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Research Article

Lie Symmetry Analysis, Exact Solutions, and Conservation Laws of Variable-Coefficients Boiti-Leon-Pempinelli Equation

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In this article, we study the generalized $(2+1)$-dimensional variable-coefficients Boiti-Leon-Pempinelli (vcBLP) equation. Using Lie's invariance infinitesimal criterion, equivalence transformations and differential invariants are derived. Applying differential invariants to construct an explicit transformation that makes vcBLP transform to the constant coefficient form, then transform to the well-known Burgers equation. The infinitesimal generators of vcBLP are obtained using the Lie group method; then, the optimal system of one-dimensional subalgebras is determined. According to the optimal system, the $(1+1)$-dimensional reduced partial differential equations (PDEs) are obtained by similarity reductions. Through (G'/G)-expansion method leads to exact solutions of vcBLP and plots the corresponding 3-dimensional figures. Subsequently, the conservation laws of vcBLP are determined using the multiplier method.

1. Introduction

Nonlinear issues are widespread in some natural disciplines, and many difficult problems in some disciplines can be reduced to solving a certain PDE or investigating some properties of a PDE [1, 2]. With the rapid development of research fields like hydrodynamics and quantum physics, it has become increasingly important to investigate the exact solutions and certain properties of nonlinear evolution equations [3–5]. Compared with the PDEs with constant coefficients, the PDEs with variable coefficients can describe richer natural phenomena and construct more detailed and complex physical models [6–8].

In this paper, we focus on the generalized $(2+1)$-dimensional Boiti-Leon-Pempinelli equation with time-part variable coefficients as

$$\begin{cases} F_1 = u_{yt} + a(t)u_x u_y + a(t)uu_{xy} + b(t)u_{xxy} + c(t)v_{xxx} = 0, \\ F_2 = v_t + d(t)v_{xx} + e(t)(uv)_x = 0, \end{cases} \tag{1}$$

where $a(t)$, $b(t)$, $c(t)$, $d(t)$, and $e(t)$ are any functions with respect to time t. It represents the development of the components in the horizontal velocity in the x and y directions when the water wave propagates in a channel of unchanging depth and infinitely small width [9]. The vcBLP is conditionally integrable, and the necessary conditions for it to be Painlevé integrable are $d'(t)e(t) = d(t)e'(t)$ and $a'(t)b(t) = a(t)b'(t)$. Some exact solutions of vcBLP were obtained in [10] by extended tanh-function method. We can find some periodic solutions and soliton solutions of vcBLP obtained with the homogeneous balance method in [11], and the conservation laws for the constant coefficients BLP were discussed in [9]. By reviewing the relevant literatures, no one has studied vcBLP using the Lie group method.

The outline of this article is as follows. In Section 2, we construct the equivalence transformations and differential invariants of vcBLP. Based on these, we give an explicit transformation to its constant coefficient form. In Section 3, the infinitesimal generators of vcBLP are obtained using the Lie group method, and then, the optimal system for the one-dimensional subalgebras is constructed. In Section 4, we obtain six sets of $(1+1)$-dimensional reduced PDEs by similarity reductions to vcBLP. In Section 5, some exact solutions of vcBLP are shown using the (G'/G)-expansion method on the basis of the reduced PDEs in above section. In Section 6, we use the multiplier method to calculate the conservation laws of vcBLP. We can find the conclusions of this paper in Section 7.

2. Equivalence Transformations and Differential Invariants of vcBLP

In this part, we construct the equivalence transformations [12] of Equation (1). The equivalence transformation of Equation (1) is a nondegenerate point transformation which from (x, y, t, u, v) to $(\tilde{x}, \tilde{y}, \tilde{t}, \tilde{u}, \tilde{v})$ [13]. It has the same differential structure but different coefficient functions $\tilde{a}(\tilde{t}), \tilde{b}(\tilde{t}), \tilde{c}(\tilde{t}), \tilde{d}(\tilde{t}), \tilde{e}(\tilde{t})$ than the original equation. First, we assume that the auxiliary conditions are

$$a_u = a_v = 0, b_u = b_v = 0, c_u = c_v = 0, d_u = d_v = 0, e_u = e_v = 0, \tag{2}$$

with the one-parameter group of equivalence transformations is determined on the basis of

$$\begin{aligned}
\tilde{x} &= x + \varepsilon \cdot \xi(x, y, t, u, v, a, b, c, d, e) + o(\varepsilon^2), \\
\tilde{y} &= y + \varepsilon \cdot \eta(x, y, t, u, v, a, b, c, d, e) + o(\varepsilon^2), \\
\tilde{t} &= t + \varepsilon \cdot \tau(x, y, t, u, v, a, b, c, d, e) + o(\varepsilon^2), \\
\tilde{u} &= u + \varepsilon \cdot \phi_1(x, y, t, u, v, a, b, c, d, e) + o(\varepsilon^2), \\
\tilde{v} &= v + \varepsilon \cdot \phi_2(x, y, t, u, v, a, b, c, d, e) + o(\varepsilon^2), \\
\tilde{a} &= a + \varepsilon \cdot \Delta_1(x, y, t, u, v, a, b, c, d, e) + o(\varepsilon^2), \\
\tilde{b} &= b + \varepsilon \cdot \Delta_2(x, y, t, u, v, a, b, c, d, e) + o(\varepsilon^2), \\
\tilde{c} &= c + \varepsilon \cdot \Delta_3(x, y, t, u, v, a, b, c, d, e) + o(\varepsilon^2), \\
\tilde{d} &= d + \varepsilon \cdot \Delta_4(x, y, t, u, v, a, b, c, d, e) + o(\varepsilon^2), \\
\tilde{e} &= e + \varepsilon \cdot \Delta_5(x, y, t, u, v, a, b, c, d, e) + o(\varepsilon^2),
\end{aligned} \tag{3}$$

and ε is the group parameter. The vector field or generators of Equation (1) which corresponds to transformations (3) as

$$\begin{aligned}
Y = {}& \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \tau \frac{\partial}{\partial t} + \phi_1 \frac{\partial}{\partial u} + \phi_2 \frac{\partial}{\partial v} + \Delta_1 \frac{\partial}{\partial a} + \Delta_2 \frac{\partial}{\partial b} \\
& + \Delta_3 \frac{\partial}{\partial c} + \Delta_4 \frac{\partial}{\partial d} + \Delta_5 \frac{\partial}{\partial e}.
\end{aligned} \tag{4}$$

Since Equation (1) is invariant under the above transformations (3) and there exists the 3rd derivatives, we have to use the 3rd prolongation $Y^{(3)}$. We define that

$$\begin{gathered}
(f^1, f^2, f^3, f^4, f^5) \equiv (a(t), b(t), c(t), d(t), e(t)), \\
(x^1, x^2, x^3) \equiv (x, y, t), (y^1, y^2) \equiv (u, v), (\psi^1, \psi^2, \psi^3) \equiv (\xi, \eta, \tau), \\
y^i_j = \frac{\partial y^i}{\partial x^j}, y^i_{jk} = \frac{\partial^2 y^i}{\partial x^j \partial x^k}, (j, k = 1, 2, 3, i = 1, 2).
\end{gathered} \tag{5}$$

On the basis of the above Equation (5) and Y, the 3rd prolongation $Y^{(3)}$ can be written as

$$\begin{aligned}
Y^{(3)} = {}& Y + \zeta^i_j \frac{\partial}{\partial y^i_j} + \zeta^i_{j\sigma} \frac{\partial}{\partial y^i_{j\sigma}} + \zeta^i_{jj\sigma} \frac{\partial}{\partial y^i_{jj\sigma}} + \tilde{\omega}^r_j \frac{\partial}{\partial f^r_{x^j}} \\
& + \bar{\omega}^r_{y^j} \frac{\partial}{\partial f^r_{y^j}}, (r = 1, 2, \cdots, 5),
\end{aligned} \tag{6}$$

where

$$\begin{gathered}
\zeta^i_j = D_j \phi_j - y^i_k D_j \psi^k, \zeta^i_{j\sigma} = D_\sigma \zeta^i_j - y^i_{jk} D_\sigma \psi^k, \zeta^i_{jj\sigma} = D_\sigma \zeta^i_{jj} - y^i_{jjk} D_\sigma \psi^k, \\
\tilde{\omega}^r_j = \tilde{D}_j(\Delta_r) - f^r_{x^k} \tilde{D}_j(\psi^k) - f^r_{y^i} \tilde{D}_j(\phi_i),
\end{gathered} \tag{7}$$

with

$$D_j = \frac{\partial}{\partial x^j} + y^i_j \frac{\partial}{\partial y^i} + y^i_{jk} \frac{\partial}{\partial y^i_k}, \tilde{D}_j = \frac{\partial}{\partial x^j} + f^r_{x^j} \frac{\partial}{\partial f^r}, \bar{D}_j = \frac{\partial}{\partial y^j} + f^r_{y^j} \frac{\partial}{\partial f^r}. \tag{8}$$

Under transformations (3), the invariance of Equation (1) requires that to satisfy the conditions

$$\begin{gathered}
Y^{(3)}(F_1) = 0, Y^{(3)}(F_2) = 0, Y^{(3)}(a_u) = Y^{(3)}(a_v) = 0, \\
Y^{(3)}(b_u) = Y^{(3)}(b_v) = 0, Y^{(3)}(c_u) = Y^{(3)}(c_v) = 0, \\
Y^{(3)}(d_u) = Y^{(3)}(d_v) = 0, Y^{(3)}(e_u) = Y^{(3)}(e_v) = 0.
\end{gathered} \tag{9}$$

We can obtain a determining equation by bringing Equation (6), Equation (7), and Equation (8) into Equation (9). Subsequently, solving this determining equation, we get

$$\begin{gathered}
\xi = C_1 x + C_2, \eta = \rho(y), \tau = \beta(t), \phi_1 = C_3 u, \\
\phi_2 = g(y), \Delta_1 = a(C_1 - C_3 - \beta_t), \Delta_2 = b(2C_1 - \beta_t), \\
\Delta_3 = -c\left(\rho_y + g + \beta_t - 3C_1 - C_3\right), \Delta_4 = d(2C_1 - \beta_t), \Delta_5 = e(C_1 - C_3 - \beta_t),
\end{gathered} \tag{10}$$

where $\beta = \beta(t)$, $\rho = \rho(y)$, and $g = g(y)$ are arbitrary smooth functions. The corresponding infinite dimensional equivalence group is generated by the following operators:

$$\begin{gathered}Y_1=x\frac{\partial}{\partial x}+a\frac{\partial}{\partial a}+2b\frac{\partial}{\partial b}+3c\frac{\partial}{\partial c}+2d\frac{\partial}{\partial d}+e\frac{\partial}{\partial e},\\ Y_2=\frac{\partial}{\partial x},\\ Y_3=u\frac{\partial}{\partial u}-a\frac{\partial}{\partial a}+c\frac{\partial}{\partial c}-e\frac{\partial}{\partial e},\\ Y_\beta=\beta\frac{\partial}{\partial t}-a\beta_t\frac{\partial}{\partial a}-b\beta_t\frac{\partial}{\partial b}-c\beta_t\frac{\partial}{\partial c}-d\beta_t\frac{\partial}{\partial d}-e\beta_t\frac{\partial}{\partial e},\\ Y_\rho=\rho\frac{\partial}{\partial y}-c\rho_y\frac{\partial}{\partial c},\\ Y_g=gv\frac{\partial}{\partial v}-cg\frac{\partial}{\partial c}.\end{gathered}\tag{11}$$

In the following, our task is to derive the zero-order differential invariants and the first-order differential invariants. We assume that the form of the zero-order differential invariant is

$$J=J(x,y,t,a,b,c,d,e).\tag{12}$$

Applying the invariant test $Y(J)=0$ to the operators (11), we can obtain

$$Y_k(J)=0,(k=1,2,3,\beta,\rho,g),\tag{13}$$

and the above equations can be reduced to

$$\begin{gathered}\frac{\partial J}{\partial x}=0,\ \frac{\partial J}{\partial y}=0,\ \frac{\partial J}{\partial c}=0,\ \frac{\partial J}{\partial t}=0,\\ b\frac{\partial J}{\partial b}+d\frac{\partial J}{\partial d}=0,\ a\frac{\partial J}{\partial a}+e\frac{\partial J}{\partial e}=0,\end{gathered}\tag{14}$$

solving this system, we get that the zero-order differential invariants are

$$J_1^{(0)}=\frac{d}{b},\ J_2^{(0)}=\frac{e}{a},\tag{15}$$

with invariant equations are

$$a=0,b=0,d=0,e=0.\tag{16}$$

Next, we derive the interesting first-order differential invariants. Similar to the above case, we suppose that the first-order differential invariant is the form

$$J=J(x,y,t,a,b,c,d,e,a_t,b_t,c_t,d_t,e_t).\tag{17}$$

In order to get the first-order differential invariants, we need to make the first prolongation of the operators $Y_k,(k=1,2,3,\beta,\rho,g)$ using $Y^{(1)}=Y+\tilde{\omega}_j^r(\partial/\partial f_{x^j}^r)$, and Equation (11) can be rewritten as

$$\begin{gathered}Y_1^{(1)}=x\frac{\partial}{\partial x}+a\frac{\partial}{\partial a}+2b\frac{\partial}{\partial b}+3c\frac{\partial}{\partial c}+2d\frac{\partial}{\partial d}+e\frac{\partial}{\partial e}\\ +a_t\frac{\partial}{\partial a_t}+2b_t\frac{\partial}{\partial b_t}+3c_t\frac{\partial}{\partial c_t}+2d_t\frac{\partial}{\partial d_t}+e_t\frac{\partial}{\partial e_t},\\ Y_2^{(1)}=\frac{\partial}{\partial x},\\ Y_3^{(1)}=u\frac{\partial}{\partial u}-a\frac{\partial}{\partial a}+c\frac{\partial}{\partial c}-e\frac{\partial}{\partial e}-a_t\frac{\partial}{\partial a_t}+c_t\frac{\partial}{\partial c_t}-e_t\frac{\partial}{\partial e_t},\\ Y_\beta^{(1)}=\beta\frac{\partial}{\partial t}-a\beta_t\frac{\partial}{\partial a}-b\beta_t\frac{\partial}{\partial b}-c\beta_t\frac{\partial}{\partial c}-d\beta_t\frac{\partial}{\partial d}-e\beta_t\frac{\partial}{\partial e}\\ -(a\beta_{tt}+2a_t\beta_t)\frac{\partial}{\partial a_t}-(b\beta_{tt}+2b_t\beta_t)\frac{\partial}{\partial b_t}\\ -(c\beta_{tt}+2c_t\beta_t)\frac{\partial}{\partial c_t}-(d\beta_{tt}+2d_t\beta_t)\frac{\partial}{\partial d_t}-(e\beta_{tt}+2e_t\beta_t)\frac{\partial}{\partial e_t},\\ Y_\rho^{(1)}=\rho\frac{\partial}{\partial y}-c\rho_y\frac{\partial}{\partial c}-c_t\rho_y\frac{\partial}{\partial c_t},\\ Y_g^{(1)}=vg\frac{\partial}{\partial v}-cg\frac{\partial}{\partial c}-c_tg\frac{\partial}{\partial c_t}.\end{gathered}\tag{18}$$

The invariance test associated with Equation (18) is

$$Y_k^{(1)}(J)=0,(k=1,2,3,\beta,\rho,g),\tag{19}$$

solving this system yields

$$J_1^{(1)}=\frac{b(ac_t-a_tc)}{c(ab_t-a_tb)},J_2^{(1)}=\frac{ad_t-a_td}{ab_t-a_tb},J_3^{(1)}=\frac{b(ae_t-a_te)}{a(ab_t-a_tb)},\tag{20}$$

with invariant equations are

$$ab_t-a_tb=0,ac_t-a_tc=0,ad_t-a_td=0,ae_t-a_te=0.\tag{21}$$

Based on the above facts, we use differential invariants to give the transformation of vcBLP to its constant coefficient form. We take the constant coefficient form of Equation (1) as

$$\begin{gathered}u_{yt}+m_1u_xu_y+m_1uu_{xy}+m_2u_{xxy}+m_3v_{xxx}=0,\\ v_t+m_4v_{xx}+m_5(uv)_x=0,\end{gathered}\tag{22}$$

where $m_i,(i=1,\cdots,5)$ are arbitrary nonzero constants. To obtain the transformation from Equation (1) to Equation (22), we need a necessary condition that the coefficient functions of Equation (1) must satisfy the following equations

$$\frac{d}{b}=\frac{m_4}{m_2},\frac{e}{a}=\frac{m_5}{m_1},ab_t-a_tb=0,ac_t-a_tc=0,ad_t-a_td=0,ae_t-a_te=0.\tag{23}$$

Under this condition, the more general form of the coefficient functions are

$$a(t)=\frac{m_1\gamma_3}{m_5}c(t),\ b(t)=\frac{m_2\gamma_4}{m_4}c(t),\ d(t)=\gamma_4 c(t),\ e(t)=\gamma_3 c(t), \tag{24}$$

therefore, Equation (1) becomes

$$\begin{gathered}u_{yt}+\frac{m_1\gamma_3}{m_5}c(t)u_x u_y+\frac{m_1\gamma_3}{m_5}c(t)uu_{xy}+\frac{m_2\gamma_4}{m_4}c(t)u_{xxy}+c(t)v_{xxx}=0,\\ v_t+\gamma_4 c(t)v_{xx}+\gamma_3 c(t)(uv)_x=0,\end{gathered} \tag{25}$$

and there exists transformation

$$\begin{gathered}u=\frac{m_5}{m_4}\cdot\frac{\gamma_1\gamma_4}{\gamma_3}\tilde{u}(\tilde{x},\tilde{y},\tilde{t}),\ v=\delta'(y)\tilde{v}(\tilde{x},\tilde{y},\tilde{t}),\\ \tilde{x}=\gamma_1 x+\gamma_2,\ \tilde{y}=\frac{m_4^2}{m_3m_5}\cdot\frac{\gamma_3}{\gamma_4^2}\delta(y),\ \tilde{t}=\frac{\gamma_1^2\gamma_4}{m_4}\int c(t)dt+\gamma_5,\end{gathered} \tag{26}$$

where $\gamma_i, (i=1,\cdots,5)$ are arbitrary constants and $\delta(y)$ is an arbitrary function. It is easy to verify that the above transformation maps Equation (1) into

$$\tilde{u}_{\tilde{y}\tilde{t}}+m_1\tilde{u}_{\tilde{x}}\tilde{u}_{\tilde{y}}+m_1\tilde{u}\tilde{u}_{\tilde{x}\tilde{y}}+m_2\tilde{u}_{\tilde{x}\tilde{x}\tilde{y}}+m_3\tilde{v}_{\tilde{x}\tilde{x}\tilde{x}}=0, \tag{27}$$

$$\tilde{v}_{\tilde{t}}+m_4\tilde{v}_{\tilde{x}\tilde{x}}+m_5(\tilde{u}\tilde{v})_{\tilde{x}}=0, \tag{28}$$

so the constant coefficient form in other literature [9] is a special case of Equation (1), and it is easy to find that Equation (24) satisfies the conditions for integrability.

We take $\tilde{v}=\tilde{u}, \tilde{y}=\tilde{x}$, then the above Equation (27) and Equation (28) become

$$\tilde{u}_{\tilde{x}\tilde{t}}+m_1\tilde{u}_{\tilde{x}}^2+m_1\tilde{u}\tilde{u}_{\tilde{x}\tilde{x}}+(m_2+m_3)\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}}=0, \tag{29}$$

$$\tilde{u}_{\tilde{t}}+2m_5\tilde{u}\tilde{u}_{\tilde{x}}+m_4\tilde{u}_{\tilde{x}\tilde{x}}=0, \tag{30}$$

subsequently, we integrate Equation (29) once with respect to x yields

$$\tilde{u}_{\tilde{t}}+m_1\tilde{u}\tilde{u}_{\tilde{x}}+(m_2+m_3)\tilde{u}_{\tilde{x}\tilde{x}}=0, \tag{31}$$

therefore, when $m_1=2m_5, m_4=m_2+m_3$, the above Equation (27) and Equation (28) can be converted into the well-known constant coefficient Burgers equation.

3. Lie Classical Symmetry Analysis of vcBLP

Lie group method is an effective way to find invariant solutions and to explore certain properties by reducing the dimensionality of the equations [14]. It has been described in sufficient detail in many literatures [15–18]. To begin with, we suppose that the one-parameter (ε) Lie group in Equation (1) is

$$\begin{aligned}x^*&=x+\varepsilon\cdot\xi(x,y,t,u,v)+o(\varepsilon^2),\\ y^*&=y+\varepsilon\cdot\eta(x,y,t,u,v)+o(\varepsilon^2),\\ t^*&=t+\varepsilon\cdot\tau(x,y,t,u,v)+o(\varepsilon^2),\\ u^*&=u+\varepsilon\cdot\phi_1(x,y,t,u,v)+o(\varepsilon^2),\\ v^*&=v+\varepsilon\cdot\phi_2(x,y,t,u,v)+o(\varepsilon^2),\end{aligned} \tag{32}$$

and Equation (1) remains invariant under transformations (32).

The vector field or infinitesimal generator of Equation (1) which corresponds to transformations (32) is

$$\begin{aligned}V=&\,\xi(x,y,t,u,v)\frac{\partial}{\partial x}+\eta(x,y,t,u,v)\frac{\partial}{\partial y}+\tau(x,y,t,u,v)\frac{\partial}{\partial t}\\ &+\phi_1(x,y,t,u,v)\frac{\partial}{\partial u}+\phi_2(x,y,t,u,v)\frac{\partial}{\partial v},\end{aligned} \tag{33}$$

its 3rd prolongation is written as $\mathrm{Pr}^{(3)}V$. Equation (1) remains invariant under transformations (32), which requires that

$$\mathrm{Pr}^{(3)}V(F_1)\Big|_{F_1=0}=0,\ \mathrm{Pr}^{(3)}V(F_2)\Big|_{F_2=0}=0. \tag{34}$$

Expanding Equation (34), the invariant conditions are redefined as

$$\begin{cases}\varphi_{yt}^1+\tau\left[a'(t)(uu_y)_x+b'(t)u_{xxy}+c'(t)v_{xxx}\right]+a(t)\left(\varphi_x^1u_y+\varphi_y^1u_x+\varphi_{xy}^1u+\phi_1u_{xy}\right)+b(t)\varphi_{xxy}^1+c(t)\varphi_{xxx}^2=0,\\ \varphi_t^2+\tau\left[d'(t)v_{xx}+e'(t)(uv)_x\right]+e(t)\left(\phi_1v_x+\phi_2u_x+\varphi_x^1v+\varphi_x^2u\right)+d(t)\varphi_{xx}^2=0,\end{cases} \tag{35}$$

where

$$\begin{aligned}\varphi_x^1&=D_x(\phi_1)-D_x(\xi)u_x-D_x(\eta)u_y-D_x(\tau)u_t,\ \varphi_x^2\\ &=D_x(\phi_2)-D_x(\xi)v_x-D_x(\eta)v_y-D_x(\tau)v_t,\end{aligned}$$

$$\begin{aligned}\varphi_y^1&=D_y(\phi_1)-D_y(\xi)u_x-D_y(\eta)u_y-D_y(\tau)u_t,\ \varphi_t^2\\ &=D_t(\phi_2)-D_t(\xi)v_x-D_t(\eta)v_y-D_t(\tau)v_t,\end{aligned}$$

$$\begin{aligned}\varphi^1_{xx} &= D_x\left(\varphi^1_x\right) - D_x(\xi)u_{xx} - D_x(\eta)u_{xy} - D_x(\tau)u_{xt}, \varphi^2_{xx} \\ &= D_x\left(\varphi^2_x\right) - D_x(\xi)v_{xx} - D_x(\eta)v_{xy} - D_x(\tau)v_{xt}, \\ \varphi^1_{xy} &= D_y\left(\varphi^1_x\right) - D_x(\xi)u_{xy} - D_x(\eta)u_{yy} - D_x(\tau)u_{yt}, \varphi^1_{yt} \\ &= D_t\left(\varphi^1_y\right) - D_y(\xi)u_{xt} - D_y(\eta)u_{yt} - D_y(\tau)u_{tt}, \\ \varphi^1_{xxy} &= D_y\left(\varphi^1_{xx}\right) - D_x(\xi)u_{xxy} - D_x(\eta)u_{xyy} - D_x(\tau)u_{xyt}, \varphi^2_{xxx} \\ &= D_x\left(\varphi^2_{xx}\right) - D_x(\xi)v_{xxx} - D_x(\eta)v_{xxy} - D_x(\tau)v_{xxt},\end{aligned} \tag{36}$$

with D_x, D_y, and D_t are the total differentiation of x, y, and t, respectively.

Solving Equation (35) yields the following results as

$$\begin{aligned}&\xi = C_1x + C_2, \eta = C_1y + C_3, \phi_1 = -C_1u, \\ &\tau = \frac{C_4}{b(t)} + \frac{2C_1}{b(t)} \cdot \int b(t)dt, \phi_2 = -C_1v,\end{aligned} \tag{37}$$

where $C_i(i = 1, 2, 3, 4)$ are any constants. Furthermore, the coefficient functions $a(t)$, $b(t)$, $c(t)$, $d(t)$, and $e(t)$ in Equation (1), which depend on time t and have to satisfy the conditions

$$\begin{aligned}&a'(t)\tau + a(t)\tau_t - 2C_1a(t) = 0, \\ &b'(t)\tau + b(t)\tau_t - 2C_1b(t) = 0, \\ &c'(t)\tau + c(t)\tau_t - 2C_1c(t) = 0, \\ &d'(t)\tau + d(t)\tau_t - 2C_1d(t) = 0, \\ &e'(t)\tau + e(t)\tau_t - 2C_1e(t) = 0.\end{aligned} \tag{38}$$

Thus, the infinitesimal generators of Equation (1) are expanded by the below vector field

$$\begin{aligned}&V_1 = \frac{\partial}{\partial x}, V_2 = \frac{\partial}{\partial y}, V_3 = \frac{1}{b(t)}\frac{\partial}{\partial t}, \\ &V_4 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{2}{b(t)}\int b(t)dt\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}.\end{aligned} \tag{39}$$

Depending on the $V_i(i = 1, 2, 3, 4)$, we have the following four groups

$$\begin{aligned}&G_1 : (x, y, t, u, v) \longrightarrow (x + \varepsilon, y, t, u, v), \\ &G_2 : (x, y, t, u, v) \longrightarrow (x, y + \varepsilon, t, u, v), \\ &G_3 : (x, y, t, u, v) \longrightarrow \left(x, y, t + \frac{\varepsilon}{b(t)}, u, v\right), \\ &G_4 : (x, y, t, u, v) \longrightarrow \left(xe^{\varepsilon}, ye^{\varepsilon}, t + \frac{2\varepsilon}{b(t)}\int b(t)dt, ue^{-\varepsilon}, ve^{-\varepsilon}\right),\end{aligned} \tag{40}$$

where $G_i(i = 1, 2, 3, 4)$ are one-parameter Lie point symmetry groups. It is not difficult to find that G_1 and G_2 are space translations and G_4 is a dependent variable translation. For G_3, it is a time translation when $b(t)$ is an arbitrary constant. For the combination of G_1 and G_2 can be understood as a translation along a certain direction, the group invariant solutions are the traveling wave solutions. The most important application of the traveling wave solutions is to construct soliton solutions of the PDE. The soliton reflects a rather common nonlinear phenomenon in nature, which is mainly characterized by its superstability, i.e., the wave shape remains stable after the collision of two solitary waves with different velocities. We can also understand solitons as local traveling wave solutions of nonlinear development equations. Also, symmetry has a great connection with conservation laws in physics; for example, space translation corresponds to momentum conservation, and time translation corresponds to energy conservation.

Generally speaking, it is possible to construct it group-invariant solutions for arbitrary subgroup or subalgebra. However, the Lie group has infinitely many subgroups with the same dimension, and it is impossible to compute the group-invariant solutions of all subgroups [19]. We have to sort them into some mutual equivalence, which requires the optimal system of one-dimensional subalgebras to be computed.

To get the optimal system, we start by constructing the commutator table as Table 1 with the help of Lie algebra $[V_i, V_i] = V_iV_j - V_jV_i$ [15, 20].

With reference to Table 1, the adjoint relationship table can be acquired as Table 2, where

$$\mathrm{Ad}(\exp(\varepsilon V_i))V_j = V_j - \varepsilon\left[V_i, V_j\right] + \frac{\varepsilon^2}{2}\left[V_i, \left[V_i, V_j\right]\right] - \cdots, \tag{41}$$

with ε is an infinitesimal real number.

Through Tables 1 and 2, it is quite simple to obtain the optimal system for the one-dimensional subalgebras of Equation (1), which is given by the following forms

$$\begin{aligned}&(1) V_1 + \alpha_1V_2 + \alpha_2V_3, \\ &(2) V_2 + \alpha_3V_3, \\ &(3) V_3, \\ &(4) V_4.\end{aligned} \tag{42}$$

4. Similarity Reductions of vcBLP

In the first step, we reduce Equation (1) to the $(1 + 1)$-dimensional PDEs based on the above optimal system using the similarly reductions. The similarity variables and the $(1 + 1)$-dimensional reduced PDEs can be found in Table 3, and the expressions for the coefficient functions depending on time t can be found in Table 4. We only show the process of calculation with the example of case $V_1 + \alpha_1 V_2 + \alpha_2V_3$; the results of other cases are in Tables 3 and 4.

Table 1: Commutator table.

$[V_i, V_j]$	V_1	V_2	V_3	V_4
V_1	0	0	0	V_1
V_2	0	0	0	V_2
V_3	0	0	0	$2V_3$
V_4	$-V_1$	$-V_2$	$-2V_3$	0

Table 2: Adjoint representation table.

Ad	V_1	V_2	V_3	V_4
V_1	V_1	V_2	V_3	$V_4 - \varepsilon V_1$
V_2	V_1	V_2	V_3	$V_4 - \varepsilon V_2$
V_3	V_1	V_2	V_3	$V_4 - 2\varepsilon V_3$
V_4	$V_1 e^{\varepsilon}$	$V_2 e^{\varepsilon}$	$V_3 e^{2\varepsilon}$	V_4

For this Lie vector, its corresponding characteristic equation is

$$\frac{dx}{1} = \frac{dy}{\alpha_1} = \frac{dt}{\alpha_2 \cdot \tau(t)} = \frac{du}{0} = \frac{dv}{0}, \tag{43}$$

solving this equation to obtain the relative similarity variables are

$$X = x - \frac{1}{\alpha_2}\int \frac{1}{\tau(t)}dt,\ Y = y - \frac{\alpha_1}{\alpha_2}\int \frac{1}{\tau(t)}dt,\ u = P(X, Y),\ v = Q(X, Y). \tag{44}$$

Through Equation (44), Equation (1) is reduced to the following forms:

$$\begin{gathered} a(t)(PP_Y)_X + b(t)P_{XXY} + c(t)Q_{XXX} - \frac{1}{\alpha_2 \cdot \tau(t)}P_{XY} - \frac{\alpha_1}{\alpha_2 \cdot \tau(t)}P_{YY} = 0, \\ e(t)(PQ)_X + d(t)Q_{XX} - \frac{\alpha_1}{\alpha_2 \cdot \tau(t)}Q_Y - \frac{1}{\alpha_2 \cdot \tau(t)}Q_X = 0, \end{gathered} \tag{45}$$

here to ensure that there are only two independent variables X and Y in Equation (45), and the coefficient functions satisfy conditions (38), so the expressions for the coefficient functions are

$$a(t) = \frac{k_1}{\tau(t)},\ b(t) = \frac{k_2}{\tau(t)},\ c(t) = \frac{k_3}{\tau(t)},\ d(t) = \frac{k_4}{\tau(t)},\ e(t) = \frac{k_5}{\tau(t)}, \tag{46}$$

where $k_i (i = 1,\cdots,5)$ are arbitrary constants. Substituting Equation (46) to Equation (45) yields the reduced PDEs as

$$\begin{gathered} \alpha_2 k_1 (PP_Y)_X + \alpha_2 k_2 P_{XXY} + \alpha_2 k_3 Q_{XXX} - \alpha_1 P_{YY} - P_{XY} = 0, \\ \alpha_2 k_5 (QP)_X + \alpha_2 k_4 Q_{XX} - \alpha_1 Q_Y - Q_X = 0. \end{gathered} \tag{47}$$

5. Exact Solutions of vcBLP

In this section, the exact solutions of corresponding reduced PDEs are found for some cases in Table 3 using the (G'/G)-expansion method [21, 22]. For computational simplicity, we let $k_i = 1 (i = 1,\cdots,5)$ for Tables 3 and 4.

Case 1. (II) $V_1 + \alpha_2 V_3$

First, we assume that the traveling wave variables are [23].

$$P(X, Y) = P(\varsigma),\ Q(X, Y) = Q(\varsigma),\ \varsigma = X - VY, \tag{48}$$

where V is the traveling wave speed to be determined.

Next, using Equation (48), the $(1+1)$-dimensional reduced PDEs are transformed into ordinary differential equations (ODEs)

$$\begin{gathered} -\alpha_2 VPP'' - \alpha_2 VP'' + VP'' - \alpha_2 VP''' + \alpha_2 Q''' = 0, \\ \alpha_2 PQ' - Q' + \alpha_2 Q'' + \alpha_2 QP' = 0, \end{gathered} \tag{49}$$

where $(')$ represents the derivative of ς. Through homogeneous balance, we assume that the solutions of the ODEs (49) can be expressed as

$$\begin{gathered} P(\varsigma) = a_1\left(\frac{G'}{G}\right) + a_0,\ a_1 \neq 0, \\ Q(\varsigma) = b_1\left(\frac{G'}{G}\right) + b_0,\ b_1 \neq 0, \end{gathered} \tag{50}$$

where a_0, a_1, b_0, b_1 are coefficients to be determined and $G = G(\varsigma)$ satisfies

$$G'' + \lambda G' + \mu G = 0, \tag{51}$$

with λ and μ are arbitrary constants.

Substituting Equation (50) and Equation (51) into Equation (49), then collecting the coefficients of the same order of (G'/G) and making them equal to zero yields

$$\begin{cases} \mu b_1 + \alpha_2(\lambda\mu b_1 - \mu a_0 b_1 - \mu a_1 b_0) = 0, \\ 2\alpha_2(b_1 - a_1 b_1) = 0, \\ b_1 + \alpha_2(3\lambda b_1 - 2\lambda a_1 b_1 - a_0 b_1 - a_1 b_0) = 0, \\ \lambda b_1 + \alpha_2\left(\lambda^2 b_1 + 2\mu b_1 - \lambda a_0 b_1 - \lambda a_1 b_0 - 2\mu a_1 b_1\right) = 0. \end{cases} \tag{52}$$

Table 3: Similarity reductions.

Case	Similarity variables	Reduced PDEs
(I) $V_1 + \alpha_1 V_2$	$X = y - \alpha_1 x,\ Y = t,$	$P_{XY} - \alpha_1 a(t)(PP_X)_X + \alpha_1^2 b(t) P_{XXX} - \alpha_1^3 c(t) Q_{XXX} = 0,$
$\alpha_1 \neq 0$	$u = P(X, Y),\ v = Q(X, Y).$	$Q_Y - \alpha_1 e(t)(PQ)_X + \alpha_1^2 d(t) Q_{XX} = 0.$
(II) $V_1 + \alpha_2 V_3$	$X = x - 1/\alpha_2 \int (1/\tau(t))dt,\ Y = y,$	$\alpha_2 k_1 (PP_Y)_X + \alpha_2 k_2 P_{XXY} + \alpha_2 k_3 Q_{XXX} - P_{XY} = 0,$
$\alpha_2 \neq 0$	$u = P(X, Y),\ v = Q(X, Y).$	$\alpha_2 k_5 (PQ)_X + \alpha_2 k_4 Q_{XX} - Q_X = 0.$
(III) $V_1 + \alpha_1 V_2 + \alpha_2 V_3$	$X = x - 1/\alpha_2 \int (1/\tau(t))dt,\ Y = y - \alpha_1/\alpha_2 \int (1/\tau(t))dt,$	$\alpha_2 k_1 (PP_Y)_X + \alpha_2 k_2 P_{XXY} + \alpha_2 k_3 Q_{XXX} - \alpha_1 P_{YY} - P_{XY} = 0,$
$\alpha_1 \neq 0, \alpha_2 \neq 0$	$u = P(X, Y),\ v = Q(X, Y).$	$\alpha_2 k_5 (PQ)_X + \alpha_2 k_4 Q_{XX} - \alpha_1 Q_Y - Q_X = 0.$
(IV) $V_2 + \alpha_3 V_3$	$X = x,\ Y = y - 1/\alpha_3 \int (1/\tau(t))dt,$	$\alpha_3 k_1 (PP_Y)_X + \alpha_3 k_2 P_{XXY} + \alpha_3 k_3 Q_{XXX} - P_{YY} = 0,$
$\alpha_3 \neq 0$	$u = P(X, Y),\ v = Q(X, Y).$	$\alpha_3 k_5 (PQ)_X + \alpha_3 k_4 Q_{XX} - Q_Y = 0.$
(V) V_3	$X = x,\ Y = y,$	$a(t)(PP_Y)_X + b(t) P_{XXY} + c(t) Q_{XXX} = 0,$
	$u = P(X, Y),\ v = Q(X, Y).$	$d(t) Q_{XX} + e(t) PQ_X + e(t) QP_X = 0.$
(VI) V_4	$X = xe^{-\int (1/\tau(t))dt},\ Y = ye^{-\int (1/\tau(t))dt},$	$-XP_{XY} + k_2 P_{XXY} + k_3 Q_{XXX} - YP_{YY} - 2P_Y = 0,$
	$u = P(X, Y)e^{\int (1/\tau(t))dt},\ v = Q(X, Y)e^{\int (1/\tau(t))dt}.$	$k_4 Q_{XX} - XQ_X - YQ_Y - Q = 0.$

Solving Equation (52), we get

$$a_0 = \frac{1}{2}\lambda + \frac{1}{\alpha_2},\ a_1 = 1,\ b_0 = \frac{1}{4}V\lambda,\ b_1 = \frac{1}{2}V, \quad (53)$$

with $V, \lambda, \mu, \alpha_2$ are any constants.

By applying Equation (53), Equation (50) can be rewritten as

$$\begin{aligned} P(\varsigma) &= \frac{G'}{G} + \frac{1}{2}\lambda + \frac{1}{\alpha_2}, \\ Q(\varsigma) &= \frac{1}{2}V\left(\frac{G'}{G}\right) + \frac{1}{4}V\lambda, \end{aligned} \quad (54)$$

where $\varsigma = X - VY$. Substituting the solutions of the Equation (51) into Equation (54), we obtain that the three types of exact solutions of reduced PDEs are as follows [24]:

When $\lambda^2 - 4\mu > 0$,

$$\begin{aligned} P_1(\varsigma) &= \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\left(\frac{l_1 \sinh (1/2)\sqrt{\lambda^2 - 4\mu}\varsigma + l_2 \cosh (1/2)\sqrt{\lambda^2 - 4\mu}\varsigma}{l_1 \cosh (1/2)\sqrt{\lambda^2 - 4\mu}\varsigma + l_2 \sinh (1/2)\sqrt{\lambda^2 - 4\mu}\varsigma}\right) + \frac{1}{\alpha_2}, \\ Q_1(\varsigma) &= \frac{1}{4}V\sqrt{\lambda^2 - 4\mu}\left(\frac{l_1 \sinh (1/2)\sqrt{\lambda^2 - 4\mu}\varsigma + l_2 \cosh (1/2)\sqrt{\lambda^2 - 4\mu}\varsigma}{l_1 \cosh (1/2)\sqrt{\lambda^2 - 4\mu}\varsigma + l_2 \sinh (1/2)\sqrt{\lambda^2 - 4\mu}\varsigma}\right), \end{aligned} \quad (55)$$

where $\varsigma = X - VY$, l_1, and l_2 are any constants.

When $\lambda^2 - 4\mu < 0$,

$$\begin{aligned} P_2(\varsigma) &= \frac{1}{2}\sqrt{4\mu - \lambda^2}\left(\frac{-l_1 \sin (1/2)\sqrt{4\mu - \lambda^2}\varsigma + l_2 \cos (1/2)\sqrt{4\mu - \lambda^2}\varsigma}{l_1 \cos (1/2)\sqrt{4\mu - \lambda^2}\varsigma + l_2 \sin (1/2)\sqrt{4\mu - \lambda^2}\varsigma}\right) + \frac{1}{\alpha_2}, \\ Q_2(\varsigma) &= \frac{1}{4}V\sqrt{4\mu - \lambda^2}\left(\frac{-l_1 \sin (1/2)\sqrt{4\mu - \lambda^2}\varsigma + l_2 \cos (1/2)\sqrt{4\mu - \lambda^2}\varsigma}{l_1 \cos (1/2)\sqrt{4\mu - \lambda^2}\varsigma + l_2 \sin (1/2)\sqrt{4\mu - \lambda^2}\varsigma}\right), \end{aligned} \quad (56)$$

where $\varsigma = X - VY$, l_1, and l_2 are any constants.

When $\lambda^2 - 4\mu = 0$,

$$\begin{aligned} P_3(\varsigma) &= \frac{l_2}{l_1 + l_2\varsigma} + \frac{1}{\alpha_2}, \\ Q_3(\varsigma) &= \frac{1}{2}\frac{l_2 V}{l_1 + l_2\varsigma}, \end{aligned} \quad (57)$$

where $\varsigma = X - VY$, l_1, and l_2 are any constants.

By substituting the corresponding similarity variables in Table 3 into the above solutions, the exact solutions of vcBLP are obtained as follows:

When $\lambda^2 - 4\mu > 0$,

$$\begin{aligned} u_1(\varsigma) &= \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\left(\frac{l_1 \sinh (1/2)\sqrt{\lambda^2 - 4\mu}\varsigma + l_2 \cosh (1/2)\sqrt{\lambda^2 - 4\mu}\varsigma}{l_1 \cosh (1/2)\sqrt{\lambda^2 - 4\mu}\varsigma + l_2 \sinh (1/2)\sqrt{\lambda^2 - 4\mu}\varsigma}\right) + \frac{1}{\alpha_2}, \\ v_1(\varsigma) &= \frac{1}{4}V\sqrt{\lambda^2 - 4\mu}\left(\frac{l_1 \sinh (1/2)\sqrt{\lambda^2 - 4\mu}\varsigma + l_2 \cosh (1/2)\sqrt{\lambda^2 - 4\mu}\varsigma}{l_1 \cosh (1/2)\sqrt{\lambda^2 - 4\mu}\varsigma + l_2 \sinh (1/2)\sqrt{\lambda^2 - 4\mu}\varsigma}\right), \end{aligned} \quad (58)$$

where $\varsigma = x - (1/\alpha_2) \cdot \int (1/\tau(t))dt - Vy$, l_1, and l_2 are any constants.

When $\lambda^2 - 4\mu < 0$,

$$\begin{aligned} u_2(\varsigma) &= \frac{1}{2}\sqrt{4\mu - \lambda^2}\left(\frac{-l_1 \sin (1/2)\sqrt{4\mu - \lambda^2}\varsigma + l_2 \cos (1/2)\sqrt{4\mu - \lambda^2}\varsigma}{l_1 \cos (1/2)\sqrt{4\mu - \lambda^2}\varsigma + l_2 \sin (1/2)\sqrt{4\mu - \lambda^2}\varsigma}\right) + \frac{1}{\alpha_2}, \\ v_2(\varsigma) &= \frac{1}{4}V\sqrt{4\mu - \lambda^2}\left(\frac{-l_1 \sin (1/2)\sqrt{4\mu - \lambda^2}\varsigma + l_2 \cos (1/2)\sqrt{4\mu - \lambda^2}\varsigma}{l_1 \cos (1/2)\sqrt{4\mu - \lambda^2}\varsigma + l_2 \sin (1/2)\sqrt{4\mu - \lambda^2}\varsigma}\right), \end{aligned} \quad (59)$$

where $\varsigma = x - (1/\alpha_2) \cdot \int (1/\tau(t))dt - Vy$, l_1, and l_2 are any constants.

Table 4: The expressions of the coefficient functions.

Case	The expressions of the coefficient functions
(I) $V_1 + \alpha_1 V_2$ $\alpha_1 \neq 0$	$a(t)$, $b(t)$, $c(t)$, $d(t)$, and $e(t)$ are any functions that depend on t.
(II) $V_1 + \alpha_2 V_3$ $\alpha_2 \neq 0$	$a(t) = k_1/\tau(t)$, $b(t) = k_2/\tau(t)$, $c(t) = k_3/\tau(t)$, $d(t) = k_4/\tau(t)$, $e(t) = k_5/\tau(t)$.
(III) $V_1 + \alpha_1 V_2 + \alpha_2 V_3$ $\alpha_1 \neq 0, \alpha_2 \neq 0$	$a(t) = k_1/\tau(t)$, $b(t) = k_2/\tau(t)$, $c(t) = k_3/\tau(t)$, $d(t) = k_4/\tau(t)$, $e(t) = k_5/\tau(t)$.
(IV) $V_2 + \alpha_3 V_3$ $\alpha_3 \neq 0$	$a(t) = k_1/\tau(t)$, $b(t) = k_2/\tau(t)$, $c(t) = k_3/\tau(t)$, $d(t) = k_4/\tau(t)$, $e(t) = k_5/\tau(t)$.
(V) V_3	$a(t)$, $b(t)$, $c(t)$, $d(t)$, and $e(t)$ are any functions that depend on t.
(VI) V_4	$a(t) = e(t) = 0, b(t) = \dfrac{k_2}{\tau(t)} e^{\int 2/\tau(t)dt}, c(t) = \dfrac{k_3}{\tau(t)} e^{\int 2/\tau(t)dt}, d(t) = \dfrac{k_4}{\tau(t)} e^{\int 2/\tau(t)dt}.$

When $\lambda^2 - 4\mu = 0$,

$$u_3(\varsigma) = \frac{l_2}{l_1 + l_2\varsigma} + \frac{1}{\alpha_2}, \quad v_3(\varsigma) = \frac{1}{2}\frac{l_2 V}{l_1 + l_2\varsigma}, \tag{60}$$

where $\varsigma = x - (1/\alpha_2) \cdot \int(1/\tau(t))dt - Vy$, l_1, and l_2 are any constants.

For the other cases, we also obtained three types of exact solutions of the reduced PDEs by the method of (G'/G) -expansion, and we will not repeat the calculation process, just list the results of the calculation.

Case 2. (III) $V_1 + \alpha_1 V_2 + \alpha_2 V_3$.

When $\lambda^2 - 4\mu > 0$,

$$u_1(\varsigma) = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\left(\frac{l_1 \sinh (1/2)\sqrt{\lambda^2 - 4\mu}\varsigma + l_2 \cosh (1/2)\sqrt{\lambda^2 - 4\mu}\varsigma}{l_1 \cosh (1/2)\sqrt{\lambda^2 - 4\mu}\varsigma + l_2 \sinh (1/2)\sqrt{\lambda^2 - 4\mu}\varsigma}\right) + \frac{1 - \alpha_1 V}{\alpha_2}, \tag{61}$$

$$v_1(\varsigma) = \frac{1}{4}V\sqrt{\lambda^2 - 4\mu}\left(\frac{l_1 \sinh (1/2)\sqrt{\lambda^2 - 4\mu}\varsigma + l_2 \cosh (1/2)\sqrt{\lambda^2 - 4\mu}\varsigma}{l_1 \cosh (1/2)\sqrt{\lambda^2 - 4\mu}\varsigma + l_2 \sinh (1/2)\sqrt{\lambda^2 - 4\mu}\varsigma}\right), \tag{62}$$

where $\varsigma = x - 1/\alpha_2 \int(1/\tau(t))dt - (y - \alpha_1/\alpha_2 \int(1/\tau(t))dt) \cdot V$, l_1, l_2, and V are any constants. We choose parameters $\lambda = 3$, $\mu = 2$, $\alpha_1 = 1$, $\alpha_2 = 1$, $l_1 = 2$, $l_2 = 1$, $V = 2$, and $\tau(t) = t$; the images of (61) and (62) are, respectively, in Figures 1 and 2.

When $\lambda^2 - 4\mu < 0$,

$$u_2(\varsigma) = \frac{1}{2}\sqrt{4\mu - \lambda^2}\left(\frac{-l_1 \sin (1/2)\sqrt{4\mu - \lambda^2}\varsigma + l_2 \cos (1/2)\sqrt{4\mu - \lambda^2}\varsigma}{l_1 \cos (1/2)\sqrt{4\mu - \lambda^2}\varsigma + l_2 \sin (1/2)\sqrt{4\mu - \lambda^2}\varsigma}\right) + \frac{1 - \alpha_1 V}{\alpha_2}, \tag{63}$$

$$v_2(\varsigma) = \frac{1}{4}V\sqrt{4\mu - \lambda^2}\left(\frac{-l_1 \sin (1/2)\sqrt{4\mu - \lambda^2}\varsigma + l_2 \cos (1/2)\sqrt{4\mu - \lambda^2}\varsigma}{l_1 \cos (1/2)\sqrt{4\mu - \lambda^2}\varsigma + l_2 \sin (1/2)\sqrt{4\mu - \lambda^2}\varsigma}\right), \tag{64}$$

where $\varsigma = x - 1/\alpha_2 \int(1/\tau(t))dt - (y - \alpha_1/\alpha_2 \int(1/\tau(t))dt) \cdot V$, l_1, l_2, and V are any constants. We choose parameters $\lambda = 4$, $\mu = 5$, $\alpha_1 = 2$, $\alpha_2 = 1$, $l_1 = 3.5$, $l_2 = 3$, $V = 1$, and $\tau(t) = t^2$; the images of (63) and (64) are, respectively, in Figures 3 and 4.

When $\lambda^2 - 4\mu = 0$,

$$u_3(\varsigma) = \frac{l_2}{l_1 + l_2\varsigma} + \frac{1}{\alpha_2} - \frac{\alpha_1}{\alpha_2}V, \tag{65}$$

$$v_3(\varsigma) = \frac{1}{2}\frac{l_2 V}{l_1 + l_2\varsigma}, \tag{66}$$

where $\varsigma = x - 1/\alpha_2 \int(1/\tau(t))dt - (y - \alpha_1/\alpha_2 \int(1/\tau(t))dt) \cdot V$, l_1, l_2, and V are any constants. We choose parameters $\alpha_1 = 2$, $\alpha_2 = 1$, $l_1 = 3$, $l_2 = 2$, $V = 1$, and $\tau(t) = t$; the images of (65) and (66) are, respectively, in Figures 5 and 6.

Case 3. (IV) $V_2 + \alpha_3 V_3$.

When $\lambda^2 - 4\mu > 0$,

$$u_1(\varsigma) = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\left(\frac{l_1 \sinh (1/2)\sqrt{\lambda^2 - 4\mu}\varsigma + l_2 \cosh (1/2)\sqrt{\lambda^2 - 4\mu}\varsigma}{l_1 \cosh (1/2)\sqrt{\lambda^2 - 4\mu}\varsigma + l_2 \sinh (1/2)\sqrt{\lambda^2 - 4\mu}\varsigma}\right) + a_0 - \frac{1}{2}\lambda,$$
$$v_1(\varsigma) = \left(\frac{1}{8}\alpha_3\lambda - \frac{1}{4}\alpha_3 a_0\right)\sqrt{\lambda^2 - 4\mu}\left(\frac{l_1 \sinh (1/2)\sqrt{\lambda^2 - 4\mu}\varsigma + l_2 \cosh (1/2)\sqrt{\lambda^2 - 4\mu}\varsigma}{l_1 \cosh (1/2)\sqrt{\lambda^2 - 4\mu}\varsigma + l_2 \sinh (1/2)\sqrt{\lambda^2 - 4\mu}\varsigma}\right), \tag{67}$$

where $\varsigma = x - (1/2)\alpha_3(\lambda - 2a_0) \cdot (y - 1/\alpha_3 \int(1/\tau(t))dt)$, l_1, l_2, and a_0 are any constants.

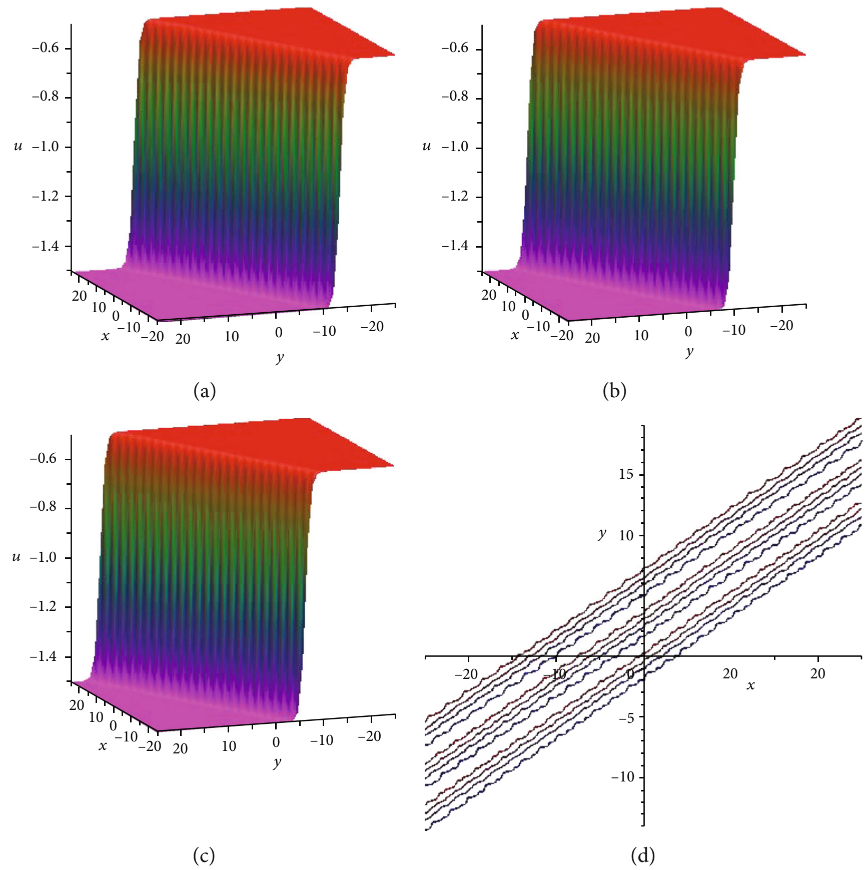

FIGURE 1: Evolution of Equation (61) at (a) $t = 10^{-1}$, (b) $t = 10^{2}$, and (c) $t = 10^{5}$. (d) Contour plot.

When $\lambda^2 - 4\mu < 0$,

$$u_2(\varsigma) = \frac{1}{2}\sqrt{4\mu - \lambda^2}\left(\frac{-l_1 \sin (1/2)\sqrt{4\mu - \lambda^2}\varsigma + l_2 \cos (1/2)\sqrt{4\mu - \lambda^2}\varsigma}{l_1 \cos (1/2)\sqrt{4\mu - \lambda^2}\varsigma + l_2 \sin (1/2)\sqrt{4\mu - \lambda^2}\varsigma}\right) + a_0 - \frac{1}{2}\lambda,$$
$$v_2(\varsigma) = \left(\frac{1}{8}\alpha_3\lambda - \frac{1}{4}\alpha_3 a_0\right)\sqrt{4\mu - \lambda^2}\left(\frac{-l_1 \sin (1/2)\sqrt{4\mu - \lambda^2}\varsigma + l_2 \cos (1/2)\sqrt{4\mu - \lambda^2}\varsigma}{l_1 \cos (1/2)\sqrt{4\mu - \lambda^2}\varsigma + l_2 \sin (1/2)\sqrt{4\mu - \lambda^2}\varsigma}\right), \tag{68}$$

where $\varsigma = x - (1/2)\alpha_3(\lambda - 2a_0) \cdot (y - 1/\alpha_3 \int (1/\tau(t))dt)$, l_1, l_2, and a_0 are any constants.

When $\lambda^2 - 4\mu = 0$,

$$u_3(\varsigma) = a_0 - \frac{\lambda}{2} + \frac{l_2}{l_1 + l_2\varsigma},$$
$$v_3(\varsigma) = \frac{1}{4}\frac{l_2\alpha_3(\lambda - 2a_0)}{l_1 + l_2\varsigma}, \tag{69}$$

where $\varsigma = x - (1/2)\alpha_3(\lambda - 2a_0) \cdot (y - 1/\alpha_3 \int (1/\tau(t))dt)$, l_1, l_2, and a_0 are any constants.

We can easily find that all the above solutions are traveling wave solutions when $\tau(t)$ takes any constant, and all the above solutions are group-invariant solutions when $\tau(t)$ is an arbitrary function of t.

6. Conservation Laws of vcBLP

The conservation law is extremely valuable for studying the integrability and exploring the exact solutions of PDE [25–27]. We can use it to explain many physical phenomena described by PDE [28–30]. In this section, we use the multiplier method [31–33] to calculate the conservation laws of vcBLP. The first order multipliers $\Lambda_1 = \Lambda_1(x, y, t, u, v, u_x, u_y, u_t, v_x, v_y, v_t)$ and $\Lambda_2 = \Lambda_2(x, y, t, u, v, u_x, u_y, u_t, v_x, v_y, v_t)$ of vcBLP can be obtained by the following equations with

$$\begin{cases} \dfrac{\delta}{\delta u}\left[\Lambda_1 \cdot \left(u_{yt} + a(t)u_x u_y + a(t)uu_{xy} + b(t)u_{xxy} + c(t)v_{xxx}\right) + \Lambda_2 \cdot \left(v_t + d(t)v_{xx} + e(t)(uv)_x\right)\right] = 0, \\ \dfrac{\delta}{\delta v}\left[\Lambda_1 \cdot \left(u_{yt} + a(t)u_x u_y + a(t)uu_{xy} + b(t)u_{xxy} + c(t)v_{xxx}\right) + \Lambda_2 \cdot \left(v_t + d(t)v_{xx} + e(t)(uv)_x\right)\right] = 0, \end{cases} \tag{70}$$

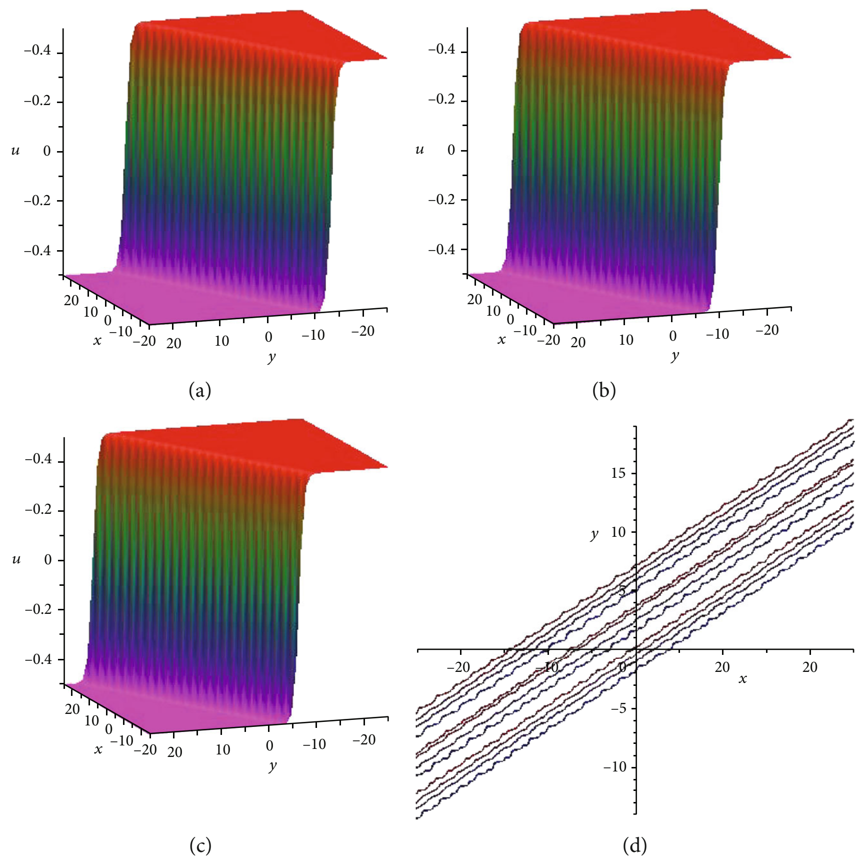

Figure 2: Evolution of Equation (62) at (a) $t = 10^{-1}$, (b) $t = 10^2$, and (c) $t = 10^5$. (d) Contour plot.

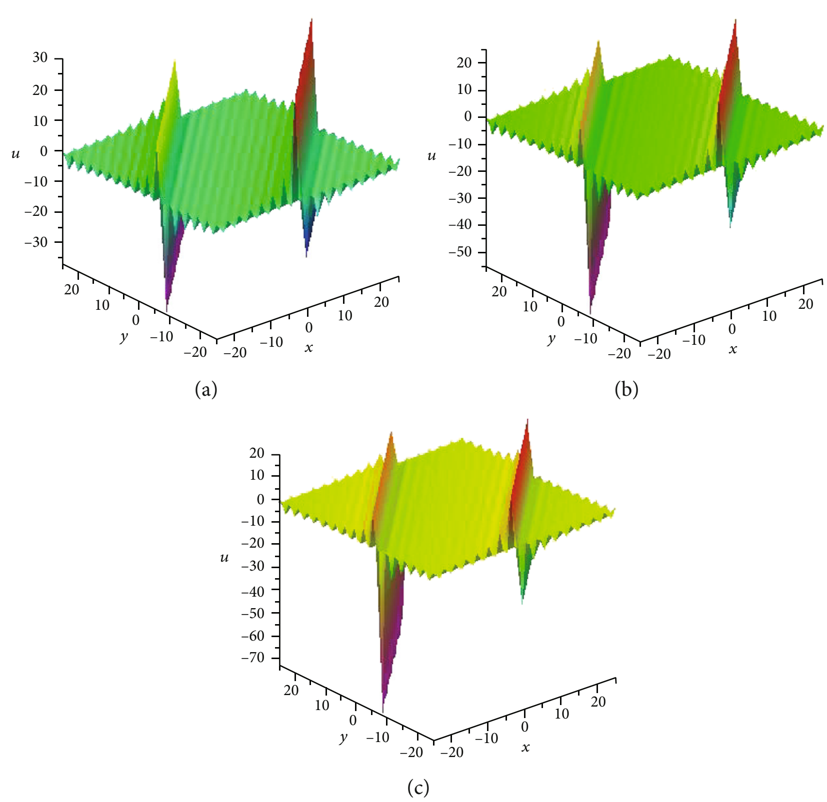

Figure 3: Evolution of Equation (63) at (a) $t = 0.293$, (b) $t = 0.2938$, and (c) $t = 0.2942$.

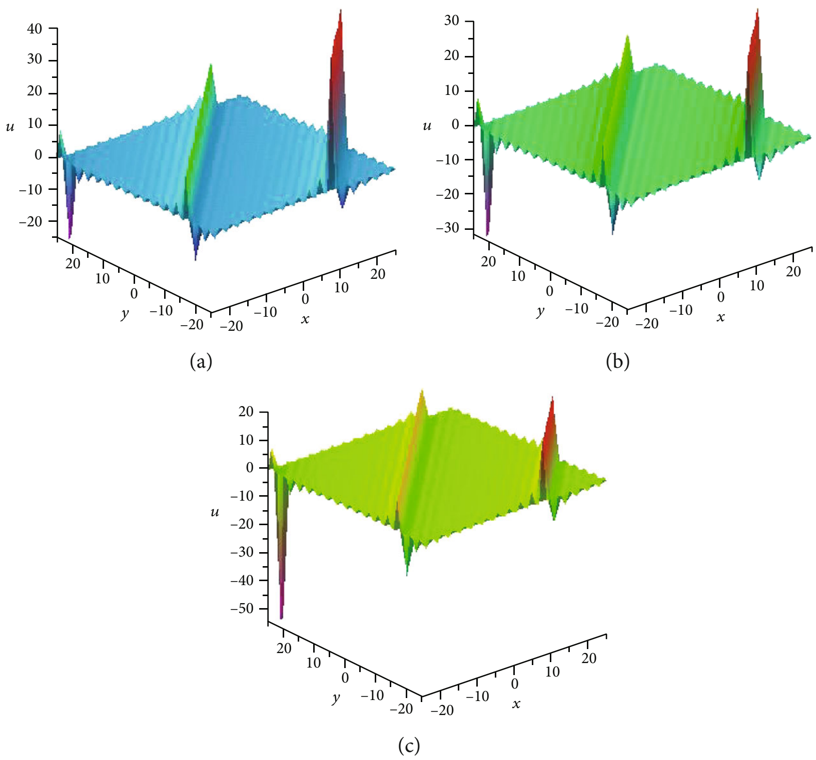

FIGURE 4: Evolution of Equation (64) at (a) $t = 0.479$, (b) $t = 0.48$, and (c) $t = 0.4815$.

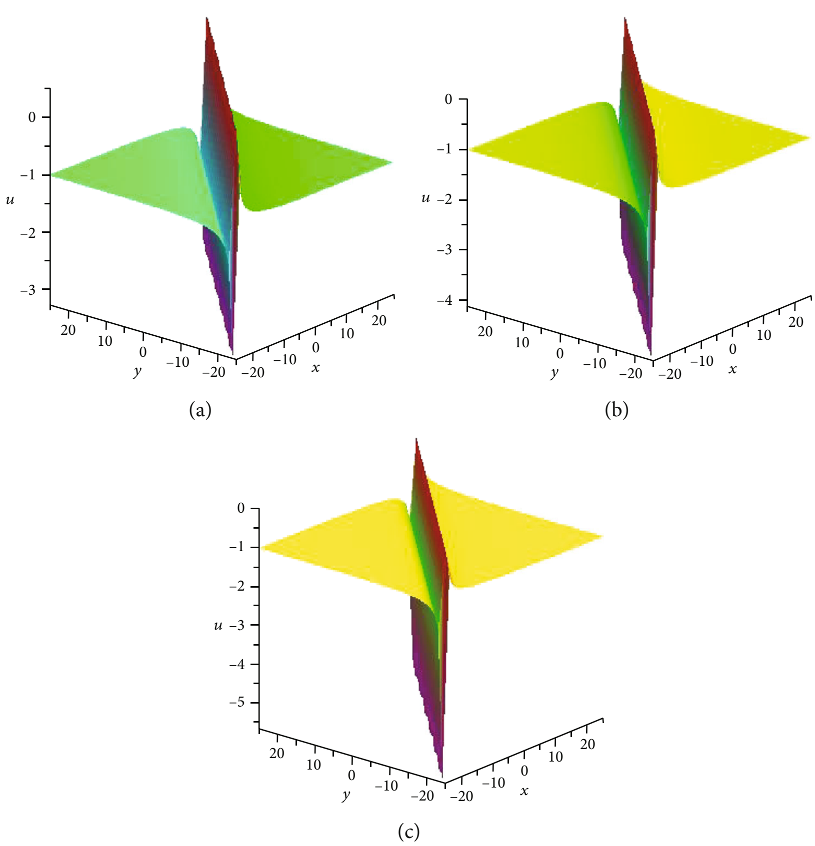

FIGURE 5: Evolution of Equation (65) at (a) $t = 0.4$, (b) $t = 0.45$, and (c) $t = 0.5$.

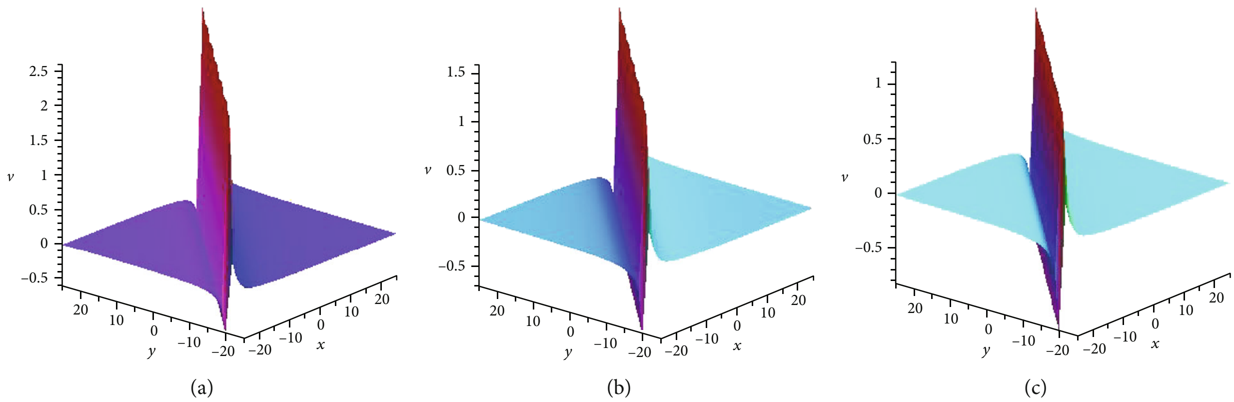

Figure 6: Evolution of Equation (66) at (a) $t = 16$, (b) $t = 18$, and (c) $t = 20$.

where $\delta/\delta u$ and $\delta/\delta v$ are Euler operators.

By expanding Equation (70) and decomposing them according to the derivatives of u, v, we can obtain a system and solve this system can obtain

$$\Lambda_1 = C_1 x^2 + C_2 x^2 H(t) + G(y), \Lambda_2 = R(y), \tag{71}$$

where C_1 and C_2 are arbitrary constants, $R(y)$ and $G(y)$ are arbitrary functions about y, and $H(t)$ is an arbitrary function about t.

Therefore, we can obtain the following low-order conservation laws and the corresponding multipliers. The details are discussed as below.

Case 1. $\Lambda_1 = G(y), \Lambda_2 = R(y)$.

$$\begin{aligned} C^t &= G(y)u_y + R(y)v, \\ C^x &= G(y)a(t)uu_y + G(y)b(t)u_{xy} + G(y)c(t)v_{xx} \\ &\quad + R(y)d(t)v_x + R(y)e(t)uv, \\ C^y &= 0. \end{aligned} \tag{72}$$

Case 2. $\Lambda_1 = x^2 + G(y), \Lambda_2 = R(y)$.

$$\begin{aligned} C^t &= G(y)u_y + x^2 u_y + R(y)v, \\ C^x &= R(y)e(t)uv + G(y)a(t)uu_y + G(y)c(t)v_{xx} \\ &\quad + c(t)x^2 v_{xx} - 2c(t)xv_x + 2c(t)v \\ &\quad + G(y)b(t)u_{xy} + R(y)d(t)v_x, \\ C^y &= a(t)x^2 uu_x + b(t)x^2 u_{xx}, \end{aligned} \tag{73}$$

Case 3. $\Lambda_1 = x^2 H(t) + G(y), \Lambda_2 = R(y)$.

$$\begin{aligned} C^t &= G(y)u_y + R(y)v, \\ C^x &= G(y)a(t)uu_y + G(y)b(t)u_{xy} + G(y)c(t)v_{xx} \\ &\quad + H(t)c(t)x^2 v_{xx} - 2H(t)c(t)xv_x \\ &\quad + 2H(t)c(t)v + R(y)d(t)v_x + R(y)e(t)uv, \\ C^y &= H(t)x^2 u_t + H(t)a(t)x^2 uu_x + H(t)b(t)x^2 u_{xx}. \end{aligned} \tag{74}$$

Case 4. $\Lambda_1 = x^2 + G(y) + x^2 H(t), \Lambda_2 = R(y)$.

$$\begin{aligned} C^t &= x^2 u_y + G(y)u_y + R(y)v, \\ C^x &= c(t)x^2 v_{xx} - 2c(t)xv_x + 2c(t)v + G(y)a(t)uu_y \\ &\quad + G(y)b(t)u_{xy} + G(y)c(t)v_{xx} + H(t)c(t)x^2 v_{xx} \\ &\quad - 2H(t)c(t)xv_x + 2H(t)c(t)v + R(y)d(t)v_x \\ &\quad + R(y)e(t)uv, \\ C^y &= a(t)x^2 uu_x + b(t)x^2 u_{xx} + H(t)x^2 u_t \\ &\quad + H(t)a(t)x^2 uu_x + H(t)b(t)x^2 u_{xx}. \end{aligned} \tag{75}$$

The results obtained above have been verified using Maple software to ensure that $(\partial/\partial t)C^t + (\partial/\partial x)C^x + (\partial/\partial y)C^y = 0$ holds.

7. Conclusions of This Article

In this paper, the zero-order and first-order differential invariants were determined by using the equivalence group of vcBLP. With the help of these, the explicit transformation to its constant coefficient form was given. Subsequently, we have successfully performed the Lie symmetry analysis of vcBLP, obtained some exact solutions, and plotted the corresponding 3-dimensional figures to describe the evolution of the solutions. Moreover, the four conservation laws of vcBLP were obtained using the multiplier method.

Data Availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] Y. D. Shang and Q. Chen, "The generalized Cole-Hopf transformation for a generalized Burgers-Fisher equation with spatiotemporal variable coefficients," *Applied Mathematics Letters*, vol. 117, article 107074, 2021.

[2] M. R. Ali and W. X. Ma, "New exact solutions of nonlinear (3 + 1)-dimensional Boiti-Leon-Manna- Pempinelli equation," *Advances in Mathematical Physics*, vol. 2019, Article ID 9801638, 7 pages, 2019.

[3] S. Kumar, D. Jyoti, K. S. Nisar, and M. Zakarya, "New exact static solutions of Einstein-Maxwell field equations with a magnetic dipole," *Results in Physics*, vol. 24, article 104136, 2021.

[4] A. G. Kushner and R. I. Matviichuk, "Dynamics and exact solutions of non-evolutionary partial differential equations," *Differential Geometry and its Applications*, vol. 76, article 101761, 2021.

[5] S. C. Anco and D. The, "Symmetries, conservation laws, and cohomology of Maxwell's equations using potentials," *Acta Applicandae Mathematicae*, vol. 89, no. 1-3, pp. 1–52, 2005.

[6] M. R. Ali and R. Sadat, "Lie symmetry analysis, new group invariant for the (3 + 1)-dimensional and variable coefficients for liquids with gas bubbles models," *Chinese Journal of Physics*, vol. 71, pp. 539–547, 2021.

[7] S. X. Yang, Z. Zhang, and B. Li, "Soliton molecules and some novel types of hybrid solutions to (2 + 1)-dimensional variable-coefficient Caudrey-Dodd-Gibbon-Kotera-Sawada equation," *Advances in Mathematical Physics*, vol. 2020, Article ID 2670710, 9 pages, 2020.

[8] Y. H. Wang, "Nonautonomous lump solutions for a variable-coefficient Kadomtsev-Petviashvili equation," *Applied Mathematics Letters*, vol. 119, article 107201, 2021.

[9] G. W. Wang, J. Vega-Guzman, A. Biswas, A. Kamis Alzahrani, and A. H. Kara, "(2 + 1)-dimensional Boiti-Leon-Pempinelli equation - domain walls, invariance properties and conservation laws," *Physics Letters A*, vol. 384, no. 10, article 126255, 2020.

[10] W. G. Ni and C. Q. Dai, "Note on same result of different ansatz based on extended tanh-function method for nonlinear models," *Applied Mathematics and Computation*, vol. 270, pp. 434–440, 2015.

[11] R. M. el-Shiekh, "Periodic and solitary wave solutions for a generalized variable-coefficient Boiti-Leon-Pempinlli system," *Computers & Mathematics with Applications*, vol. 73, no. 7, pp. 1414–1420, 2017.

[12] L. V. Ovsyannikov, *Group Analysis of Differential Equations*, Academic, New York, USA, 1982.

[13] M. S. Bruzón, R. de la Rosa, and R. Tracinà, "Exact solutions via equivalence transformations of variable-coefficient fifth-order KdV equations," *Applied Mathematics and Computation*, vol. 325, pp. 239–245, 2018.

[14] J. Zheng, "Lie symmetry analysis and invariant solutions of a nonlinear Fokker-Planck equation describing cell population growth," *Advances in Mathematical Physics*, vol. 2020, Article ID 4975943, 7 pages, 2020.

[15] W. Sinkala and T. F. Nkalashe, "Lie symmetry analysis of a first-order feedback model of option pricing," *Advances in Mathematical Physics*, vol. 2015, Article ID 361785, 9 pages, 2015.

[16] M. S. Liu and H. Dong, "On the existence of solution, Lie symmetry analysis and conservation law of magnetohydrodynamic equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 87, article 105277, 2020.

[17] F. Y. Liu, Y. T. Gao, X. Yu, C. C. Ding, G. F. Deng, and T. T. Jia, "Painleve analysis, Lie group analysis and soliton-cnoidal, resonant, hyperbolic function and rational solutions for the modified Korteweg-de Vries- Calogero-Bogoyavlenskii-Schiff equation in fluid mechanics/plasma physics," *Chaos, Solitons & Fractals*, vol. 144, article 110559, 2021.

[18] A. Paliathanasis, K. Krishnakumar, K. M. Tamizhmani, and P. Leach, "Lie symmetry analysis of the Black-Scholes-Merton model for European options with stochastic volatility," *Mathematics*, vol. 4, no. 2, article 28, 2016.

[19] M. Devi, S. Yadav, and R. Arora, "Optimal system, invariance analysis of fourth-order nonlinear ablowitz-Kaup- Newell-Segur water wave dynamical equation using lie symmetry approach," *Applied Mathematics and Computation*, vol. 404, article 126230, 2021.

[20] N. Benoudina, Y. Zhang, and C. M. Khalique, "Lie symmetry analysis, optimal system, new solitary wave solutions and conservation laws of the Pavlov equation," *Communications in Nonlinear Science and Numerical Simulation*, vol. 94, article 105560, 2021.

[21] M. L. Wang, X. Li, and J. Zhang, "The $(G'G)$-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics," *Physics Letters A*, vol. 372, no. 4, pp. 417–423, 2008.

[22] S. Sahoo, S. Saha Ray, and M. A. Abdou, "New exact solutions for time-fractional Kaup-Kupershmidt equation using improved (G'/G)-expansion and extended (G'/G)-expansion methods," *Alexandria Engineering Journal*, vol. 59, no. 5, pp. 3105–3110, 2020.

[23] M. Li, B. Wang, T. Xu, and L. Wang, "Quantitative analysis on the bifurcations and exact travelling wave solutions of a generalized fourth-order dispersive nonlinear Schrodinger equation in Heisenberg spin chain," *Chaos, Solitons & Fractals*, vol. 145, article 110767, 2021.

[24] Z. P. Yang, W. P. Zhong, W. Y. Zhong, and M. R. Belić, "New traveling wave and soliton solutions of the sine-Gordon equation with a variable coefficient," *Optik*, vol. 198, article 163247, 2019.

[25] S. C. Anco, M. L. Gandarias, and E. Recio, "Conservation laws, symmetries, and line soliton solutions of generalized KP and Boussinesq equations with p-power nonlinearities in two dimensions," *Theoretical and Mathematical Physics*, vol. 197, no. 1, pp. 1393–1411, 2018.

[26] S. Wael, A. R. Seadawy, O. H. el-Kalaawy, S. M. Maowad, and D. Baleanu, "Symmetry reduction, conservation laws and acoustic wave solutions for the extended Zakharov-Kuznetsov dynamical model arising in a dust plasma," *Results in Physics*, vol. 19, article 103652, 2020.

[27] A. K. Halder, P. G. L. Leach, and A. Paliathanasis, "Similarity solutions and conservation laws for the Bogoyavlensky-Konopelchenko equation by Lie point symmetries," *Quaestiones Mathematicae*, vol. 44, no. 6, pp. 815–827, 2020.

[28] X. P. Xin, Y. Liu, and X. Liu, "Nonlocal symmetries, exact solutions and conservation laws of the coupled Hirota equations," *Applied Mathematics Letters*, vol. 55, pp. 63–71, 2016.

[29] G. Wang, X. Yong, Y. Huang, and J. Tian, "Symmetry, pulson solution, and conservation laws of the Holm-Hone equation," *Advances in Mathematical Physics*, vol. 2019, Article ID 4364108, 6 pages, 2019.

[30] U. S. al-Ali, A. H. Bokhari, A. H. Kara, and F. D. Zaman, "On the symmetries and conservation laws of the multidimensional nonlinear damped wave equations," *Advances in Mathematical Physics*, vol. 2017, Article ID 9401205, 11 pages, 2017.

[31] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Springer, New York, NY, USA, 1986.

[32] S. C. Anco, "On the incompleteness of Ibragimov's Conservation law theorem and its equivalence to a standard formula using symmetries and adjoint-symmetries," *Symmetry*, vol. 9, no. 3, article 33, 2017.

[33] S. C. Anco, "Symmetry properties of conservation laws," *International Journal of Modern Physics B*, vol. 30, no. 28n29, article 1640003, 2016.