.0055 \approx 0.00553$ and is not less than -0.00096
- (7) Depending on simulation study, there is no relation between θ and ϕ and the behaviour of Q
- (8) The signs of x and y may be the same or different in many cases, and there are no relation between x and y and the behaviour of W
- (9) In all cases of M , the relation $W < Q < S$ was verified

5. Conclusion

A statistical method is used to predict the behaviour of a quantum model consisting of a qubit interacting with a single-mode cavity field is addressed. The qubit is prepared in the excited state while the field is in the binomial distribution. The wave function of the proposed model is obtained and studied numerically. A von Neumann entropy is used to investigate the behaviour of the entanglement between the field and the qubits. Moreover, the atomic Q and Wigner functions are used to identify the behaviour of the distribution in a phase space. A numerical study is performed to

estimate the specific dependency of the binomial distribution state. The results of entanglement were compared with the atomic Q and Wigner functions (see Tables 1–4 and Figures 1–6). Finally, different values of the parameters $\tau, \eta, \theta, \phi, x,$ and y are simulated and then the maximum and minimum values of $S, Q,$ and W are obtained. In brief, depending on the statistical study which was presented on Section 4, the values of S and Q are always positive but W may take negative values. Furthermore, there is no affection of $\tau, \eta, \theta, \phi, x, y,$ and M on the behaviour of $S, Q,$ and W functions.

Data Availability

The data used are generated from the mathematical models in the article.

Conflicts of Interest

The authors have no conflicts of interest regarding the publication of the paper.

Acknowledgments

This research was funded by the Deanship of Scientific Research, Taif University, KSA (research group number 1-1441-100).

References

- [1] Saifullah et al., “Feedback control of probability amplitudes for two-level atom in optical field,” *Optics Communications*, vol. 281, no. 4, pp. 640–643, 2008.
- [2] S. V. Prants and V. Y. Sirotkin, “Effects of the Rabi oscillations on the atomic motion in a standing-wave cavity,” *Physical Review A*, vol. 64, no. 3, 2001.
- [3] M. S. Abdalla, E. M. Khalil, and A.-S. F. Obada, “Statistical properties of a two-photon cavity mode in the presence of degenerate parametric amplifier,” *Annals of Physics*, vol. 322, no. 11, pp. 2554–2568, 2007.
- [4] E. M. Khalil, M. S. Abdalla, and A.-S. F. Obada, “Entropy and variance squeezing of two coupled modes interacting with a two-level atom: frequency converter type,” *Annals of Physics*, vol. 321, no. 2, pp. 421–434, 2006.
- [5] S. Bougouffa, “Entanglement dynamics of two-bipartite system under the influence of dissipative environments,” *Optics Communication*, vol. 283, no. 14, pp. 2989–2996, 2010.
- [6] N. H. Abdel-Wahab and M. F. Mourad, “On the interaction between two two-level atoms and a two mode cavity field in the presence of detuning and cross-Kerr nonlinearity,” *Physica Scripta*, vol. 84, no. 1, article 015401, 2011.
- [7] A. Slaoui, A. Salah, and M. Daoud, “Influence of Stark-shift on quantum coherence and non-classical correlations for two two-level atoms interacting with a single-mode cavity field,” *Physica A: Statistical Mechanics and its Applications*, vol. 558, article 124946, 2020.
- [8] K. El Anouz, A. El Allati, A. Salah, and F. Saif, “Quantum fisher information: probe to measure fractional evolution,” *International Journal of Theoretical Physics*, vol. 59, no. 5, pp. 1460–1474, 2020.
- [9] N. H. Abdel-Wahab and A. Salah, “On the interaction between a time-dependent field and a two-level atom,” *Modern Physics Letters A*, vol. 34, no. 10, article 1950081, 2019.
- [10] A.-S. F. Obada, M. M. A. Ahmed, A. M. Farouk, and A. Salah, “A moving three-level Λ -type atom in a dissipative cavity,” *The European Physical Journal D*, vol. 71, no. 12, pp. 1–13, 2017.
- [11] N. H. Abd el-Wahab and A. Salah, “The influence of the classical homogenous gravitational field on interaction of a three-level atom with a single mode cavity field,” *Modern Physics Letters B*, vol. 29, no. 29, article 1550175, 2015.
- [12] J. K. Kalaga, A. Kowalewska-Kudłaszyk, W. Leoński, and A. Barasiński, “Quantum correlations and entanglement in a model comprised of a short chain of nonlinear oscillators,” *Physical Review A*, vol. 94, no. 3, 2016.
- [13] L.-T. Shen, Z.-B. Yang, H.-Z. Wu, X.-Y. Chen, and S.-B. Zheng, “Control of two-atom entanglement with two thermal fields in coupled cavities,” *Journal of the Optical Society of America B*, vol. 29, no. 9, article 2379, 2012.
- [14] J. K. Kalaga, W. Leonski, and R. Szczesniak, “Quantum steering and entanglement in three-mode triangle Bose–Hubbard system,” *Quantum Information Processing*, vol. 16, no. 11, p. 265, 2017.
- [15] S. L. Braunstein, *Quantum Computing: Where Do We Want to Go Tomorrow?*, Wiley-VCH, New York, 1999.
- [16] A.-S. F. Obada, M. M. A. Ahmed, and E. M. Khalil, “Generation of a nonlinear two-mode Stark shift through the adiabatic elimination method,” *Journal of Modern Optics*, vol. 53, no. 8, pp. 1149–1163, 2006.
- [17] E. M. Khalil, K. Berrada, S. Abdel-Khalek, A. Al-Barakaty, and J. Perina, “Entanglement and entropy squeezing in the system of two qubits interacting with a two-mode field in the context of power low potentials,” *Scientific Reports*, vol. 10, no. 1, article 76059, 2020.
- [18] R. Lo Franco, G. Compagno, A. Messina, and A. Napoli, “Generating and revealing a quantum superposition of electromagnetic-field binomial states in a cavity,” *Physical Review A*, vol. 76, no. 1, article 011804, 2007.
- [19] H. Mabuchi and A. C. Doherty, “Cavity quantum electrodynamics: coherence in context,” *Science*, vol. 298, no. 5597, pp. 1372–1377, 2002.
- [20] C. Monroe, “Quantum information processing with atoms and photons,” *Nature*, vol. 416, no. 6877, pp. 238–246, 2002.
- [21] G. Jaeger, A. V. Sergienko, B. E. A. Saleh, and M. C. Teich, “Entanglement, mixedness, and spin-flip symmetry in multiple-qubit systems,” *Physical Review A*, vol. 68, no. 2, article 022318, 2003.
- [22] A.-S. F. Obada, S. Abdel-Khalek, E. M. Khalil, and S. I. Ali, “Effects of Stark shift and decoherence terms on the dynamics of phase-space entropy of the multiphoton Jaynes Cummings model,” *Physica Scripta*, vol. 86, no. 5, article 055009, 2012.
- [23] E. Wigner, “On the quantum correction for thermodynamic equilibrium,” *Physics Review*, vol. 40, no. 5, pp. 749–759, 1932.
- [24] M. Y. Abd-Rabbou, E. M. Khalil, M. M. A. Ahmed, and A.-S. F. Obada, “External classical field and damping effects on a moving two level atom in a cavity field interaction with Kerr-like medium,” *International Journal of Theoretical Physics*, vol. 58, pp. 4012–4024, 2019.
- [25] T. Morris, I. White, and M. Crowther, “Using simulation studies to evaluate statistical methods,” *Statistics in Medicine*, vol. 38, no. 11, pp. 2074–2102, 2019.

- [26] A.-L. Boulesteix, R. H. H. Groenwold, M. Abrahamowicz et al., "Introduction to statistical simulations in health research," *Epidemiology Communication*, vol. 10, no. 12, article e039921, 2020.
- [27] M. M. Amein, "Estimation for unknown parameters of the Burr type-XII distribution based on an adaptive progressive type-II censoring scheme," *Journal of Mathematics and Statistics*, vol. 12, no. 2, pp. 119–126, 2016.
- [28] M. M. Mohie El-Din, M. M. Amein, A. R. Shafay, and S. Mohamed, "Bayesian prediction based on an adaptive type-II progressive censored data from the generalized exponential distribution," *International Journal of Computer Applications*, vol. 179, no. 46, pp. 1–6, 2018.
- [29] M. M. Amein and N. Sayed-Ahmed, "Estimation for unknown parameters of the extended Burr type-XII distribution based on type-I hybrid progressive censoring scheme," *International Journal of Applied Engineering Research*, vol. 13, no. 3, article 1761, 2018.
- [30] M. M. Amein, N. Sayed-Ahmed, and O. S. Bakri, "Interval estimation for Burr-X distribution under type-I hybrid progressive censoring scheme," *Journal of Statistics Applications & Probability*, vol. 7, no. 1, p. 1, 2020.