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*Research Article*

# Numerical Study of the Inverse Problem of Generalized Burgers–Fisher and Generalized Burgers–Huxley Equations

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In this paper, the boundary value inverse problem related to the generalized Burgers–Fisher and generalized Burgers–Huxley equations is solved numerically based on a spline approximation tool. B-splines with quasilinearization and Tikhonov regularization methods are used to obtain new numerical solutions to this problem. First, a quasilinearization method is used to linearize the equation in a specific time step. Then, a linear combination of B-splines is used to approximate the largest order of derivatives in the equation. By integrating from this linear combination, some approximations have been obtained for each of the functions and derivatives with respect to time and space. The boundary and additional conditions of the problem are also applied in these approximations. The Tikhonov regularization method is used to solve the system of linear equations using noisy data. Several numerical examples are provided to illustrate the accuracy and efficiency of the method.

## 1. Introduction

Most of the physical problems arising in various fields of physical science and engineering are modeled by nonlinear partial differential equations (NLPDEs) [1]. Two of the most famous NLPDEs are the generalized Burgers–Huxley and generalized Burgers–Fisher equations [2]. These equations describe the interaction between diffusion, convection, and reaction [3].

The generalized Burgers–Huxley and generalized Burgers–Fisher equations are of the form

$$u_t = \varepsilon u_{xx} - \alpha u^{\delta} u_x + \beta u\left(1 - u^{\delta}\right)\left(\eta u^{\delta} - \gamma\right), \quad a < x < b, t > 0, \tag{1}$$

with the initial condition

$$u(x, 0) = f(x), \quad a \le x \le b, \tag{2}$$

and Dirichlet boundary conditions

$$u(a, t) = q(t), \quad t \ge 0, \tag{3}$$

$$u(b, t) = g(t), \quad t \ge 0. \tag{4}$$

Also, in order to determine $q$, we consider an additional condition given at the interior point, $x = l$ of the region

$$u(l, t) = p(t), \quad a < l < b, t \ge 0, \tag{5}$$

where $\varepsilon$, $\alpha$, $\beta$, $\gamma$, $\delta$, and $\eta$ are constants such that $0 < \varepsilon \le 1$, $\beta \ge 0$, $\delta > 0$, $\gamma \in (0, 1)$, and $\eta = 0, 1$, and $g$ and $f$ are considered known functions, while $q$ and $u$ are unknown functions.

If $\eta = 1$, (1) describes the generalized Burgers–Huxley equation, and in the case that $\eta = \gamma = 0$, (1) describes the generalized Burgers–Fisher equation.

In some cases, the exact solitary wave solutions of equation (1) are obtained using the relevant nonlinear transformations [4]. In the case that $\eta = 1$ and $\varepsilon = 1$, the exact solution of the generalized Burgers–Huxley equation (1) is taken from [2], given by

$$u(x,t)=\left(\frac{\gamma}{2}+\frac{\gamma}{2}\tanh\left(w_1(x-w_2t)\right)\right)^{1/\delta}, \tag{6}$$

where

$$w_1=\frac{\nu\gamma\delta}{4(1+\delta)},$$

$$w_2=\frac{\alpha\gamma}{1+\delta}-\frac{\nu(1+\delta-\gamma)}{2(1+\delta)}, \tag{7}$$

and $\nu=-\alpha+\sqrt{\alpha^2+4\beta(1+\delta)}$.

Note that, in here, to get the exact solution, we first assume that $u=w^{1/\delta}$. Then, by assuming $w(x,t)=w(x-ct)=w(\zeta)$, the equation transforms into an ordinary differential equation as the form $d^2w/d\zeta^2=a^2(2w-\gamma)w(w-\gamma)$, which can be easily solvable.

If $\eta=0$, $\varepsilon=1$, and $\gamma=-1$, the exact solution of the generalized Burgers–Fisher equation (1) is taken from [2], given by

$$u(x,t)=\left(\frac{1}{2}+\frac{1}{2}\tanh\left(\theta_1(x-\theta_2t)\right)\right)^{1/\delta}, \tag{8}$$

where

$$\theta_1=\frac{-\alpha\delta}{2(1+\delta)},$$

$$\theta_2=\frac{\alpha}{1+\delta}+\frac{\beta(1+\delta)}{\alpha}. \tag{9}$$

The boundary conditions are taken from the exact solution.

Burgers' equation was first introduced by Bateman [5] when he mentioned it as worthy of study and gave its steady solutions. Later on, Burgers [6] treated it as a mathematical model for turbulence and after whom such an equation is widely referred to as Burgers' equation. The study of Burgers' equation is important since it arises in the approximate theory of flow through a shock wave propagating in a viscous fluid and in the modeling of turbulence [7]. The generalized Burgers–Huxley equation describes a wide class of physical nonlinear phenomena, for instance, a prototype model for describing the interaction between reaction mechanisms, convection effects, and diffusion transports [8]. It has found its applications in many fields such as biology, metallurgy, chemistry, combustion, mathematics, and engineering [8, 9]. The generalized Burgers–Fisher equation has been found in many applications in fields such as gas dynamics, number theory, heat conduction, and elasticity [10]. The following are some works on these equations. Yadav and Jiwari [11] developed a finite element analysis and approximation of the Burgers–Fisher equation. Jiwrai and Mittal [12] presented a high-order numerical scheme for the singularly perturbed Burgers–Huxley equation. Also, they have a numerical study of the Burgers–Huxley equation by the differential quadrature method [13]. The Lie symmetry analysis and explicit solutions for the time fractional generalized Burgers–Huxley equation were studied by Inc et al. [14]. Korpinar et al. [15] studied the exact special solutions for the stochastic regularized long wave–Burgers equation. Dhawan et al. have a contemporary review of techniques for the solution of the nonlinear Burgers equation [16] (also, see [17, 18]).

In this article, for the first time, a boundary value inverse problem for the generalized Burgers–Huxley and generalized Burgers–Fisher equations will be studied. For this purpose, first, a quasilinearization method is used to linearize the equation in a specific time step. Then, a linear combination of B-splines is used to approximate the largest order of derivatives in the equation. By integrating from this linear combination, some new approximations have been obtained for each of the functions and derivatives with respect to time and space. In this new method, the boundary and additional conditions of the problem are also applied in these approximations. Then, the Tikhonov regularization method is used to solve the system of linear equations using noisy data. In the end, several numerical examples are provided and 2D and 3D graphical illustrations are reported to show the accuracy and efficiency of the method.

The rest of the article is organized as follows. In the first subsection of Section 2, the B-spline functions and their first- and second-order integrals are introduced. In the continuation of this section, the quasilinearization method is presented. The solution method is presented to solve the inverse problem (1), (2), (4), and (5) in Section 3. Some numerical experiments are given with graphical and tabular illustrations in Section 4. The conclusion of the presented method is given at the end of the paper in Section 5.

## 2. Preliminaries

In this section, first, the spline approximation, used in this article, is introduced and then the quasilinearization approximation will be obtained.

### 2.1. Cubic B-Spline.

In this approach, the space derivatives are approximated using the cubic B-spline method. A mesh $\Omega$, which is equally divided by knots $x_i$ into $M$ subintervals $[x_i,x_{i+1}]$, $i=0,1,\cdots,M-1$, such that $\Omega: a=x_0<x_1<\cdots<x_M=b$, is used. Also, let $S_4(\Omega)$ be the space of cubic splines on $\Omega$. The corresponding set of cubic B-splines $\{B_{-1},B_0,\cdots,B_{M+1}\}$, which is a basis for $S_4(\Omega)$, is defined using the recursive relation [19]:

$$b_{j,p}(x)=\frac{x-x_j}{x_{j+p}-x_j}b_{j,p-1}(x)+\frac{x_{j+p+1}-x}{x_{j+p+1}-x_{j+1}}b_{j+1,p-1}(x), \tag{10}$$

starting from

$$b_{j,0}(x)=\begin{cases}1, & x_j\le x<x_{j+1},\\ 0, & \text{otherwise},\end{cases} \tag{11}$$

where $j=-3,-2,\cdots,M-1$, $x_{-3}=x_{-2}=x_{-1}=a$, $x_{M+1}=x_{M+2}=x_{M+3}=b$, $p=1,2,\cdots$, and $B_k(x)=b_{k-2,3}(x)$, $k=-1,0,\cdots,M+1$, under the convention that fractions with zero denominators have the value zero. With the above definition, all the B-splines take the value zero at the endpoint $b$. Therefore, in order to avoid asymmetry over the interval $[a, b]$, it is common to assume the B-splines to be left continuous at $b$. We will follow suit.

Using induction on recurrence relation (10), we deduce immediately the following basic properties of a B-spline [20]:

(i) A B-spline is right continuous; i.e., the value at a point $x$ is obtained by taking the limit from the right

(ii) A B-spline is locally supported on the interval given by the extreme knots used in its definition. More precisely,

$$B_j(x) = 0, \quad x \notin [x_j, x_{j+4}). \tag{12}$$

(iii) A B-spline is nonnegative everywhere and positive inside its support, i.e.,

$$\begin{aligned} B_j(x) &\geq 0, \quad x \in \mathbb{R}, \\ B_j(x) &> 0, \quad x \in (x_j, x_{j+4}). \end{aligned} \tag{13}$$

(iv) From recurrence relation (10), one can find that the following formula for cubic B-splines:

$$B_j(x) = \begin{cases} \dfrac{(x-x_j)^3}{(x_{j+1}-x_j)(x_{j+2}-x_j)(x_{j+3}-x_j)}, & x \in [x_j, x_{j+1}), \\ \dfrac{(x-x_j)^2(x_{j+2}-x)}{(x_{j+2}-x_j)(x_{j+2}-x_{j+1})(x_{j+3}-x_j)} \\ \quad + \dfrac{(x-x_j)(x_{j+3}-x)(x-x_{j+1})}{(x_{j+3}-x_j)(x_{j+3}-x_{j+1})(x_{j+2}-x_{j+1})} \\ \quad + \dfrac{(x_{j+4}-x)(x-x_{j+1})^2}{(x_{j+2}-x_{j+1})(x_{j+3}-x_{j+1})(x_{j+4}-x_{j+1})}, & x \in [x_{j+1}, x_{j+2}), \\ \dfrac{(x-x_j)(x_{j+3}-x)^2}{(x_{j+3}-x_j)(x_{j+3}-x_{j+1})(x_{j+3}-x_{j+2})} \\ \quad + \dfrac{(x-x_{j+1})(x_{j+3}-x)(x_{j+4}-x)}{(x_{j+3}-x_{j+1})(x_{j+3}-x_{j+2})(x_{j+4}-x_{j+1})} \\ \quad + \dfrac{(x_{j+4}-x)^2(x-x_{j+2})}{(x_{j+4}-x_{j+1})(x_{j+4}-x_{j+2})(x_{j+3}-x_{j+2})}, & x \in [x_{j+2}, x_{j+3}), \\ \dfrac{(x_{j+4}-x)^3}{(x_{j+4}-x_{j+1})(x_{j+4}-x_{j+2})(x_{j+4}-x_{j+3})}, & x \in [x_{j+3}, x_{j+4}), \\ 0, \quad \text{o.w.}, \end{cases} \tag{14}$$

for $j = -3, -2, \cdots, M-1$.

Many other properties can be found in [19, 20] and references therein.

*2.2. Spline Approximation.* Now, let $f \in C[a, b]$; we consider a linear combination of B-splines $S_M(f)(x)$, as an approximation of $f(x)$, as follows:

$$S_M(f)(x) = \sum_{k=-1}^{M+1} c_k B_k(x) = C_M^T \Pi_M(x), \tag{15}$$

where $C_M = (c_{-1}, c_0, \cdots, c_{M+1})^T$ and $\Pi_M(x) = (B_{-1}(x), B_0(x), \cdots, B_{M+1}(x))^T$. Furthermore, in order to achieve a square system in numerical computations, the set of the nodes $\Omega^* = (\xi_i)_{i=-1}^{M+1}$ is used, where

$$\begin{cases} \xi_{-1} = x_0, \xi_0 = x_0 + \dfrac{h}{2}, \\ \xi_1 = x_1, \xi_2 = x_2, \cdots, \xi_{M-2} = x_{M-2}, \\ \xi_{M-1} = x_{M-1}, \xi_M = x_M - \dfrac{h}{2}, \xi_{M+1} = x_M, \end{cases} \tag{16}$$

where $h = (b - q)/M$.

*Definition 1.* Assume that $B$, $I_1B$, and $I_2B$ are $(M+3)$-square matrices defined by

$$\begin{aligned} (B)_{i,j} &= B_i(\xi_j), \\ (I_1B)_{i,j} &= \int_a^{\xi_j} B_i(y)\,dy, \\ (I_2B)_{i,j} &= \int_a^{\xi_j} \int_a^{z} B_i(y)\,dy\,dz, \end{aligned} \tag{17}$$

where $i, j = -1, 0, \cdots, M+1$. According to the definition of $B_k$ , we have

$$B = \left( \begin{array}{ccccccccccc} 1 & \frac{1}{8} & & & & & & & & & \\ & \frac{19}{32} & \frac{1}{4} & & & & & & & & \\ & \frac{25}{96} & \frac{7}{12} & \frac{1}{6} & & & & & & & \\ & \frac{1}{48} & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & & & & & & \\ & & & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & & & & & \\ & & & & \ddots & \ddots & \ddots & & & & \\ & & & & & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & & & \\ & & & & & & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \frac{1}{48} & \\ & & & & & & & \frac{1}{6} & \frac{7}{12} & \frac{25}{96} & \\ & & & & & & & & \frac{1}{4} & \frac{19}{32} & \\ & & & & & & & & & \frac{1}{8} & 1 \end{array} \right). \tag{18}$$

The matrices $I_1B$ and $I_2B$ are listed in the appendix.

Thus, we can write

$$\int_{a}^{\xi_j} S_M(f)(y)dy = C_n^T I_1^j, \tag{19}$$

$$\int_{ya}^{\xi_j} \int_{a}^{z} S_M(f)(y)dydz = C_n^T I_2^j, \tag{20}$$

where $I_v^j$ is the $j$th column of matrix $I_v B$, $v = 1, 2$.

*2.3. The Quasilinearization Method.* In equation (1), we have three nonlinear terms such as $u^\delta u_x$, $u^\delta u$, and $u^{2\delta} u$. In this section, a quasilinearization method is presented to linearize these terms. The quasilinearization technique is an application of the Newton–Raphson–Kantorovich approximation in function space [21–24].

Let $0 \le t \le T$ and $t_n = n\Delta t$, $n = 0, 1, \cdots, N$, are the equal parts of $[0, T]$, where $\Delta t = T/N$. Also, assume that $t \in [t_n, t_{n+1}]$, $u, v \in C[a, b] \times C[0, T]$, and $h(u, v) = u^\zeta v$. Using two-variable Taylor series for $h$ in some open neighborhood around $(u, v) = (u^n, v^n)$, there is $c = (c_1, c_2)$, where $c_1, c_2 \in C[a, b] \times C[0, T]$, so that

$$h(x) = h(a) + (x - a) \cdot \nabla h(a) + (x - a) \cdot H(c) \cdot (x - a), \tag{21}$$

where $x = (u, v)$, $a = (u^n, v^n)$, $u^n = u(x, t_n)$, $v^n = v(x, t_n)$, and $H$ is the Hessian matrix:

$$H(c) = \begin{pmatrix} h_{c_1c_1}(c) & h_{c_1c_2}(c) \\ h_{c_1c_2}(c) & h_{c_2c_2}(c) \end{pmatrix}. \tag{22}$$

Upon ignoring two-order terms, equation (21) becomes

$$h(x) \approx h(a) + (x - a) \cdot \nabla h(a). \tag{23}$$

Therefore,

$$\begin{aligned} h(u, v) &\approx \left(u^\zeta\right)^n v^n + (u - u^n, v - v^n) \cdot \left(\zeta \left(u^{\zeta-1}\right)^n v^n, \left(u^\zeta\right)^n\right) \\ &= \zeta \left(u^{\zeta-1}\right)^n v^n u - \zeta \left(u^\zeta\right)^n v^n + \left(u^\zeta\right)^n v. \end{aligned} \tag{24}$$

By placing $(\zeta, v) = (\delta, u_x)$, $(\zeta, v) = (\delta, u)$, and $(\zeta, v) = (2\delta, u)$ in (24), we obtain linear approximations for $u^\delta u_x$, $u^\delta u$, and $u^{2\delta} u$, respectively, as follows:

$$u^\delta u_x \approx \delta \left(u^{\delta-1}\right)^n (u_x)^n u - \delta \left(u^\delta\right)^n (u_x)^n + \left(u^\delta\right)^n u_x, \tag{25}$$

$$u^\delta u \approx \delta \left(u^\delta\right)^n u - \delta \left(u^{\delta+1}\right)^n + \left(u^\delta\right)^n u, \tag{26}$$

$$u^{2\delta} u \approx 2\delta \left(u^{2\delta}\right)^n u - 2\delta \left(u^{2\delta+1}\right)^n + \left(u^{2\delta}\right)^n u. \tag{27}$$

## 3. Solution Method for the Burgers–Huxley and Burgers–Fisher Equations

In this section, the inverse problem (1)–(5) is solved using $S_M$ as an approximation tool. Assume that in (16), $l = \xi_v$, $v \in \{-1, 0, \cdots, M + 1\}$.

To discretize (1), the method of [25, 26] is used. We assume that $u_{txx}(x, t)$ can be expanded in terms of linear combination of cubic B-splines (15) as follows:

$$u_{txx}(x, t) = \sum_{k=-1}^{M+1} c_k^n B_k(x) = C_M^T \Pi_M(x), \tag{28}$$

where $t \in [t_n, t_{n+1}]$, and the row vector $C_M^T$ is assumed constant in the subinterval $[t_n, t_{n+1}]$. By integrating (28) with respect to $t$ from $t_n$ to $t$, we obtain

$$u_{xx}(x, t) = u_{xx}(x, t_n) + (t - t_n) C_M^T \Pi_M(x). \tag{29}$$

Also, by integrating (28) with respect to $x$ from $l$ to $x$, we have

$$u_{tx}(x, t) = u_{tx}(l, t) + \sum_{k=-1}^{M+1} c_k^n \int_{l}^{x} B_k(y)dy. \tag{30}$$

Integrating (30) with respect to $x$ from $l$ to $x$ gives

$$\begin{aligned} u_x(x, t) = &\, u_x(x, t_n) + u_x(l, t) - u_x(l, t_n) \\ &+ (t - t_n) \sum_{k=-1}^{M+1} c_k^n \int_{l}^{x} B_k(y)dy. \end{aligned} \tag{31}$$

Again, by integrating (31) with respect to $x$ from $l$ to $x$, we gain

$$\begin{aligned} u(x, t) = &\, u(x, t_n) + p(t) - p(t_n) + (x - l)[u_x(l, t) - u_x(l, t_n)] \\ &+ (t - t_n) \sum_{k=-1}^{M+1} c_k^n \int_{l}^{x} \int_{l}^{z} B_k(y)dydz. \end{aligned} \tag{32}$$

Putting $x = b$ in (32), we get

$$\begin{aligned} u_x(l, t) - u_x(l, t_n) = &\, \frac{1}{b - l}[g(t) - g(t_n) - p(t) + p(t_n) \\ &- (t - t_n) \sum_{k=-1}^{M+1} c_k^n \int_{l}^{b} \int_{l}^{z} B_k(y)dydz]. \end{aligned} \tag{33}$$

Substituting equation (33) into (31) and (32) and using (4) and (5) held

$$\begin{aligned} u_x(x,t) = u_x(x,t_n) &+ \frac{1}{b-l}\left[g(t)-g(t_n)-p(t)+p(t_n)\right] \\ &+ (t-t_n)\sum_{k=-1}^{M+1} c_k^n \\ &\cdot \left(\int_l^x B_k(y)dy - \frac{1}{b-l}\int_l^b\int_l^z B_k(y)dydz\right), \end{aligned} \tag{34}$$

$$\begin{aligned} u(x,t) = u(x,t_n) + p(t) - p(t_n) &+ \frac{x-l}{b-l} \\ &\cdot \left[g(t)-g(t_n)-p(t)+p(t_n)\right] \\ &+ (t-t_n)\sum_{k=-1}^{M+1} c_k^n \\ &\cdot \left(\int_l^x\int_l^z B_k(y)dydz - \int_l^b\int_l^z B_k(y)dydz\right). \end{aligned} \tag{35}$$

By integrating (28) twice with respect to $x$ from $l$ to $x$ and using (5), we obtain

$$u_t(x,t) = \dot{p}(t) + (x-l)u_{tx}(l,t) + \sum_{k=-1}^{M+1} c_k^n \int_l^x\int_l^z B_k(y)dydz, \tag{36}$$

where $\dot{}$ denotes the differentiation with respect to $t$. By substituting $x = b$ in equation (36) and using (4), we get

$$u_{tx}(l,t) = \frac{1}{b-l}\left[\dot{g}(t) - \dot{p}(t) - \sum_{k=-1}^{M+1} c_k^n \int_l^b\int_l^z B_k(y)dydz\right]. \tag{37}$$

Substituting equation (37) into (36) held

$$\begin{aligned} u_t(x,t) = \dot{p}(t) + \frac{x-l}{b-l}\left[\dot{g}(t)-\dot{p}(t)\right] &+ \sum_{k=-1}^{M+1} c_k^n \\ &\cdot \left(\int_l^x\int_l^z B_k(y)dydz - \frac{x-l}{b-l}\int_l^b\int_l^z B_k(y)dydz\right). \end{aligned} \tag{38}$$

Since

$$\begin{aligned} \int_l^x\int_l^z B_k(y)dydz = \int_a^x\int_a^z B_k(y)dydz &- (x-l)\int_a^l B_k(y)dy \\ &- \int_a^l\int_a^z B_k(y)dydz, \end{aligned} \tag{39}$$

from (34), (35), and (38), we obtain

$$\begin{aligned} u_x(x,t) = u_x(x,t_n) &+ \frac{1}{b-l} \\ &\cdot \left[g(t)-g(t_n)-p(t)+p(t_n)\right] \\ &+ (t-t_n)\sum_{k=-1}^{M+1} c_k^n \mathscr{I}, \end{aligned} \tag{40}$$

$$\begin{aligned} u(x,t) = u(x,t_n) + p(t) - p(t_n) &+ \frac{x-l}{b-l} \\ &\cdot \left[g(t)-g(t_n)-p(t)+p(t_n)\right] \\ &+ (t-t_n)\sum_{k=-1}^{M+1} c_k^n \mathscr{J}, \end{aligned} \tag{41}$$

$$u_t(x,t) = \dot{p}(t) + \frac{x-l}{b-l}\left[\dot{g}(t)-\dot{p}(t)\right] + \sum_{k=-1}^{M+1} c_k^n \mathscr{J}, \tag{42}$$

where

$$\begin{aligned} \mathscr{I} &= \int_a^x B_k(y)dy - \frac{1}{b-l}\left(\int_a^b\int_a^z B_k(y)dydz - \int_a^l\int_a^z B_k(y)dydz\right), \\ \mathscr{J} &= \int_a^x\int_a^z B_k(y)dydz + \frac{x-b}{b-l}\int_a^l\int_a^z B_k(y)dydz \\ &\quad - \frac{x-l}{b-l}\int_a^b\int_a^z B_k(y)dydz. \end{aligned} \tag{43}$$

Further, by discretizing (29), (40), (41), and (42), assuming $x \rightarrow \xi_j$ and $t \rightarrow t_{n+1}$, and using (19) and (20), we get

$$(u_{xx})_i^{n+1} = (u_{xx})_i^n + \Delta t C_M^T \Pi_M(\xi_i), \tag{44}$$

$$(u_x)_i^{n+1} = (u_x)_i^n + \frac{1}{b-l}\varphi_n + \Delta t C_M^T L^i, \tag{45}$$

$$(u_t)_i^{n+1} = \dot{p}(t_{n+1}) + d_i\left[\dot{g}(t_{n+1}) - \dot{p}(t_{n+1})\right] + C_M^T S^i, \tag{46}$$

$$u_i^{n+1} = u_i^n + p(t_{n+1}) - p(t_n) + d_i\varphi_n + \Delta t C_M^T S^i, \tag{47}$$

where

$$\begin{aligned} S^i &= I_2^i + v_i I_2^\upsilon - d_i I_2^{M+1}, \\ L^i &= I_1^i - \frac{1}{b-l}\left(I_2^{M+1} - I_2^\upsilon\right), \\ v_i &= \frac{\xi_i - b}{b-l}, \\ d_i &= \frac{\xi_i - l}{b-l}, \\ \varphi_n &= g(t_{n+1}) - g(t_n) - p(t_{n+1}) + p(t_n), \\ (u_{xx})_i^{n+1} &= u_{xx}(x_i, t_{n+1}), \\ (u_x)_i^{n+1} &= u_x(x_i, t_{n+1}), \end{aligned}$$

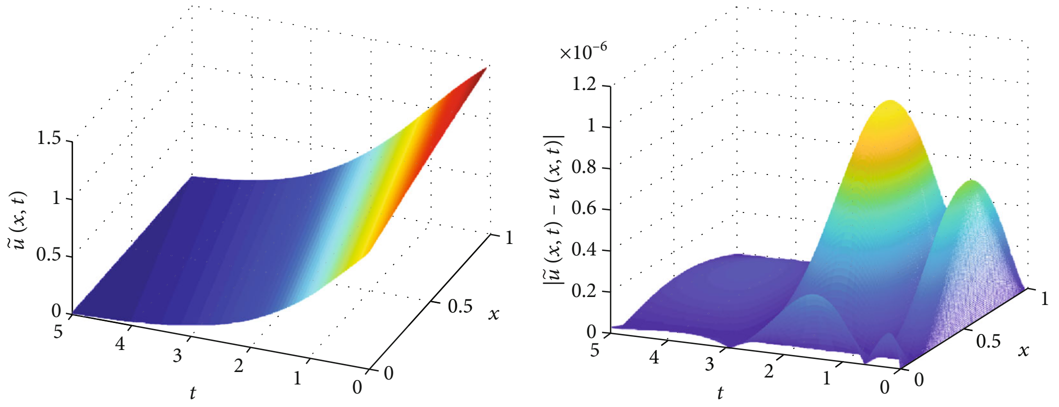

FIGURE 1: The exact solution (left) and the absolute error (right) of Example 1 with $\Delta t = 0.001$ and $h = 0.01$, without noise.

$$
\begin{aligned}
(u_t)_i^{n+1} &= u_t(x_i, t_{n+1}),\\
(u)_i^{n+1} &= u(x_i, t_{n+1}).
\end{aligned}
\tag{48}
$$

By substituting quasilinearization formulas (25)–(27) in (1), we get

$$
\begin{aligned}
u_t = \varepsilon u_{xx} &- \alpha\left(\delta (u^n)^{\delta-1}(u_x)^n u + (u^n)^{\delta} u_x\right)\\
&+ \beta\left[(\eta+\gamma)(\delta+1)(u^n)^{\delta} - \eta(2\delta+1)(u^n)^{2\delta} - \gamma\right]u\\
&+ \alpha\delta (u^n)^{\delta}(u_x)^n + \beta\left[2\eta\delta (u^n)^{2\delta+1} - (\eta+\gamma)\delta (u^n)^{\delta+1}\right].
\end{aligned}
\tag{49}
$$

Finally, substituting the approximation formulas (44)–(47) into (49) yields

$$
C_M^T Z_i^n = \sigma_i^n, \tag{50}
$$

where

$$
\begin{aligned}
Z_i^n &= \Delta t\left(\alpha (u_i^n)^{\delta} L^i - \varepsilon \Pi_M(\xi_i)\right)\\
&\quad + (1 - \Delta t w_i^n) S^i,
\end{aligned}
$$

$$
\begin{aligned}
\sigma_i^n &= r_i^n + \varepsilon (u_{xx})_i^n - \dot{p}(t_{n+1})\\
&\quad - d_i(\dot{g}(t_{n+1}) - \dot{p}(t_{n+1}))\\
&\quad - \alpha (u_i^n)^{\delta}\left((u_x)_i^n + \frac{1}{b-l}\varphi_n\right)\\
&\quad + w_i^n (u_i^n + p(t_{n+1})\\
&\quad - p(t_n) + d_i\varphi_n),
\end{aligned}
$$

$$
\begin{aligned}
r_i^n &= \alpha\delta (u_i^n)^{\delta}(u_x)_i^n + \beta\\
&\quad \cdot \left[2\eta\delta (u_i^n)^{2\delta+1} - (\eta+\gamma)\delta (u_i^n)^{\delta+1}\right],
\end{aligned}
$$

$$
\begin{aligned}
w_i^n = \beta\Big[&(\eta+\gamma)(\delta+1)(u_i^n)^{\delta}\\
&- \eta(2\delta+1)(u_i^n)^{2\delta} - \gamma\Big]\\
&- \alpha\delta (u_i^n)^{\delta-1}(u_x)_i^n.
\end{aligned}
\tag{51}
$$

By organizing (50) with respect to $i = -1, 0, \cdots, M+1$, we obtain

$$
Z^n C_M = R^n, \tag{52}
$$

where

$$
\begin{aligned}
Z^n &= \left(Z_{-1}^n, Z_0^n, \cdots, Z_{M+1}^n\right)^T,\\
R^n &= \left(\sigma_{-1}^n, \sigma_0^n, \cdots, \sigma_{M+1}^n\right)^T.
\end{aligned}
\tag{53}
$$

Note that for $n = 0$, we use equation (2) as $u_{xx}(x_i, t_0) = f''(x_i)$, $u_x(x_i, t_0) = f'(x_i)$, and $u(x_i, t_0) = f(x_i)$; otherwise, $u_{xx}(x_i, t_n)$, $u_x(x_i, t_n)$, and $u(x_i, t_n)$, are updated using (44), (45), and (47), respectively.

## 4. Numerical Examples

All examples in this section are solved once with the exact values of the right-hand metallurgy side vector $R^0$ and again by adding noise to it. We add the noise to the vector $R^n$ in the form $R_\varepsilon^n = R^n + \vartheta \times \mathrm{randn}(M+3)$, where $\vartheta$ is an absolute noise level and $\mathrm{randn}(M+3)$ is a normal distribution vector with zero mean and unit standard deviation, and it is realized using the MATLAB function randn. In this article, we consider four noise levels $\vartheta = 0.0001$, $0.001$, $0.01$, and $0.1$.

In the case that noise is added to the system (52), we will use the Tikhonov regularization method [27] to solve the system. By this technique, we have a minimization problem as follows:

$$
\min_{x\in\mathbb{R}^{M+3}} \|Z^n C_M - R_\vartheta^n\|_2^2 + \lambda \|C_M\|_2^2, \tag{54}
$$

where $\lambda > 0$ is the regularization parameter, which controls the trade-off between fidelity to the data and smoothness of

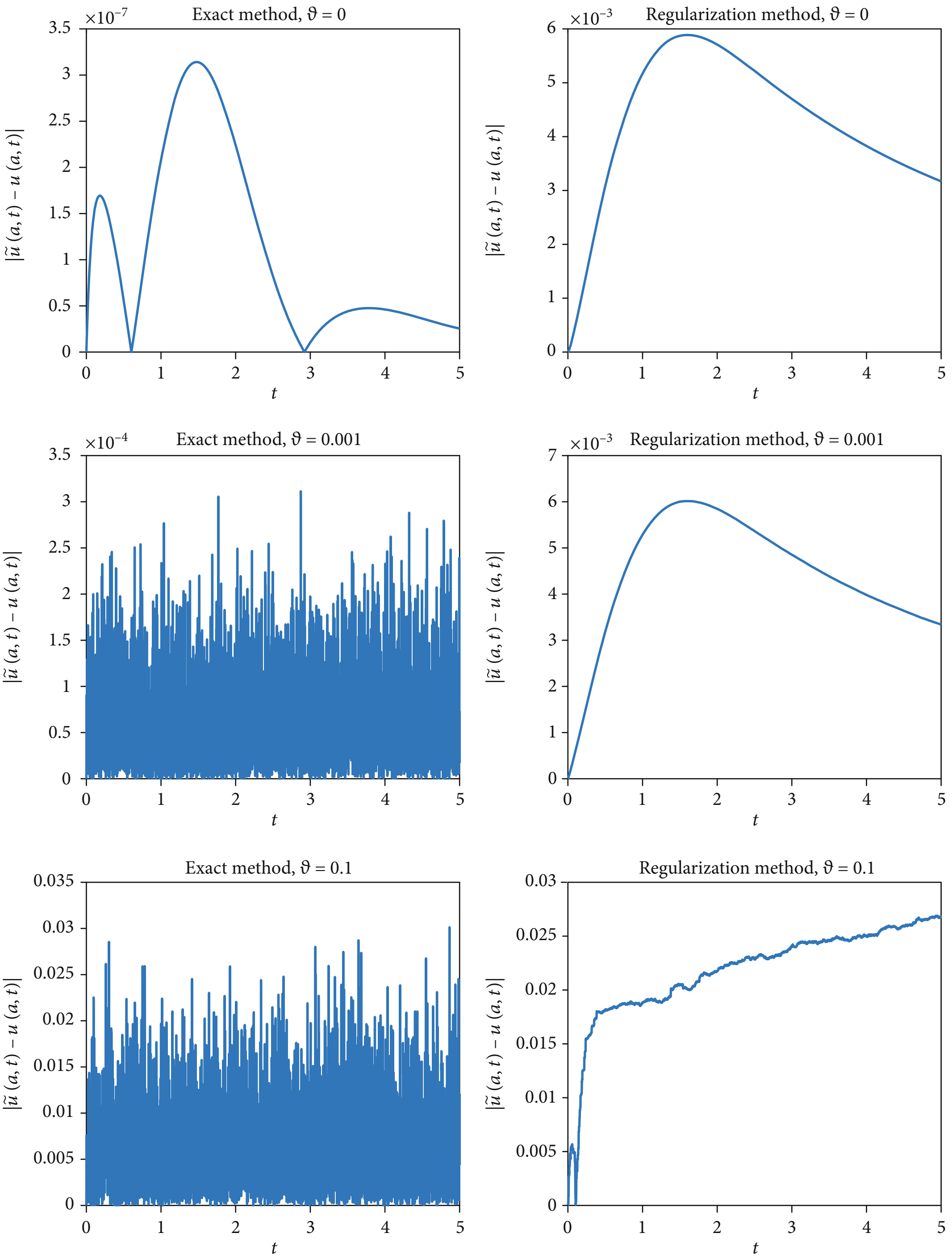

FIGURE 2: The absolute errors $|\tilde{u}(a,t) - u(a,t)|$, with the exact and regularization methods and different values of noises for Example 1 using $\Delta t = 0.001$ and $h = 0.05$.

TABLE 1: $L_\infty$ errors of Example 1 for different values of $\Delta t$ and $\vartheta$ with $h = 0.05$.

| $\vartheta$ | Method | $\Delta t = \frac{1}{10}$ | $\Delta t = \frac{1}{100}$ | $\Delta t = \frac{1}{500}$ | $\Delta t = \frac{1}{1000}$ |
|---|---|---|---|---|---|
| 0 | Exact | $2.814715e-05$ | $2.583762e-06$ | $6.155098e-07$ | $3.141230e-07$ |
| 0.0001 | Exact | $3.621854e-05$ | $1.145132e-05$ | $1.462985e-05$ | $3.314704e-05$ |
| 0.0001 | Regularization | $4.802055e-05$ | $1.106724e-05$ | $4.894096e-03$ | $5.889474e-03$ |
| 0.001 | Exact | $1.049879e-04$ | $1.147022e-04$ | $1.588327e-04$ | $3.328792e-04$ |
| 0.001 | Regularization | $1.198265e-04$ | $1.262147e-04$ | $4.903954e-03$ | $6.321994e-03$ |
| 0.01 | Exact | $7.991674e-04$ | $1.013317e-03$ | $1.543202e-03$ | $3.522300e-03$ |
| 0.01 | Regularization | $6.676744e-04$ | $1.020415e-03$ | $5.201666e-03$ | $6.770758e-03$ |
| 0.1 | Exact | $1.134644e-02$ | $1.124839e-02$ | $1.672706e-02$ | $3.645521e-02$ |
| 0.1 | Regularization | $8.608509e-03$ | $8.209530e-03$ | $1.927866e-02$ | $2.575360e-02$ |

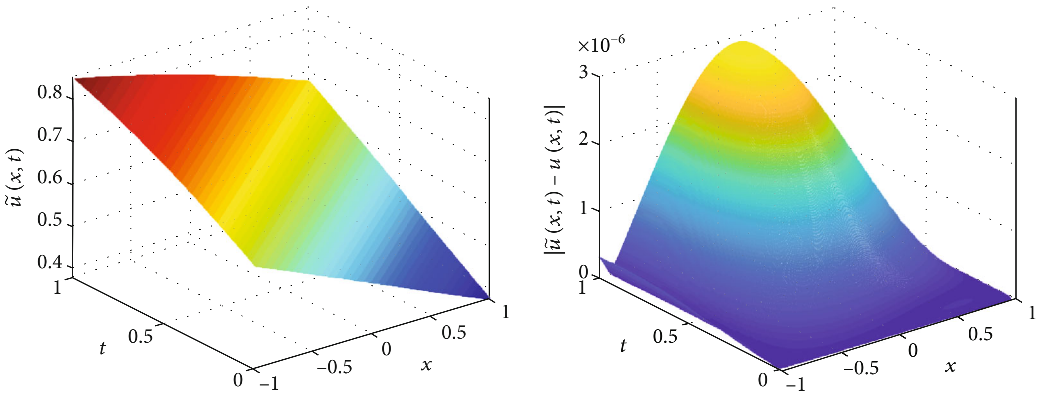

FIGURE 3: The exact solution (left) and the absolute error (right) of Example 2 with $\Delta t = 0.001$ and $h = 0.01$, without noise.

TABLE 2: $L_\infty$ errors of Example 2 for different values of $\Delta t$ and $\vartheta$ with $h = 0.05$.

| $\vartheta$ | Method | $\Delta t = \frac{1}{10}$ | $\Delta t = \frac{1}{100}$ | $\Delta t = \frac{1}{500}$ | $\Delta t = \frac{1}{1000}$ |
|---|---|---|---|---|---|
| 0 | Exact | $3.612476e-05$ | $2.959095e-06$ | $6.619158e-07$ | $3.353989e-07$ |
| 0.0001 | Exact | $4.699528e-05$ | $1.248949e-05$ | $2.062944e-05$ | $3.238930e-05$ |
| 0.0001 | Regularization | $2.624024e-05$ | $1.202468e-05$ | $2.387568e-04$ | $2.749732e-04$ |
| 0.001 | Exact | $1.183073e-04$ | $1.362642e-04$ | $1.925736e-04$ | $4.107127e-04$ |
| 0.001 | Regularization | $1.363059e-04$ | $8.611165e-05$ | $2.450763e-04$ | $3.206762e-04$ |
| 0.01 | Exact | $1.525685e-03$ | $1.297835e-03$ | $1.956289e-03$ | $3.485143e-03$ |
| 0.01 | Regularization | $7.281806e-04$ | $6.885447e-04$ | $7.839071e-04$ | $2.761185e-03$ |
| 0.1 | Exact | $7.399248e-03$ | $1.083570e-02$ | $2.062804e-02$ | $3.805156e-02$ |
| 0.1 | Regularization | $9.402128e-03$ | $9.843907e-03$ | $1.193822e-02$ | $2.510321e-02$ |

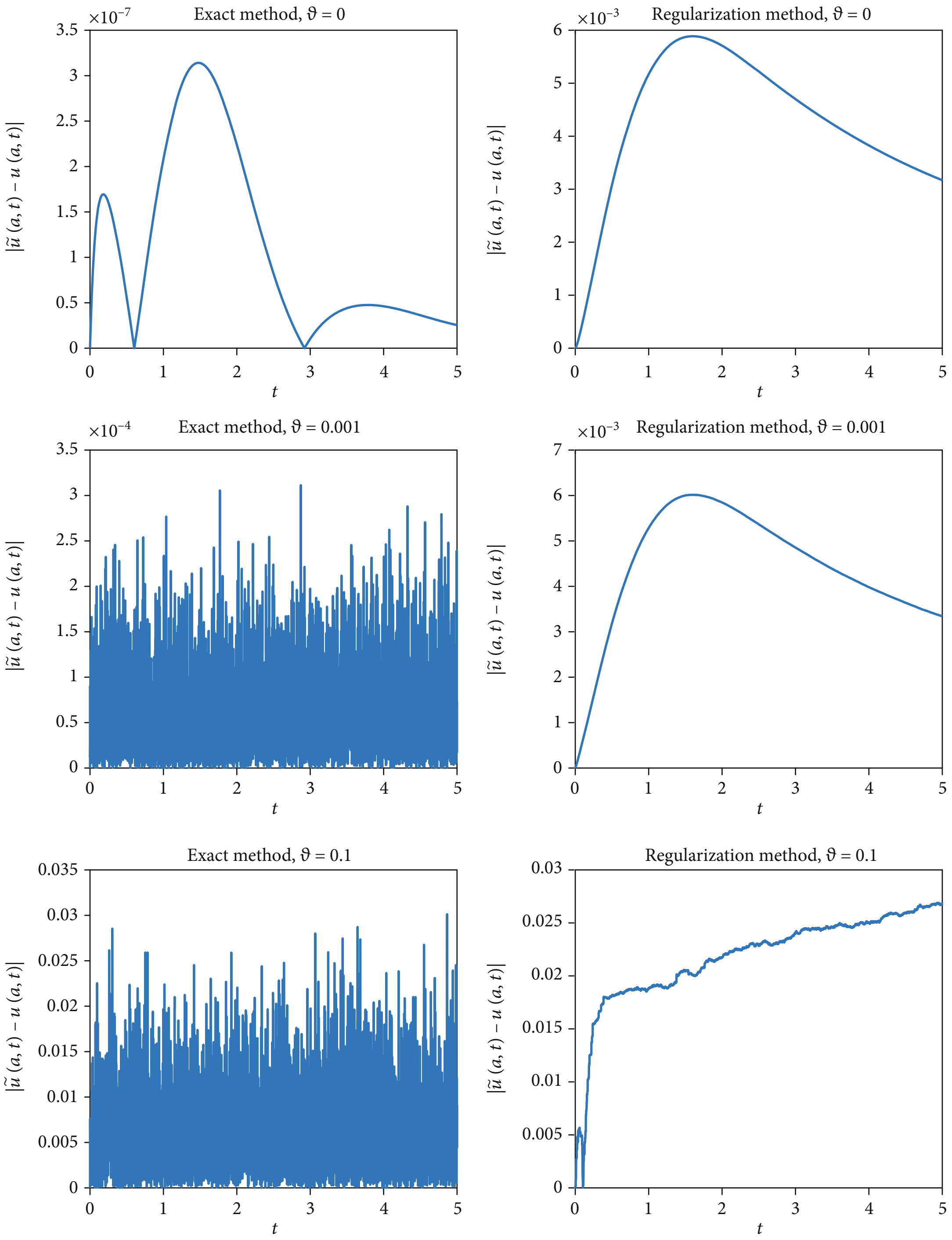

Figure 4: The absolute errors $|\tilde{u}(a,t) - u(a,t)|$, with the exact and regularization methods and different values of noises for Example 2 using $\Delta t = 0.001$ and $h = 0.05$.

Table 3: $L_\infty$ errors of Example 3 for different values of $\Delta t$ and $\vartheta$ with $h = 0.05$.

| $\vartheta$ | Method | $\Delta t = \frac{1}{10}$ | $\Delta t = \frac{1}{100}$ | $\Delta t = \frac{1}{500}$ | $\Delta t = \frac{1}{1000}$ |
|---|---|---|---|---|---|
| 0 | Exact | $4.816528e-06$ | $4.792704e-07$ | $9.580090e-08$ | $4.789637e-08$ |
| 0.0001 | Exact | $2.351403e-05$ | $2.514832e-05$ | $4.071947e-05$ | $7.494035e-05$ |
| 0.0001 | Regularization | $2.801648e-05$ | $5.715366e-05$ | $1.121469e-04$ | $1.220928e-04$ |
| 0.001 | Exact | $1.306283e-04$ | $2.291344e-04$ | $3.901340e-04$ | $7.325559e-04$ |
| 0.001 | Regularization | $7.512398e-05$ | $1.175376e-04$ | $1.169995e-04$ | $2.451920e-04$ |
| 0.01 | Exact | $2.557327e-03$ | $2.312663e-03$ | $3.491372e-03$ | $6.974782e-03$ |
| 0.01 | Regularization | $1.193753e-03$ | $4.763686e-04$ | $4.462170e-04$ | $2.318261e-03$ |
| 0.1 | Exact | $1.812669e-02$ | $2.584793e-02$ | $3.202167e-02$ | $7.193698e-02$ |
| 0.1 | Regularization | $7.477861e-03$ | $1.372036e-02$ | $4.073468e-03$ | $2.030713e-02$ |

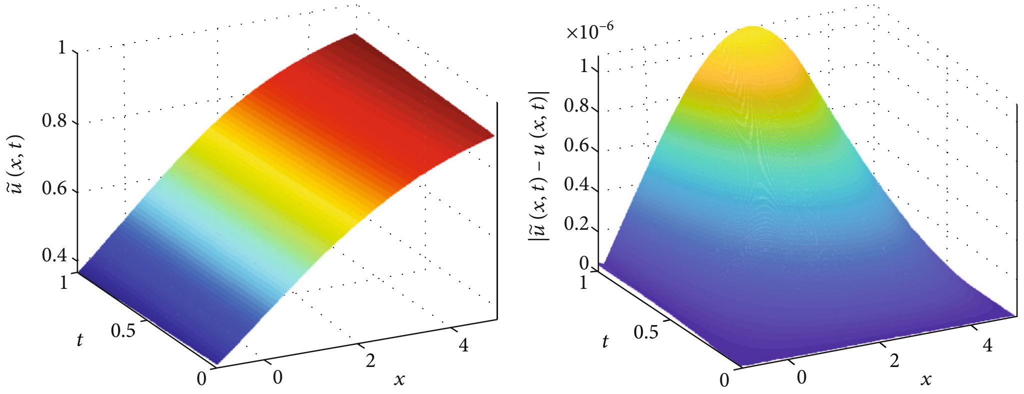

Figure 5: The exact solution (left) and the absolute error (right) of Example 3 with $\Delta t = 0.001$ and $h = 0.01$, without noise.

the solution. In this word, the generalized cross-validation (GCV) method [28] is used to determine the regularization parameter $\lambda$. In our computations, we will use the MATLAB codes developed by Hansen [29] for solving the ill-conditioned systems.

In numerical examples, we suppose that $\tilde{u}(x, t)$ denotes the exact solution and $u(x, t)$ denotes the estimated solution.

The versatility and accuracy of the methods are measured using the maximum absolute error norm $L_\infty$, defined by [30]:

$$L_\infty = \max_{0 \le n \le N} |\tilde{u}(a, t_n) - u(a, t_n)|. \tag{55}$$

In all examples and for all different values of $n$ and $h$, the conditional numbers of the coefficient matrices $Z^n$ are less than 1000 but their smallest singular values are about $10^{-5}$ and relatively small. For this reason, we expect the ill-posedness of the systems to increase with increasing $\vartheta$.

In all examples, solving the system by the decomposition method (Cholesky et al.) is called the "exact method" and solving the system using the Tikhonov regularization method is called the "regularization method."

It is notable that we perform all of the computations by MATLAB® R2019a software (V9.6.0.1072779, 64-bit (win64), License Number: 968398, MathWorks Inc., Natick, MA) running on a Sony VAIO Laptop (Intel® Core™ i5-2410M Processor 2.30 GHz with Turbo Boost up to 2.90 GHz, 8 GB of RAM, 64-bit) PC.

*Example 1.* We consider the problem (1)–(5) in the domain $[0, 1]$ with $\varepsilon = 1$, $l = 0.1$, $T = 5$, $\eta = 1$, $\alpha = 1$, $\beta = 1$, $\gamma = 2$, and $\delta = 1$. The exact solution will be obtained using equation (6).

The exact solution and the absolute error using $\Delta t = 0.001$ and $h = 0.01$ are depicted in Figure 1. Also, the absolute errors $|\tilde{u}(a, t) - u(a, t)|$, by applying the exact and regularization methods and different values of $\vartheta$ with $\Delta t = 0.001$ and $h = 0.05$, are shown in Figure 2. In Table 1, the maximum absolute errors $L_\infty$ are tabulated using $h = 0.05$ and different values of $\vartheta$ and $\Delta t$.

*Example 2.* In this example, we consider the problem (1)–(5) with $\varepsilon = 1$, $l = -0.9$, $T = 1$, $\eta = 0$, $\alpha = 1$, $\beta = 1$, $\gamma = -1$, and $\delta = 1$ in the domain $[-1, 1]$. The exact solution will be obtained using equation (8).

In Figure 3, the exact solution and the absolute error using $\Delta t = 0.001$ and $h = 0.01$ are presented. In addition, the absolute errors $|\tilde{u}(a, t) - u(a, t)|$, using the exact and regularization methods and different values of $\vartheta$ with $\Delta t = 0.001$ and $h = 0.05$, are displayed in Figure 4. The $L_\infty$ are shown using different values of $\vartheta$ and $\Delta t$ and $h = 0.05$ in Table 2.

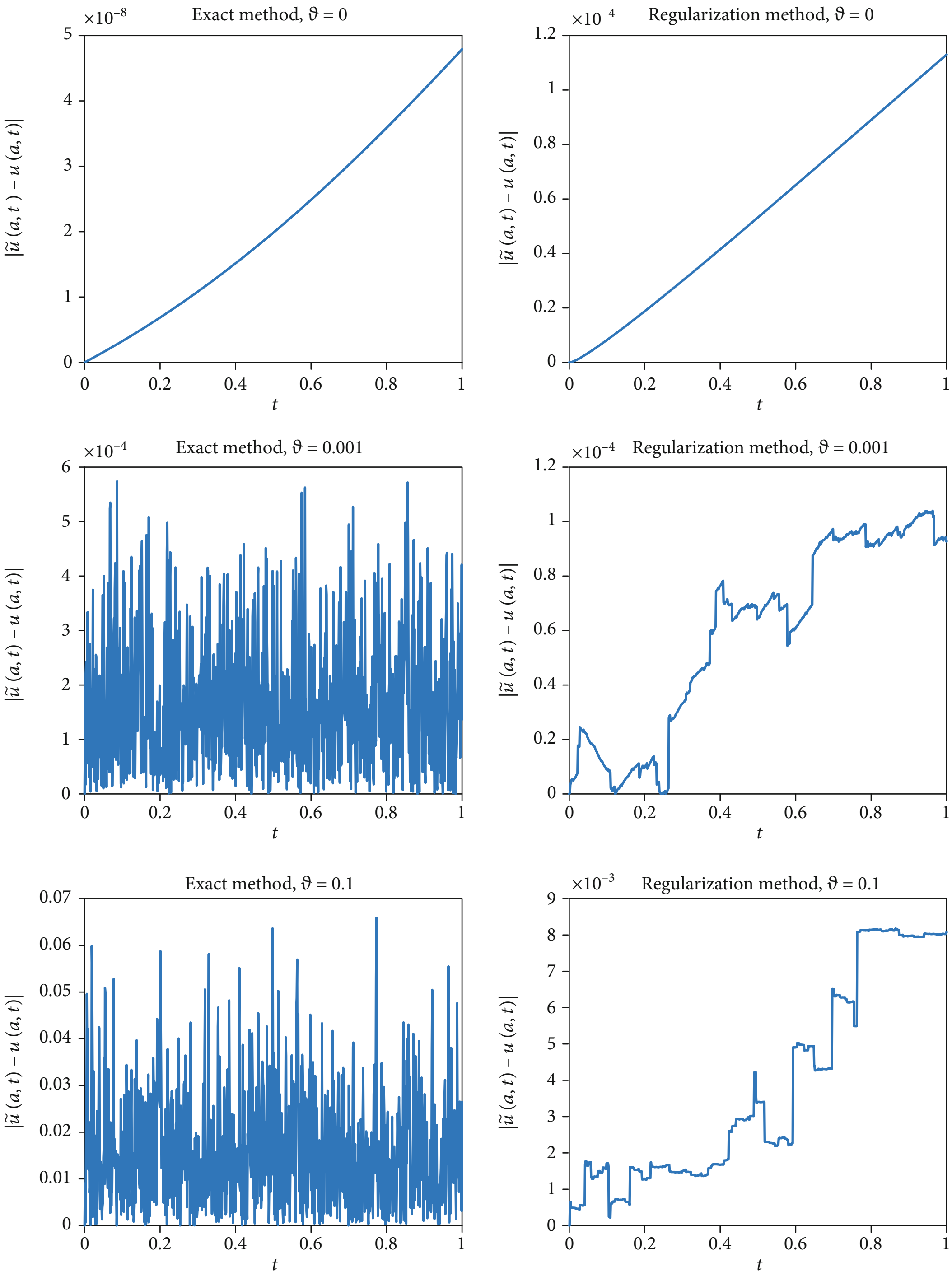

FIGURE 6: The absolute errors $|\tilde{u}(a,t) - u(a,t)|$, with the exact and regularization methods and different values of noises for Example 3 using $\Delta t = 0.001$ and $h = 0.05$.

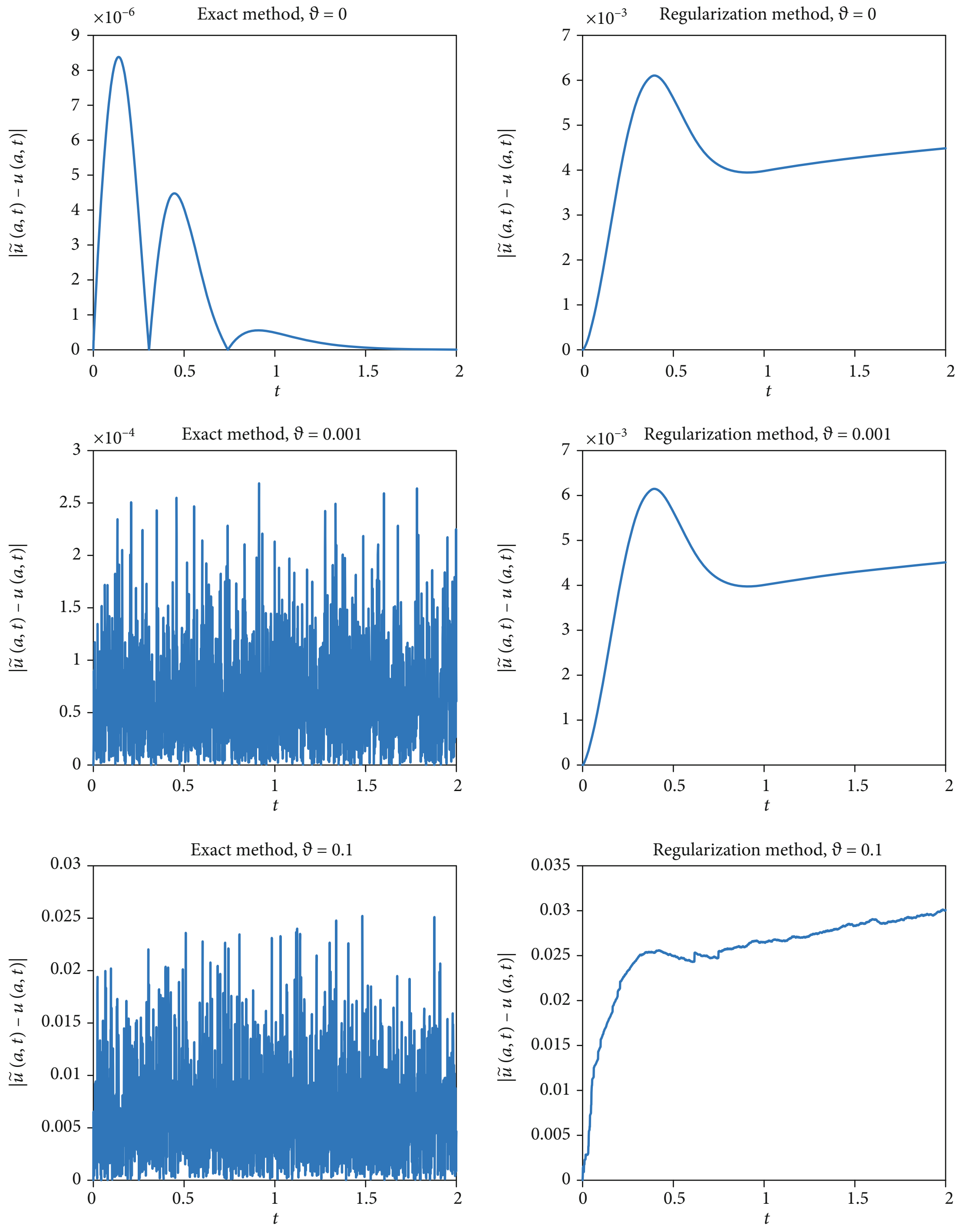

FIGURE 7: The absolute errors $|\tilde{u}(a,t) - u(a,t)|$, with the exact and regularization methods and different values of noises for Example 4 using $\Delta t = 0.001$ and $h = 0.05$.

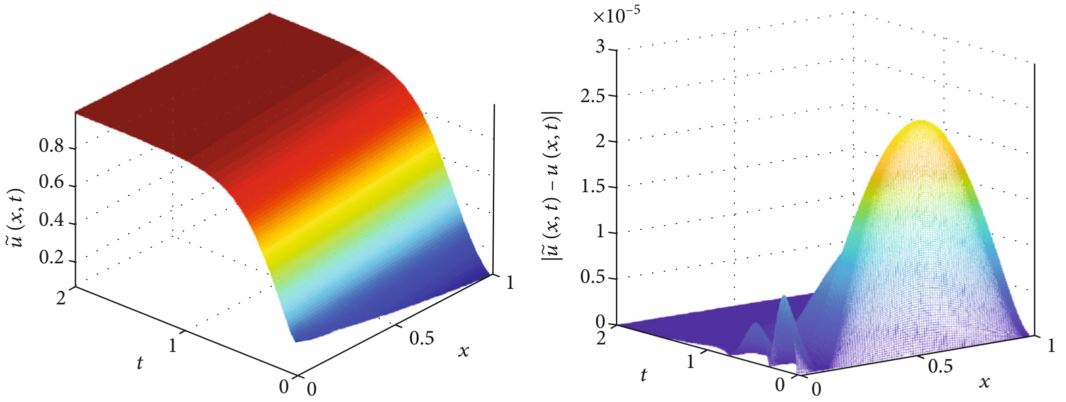

FIGURE 8: The exact solution (left) and the absolute error (right) of Example 4 with $\Delta t = 0.001$ and $h = 0.01$, without noise.

TABLE 4: $L_\infty$ errors of Example 4 for different values of $\Delta t$ and $\vartheta$ with $h = 0.05$.

| $\vartheta$ | Method | $\Delta t = \frac{1}{10}$ | $\Delta t = \frac{1}{100}$ | $\Delta t = \frac{1}{500}$ | $\Delta t = \frac{1}{1000}$ |
|---|---|---|---|---|---|
| 0 | Exact | $5.686819e-03$ | $1.263590e-04$ | $1.769301e-05$ | $8.381687e-06$ |
| 0.0001 | Exact | $5.685089e-03$ | $1.327687e-04$ | $3.060667e-05$ | $3.335971e-05$ |
| 0.0001 | Regularization | $5.332308e-03$ | $1.069778e-04$ | $5.492670e-03$ | $6.108492e-03$ |
| 0.001 | Exact | $5.701673e-03$ | $1.770599e-04$ | $1.672482e-04$ | $2.856058e-04$ |
| 0.001 | Regularization | $5.308533e-03$ | $1.018102e-04$ | $5.505681e-03$ | $6.212021e-03$ |
| 0.01 | Exact | $6.142029e-03$ | $9.728911e-04$ | $1.744781e-03$ | $2.566482e-03$ |
| 0.01 | Regularization | $5.132815e-03$ | $1.078324e-03$ | $4.795718e-03$ | $6.132190e-03$ |
| 0.1 | Exact | $1.584424e-02$ | $1.078511e-02$ | $1.603969e-02$ | $3.379878e-02$ |
| 0.1 | Regularization | $7.376052e-03$ | $7.101563e-03$ | $1.218363e-02$ | $3.157183e-02$ |

*Example 3.* Let $a = -1$, $b = 5$, $\varepsilon = 1$, $l = -0.9$, $T = 1$, $\eta = 1$, $\alpha = 0$, $\beta = 1$, and $\delta = 1$, in the problem (1)–(5). The exact solution of this example is given as [31]

$$u(x,t) = \frac{\gamma b_1 e^{(1/2)\left(\sqrt{2}\gamma x+\gamma^2 t\right)} + b_2 e^{(1/2)(\sqrt{2}x+t)}}{b_1 e^{(1/2)(\sqrt{2}\gamma x+\gamma^2 t)} + b_2 e^{(1/2)(\sqrt{2}x+t)} + b_3 e^{\gamma t}}, \qquad (56)$$

where $b_1$, $b_2$, and $b_3$ are arbitrary constants. For the computation, we take $\gamma = 1/2$, $b_1 = 1$, $b_2 = 1$, and $b_3 = 1$.

The error norms $L_\infty$ are tabulated using different values of $\vartheta$ and $\Delta t$ and $h = 0.05$ in Table 3. The exact solution and the absolute error using $\Delta t = 0.001$ and $h = 0.01$ are presented in Figure 5. Moreover, the absolute errors $|\tilde{u}(a,t) - u(a,t)|$, using the exact and regularization methods and different values of $\vartheta$ with $\Delta t = 0.001$ and $h = 0.05$, are shown in Figure 6.

*Example 4.* We consider the problem (1)–(5) with $\varepsilon = 1$, $l = 0.1$, $T = 3$, $\eta = 0$, $\alpha = 0$, $\gamma = -1$, $\delta = 1$, $a = 0$, and $b = 1$. The exact solution is given by [32] as follows:

$$u(x,t) = \left(1 + e^{\sqrt{\beta/6}x-(5\beta/6)t}\right)^{-2}, \qquad (57)$$

and we assume that $\beta = 6$.

In Figure 7, the absolute errors $|\tilde{u}(a,t) - u(a,t)|$, using the exact and regularization methods and different values of $\vartheta$ with $\Delta t = 0.001$ and $h = 0.05$, are depicted. In Figure 8, the exact solution and the absolute error using $\Delta t = 0.001$ and $h = 0.01$ are presented. The maximum absolute errors $L_\infty$ are tabulated using $h = 0.05$ and different values of $\vartheta$ and $\Delta t$ in Table 4.

## 5. Conclusions

The boundary value inverse problem related to the generalized Burgers–Fisher and generalized Burgers–Huxley equations was solved numerically. We considered the equation in a small time interval and then applied quasilinearization in time. We approximated the largest order of derivatives in the equation using a linear combination of B-splines. By integrating several times with respect to the time and space variables, we obtain approximations for the function and its partial derivatives. By substituting quasilinearization and the obtained approximations in the equation, a desired numerical scheme was obtained. In numerical examples, we saw that the obtained linear system from the numerical scheme has a relatively small condition number. The numerical results show that the solutions are very accurate. By adding large noise levels to the system, it was observed that the solutions were still appropriate.

## Appendix

The matrices $I_1B$ and $I_2B$ are listed below.

$$
\begin{gathered}
I_1B = h\begin{pmatrix}
0 & \frac{15}{64} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
0 & \frac{55}{256} & \frac{7}{16} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
0 & \frac{37}{768} & \frac{13}{48} & \frac{17}{24} & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & \cdots & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} \\
0 & \frac{1}{384} & \frac{1}{24} & \frac{1}{2} & \frac{23}{24} & 1 & 1 & \cdots & 1 & 1 & 1 \\
 & & & \frac{1}{24} & \frac{1}{2} & \frac{23}{24} & 1 & \cdots & 1 & 1 & 1 \\
 & & & & \ddots & \ddots & \ddots & \ddots & & & 1 \\
 & & & & & \frac{1}{24} & \frac{1}{2} & \frac{23}{24} & 1 & 1 & 1 \\
 & & & & & & \frac{1}{24} & \frac{1}{2} & \frac{23}{24} & \frac{383}{384} & 1 \\
 & & & & & & & \frac{1}{24} & \frac{23}{48} & \frac{539}{768} & \frac{3}{4} \\
 & & & & & & & & \frac{1}{16} & \frac{73}{256} & \frac{1}{2} \\
 & & & & & & & & & \frac{1}{64} & \frac{1}{4}
\end{pmatrix}, \\
I_2B = h^2\begin{pmatrix}
0 & \frac{49}{640} & \frac{4}{20} & \frac{9}{20} & \frac{14}{20} & \frac{19}{20} & \frac{24}{20} & \cdots & \frac{4+5(2n-2)}{20} & \frac{20n-7}{40} & \frac{4+5(2n-1)}{20} \\
0 & \frac{107}{2560} & \frac{17}{80} & \frac{7}{10} & \frac{12}{10} & \frac{17}{10} & \frac{22}{10} & \cdots & \frac{7+5(2n-3)}{10} & \frac{20n-11}{20} & \frac{7+5(2n-2)}{10} \\
0 & \frac{49}{7680} & \frac{19}{240} & \frac{73}{120} & \frac{27}{20} & \frac{42}{20} & \frac{57}{20} & \cdots & \frac{27+15(2n-4)}{20} & \frac{60n-51}{40} & \frac{27+15(2n-3)}{20} \\
0 & \frac{1}{3840} & \frac{1}{120} & \frac{7}{30} & \frac{121}{120} & 2 & 3 & \cdots & 2n-3 & \frac{4n-5}{2} & 2n-2 \\
 & & & \frac{1}{120} & \frac{7}{30} & \frac{121}{120} & 2 & & 2n-4 & \frac{4n-7}{2} & 2n-3 \\
 & & & & \ddots & \ddots & \ddots & \ddots & & & \\
 & & & & & \frac{1}{120} & \frac{7}{30} & \frac{121}{120} & 2 & \frac{5}{2} & 3 \\
 & & & & & & \frac{1}{120} & \frac{7}{30} & \frac{121}{120} & \frac{5761}{3840} & 2 \\
 & & & & & & & \frac{1}{120} & \frac{11}{48} & \frac{4081}{7680} & \frac{9}{10} \\
 & & & & & & & & \frac{1}{80} & \frac{47}{512} & \frac{3}{10} \\
 & & & & & & & & & \frac{1}{640} & \frac{1}{20}
\end{pmatrix}.
\end{gathered}
\tag{A.1}
$$

## Data Availability

All results have been obtained by conducting the numerical procedure, and the ideas can be shared with the researchers.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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