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Review Article

An Efficient Explicit Decoupled Group Method for Solving Two–Dimensional Fractional Burgers' Equation and Its Convergence Analysis

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In this paper, the Crank–Nicolson (CN) and rotated four-point fractional explicit decoupled group (EDG) methods are introduced to solve the two-dimensional time–fractional Burgers' equation. The EDG method is derived by the Taylor expansion and 45° rotation of the Crank–Nicolson method around the x and y axes. The local truncation error of CN and EDG is presented. Also, the stability and convergence of the proposed methods are proved. Some numerical experiments are performed to show the efficiency of the presented methods in terms of accuracy and CPU time.

1. Introduction

Fractional calculus is a generalization of the integration and derivation of integer order operators to fractional order that allows us to describe a real system more accurately than integers. Although the fractional order of a real system may be low, it is yet considered as a fractional system. An important feature of fractional calculus is its nonlocality. The fractional derivative (and integrals) of a function is given by a definite integral, thus, it depends on the value of the function over the entire interval [1]. Researchers confirm the existence of interesting phenomena in nature, which cannot be modeled by classical differential equations. To cope with this problem, the nonlocality property of fractional derivative could be a beneficial tool to study our considered system. Fractional calculus has recently been used in various scientific and engineering fields [2–4]. Some fractional calculus applications in modeling and design of control systems are introduced in [5]. The most recent developments and trends in the use of fractional calculus in biomedicine and biology are presented by Ionescu et al. [6]. Based on the fractional calculate, Tang et al. [7] proposed a new four-element creep model; this model accords well with the experimental data of Changshan rock salt. Fractional calculus has an extraordinary potential in signal denoising [8]. Gong et al. discussed the generation conditions of chaotic behavior and proposed the adaptive synchronization control method for a class of fractional-order financial system [9]. Numerous definitions of fractional derivative have been introduced in the literature, amongst are Riemann–Liouville, Caputo, and Caputo–Fabrizio [10]. The Caputo–Fabrizio fractional derivative on the contrary of other derivatives has a nonsingular kernel. Hence, it has been considered by many researchers [11–15]. Since to obtain the exact solution of fractional differential equations is very difficult and sometimes impossible, it is usually approximated by a numerical method such as finite difference method [16–18], finite volume methods [19], and spectral method [20].

Burgers' equation as a nonlinear partial differential equation is widely used in various areas such as fluid mechanics, gas dynamics, and traffic flow which combine nonlinear propagation effects with diffusive one.

In this paper, we consider the following two dimensional time-fractional Burgers' equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + u\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + f(x, y, t), \quad (x, y) \in \Omega, 0 < t < T, \tag{1}$$

with the initial condition

$$u(x, y, 0) = u_0(x, y), (x, y) \in \Omega, \tag{2}$$

and the boundary condition

$$u(x, y, t) = 0, (x, y) \in \partial\Omega, 0 \le t \le T, \tag{3}$$

where $\nu = 1/\mathrm{Re}$, Re is Reynolds number characterizing the strength of viscosity, $\Omega = (0, 1) \times (0, 1)$, $0 < \alpha < 1$, and the term $\partial^\alpha u/\partial t^\alpha$ denotes the α order Caputo-Fabrizio fractional derivative of the function $u(x, y, t)$ defined as:

$$\frac{\partial^\alpha u(x, y, t)}{\partial t^\alpha} = \frac{M(\alpha)}{1-\alpha}\int_0^t \frac{\partial u(x, y, s)}{\partial s} e^{-\frac{\alpha}{1-\alpha}(t-s)} ds, \tag{4}$$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$.

We apply the finite difference method to solve Equations (1)–(3). In finite difference, to find the value corresponding to each grid point, it is used natural ordering (indexing the grid of point from left to right and bottom to top by point to point) or group to group. There are several different methods to order the interior mesh point such as natural ordering, diagonal ordering, and alternating diagonal ordering [21]. Different linear systems would be produced by different arrangements of the grid points.

We propose two finite difference schemes. The first scheme is given by the Crank–Nicolson difference method (by natural ordering). In this scheme, to obtain a more accurate numerical solution, we should use a smaller mesh size, which requires more storage space and computing time. In order to fix this problem and accelerate the convergence, we use the explicit decoupled group (EDG) method introduced by Abdullah in 1990 [22]. Many studies have been done in reference to the EDG method for examples [23–26]. The EDG method is based on rotating the Crank–Nicolson difference scheme and group ordering of grid points. Applying the EDG method to the Crank–Nicolson difference scheme result in a new scheme in which the dimension of the system is half of the dimension of the system generated by the Crank–Nicolson difference scheme. On this account, half of the grid points are obtained and the rest can be calculated directly. Consequently, the EDG method can be favourably used to reduce the computational cost. In addition, it is worth to notice that we can take advantage of parallel computers to run it.

The rest of the paper is organized as follows. In Section 2, the Crank–Nicolson difference scheme will be applied to Equations (1)–(3), and also, we give the truncation error. In Section 3, we describe the formulation of the EDG method. The stability of these schemes is discussed in Section 4. In Section 5, we analyze the convergence of these schemes. Numerical examples are carried out to verify the high efficiency of our method in Section 6. Finally, the paper ends with a brief conclusion in Section 7.

2. Ù‘The Crank–Nicolson Difference Scheme

For the numerical solution of Equations (1)–(3), we introduce a uniform grid of mesh points (x_i, y_j, t_n) with $x_i = i\Delta x$, $i = 0, 1, \cdots, I$, $y_j = j\Delta y$, $j = 0, 1, \cdots, J$, and $t_n = n\Delta t$, $n = 0, 1, \cdots, N$.

Using the Crank–Nicolson approximation to Equations (1)–(3), we have

$$\left(\frac{\partial^\alpha u}{\partial t^\alpha}\right)_{i,j}^{n+1/2} + \frac{(uu_x)_{i,j}^{n+1} + (uu_x)_{i,j}^{n}}{2} + \frac{(uu_y)_{i,j}^{n+1} + (uu_y)_{i,j}^{n}}{2} = \nu\left(\frac{(u_{xx})_{i,j}^{n+1} + (u_{xx})_{i,j}^{n}}{2} + \frac{(u_{yy})_{i,j}^{n+1} + (u_{yy})_{i,j}^{n}}{2}\right) + f_{i,j}^{n+1/2}. \tag{5}$$

We use the following linearization technique for nonlinear term $(uu_x)^{n+1}$ and $(uu_y)^{n+1}$ [27]

$$\begin{aligned}(uu_x)^{n+1} &\approx u^{n+1}u_x^n + u^n u_x^{n+1} - u^n u_x^n,\\ (uu_y)^{n+1} &\approx u^{n+1}u_y^n + u^n u_y^{n+1} - u^n u_y^n.\end{aligned} \tag{6}$$

Substituting the above approximation into Equation (5), we yield

$$\left(\frac{\partial^\alpha u}{\partial t^\alpha}\right)_{i,j}^{n+1/2} + \frac{u_{i,j}^{n+1}(u_x)_{i,j}^{n} + u_{i,j}^{n}(u_x)_{i,j}^{n+1}}{2} + \frac{u_{i,j}^{n+1}(u_y)_{i,j}^{n} + u_{i,j}^{n}(u_y)_{i,j}^{n+1}}{2} = \nu\left(\frac{(u_{xx})_{i,j}^{n+1} + (u_{xx})_{i,j}^{n}}{2} + \frac{(u_{yy})_{i,j}^{n+1} + (u_{yy})_{i,j}^{n}}{2}\right) + f_{i,j}^{n+1/2}. \tag{7}$$

A discrete approximation to the ${}_0^{CF}D_t^\alpha u(x, y, t)$ at $(x_i, y_j, t_{n+1/2})$ can be obtained by the following approximation

$$\begin{aligned}
\frac{\partial^\alpha u\left(x_i, y_j, t_{n+1/2}\right)}{\partial t^\alpha} &= \frac{1}{1-\alpha}\int_0^{t_{n+\frac{1}{2}}} \frac{\partial u\left(x_i, y_j, s\right)}{\partial s} e^{-\frac{\alpha}{1-\alpha}\left(t_{n+\frac{1}{2}}-s\right)} ds \\
&= \frac{1}{1-\alpha}\left[\int_0^{t_n} \frac{\partial u\left(x_i, y_j, s\right)}{\partial s} e^{-\frac{\alpha}{1-\alpha}\left(t_{n+\frac{1}{2}}-s\right)} ds + \int_{t_n}^{t_{n+\frac{1}{2}}} \frac{\partial u\left(x_i, y_j, s\right)}{\partial s} e^{-\frac{\alpha}{1-\alpha}\left(t_{n+\frac{1}{2}}-s\right)} ds\right] \\
&= \frac{1}{1-\alpha}\left[\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left(\frac{u_{i,j}^k - u_{i,j}^{k-1}}{\Delta t} + \left(s - t_{k-\frac{1}{2}}\right) u_{tt}\left(x_i, y_j, c_k\right)\right) e^{-\frac{\alpha}{1-\alpha}\left(t_{n+\frac{1}{2}}-s\right)} ds \right. \\
&\quad \left. + \int_{t_n}^{t_{n+\frac{1}{2}}}\left(\frac{u_{i,j}^{n+1/2} - u_{i,j}^n}{\Delta t/2} + O(\Delta t)\right) e^{-\frac{\alpha}{1-\alpha}\left(t_{n+\frac{1}{2}}-s\right)} ds\right] \\
&= \frac{1}{1-\alpha}\left[\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left(\frac{u_{i,j}^k - u_{i,j}^{k-1}}{\Delta t} + \left(s - t_{k-\frac{1}{2}}\right) u_{tt}\left(x_i, y_j, c_k\right)\right) e^{-\frac{\alpha}{1-\alpha}\left(t_{n+\frac{1}{2}}-s\right)} ds \right. \\
&\quad \left. + \int_{t_n}^{t_{n+\frac{1}{2}}}\left(\frac{u_{i,j}^{n+1} + u_{i,j}^n/2 - u_{i,j}^n}{\Delta t/2} + O(\Delta t)\right) e^{-\frac{\alpha}{1-\alpha}\left(t_{n+\frac{1}{2}}-s\right)} ds\right],
\end{aligned} \tag{8}$$

where $c_k \in (t_{k-1}, t_k)$. Then,

$$\begin{aligned}
\frac{\partial^\alpha u\left(x_i, y_j, t_{n+1/2}\right)}{\partial t^\alpha} &= \frac{1}{1-\alpha}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} \frac{u_{i,j}^k - u_{i,j}^{k-1}}{\Delta t} e^{-\frac{\alpha}{1-\alpha}\left(t_{n+\frac{1}{2}}-s\right)} ds \\
&\quad + \frac{1}{1-\alpha}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left(s - t_{k-\frac{1}{2}}\right) u_{tt}\left(x_i, y_j, c_k\right) e^{-\frac{\alpha}{1-\alpha}\left(t_{n+\frac{1}{2}}-s\right)} ds \\
&\quad + \frac{1}{1-\alpha}\int_{t_n}^{t_{n+\frac{1}{2}}}\left(\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} + O(\Delta t)\right) e^{-\frac{\alpha}{1-\alpha}\left(t_{n+\frac{1}{2}}-s\right)} ds \\
&= \frac{1}{\alpha\Delta t}\left\{\left(u_{i,j}^{n+1} - u_{i,j}^n\right)\left(1 - e^{-\frac{\alpha}{1-\alpha}\left(\frac{1}{2}\right)\Delta t}\right) + \sum_{k=1}^{n}\left[u_{i,j}^k - u_{i,j}^{k-1}\right] \right. \\
&\quad \left. \cdot \left[e^{-\frac{\alpha}{1-\alpha}\left(n-k+\frac{1}{2}\right)\Delta t} - e^{-\frac{\alpha}{1-\alpha}\left(n-k+\frac{3}{2}\right)\Delta t}\right]\right\} + R_1 + R_2,
\end{aligned} \tag{9}$$

which

$$R_1 = \frac{1}{1-\alpha}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left(s - t_{k-\frac{1}{2}}\right) u_{tt}\left(x_i, y_j, c_k\right) e^{-\frac{\alpha}{1-\alpha}\left(t_{n+\frac{1}{2}}-s\right)} ds, \tag{10}$$

and

$$R_2 = \frac{1}{\alpha}\left(1 - e^{-\frac{\alpha}{1-\alpha}\left(\frac{1}{2}\right)\Delta t}\right) O(\Delta t). \tag{11}$$

Therefore, we have

$$\begin{aligned}
R_1 &= \frac{1}{1-\alpha}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left(s - t_{k-\frac{1}{2}}\right) u_{tt}\left(x_i, y_j, c_k\right) e^{-\frac{\alpha}{1-\alpha}\left(t_{n+\frac{1}{2}}-s\right)} ds \\
&= \frac{1}{1-\alpha}\sum_{k=1}^{n} u_{tt}\left(x_i, y_j, c_k\right)\int_{t_{k-1}}^{t_k}\left(s - t_{k-\frac{1}{2}}\right) e^{-\frac{\alpha}{1-\alpha}\left(t_{n+\frac{1}{2}}-s\right)} ds \\
&\le \frac{\max_{1\le k\le n}\left|u_{tt}\left(x_i, y_j, c_k\right)\right|\Delta t}{2(1-\alpha)}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} e^{-\frac{\alpha}{1-\alpha}\left(t_{n+\frac{1}{2}}-s\right)} ds \\
&= \frac{\max_{1\le k\le n}\left|u_{tt}\left(x_i, y_j, c_k\right)\right|\Delta t}{2\alpha}\sum_{k=1}^{n}\left(e^{-\frac{\alpha}{1-\alpha}\left(n-k+\frac{1}{2}\right)\Delta t} - e^{-\frac{\alpha}{1-\alpha}\left(n-k+\frac{3}{2}\right)\Delta t}\right) \\
&= \frac{\max_{1\le k\le n}\left|u_{tt}\left(x_i, y_j, c_k\right)\right|\Delta t}{2\alpha}\left(e^{-\frac{\alpha}{1-\alpha}\left(\frac{1}{2}\right)\Delta t} - e^{-\frac{\alpha}{1-\alpha}\left(n+\frac{1}{2}\right)\Delta t}\right) \\
&\approx \frac{\max_{1\le k\le n}\left|u_{tt}\left(x_i, y_j, c_k\right)\right|\Delta t}{2\alpha}\left(\frac{\alpha}{1-\alpha}\left(n + \frac{1}{2} - \frac{1}{2}\right)\Delta t\right) \\
&= \frac{\max_{1\le k\le n}\left|u_{tt}\left(x_i, y_j, c_k\right)\right|\Delta t}{2(1-\alpha)} n\Delta t \\
&= \frac{\max_{1\le k\le n}\left|u_{tt}\left(x_i, y_j, c_k\right)\right| T\Delta t}{2(1-\alpha)} = O(\Delta t).
\end{aligned} \tag{12}$$

By setting

$$w_k = e^{-\frac{\alpha}{1-\alpha}\left(k-\frac{1}{2}\right)\Delta t} - e^{-\frac{\alpha}{1-\alpha}\left(k+\frac{1}{2}\right)\Delta t}, \tag{13}$$

finally, we obtain

$$\begin{aligned}
\frac{\partial^\alpha u\left(x_i, y_j, t_{n+1/2}\right)}{\partial t^\alpha} &= \frac{1}{\alpha\Delta t}\left[-w_n u_{i,j}^0 + \sum_{k=1}^{n-1}(w_{n-k+1} - w_{n-k}) u_{i,j}^k \right. \\
&\quad \left. + w_1 u_{i,j}^n + \left(u_{i,j}^{n+1} - u_{i,j}^n\right)\left(1 - e^{-\frac{\alpha}{1-\alpha}\left(\frac{1}{2}\right)\Delta t}\right)\right] \\
&\quad + O(\Delta t).
\end{aligned} \tag{14}$$

Besides, utilizing the Taylor expansion, we have

$$\begin{aligned}
\frac{\partial^2 u\left(x_i, y_j, t_{n+1/2}\right)}{\partial x^2} &= \frac{1}{2}\left[\frac{u_{i+1,j}^{n+1} - 2u_{i,j}^{n+1} + u_{i-1,j}^{n+1}}{(\Delta x)^2} + \frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{(\Delta x)^2}\right] \\
&\quad + O\left(\Delta t^2 + \Delta x^2\right),
\end{aligned} \tag{15}$$

$$\begin{aligned}
\frac{\partial^2 u\left(x_i, y_j, t_{n+1/2}\right)}{\partial y^2} &= \frac{1}{2}\left[\frac{u_{i,j+1}^{n+1} - 2u_{i,j}^{n+1} + u_{i,j-1}^{n+1}}{(\Delta y)^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{(\Delta y)^2}\right] \\
&\quad + O\left(\Delta t^2 + \Delta y^2\right),
\end{aligned} \tag{16}$$

$$u\left(x_i, y_j, t_{n+\frac{1}{2}}\right)\frac{\partial u\left(x_i, y_j, t_{k+1/2}\right)}{\partial x} = \frac{1}{4\Delta x}\left[u_{i,j}^{n}\left(u_{i+1,j}^{n+1} - u_{i-1,j}^{n+1}\right) + u_{i,j}^{n+1}\left(u_{i+1,j}^{n} - u_{i-1,j}^{n}\right)\right] + O\left(\Delta t^2 + \Delta x^2\right), \tag{17}$$

$$u\left(x_i, y_j, t_{n+\frac{1}{2}}\right)\frac{\partial u\left(x_i, y_j, t_{n+1/2}\right)}{\partial y} = \frac{1}{4\Delta y}\left[u_{i,j}^{n}\left(u_{i,j+1}^{n+1} - u_{i,j-1}^{n+1}\right) + u_{i,j}^{n+1}\left(u_{i,j+1}^{n} - u_{i,j-1}^{n}\right)\right] + O\left(\Delta t^2 + \Delta y^2\right). \tag{18}$$

Using the Equations (14)–(18), we derive the following difference scheme which is accurate of the order $O(\Delta t + \Delta x^2 + \Delta y^2)$,

$$\begin{aligned}
&\frac{1}{\alpha\Delta t}\left[-w_n U_{i,j}^{0} + \sum_{k=1}^{n-1}(w_{n-k+1} - w_{n-k})U_{i,j}^{k} + w_1 U_{i,j}^{n} + \left(U_{i,j}^{n+1} - U_{i,j}^{n}\right)D_0\right] \\
&= \frac{\nu}{2}\left[\frac{U_{i+1,j}^{n+1} - 2U_{i,j}^{n+1} + U_{i-1,j}^{n+1}}{(\Delta x)^2} + \frac{U_{i+1,j}^{n} - 2U_{i,j}^{n} + U_{i-1,j}^{n}}{(\Delta x)^2}\right] \\
&+ \frac{\nu}{2}\left[\frac{U_{i,j+1}^{n+1} - 2U_{i,j}^{n+1} + U_{i,j-1}^{n+1}}{(\Delta y)^2} + \frac{U_{i,j+1}^{n} - 2U_{i,j}^{n} + U_{i,j-1}^{n}}{(\Delta y)^2}\right] \\
&- \frac{1}{4\Delta x}\left[U_{i,j}^{n}\left(U_{i+1,j}^{n+1} - U_{i-1,j}^{n+1}\right) + U_{i,j}^{n+1}\left(U_{i+1,j}^{n} - U_{i-1,j}^{n}\right)\right] \\
&- \frac{1}{4\Delta y}\left[U_{i,j}^{n}\left(U_{i,j+1}^{n+1} - U_{i,j-1}^{n+1}\right) + U_{i,j}^{n+1}\left(U_{i,j+1}^{n} - U_{i,j-1}^{n}\right)\right] \\
&+ f\left(x_i, y_j, t_{n+\frac{1}{2}}\right),
\end{aligned} \tag{19}$$

where $D_0 = (1 - e^{-(\alpha/1-\alpha)(1/2)\Delta t})$ and $U_{i,j}^{n}$ represents the approximate solution of Equation (1).

After simplification, we obtain

$$\begin{aligned}
&\left(D_0 + \frac{\alpha\Delta t}{4\Delta x}\left(U_{i+1,j}^{n} - U_{i-1,j}^{n}\right) + \frac{\alpha\Delta t}{4\Delta y}\left(U_{i,j+1}^{n} - U_{i,j-1}^{n}\right) + \frac{\alpha\Delta t\nu}{(\Delta x)^2} + \frac{\alpha\Delta t\nu}{(\Delta y)^2}\right)U_{i,j}^{n+1} \\
&- \left(\frac{\alpha\Delta t\nu}{2(\Delta x)^2} - U_{i,j}^{n}\frac{\alpha\Delta t}{4\Delta x}\right)U_{i+1,j}^{n+1} - \left(\frac{\alpha\Delta t\nu}{2(\Delta x)^2} + U_{i,j}^{n}\frac{\alpha\Delta t}{4\Delta x}\right)U_{i-1,j}^{n+1} \\
&- \left(\frac{\alpha\Delta t\nu}{2(\Delta y)^2} - U_{i,j}^{n}\frac{\alpha\Delta t}{4\Delta y}\right)U_{i,j+1}^{n+1} - \left(\frac{\alpha\Delta t\nu}{2(\Delta y)^2} + U_{i,j}^{n}\frac{\alpha\Delta t}{4\Delta y}\right)U_{i,j-1}^{n+1} \\
&= \left(D_0 - \frac{\alpha\Delta t\nu}{(\Delta x)^2} - \frac{\alpha\Delta t\nu}{(\Delta y)^2} - w_1\right)U_{i,j}^{n} + \frac{\alpha\Delta t\nu}{2(\Delta x)^2}U_{i+1,j}^{n} + \frac{\alpha\Delta t\nu}{2(\Delta x)^2}U_{i-1,j}^{n} \\
&+ \frac{\alpha\Delta t\nu}{2(\Delta y)^2}U_{i,j+1}^{n} + \frac{\alpha\Delta t\nu}{2(\Delta y)^2}U_{i,j-1}^{n} + w_n U_{i,j}^{0} - \sum_{k=1}^{n-1}(w_{n-k+1} - w_{n-k})U_{i,j}^{k} + (\alpha\Delta t)f_{i,j}^{n+\frac{1}{2}}.
\end{aligned} \tag{20}$$

3. Fractional Explicit Decoupled Group Method

Another approximate formula for Equation (1) is obtained by Taylor's expansion and rotating Equation (20), 45° degrees clockwise around the $x - y$ axis. The Crank–Nicolson rotation formula for Equation (1) is as follows

$$\begin{aligned}
&\frac{1}{\alpha\Delta t}\left[-w_n U_{i,j}^{0} + \sum_{k=1}^{n-1}(w_{n-k+1} - w_{n-k})U_{i,j}^{k} + w_1 U_{i,j}^{n} + \left(U_{i,j}^{n+1} - U_{i,j}^{n}\right)D_0\right] \\
&= \frac{\nu}{4}\left[\frac{U_{i-1,j+1}^{n+1} - 2U_{i,j}^{n+1} + U_{i+1,j-1}^{n+1}}{h^2} + \frac{U_{i-1,j+1}^{n} - 2U_{i,j}^{n} + U_{i+1,j-1}^{n}}{h^2}\right] \\
&+ \frac{\nu}{4}\left[\frac{U_{i+1,j+1}^{n+1} - 2U_{i,j}^{n+1} + U_{i-1,j-1}^{n+1}}{h^2} + \frac{U_{i+1,j+1}^{n} - 2U_{i,j}^{n} + U_{i-1,j-1}^{n}}{h^2}\right] \\
&- \frac{1}{4h}\left[\frac{U_{i,j}^{n}}{2}\left(\left(U_{i+1,j+1}^{n+1} + U_{i+1,j-1}^{n+1}\right) - \left(U_{i-1,j-1}^{n+1} + U_{i-1,j+1}^{n+1}\right)\right)\right. \\
&\left.+ \frac{U_{i,j}^{n+1}}{2}\left(\left(U_{i+1,j+1}^{n} + U_{i+1,j-1}^{n}\right) - \left(U_{i-1,j-1}^{n} + U_{i-1,j+1}^{n}\right)\right)\right] \\
&- \frac{1}{4h}\left[\frac{U_{i,j}^{n}}{2}\left(\left(U_{i-1,j+1}^{n+1} + U_{i+1,j+1}^{n+1}\right) - \left(U_{i-1,j-1}^{n+1} + U_{i+1,j-1}^{n+1}\right)\right)\right. \\
&\left.+ \frac{U_{i,j}^{n+1}}{2}\left(\left(U_{i-1,j+1}^{n} + U_{i+1,j+1}^{n}\right) - \left(U_{i-1,j-1}^{n} + U_{i+1,j-1}^{n}\right)\right)\right] \\
&+ f\left(x_i, y_j, t_{n+\frac{1}{2}}\right),
\end{aligned} \tag{21}$$

where $h = \Delta x = \Delta y$. Similarly, the above-rotated difference scheme is accurate of order $O(\Delta t + \Delta x^2 + \Delta y^2)$. On simplification with $r_x = \alpha\Delta t/4h$ and $r_{xx} = \alpha\Delta t\nu/2h^2$, the following equation is obtained

$$\begin{aligned}
&\left(D_0 + r_x\left(U_{i+1,j+1}^{n} - U_{i-1,j-1}^{n}\right) + 2r_{xx}\right)U_{i,j}^{n+1} - \frac{r_{xx}}{2}U_{i+1,j-1}^{n+1} \\
&- \frac{r_{xx}}{2}U_{i-1,j+1}^{n+1} - \left(\frac{r_{xx}}{2} - r_x U_{i,j}^{n}\right)U_{i+1,j+1}^{n+1} - \left(\frac{r_{xx}}{2} + r_x U_{i,j}^{n}\right)U_{i-1,j-1}^{n+1} \\
&= (D_0 - 2r_{xx} - w_1)U_{i,j}^{n} + \frac{r_{xx}}{2}U_{i+1,j-1}^{n} + \frac{r_{xx}}{2}U_{i-1,j+1}^{n} + \frac{r_{xx}}{2}U_{i+1,j+1}^{n} \\
&+ \frac{r_{xx}}{2}U_{i-1,j-1}^{n} + w_n U_{i,j}^{0} - \sum_{k=1}^{n-1}(w_{n-k+1} - w_{n-k})U_{i,j}^{k} + (\alpha\Delta t)f_{i,j}^{n+1/2}.
\end{aligned} \tag{22}$$

Utilizing Equation (22) to any group of four points on the solution domain gives a (4×4) system as follows

$$\begin{bmatrix} a_{i,j} & -c_{i,j} & 0 & 0 \\ -d_{i+1,j+1} & a_{i+1,j+1} & 0 & 0 \\ 0 & 0 & a_{i+1,j} & -b \\ 0 & 0 & -b & a_{i,j+1} \end{bmatrix}\begin{bmatrix} U_{i,j}^{n+1} \\ U_{i+1,j+1}^{n+1} \\ U_{i+1,j}^{n+1} \\ U_{i,j+1}^{n+1} \end{bmatrix} = \begin{bmatrix} rhs_{i,j} \\ rhs_{i+1,j+1} \\ rhs_{i+1,j} \\ rhs_{i,j+1} \end{bmatrix}, \tag{23}$$

where

$$a_{i,j} = D_0 + r_x\left(U_{i+1,j+1}^{n} - U_{i-1,j-1}^{n}\right) + 2r_{xx}, c_{i,j} = \frac{r_{xx}}{2} - r_x U_{i,j}^{n}, d_{i,j} = \frac{r_{xx}}{2} + r_x U_{i,j}^{n}, b = \frac{r_{xx}}{2}, \tag{24}$$

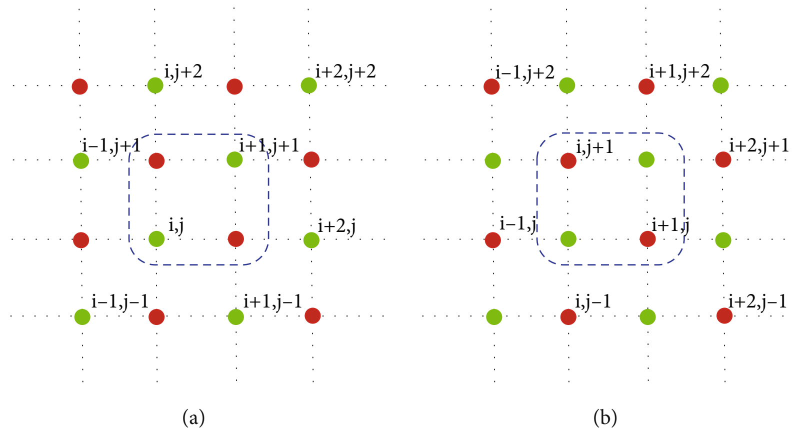


Figure 1: Grid point on $x-y$ plane. Computational molecule of Equation (27) (a) and computational molecule of Equation (28) (b).

with

$$\begin{bmatrix} rhs_{i,j} \\ rhs_{i+1,j+1} \\ rhs_{i+1,j} \\ rhs_{i,j+1} \end{bmatrix} = \begin{bmatrix} \left(b\left(U^{n+1}_{i+1,j-1} + U^{n+1}_{i-1,j+1}\right) + d_{i,j}U^{n+1}_{i-1,j-1}\right) + (D_0 - 2r_{xx} - w_1)U^n_{i,j} + b\left(U^n_{i+1,j-1} + U^n_{i-1,j+1} + U^n_{i+1,j+1} + U^n_{i-1,j-1}\right) + \tau_{i,j} \\ \left(b\left(U^{n+1}_{i+2,j} + U^{n+1}_{i,j+2}\right) + c_{i+1,j+1}U^{n+1}_{i+2,j+2}\right) + (D_0 - 2r_{xx} - w_1)U^n_{i+1,j+1} + b\left(U^n_{i+2,j} + U^n_{i,j+2} + U^n_{i+2,j+2} + U^n_{i,j}\right) + \tau_{i+1,j+1} \\ \left(bU^{n+1}_{i+2,j-1} + c_{i+1,j}U^{n+1}_{i+2,j+1} + d_{i+1,j}U^{n+1}_{i,j-1}\right) + (D_0 - 2r_{xx} - w_1)U^n_{i+1,j} + b\left(U^n_{i+2,j-1} + U^n_{i,j+1} + U^n_{i+2,j+1} + U^n_{i,j-1}\right) + \tau_{i+1,j} \\ \left(bU^{n+1}_{i-1,j+2} + c_{i,j+1}U^{n+1}_{i+1,j+2} + d_{i,j+1}U^{n+1}_{i-1,j}\right) + (D_0 - 2r_{xx} - w_1)U^n_{i+1,j} + b\left(U^n_{i+1,j} + U^n_{i-1,j+2} + U^n_{i+1,j+2} + U^n_{i-1,j}\right) + \tau_{i,j+1} \end{bmatrix}, \quad (25)$$

and

$$\begin{bmatrix} \tau_{i,j} \\ \tau_{i+1,j+1} \\ \tau_{i+1,j} \\ \tau_{i,j+1} \end{bmatrix} = \begin{bmatrix} w_n U^0_{i,j} - \sum_{k=1}^{n-1}(w_{n-k+1} - w_{n-k})U^k_{i,j} + (\alpha\Delta t)f^{n+\frac{1}{2}}_{i,j} \\ w_n U^0_{i+1,j+1} - \sum_{k=1}^{n-1}(w_{n-k+1} - w_{n-k})U^k_{i+1,j+1} + (\alpha\Delta t)f^{n+\frac{1}{2}}_{i+1,j+1} \\ w_n U^0_{i+1,j} - \sum_{k=1}^{n-1}(w_{n-k+1} - w_{n-k})U^k_{i+1,j} + (\alpha\Delta t)f^{n+\frac{1}{2}}_{i+1,j} \\ w_n U^0_{i,j+1} - \sum_{k=1}^{n-1}(w_{n-k+1} - w_{n-k})U^k_{i,j+1} + (\alpha\Delta t)f^{n+\frac{1}{2}}_{i,j+1} \end{bmatrix}. \quad (26)$$

Equation (23) leads to a decoupled system of 2×2 equations in explicit form

$$\begin{bmatrix} a_{i,j} & -c_{i,j} \\ -d_{i+1,j+1} & a_{i+1,j+1} \end{bmatrix} \begin{bmatrix} U^{n+1}_{i,j} \\ U^{n+1}_{i+1,j+1} \end{bmatrix} = \begin{bmatrix} rhs_{i,j} \\ rhs_{i+1,j+1} \end{bmatrix}, \quad (27)$$

and

$$\begin{bmatrix} a_{i+1,j} & -b \\ -b & a_{i,j+1} \end{bmatrix} \begin{bmatrix} U^{n+1}_{i+1,j} \\ U^{n+1}_{i,j+1} \end{bmatrix} = \begin{bmatrix} rhs_{i+1,j} \\ rhs_{i,j+1} \end{bmatrix}. \quad (28)$$

The computational molecule of Equations (27) and (28) is shown in Figure 1.

From Figure 1(a), it can be seen that Equation (27) is executed only by considering the green dots. On the contrary, Equation (28) only runs with red dots. Therefore, the implementation of these two equations is independent of each other, which makes the solution of Equation (1) consume less time.

In the EDG method, the grid points are divided into several groups. Each group consists of only two points of the grid (shown in Figure 2). We apply one of the Equation (27) or Equation (28) for each group in Figure 2. Therefore, half of the grid points (green dots) are calculated by the rotated finite-difference Equation (22). Before going to the next time level, we obtain other points of the grid (red dots) directly once by taking the Equation (20).

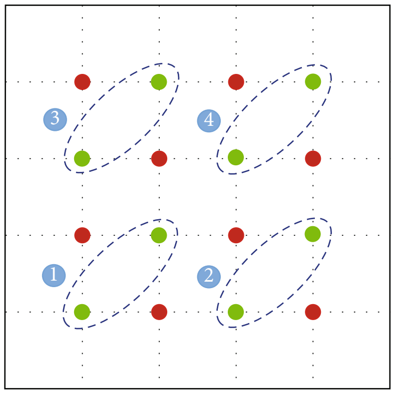


Figure 2: Group ordering for EDG method for $N = 5$.

4. Stability Analysis

In this section, the stability of the finite difference method is investigated with the Von-Neumann analysis. We first give a lemma about w_j, which will be used in the stability analysis.

Lemma 1 (see [28]). *The coefficients w_j in Equation (13) satisfy the following properties.*

$$0 \le w_j \le C\Delta t, \tag{29}$$

and

$$0 \le w_j - w_{j+1} \le C\Delta t w_j. \tag{30}$$

To investigate the stability of the difference scheme, the nonlinear term $u(u_x + u_y)$ in Equations (1)–(3) has been linearized by making the quantity u to a local constant. Thus, the nonlinear term in Equation (1) converts into $\hat{u}(u_x + u_y)$, and Equation (1) becomes:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \hat{u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + f(x, y, t). \tag{31}$$

Let $\tilde{U}^n_{i,j}$ and $\bar{U}^n_{i,j}$ be the approximate solutions of Equations (20) and (22), respectively, and define

$$\rho^n_{i,j} = U^n_{i,j} - \tilde{U}^n_{i,j}, \tag{32}$$

$$\phi^n_{i,j} = U^n_{i,j} - \bar{U}^n_{i,j},\ 1 \le i \le I,\ 1 \le j \le J,\ 1 \le n \le N. \tag{33}$$

Then, by substituting Equation (32) into Equation (20), we have

$$\begin{aligned}
&\left(\frac{\nu\alpha\Delta t}{(\Delta x)^2} + \frac{\nu\alpha\Delta t}{(\Delta y)^2} + D_0\right)\rho^{n+1}_{i,j} + \left(\frac{\alpha\Delta t\hat{u}}{4\Delta x} - \frac{\nu\alpha\Delta t}{2(\Delta x)^2}\right)\rho^{n+1}_{i+1,j} \\
&+ \left(-\frac{\alpha\Delta t\hat{u}}{4\Delta x} - \frac{\nu\alpha\Delta t}{2(\Delta x)^2}\right)\rho^{n+1}_{i-1,j} + \left(\frac{\alpha\Delta t\hat{u}}{4\Delta y} - \frac{\nu\alpha\Delta t}{2(\Delta y)^2}\right)\rho^{n+1}_{i,j+1} \\
&+ \left(-\frac{\alpha\Delta t\hat{u}}{4\Delta y} - \frac{\nu\alpha\Delta t}{2(\Delta y)^2}\right)\rho^{n+1}_{i,j-1} = \left(-\frac{\nu\alpha\Delta t}{(\Delta x)^2} - \frac{\nu\alpha\Delta t}{(\Delta y)^2} - w_1 + D_0\right)\rho^n_{i,j} \\
&+ \left(-\frac{\alpha\Delta t\hat{u}}{4\Delta x} + \frac{\nu\alpha\Delta t}{2(\Delta x)^2}\right)\rho^n_{i+1,j} + \left(\frac{\alpha\Delta t\hat{u}}{4\Delta x} + \frac{\nu\alpha\Delta t}{2(\Delta x)^2}\right)\rho^n_{i-1,j} \\
&+ \left(-\frac{\alpha\Delta t\hat{u}}{4\Delta y} + \frac{\nu\alpha\Delta t}{2(\Delta y)^2}\right)\rho^n_{i,j+1} + \left(\frac{\alpha\Delta t\hat{u}}{4\Delta y} + \frac{\nu\alpha\Delta t}{2(\Delta y)^2}\right)\rho^n_{i,j-1} + w_n\rho^0_{i,j} \\
&- \sum_{k=1}^{n-1}(w_{n-k+1} - w_{n-k})\rho^k_{i,j}.
\end{aligned} \tag{34}$$

Also by putting Equation (33) in Equation (22), we get

$$\begin{aligned}
&(D_0 + 2r_{xx})\phi^{n+1}_{i,j} - \frac{r_{xx}}{2}\left(\phi^{n+1}_{i+1,j-1} - \phi^{n+1}_{i-1,j+1}\right) - \left(\frac{r_{xx}}{2} - \hat{u}r_x\right)\phi^{n+1}_{i+1,j+1} \\
&- \left(\frac{r_{xx}}{2} + \hat{u}r_x\right)\phi^{n+1}_{i-1,j-1} = (D_0 - w_1 + 2r_{xx})\phi^n_{i,j} \\
&+ \frac{r_{xx}}{2}\left(\phi^n_{i+1,j-1} + \phi^n_{i-1,j+1}\right) + \left(\frac{r_{xx}}{2} - \hat{u}r_x\right)\phi^n_{i+1,j+1} \\
&+ \left(\frac{r_{xx}}{2} + \hat{u}r_x\right)\phi^n_{i-1,j-1} + w_n\phi^0_{i,j} - \sum_{k=1}^{n-1}(w_{n-k+1} - w_{n-k})\phi^k_{i,j}.
\end{aligned} \tag{35}$$

The Fourier series for $\rho^n(x, y)$ and $\phi^n(x, y)$ is

$$\begin{aligned}
\rho^n(x, y) &= \sum_{m_2=-\infty}^{\infty}\sum_{m_1=-\infty}^{\infty} \xi_n(m_1, m_2)e^{\imath 2\pi\{m_1x + m_2y\}}, \\
\phi^n(x, y) &= \sum_{m_2=-\infty}^{\infty}\sum_{m_1=-\infty}^{\infty} \eta_n(m_1, m_2)e^{\imath 2\pi\{m_1x + m_2y\}},
\end{aligned} \tag{36}$$

where $\imath = \sqrt{-1}$ and the amplication factors ξ_n and η_n are defined by

$$\xi_n(m_1, m_2) = \int_0^1\int_0^1 e^{-\imath 2\pi\{m_1\tau + m_2\varepsilon\}}\rho^n(\tau, \varepsilon)d\tau d\varepsilon, \tag{37}$$

$$\eta_n(m_1, m_2) = \int_0^1\int_0^1 e^{-\imath 2\pi\{m_1\tau + m_2\varepsilon\}}\phi^n(\tau, \varepsilon)d\tau d\varepsilon. \tag{38}$$

Introducing the following norm

$$\begin{aligned}
\|\rho^n\|_2 &= \left(\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\Delta x\Delta y\left|\rho^n_{i,j}\right|^2\right)^{1/2} = \left(\int_0^1\int_0^1\left|\rho^n_{i,j}\right|^2 d\tau d\varepsilon\right)^{1/2}, \\
\|\phi^n\|_2 &= \left(\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\Delta x\Delta y\left|\phi^n_{i,j}\right|^2\right)^{1/2} = \left(\int_0^1\int_0^1\left|\phi^n_{i,j}\right|^2 d\tau d\varepsilon\right)^{1/2}.
\end{aligned} \tag{39}$$

By applying the Parseval's equality

$$\begin{aligned}\int_0^1\int_0^1 |\rho^n(\tau,\varepsilon)|^2 d\tau d\varepsilon &= \sum_{m_2=-\infty}^{\infty}\sum_{m_1=-\infty}^{\infty} |\xi_n(m_1,m_2)|^2,\\ \int_0^1\int_0^1 |\phi^n(\tau,\varepsilon)|^2 d\tau d\varepsilon &= \sum_{m_2=-\infty}^{\infty}\sum_{m_1=-\infty}^{\infty} |\eta_n(m_1,m_2)|^2,\end{aligned} \tag{40}$$

we obtain

$$\begin{aligned}\|\rho^n\|_2^2 &= \sum_{m_2=-\infty}^{\infty}\sum_{m_1=-\infty}^{\infty} |\xi_n(m_1,m_2)|^2,\\ \|\phi^n\|_2^2 &= \sum_{m_2=-\infty}^{\infty}\sum_{m_1=-\infty}^{\infty} |\eta_n(m_1,m_2)|^2.\end{aligned} \tag{41}$$

According to the above analysis, we can suppose that the solution of Equations (34) and (35) has the following form

$$\begin{aligned}\rho_{i,j}^n &= \xi_n e^{\iota(\sigma_x i\Delta x+\sigma_y j\Delta y)},\\ \phi_{i,j}^n &= \eta_n e^{\iota(\sigma_x i\Delta x+\sigma_y j\Delta y)},\end{aligned} \tag{42}$$

where $\sigma_x = 2m_1\pi, \sigma_y = 2m_2\pi$.

Lemma 2. *If $w_1 \le D_0$ in Equation (20) and $\alpha \in (0,1)$, then we have*

$$|\xi_n| \le |\xi_0|, n = 1, 2, \cdots, N, \tag{43}$$

where ξ_n is defined in Equation (37).

Proof. Substituting $\rho_{i,j}^n = \xi_n e^{\iota(\sigma_x i\Delta x+\sigma_y j\Delta y)}$ into Equation (34), we have

$$\begin{aligned}&\left(\frac{\nu\alpha\Delta t}{(\Delta x)^2}+\frac{\nu\alpha\Delta t}{(\Delta y)^2}+D_0\right)\xi_{n+1}e^{\iota(\sigma_x i\Delta x+\sigma_y j\Delta y)}\\ &\quad+\left(\frac{\alpha\Delta t\widehat{u}}{4\Delta x}-\frac{\nu\alpha\Delta t}{2\Delta x^2}\right)\xi_{n+1}e^{\iota(\sigma_x (i+1)\Delta x+\sigma_y j\Delta y)}\\ &\quad+\left(-\frac{\alpha\Delta t\widehat{u}}{4\Delta x}-\frac{\nu\alpha\Delta t}{2\Delta x^2}\right)\xi_{n+1}e^{\iota(\sigma_x (i-1)\Delta x+\sigma_y j\Delta y)}\\ &\quad+\left(\frac{\alpha\Delta t\widehat{u}}{4\Delta y}-\frac{\nu\alpha\Delta t}{2\Delta y^2}\right)\xi_{n+1}e^{\iota(\sigma_x i\Delta x+\sigma_y (j+1)\Delta y)}\\ &\quad+\left(-\frac{\alpha\Delta t\widehat{u}}{4\Delta y}-\frac{\nu\alpha\Delta t}{2\Delta y^2}\right)\xi_{n+1}e^{\iota(\sigma_x i\Delta x+\sigma_y (j-1)\Delta y)}\\ &=\left(-\frac{\nu\alpha\Delta t}{(\Delta x)^2}-\frac{\nu\alpha\Delta t}{(\Delta y)^2}-w_1+D_0\right)\xi_n e^{\iota(\sigma_x i\Delta x+\sigma_y j\Delta y)}\\ &\quad+\left(-\frac{\alpha\Delta t\widehat{u}}{4\Delta x}+\frac{\nu\alpha\Delta t}{2(\Delta x)^2}\right)\xi_n e^{\iota(\sigma_x (i+1)\Delta x+\sigma_y j\Delta y)}\\ &\quad+\left(\frac{\alpha\Delta t\widehat{u}}{4\Delta x}+\frac{\nu\alpha\Delta t}{2(\Delta x)^2}\right)\xi_n e^{\iota(\sigma_x (i-1)\Delta x+\sigma_y j\Delta y)}\\ &\quad+\left(-\frac{\alpha\Delta t\widehat{u}}{4\Delta y}+\frac{\nu\alpha\Delta t}{2(\Delta y)^2}\right)\xi_n e^{\iota(\sigma_x i\Delta x+\sigma_y (j+1)\Delta y)}\\ &\quad+\left(\frac{\alpha\Delta t\widehat{u}}{4\Delta y}+\frac{\nu\alpha\Delta t}{2(\Delta y)^2}\right)\xi_n e^{\iota(\sigma_x i\Delta x+\sigma_y (j-1)\Delta y)}\\ &\quad+w_n\xi_0 e^{\iota(\sigma_x i\Delta x+\sigma_y j\Delta y)}\\ &\quad-\sum_{k=1}^{n-1}(w_{n-k+1}-w_{n-k})\xi_k e^{\iota(\sigma_x i\Delta x+\sigma_y j\Delta y)},\end{aligned} \tag{44}$$

after simplifications, we can get

$$\begin{aligned}&\xi_{n+1}\left[-\nu\alpha\Delta t\left(-\frac{1}{(\Delta x)^2}-\frac{1}{(\Delta y)^2}+\frac{\cos(\sigma_x\Delta x)}{(\Delta x)^2}+\frac{\cos(\sigma_y\Delta y)}{(\Delta y)^2}\right)\right.\\ &\quad\left.+D_0+\iota\frac{\alpha\Delta t\widehat{u}}{2}\left(\frac{\sin(\sigma_x\Delta x)}{\Delta x}+\frac{\sin(\sigma_y\Delta y)}{\Delta y}\right)\right]\\ &=\xi_n\left[\nu\alpha\Delta t\left(-\frac{1}{(\Delta x)^2}-\frac{1}{(\Delta y)^2}+\frac{\cos(\sigma_x\Delta x)}{(\Delta x)^2}+\frac{\cos(\sigma_y\Delta y)}{(\Delta y)^2}\right)\right.\\ &\quad\left.+D_0-w_1-\iota\frac{\alpha\Delta t\widehat{u}}{2}\left(\frac{\sin(\sigma_x\Delta x)}{\Delta x}+\frac{\sin(\sigma_y\Delta y)}{\Delta y}\right)\right]\\ &\quad+w_n\xi_0-\sum_{k=1}^{n-1}(w_{n-k+1}-w_{n-k})\xi_k.\end{aligned} \tag{45}$$

First, letting $n = 0$ in Equation (45), we obtain

$$|\xi_1| = |\gamma||\xi_0|, \tag{46}$$

where

$$\begin{aligned}\gamma &= \frac{\nu\alpha\Delta tA + D_0 - \iota(\alpha\Delta t\widehat{u}/2)B}{-\nu\alpha\Delta tA + D_0 + \iota(\alpha\Delta t\widehat{u}/2)B},\\ A &= -\frac{1}{(\Delta x)^2}-\frac{1}{(\Delta y)^2}+\frac{\cos(\sigma_x\Delta x)}{(\Delta x)^2}+\frac{\cos(\sigma_y\Delta y)}{(\Delta y)^2},\\ B &= \frac{\sin(\sigma_x\Delta x)}{\Delta x}+\frac{\sin(\sigma_y\Delta y)}{\Delta y}.\end{aligned} \tag{47}$$

In the above expression, it is clear that the real part of the numerator is smaller than the real part of the denominator. Thus, the magnitude of the numerator is smaller than the denominator. So we have

$$|\xi_1| \le |\xi_0|. \tag{48}$$

Now, suppose that we have proved that $|\xi_n| \le |\xi_0|, n = 1, 2, \cdots, m$.

We should prove this for $n = m + 1$. Using Equation (45), we get

$$|\xi_{m+1}| \le \left|\frac{\nu\alpha\Delta tA + D_0 - w_1 - \iota(\alpha\Delta t\hat{u}/2)B}{-\nu\alpha\Delta tA + D_0 + \iota(\alpha\Delta t\hat{u}/2)B}\right||\xi_0| + \left|\frac{w_m - \sum_{k=1}^{m-1}(w_{m-k+1} - w_{m-k})}{-\nu\alpha\Delta tA + D_0 + \iota(\alpha\Delta t\hat{u}/2)B}\right||\xi_0|. \tag{49}$$

Using Lemma 1, we have

$$|\xi_{m+1}| \le \frac{|\nu\alpha\Delta tA + D_0 - w_1 - \iota(\alpha\Delta t\hat{u}/2)B| + w_1}{|-\nu\alpha\Delta tA + D_0 + \iota(\alpha\Delta t\hat{u}/2)B|}|\xi_0|. \tag{50}$$

Consider the following two cases:

Case 3. If $\nu\alpha\Delta tA + D_0 - w_1 > 0$, then, we have

$$|\xi_{m+1}| \le \left(\frac{\nu\alpha\Delta tA + D_0}{-\nu\alpha\Delta tA + D_0}\right)|\xi_0| \le |\xi_0|. \tag{51}$$

Case 4. If $\nu\alpha\Delta tA + D_0 - w_1 \le 0$, then, we have

$$|\xi_{m+1}| \le \left(\frac{2w_1 - \nu\alpha\Delta tA - D_0}{-\nu\alpha\Delta tA + D_0}\right)|\xi_0|. \tag{52}$$

Therefore,

$$\frac{2w_1 - \nu\alpha\Delta tA - D_0}{-\nu\alpha\Delta tA + D_0} \le 1, \tag{53}$$

That is, $w_1 \le D_0$, or $|\xi_{m+1}| \le |\xi_0|$.

By mathematical induction, we finish the proof.

Theorem 5. *For $\alpha \in (0, 1)$, the finite difference scheme Equation (20) is stable if $w_1 \le D_0$.*

Proof. Let $w_1 \le D_0$, using Lemma 2 and Parseval's equality, we get

$$\begin{aligned}\|\rho^n\|_2 &= \sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\Delta y\Delta x\left|\rho_{i,j}^n\right|^2 = \Delta y\Delta x\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\left|\xi_n e^{\iota\left(\sigma_x i\Delta x + \sigma_y j\Delta y\right)}\right|^2 \\ &= \Delta y\Delta x\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}|\xi_n|^2 \le \Delta y\Delta x\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}|\xi_0|^2 \\ &= \Delta y\Delta x\sum_{j=1}^{J-1}\sum_{i=1}^{I-1}\left|\xi_0 e^{\iota\left(\sigma_x i\Delta x + \sigma_y j\Delta y\right)}\right|^2 = \left\|\rho^0\right\|_2.\end{aligned} \tag{54}$$

So the difference scheme Equation (20) is conditionally stable.

Lemma 6. *If $w_1 \le D_0$ in Equation (22) and $\alpha \in (0, 1)$, then, we have*

$$|\eta_n| \le |\eta_0|, n = 1, 2, \cdots, N, \tag{55}$$

where η_n is defined in Equation (38).

Proof. Substituting $\phi_{i,j}^n = \eta_n e^{\iota(\sigma_x i\Delta x + \sigma_y j\Delta y)}$ into Equation (35), we have

$$\begin{aligned}&(D_0 + 2r_{xx})\eta_{n+1}e^{\iota\left(\sigma_x i\Delta x + \sigma_y j\Delta y\right)} - \frac{r_{xx}}{2}\eta_{n+1}e^{\iota\left(\sigma_x(i+1)\Delta x + \sigma_y(j-1)\Delta y\right)} \\ &\quad - \frac{r_{xx}}{2}\eta_{n+1}e^{\iota\left(\sigma_x(i-1)\Delta x + \sigma_y(j+1)\Delta y\right)} \\ &\quad - \left(\frac{r_{xx}}{2} - \hat{u}r_x\right)\eta_{n+1}e^{\iota\left(\sigma_x(i+1)\Delta x + \sigma_y(j+1)\Delta y\right)} \\ &\quad - \left(\frac{r_{xx}}{2} + \hat{u}r_x\right)\eta_{n+1}e^{\iota\left(\sigma_x(i-1)\Delta x + \sigma_y(j-1)\Delta y\right)} \\ &= (D_0 - w_1 - 2r_{xx})\eta_n e^{\iota\left(\sigma_x i\Delta x + \sigma_y j\Delta y\right)} + \frac{r_{xx}}{2}\eta_n e^{\iota\left(\sigma_x(i+1)\Delta x + \sigma_y(j-1)\Delta y\right)} \\ &\quad + \frac{r_{xx}}{2}\eta_n e^{\iota\left(\sigma_x(i-1)\Delta x + \sigma_y(j+1)\Delta y\right)} + \left(\frac{r_{xx}}{2} - \hat{u}r_x\right)\eta_n e^{\iota\left(\sigma_x(i+1)\Delta x + \sigma_y(j+1)\Delta y\right)} \\ &\quad + \left(\frac{r_{xx}}{2} + \hat{u}r_x\right)\eta_n e^{\iota\left(\sigma_x(i-1)\Delta x + \sigma_y(j-1)\Delta y\right)} + w_n\eta_0 e^{\iota\left(\sigma_x i\Delta x + \sigma_y j\Delta y\right)} \\ &\quad - \sum_{k=1}^{n-1}(w_{n-k+1} - w_{n-k})\eta_k e^{\iota\left(\sigma_x i\Delta x + \sigma_y j\Delta y\right)},\end{aligned} \tag{56}$$

by simple computation and noticing that $e^{\iota\beta} + e^{-\iota\beta} = 2\cos(\beta)$ and $e^{\iota\beta} - e^{-\iota\beta} = 2\iota sin(\beta)$, we can get

$$\begin{aligned}&\eta_{n+1}(D_0 + r_{xx}(2 - A) + 2\iota\hat{u}r_xB) = \eta_n(D_0 - w_1 - r_{xx}(2 - A) - 2\iota\hat{u}r_xB) \\ &\quad + w_n\eta_0 + \sum_{k=1}^{n-1}(w_{n-k} - w_{n-k+1})\eta_k,\end{aligned} \tag{57}$$

where

$$\begin{aligned}A &= \cos\left(\sigma_x\Delta x - \sigma_y\Delta y\right) + \cos\left(\sigma_x\Delta x + \sigma_y\Delta y\right), \\ B &= \sin\left(\sigma_x\Delta x + \sigma_y\Delta y\right).\end{aligned} \tag{58}$$

First, setting $n = 0$ in Equation (57), we obtain

$$\begin{aligned}|\eta_1| &= \left|\frac{D_0 - r_{xx}(2 - A) - 2\iota\hat{u}r_xB}{D_0 + r_{xx}(2 - A) + 2\iota\hat{u}r_xB}\right||\eta_0| \\ &= \sqrt{\frac{(D_0 - r_{xx}(2 - A))^2 + (2u\wedge r_xB)^2}{(D_0 + r_{xx}(2 - A))^2 + (2u\wedge r_xB)^2}}|\eta_0| \le |\eta_0|.\end{aligned} \tag{59}$$

Now, assume that we have proved that $|\eta_n| \le |\eta_0|$, $n = 1, 2, \cdots, m$.

We should prove this for $n = m + 1$. Utilizing Equation (57), we obtain

$$\begin{aligned} |\eta_{m+1}| &\le \left|\frac{D_0 - w_1 - r_{xx}(2-A) - 2\imath\widehat{u}r_x B}{D_0 + r_{xx}(2-A) + 2\imath\widehat{u}r_x B}\right| |\eta_0| \\ &\quad + \left|\frac{w_m + \sum_{k=1}^{m-1}(w_{m-k} - w_{m-k+1})}{D_0 + r_{xx}(2-A) + 2\imath\widehat{u}r_x B}\right| |\eta_0|. \end{aligned} \tag{60}$$

According to Lemma 1, we have

$$|\eta_{m+1}| \le \frac{|D_0 - w_1 - r_{xx}(2-A) - 2\imath\widehat{u}r_x B| + w_1}{|D_0 + r_{xx}(2-A) + 2\imath\widehat{u}r_x B|} |\eta_0|. \tag{61}$$

Consider the following two cases:

Case 7. If $D_0 - w_1 - r_{xx}(2-A) > 0$, then, we have

$$|\eta_{m+1}| \le \left(\frac{D_0 - r_{xx}(2-A)}{D_0 + r_{xx}(2-A)}\right) |\eta_0| \le |\eta_0|. \tag{62}$$

Case 8. If $D_0 - w_1 - r_{xx}(2-A) \le 0$, then, we have

$$|\eta_{m+1}| \le \left(\frac{2w_1 - D_0 + r_{xx}(2-A)}{D_0 + r_{xx}(2-A)}\right) |\eta_0|. \tag{63}$$

So that,

$$\frac{2w_1 - D_0 + r_{xx}(2-A)}{D_0 + r_{xx}(2-A)} \le 1, \tag{64}$$

This means $w_1 \le D_0$, or $|\eta_{m+1}| \le |\eta_0|$.
By mathematical induction, the proof is complete.

Theorem 9. *For $\alpha \in (0, 1)$, the finite difference scheme Equation (22) is stable if $w_1 \le D_0$.*

Proof. Suppose $w_1 \le D_0$, from Lemma 6 and Parseval's equality, we get

$$\begin{aligned} \|\phi^n\|_2 &= \sum_{j=1}^{J-1}\sum_{i=1}^{I-1} \Delta y \Delta x \left|\phi_{i,j}^n\right|^2 = \Delta y \Delta x \sum_{j=1}^{J-1}\sum_{i=1}^{I-1} \left|\eta_n e^{\imath(\sigma_x i\Delta x + \sigma_y j \Delta y)}\right|^2 \\ &= \Delta y \Delta x \sum_{j=1}^{J-1}\sum_{i=1}^{I-1} |\eta_n|^2 \le \Delta y \Delta x \sum_{j=1}^{J-1}\sum_{i=1}^{I-1} |\eta_0|^2 \\ &= \Delta y \Delta x \sum_{j=1}^{J-1}\sum_{i=1}^{I-1} \left|\eta_0 e^{\imath(\sigma_x i\Delta x + \sigma_y j \Delta y)}\right|^2 = \|\phi^0\|_2. \end{aligned} \tag{65}$$

So the difference scheme Equation (22) is conditionally stable.

5. Convergence Analysis

We first introduce some notations and lemmas which will be used in the convergence analysis.

$$\begin{aligned} &\delta_x^2 v_{i,j}^n = \frac{v_{i+1,j}^n - 2v_{i,j}^n + v_{i-1,j}^n}{(\Delta x)^2}, \delta_y^2 v_{i,j}^n = \frac{v_{i,j+1}^n - 2v_{i,j}^n + v_{i,j-1}^n}{(\Delta y)^2}, \\ &\Delta_x^0 v_{i,j}^n = \frac{v_{i+1,j}^n - v_{i-1,j}^n}{2\Delta x}, \Delta_y^0 v_{i,j}^n = \frac{v_{i,j+1}^n - v_{i,j-1}^n}{2\Delta y}, \\ &(\mathbf{v}, \mathbf{w}) = \Delta x \Delta y \sum_{j=1}^{J-1}\sum_{i=1}^{I-1} v_{ij} w_{ij}, \|\mathbf{v}\| = \left[\Delta x \Delta y \sum_{j=1}^{J-1}\sum_{i=1}^{I-1} \left(v_{ij}\right)^2\right]^{1/2}, \\ &c_0 = \max_{(x,y,t)\in[0,L]\times[0,L]\times(0,T)} \left\{|u(x,y,t)|, \left|\frac{\partial u}{\partial x}(x,y,t)\right|, \left|\frac{\partial u}{\partial y}(x,y,t)\right|\right\}. \end{aligned} \tag{66}$$

It is straightforward to show

$$\left|\Delta_x^0 v, v\right| = 0, \left|\Delta_y^0 v, v\right| = 0. \tag{67}$$

Notice that in this section we suppose C stands for a positive constant independent of $\Delta t, \Delta x, \Delta y, i, j,$ and n, which may take different values at different places.

Lemma 10. *(Discrete Gronwall's inequality [29]).*
Suppose d be a nonnegetive constant, $\{z_n\}$ and $\{f_n\}$ are nonnegative sequences. Let

$$z_n \le d + \sum_{0\le k<n} f_k z_k, n \ge 0, \tag{68}$$

then

$$z_n \le d \ \exp\left(\sum_{0\le j<n} f_j\right), n \ge 0. \tag{69}$$

Theorem 11. *The Crank–Nicolson scheme Equation (20) is convergent and the order of convergence is $O(\Delta t + \Delta x^2 + \Delta y^2)$.*

Proof. Let $e_{i,j}^n$ be the error at (x_i, y_j, t_n) as defined below

$$\begin{aligned} e_{i,j}^n &= u\left(x_i, y_j, t_n\right) - U_{i,j}^n \\ &= u_{i,j}^n - U_{i,j}^n, 1 \le j \le J, 1 \le i \le I, 1 \le n \le N. \end{aligned} \tag{70}$$

Substituting Equation (70) in Equation (19), we get the following error equations

Table 1: Absolute errors of Example 13 at $T = 1$ for Re = 10, $\Delta t = 0.01$, $\alpha = 0.1$, and $\alpha = 0.3$.

N	$\alpha = 0.1$				$\alpha = 0.3$			
	CN	Time	EDG	Time	CN	Time	EDG	Time
9	$4.2593e-03$	0.75	$6.1387e-03$	0.74	$3.9001e-03$	0.22	$6.0143e-03$	0.11
19	$9.8379e-04$	0.93	$1.6263e-03$	0.69	$9.0356e-04$	1.01	$1.6124e-03$	0.51
49	$1.9672e-04$	9.82	$2.6073e-04$	4.69	$1.8465e-04$	10.61	$2.6040e-04$	4.73
99	$9.1297e-05$	123.15	$9.7575e-05$	39.87	$8.8260e-05$	118.02	$9.3869e-05$	42.01

Table 2: Absolute errors of Example 13 at $T = 1$ for Re = 10, $\Delta t = 0.01$, $\alpha = 0.7$, and $\alpha = 0.9$.

N	$\alpha = 0.7$				$\alpha = 0.9$			
	CN	Time	EDG	Time	CN	Time	EDG	Time
9	$2.7474e-03$	0.16	$5.6163e-03$	0.12	$1.8409e-03$	0.17	$5.2940e-03$	0.12
19	$6.6801e-04$	0.84	$1.5736e-03$	0.60	$4.8985e-04$	0.86	$1.5707e-03$	0.59
49	$1.5163e-04$	10.26	$2.5934e-04$	5.68	$1.4882e-04$	10.38	$2.5869e-04$	5.35
99	$8.2965e-05$	119.25	$8.6473e-05$	41.44	$1.0295e-04$	121.48	$1.0451e-04$	42.37

$$
\begin{aligned}
-w_n e_{ij}^0 &+ \sum_{k=1}^{n-1}(w_{n-k+1}-w_{n-k})e_{ij}^k + w_1 e_{ij}^n + \left(e_{ij}^{n+1}-e_{ij}^n\right)D_0 \\
&= \frac{\nu\alpha\Delta t}{2}\left(\delta_x^2\left(e_{i,j}^{n+1}+e_{i,j}^n\right)+\delta_y^2\left(e_{i,j}^{n+1}+e_{i,j}^n\right)\right) \\
&\quad - \frac{\alpha\Delta t}{2}(\sigma_4+\sigma_5)+\alpha\Delta t R_{i,j}^{n+\frac{1}{2}}, \\
&\quad 1\le j\le J-1, 1\le i\le I-1, 1\le n\le N, \\
e_{0,j}^n &= e_{I,j}^n = 0, 1\le n\le N, 0\le j\le J, \\
e_{i,0}^n &= e_{i,J}^n = 0, 1\le n\le N, 0\le i\le I, \\
e_{i,j}^0 &= 0, 0\le j\le J, 1\le i\le I,
\end{aligned} \tag{71}
$$

where

$$
\begin{aligned}
R_{i,j}^{n+\frac{1}{2}} &= O\left(\Delta t+\Delta x^2+\Delta y^2\right), \\
\sigma_4 &= e_{i,j}^n\Delta_x^0 u_{i,j}^{n+1} + u_{i,j}^n\Delta_x^0 e_{i,j}^{n+1} + e_{i,j}^{n+1}\Delta_x^0 u_{i,j}^n + u_{i,j}^{n+1}\Delta_x^0 e_{i,j}^n, \\
\sigma_5 &= e_{i,j}^n\Delta_y^0 u_{i,j}^{n+1} + u_{i,j}^n\Delta_y^0 e_{i,j}^{n+1} + e_{i,j}^{n+1}\Delta_y^0 u_{i,j}^n + u_{i,j}^{n+1}\Delta_y^0 e_{i,j}^n.
\end{aligned} \tag{72}
$$

Multiplying Equation (71) by $\Delta x\Delta y(e_{i,j}^{n+1}+e_{i,j}^n)$ and summing up for i from 1 to I − 1 and j from 1 to $J-1$, we obtain

$$
\begin{aligned}
&\left|\sum_{k=1}^{n-1}(w_{n-k+1}-w_{n-k})e^k + w_1 e^n + \left(e^{n+1}-e^n\right)D_0, e^{n+1}+e^n\right| \\
&= \frac{\nu\alpha\Delta t}{2}\left(\left|\delta_x^2\left(e^{n+1}+e^n\right), e^{n+1}+e^n\right| + \left|\delta_y^2\left(e^{n+1}+e^n\right), e^{n+1}+e^n\right|\right) \\
&\quad - \frac{\alpha\Delta t}{2}\left|\sigma_4+\sigma_5, e^{n+1}+e^n\right| + \alpha\Delta t\left|R^{n+\frac{1}{2}}, e^{n+1}+e^n\right|, 1\le n\le N.
\end{aligned} \tag{73}
$$

It is clear that $|\delta_x^2(e^{n+1}+e^n), e^{n+1}+e^n|\le 0$ and $|\delta_y^2(e^{n+1}+e^n), e^{n+1}+e^n|\le 0$, so we have

$$
\begin{aligned}
D_0\left\|e^{n+1}\right\|^2 &\le (D_0-w_1)\|e^n\|^2 + w_1\left\|e^{n+1}, e^n\right\| \\
&\quad + \left|\sum_{k=1}^{n-1}(w_{n-k}-w_{n-k+1})e^k, e^{n+1}+e^n\right| \\
&\quad + \frac{\alpha\Delta t}{2}\left\|\sigma_4+\sigma_5, e^{n+1}+e^n\right\| \\
&\quad + \alpha\Delta t\left|R^{n+\frac{1}{2}}, e^{n+1}+e^n\right|, 1\le n\le N.
\end{aligned} \tag{74}
$$

Now, we estimate the third, fourth, and fifth terms of the right-hand side of Equation (74), respectively

$$
\begin{aligned}
&\left|\sum_{k=1}^{n-1}(w_{n-k}-w_{n-k+1})e^k, e^{n+1}+e^n\right| \\
&= \sum_{k=1}^{n-1}(w_{n-k}-w_{n-k+1})\left|e^k, e^{n+1}\right| + \sum_{k=1}^{n-1}(w_{n-k}-w_{n-k+1})\left|e^k, e^n\right|.
\end{aligned} \tag{75}
$$

Using Young inequality $ab\le\varepsilon a^2+(1/4\varepsilon)b^2$, $a, b\in\mathbb{R}$, and Lemma 1, we have

$$
\begin{aligned}
\left|\sum_{k=1}^{n-1}(w_{n-k}-w_{n-k+1})e^k, e^{n+1}+e^n\right| &\le C\Delta t^2\sum_{k=1}^{n-1}\left\|e^k\right\|^2 \\
&\quad + C\Delta t\left(\|e^n\|^2+\left\|e^{n+1}\right\|^2\right).
\end{aligned} \tag{76}
$$

For the fourth term, utilizing Young inequality and Equation (67), we obtain

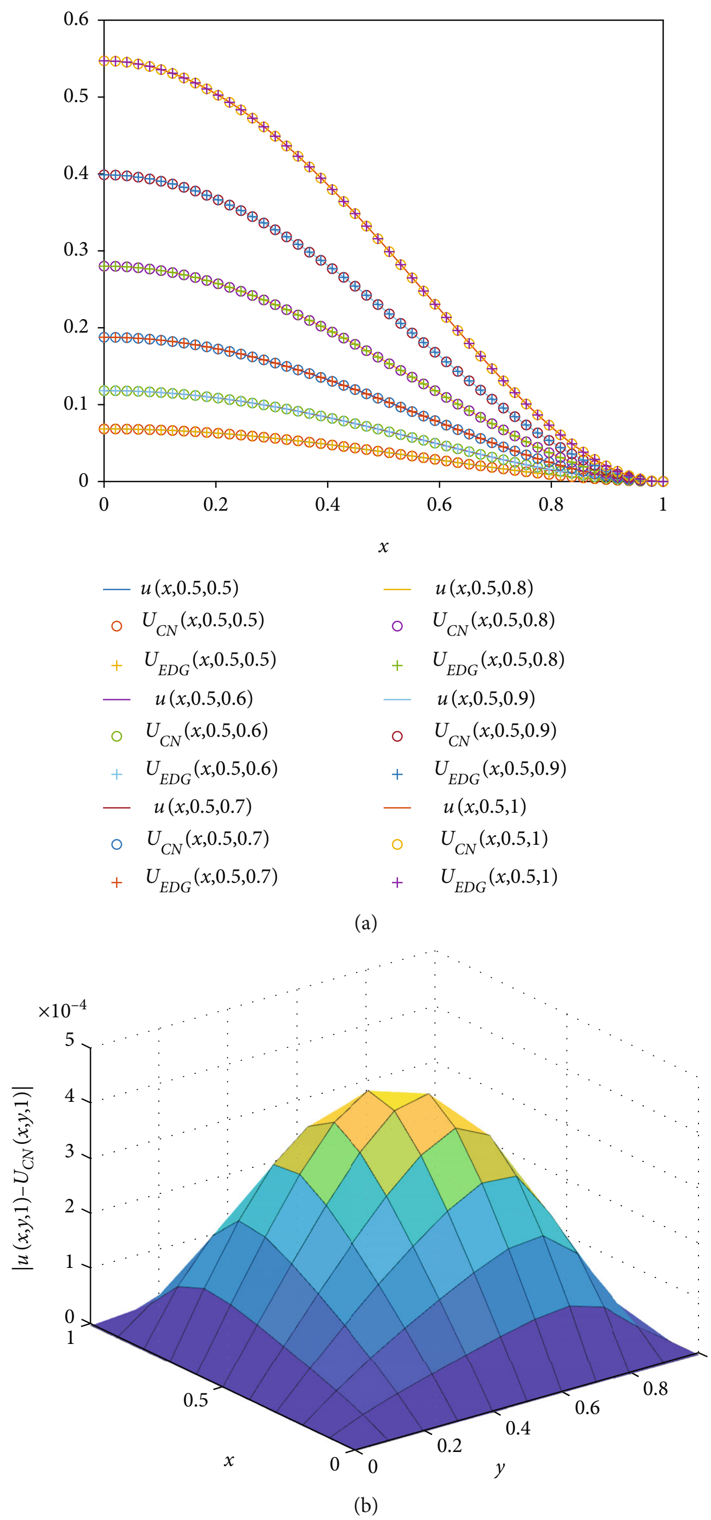
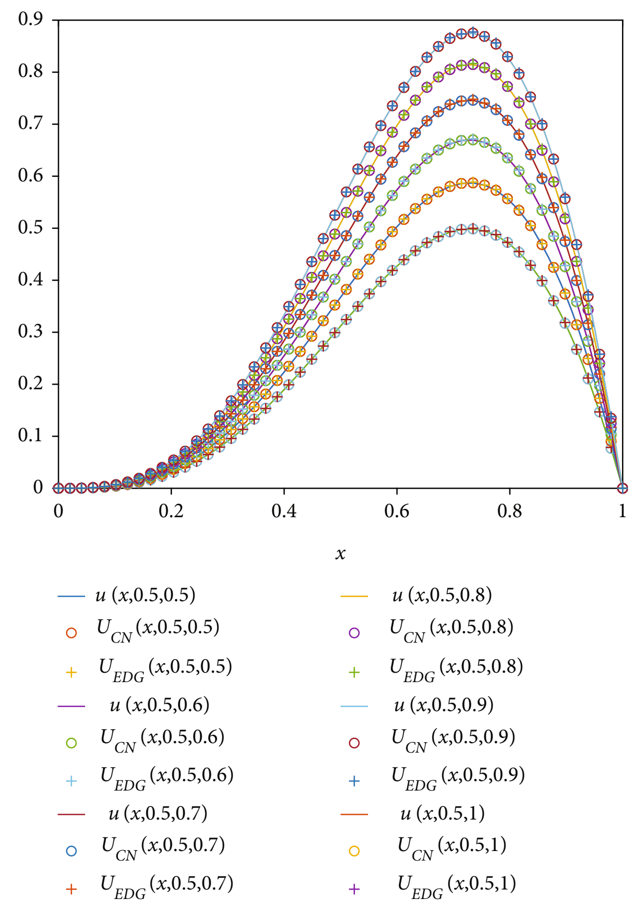
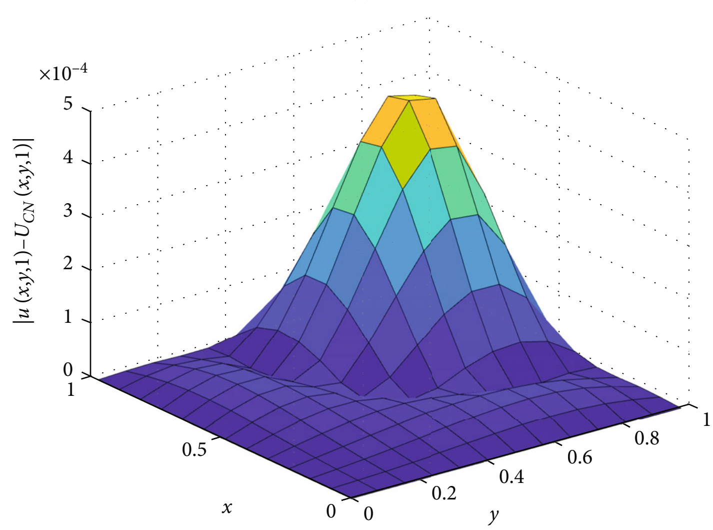


Figure 3: The exact, Crank–Nicolson, and EDG solutions in different values of t and $y = 0.5$ (a) and the absolute error of Crank–Nicolson method at $T = 1$ (b) of Example 13 for $\Delta t = 0.01$, $\mathrm{Re} = 100$, $N = 49$, and $\alpha = 0.5$.



(a)



(b)

FIGURE 4: The exact, Crank–Nicolson and EDG solutions in different values of t and $y = 0.5$ (a) and the absolute error of Crank–Nicolson method at $T = 1$ (b) of Example 14 for $\Delta t = 0.01$, $\mathrm{Re} = 50$, $N = 49$, and $\alpha = 0.5$.

Table 3: Absolute errors of Example 14 at $T = 1$ for Re = 100, $\Delta t = 0.01$, $\alpha = 0.1$, and $\alpha = 0.3$.

N	$\alpha = 0.1$ CN	Time	EDG	Time	$\alpha = 0.3$ CN	Time	EDG	Time
9	$4.1318e-03$	0.40	$1.0556e-02$	0.26	$3.7717e-03$	0.53	$9.8104e-03$	0.28
19	$8.8813e-04$	1.04	$2.6531e-03$	0.73	$8.1156e-04$	1.17	$2.5458e-03$	0.73
49	$1.3364e-04$	12.14	$4.1507e-04$	4.81	$1.2247e-04$	15.24	$4.0109e-04$	4.86
99	$3.4588e-05$	122.31	$1.0389e-04$	40.55	$3.1779e-05$	130.36	$1.0045e-04$	40.71

Table 4: Absolute errors of Example 14 at $T = 1$ for Re = 100, $\Delta t = 0.01$, $\alpha = 0.7$, and $\alpha = 0.9$.

N	$\alpha = 0.7$ CN	Time	EDG	Time	$\alpha = 0.9$ CN	Time	EDG	Time
9	$2.9459e-03$	0.57	$8.2550e-03$	0.34	$2.4730e-03$	0.96	$7.5148e-03$	0.26
19	$6.3611e-04$	1.39	$2.3322e-03$	0.84	$5.4346e-04$	1.56	$2.2536e-03$	0.84
49	$9.7981e-05$	15.19	$3.7198e-04$	5.69	$8.6200e-05$	17.26	$3.5924e-04$	5.32
99	$2.5774e-05$	126.65	$9.3273e-05$	44.77	$2.4146e-05$	136.80	$9.1092e-05$	43.01

Table 5: Absolute errors of Example 15 at $T = 1$ for Re = 5, $\Delta t = 0.01$, $\alpha = 0.1$, and $\alpha = 0.3$.

N	$\alpha = 0.1$ CN	Time	EDG	Time	$\alpha = 0.3$ CN	Time	EDG	Time
9	$2.1509-03$	0.51	$7.1351e-03$	0.24	$2.0937e-03$	0.19	$6.9914e-03$	0.12
19	$4.8726e-04$	1.09	$1.7356e-03$	0.59	$4.7437e-04$	0.91	$1.7078e-03$	0.56
49	$7.3239e-05$	11.88	$2.6457e-04$	4.86	$7.1213e-05$	11.80	$2.6032e-04$	5.04
99	$1.7989e-05$	121.06	$6.4817e-05$	40.01	$1.7399e-05$	120.95	$6.3688e-05$	40.27

Table 6: Absolute errors of Example 15 at $T = 1$ for Re = 5, $\Delta t = 0.01$, $\alpha = 0.7$, and $\alpha = 0.9$.

N	$\alpha = 0.7$ CN	Time	EDG	Time	$\alpha = 0.9$ CN	Time	EDG	Time
9	$1.9317e-03$	0.21	$6.5894e-03$	0.12	$1.8282e-03$	0.23	$6.3412e-03$	0.13
19	$4.3742e-04$	0.97	$1.6292e-03$	0.60	$4.1260e-04$	1.01	$1.5796e-03$	0.61
49	$6.4857e-05$	12.21	$2.4776e-04$	5.29	$5.9238e-05$	12.58	$2.3834e-04$	5.35
99	$1.5095e-05$	123.57	$5.9878e-05$	41.38	$1.2086e-05$	124.23	$5.5847e-05$	41.61

$$\left|\left\langle\sigma_4, e^{n+1}+e^n\right\rangle\right| \le c_0\left(1+\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5\right)\left\|e^{n+1}\right\|^2 + c_0\left(1+\frac{1}{4\varepsilon_1}+\frac{1}{4\varepsilon_2}+\frac{1}{4\varepsilon_3}+\frac{1}{4\varepsilon_4}+\frac{1}{4\varepsilon_5}\right)\left\|e^n\right\|^2. \tag{77}$$

Similarly, we get

$$\left|\left\langle\sigma_5, e^{n+1}+e^n\right\rangle\right| \le c_0\left(1+\varepsilon_6+\varepsilon_7+\varepsilon_8+\varepsilon_9+\varepsilon_{10}\right)\left\|e^{n+1}\right\|^2 + c_0\left(1+\frac{1}{4\varepsilon_6}+\frac{1}{4\varepsilon_7}+\frac{1}{4\varepsilon_8}+\frac{1}{4\varepsilon_9}+\frac{1}{4\varepsilon_{10}}\right)\left\|e^n\right\|^2. \tag{78}$$

Also for the fifth term, we have

$$\left|R^{n+\frac{1}{2}}, e^{n+1}+e^n\right| \le \left(\varepsilon_{11}+\frac{1}{4\varepsilon_{12}}\right)\left\|R^{n+\frac{1}{2}}\right\|^2 + \varepsilon_{12}\left\|e^{n+1}\right\|^2 + \frac{1}{4\varepsilon_{11}}\left\|e^n\right\|^2. \tag{79}$$

For the second term on the right-hand side of Equation (74), we get

$$w_1\left|\left|e^{n+1}, e^n\right|\right| \le w_1\left(\varepsilon_{13}\left\|e^{n+1}\right\|^2 + \frac{1}{4\varepsilon_{13}}\left\|e^n\right\|^2\right). \tag{80}$$

Combining Equations (76)–(80) in Equation (74) and using Lemma 1, we obtain

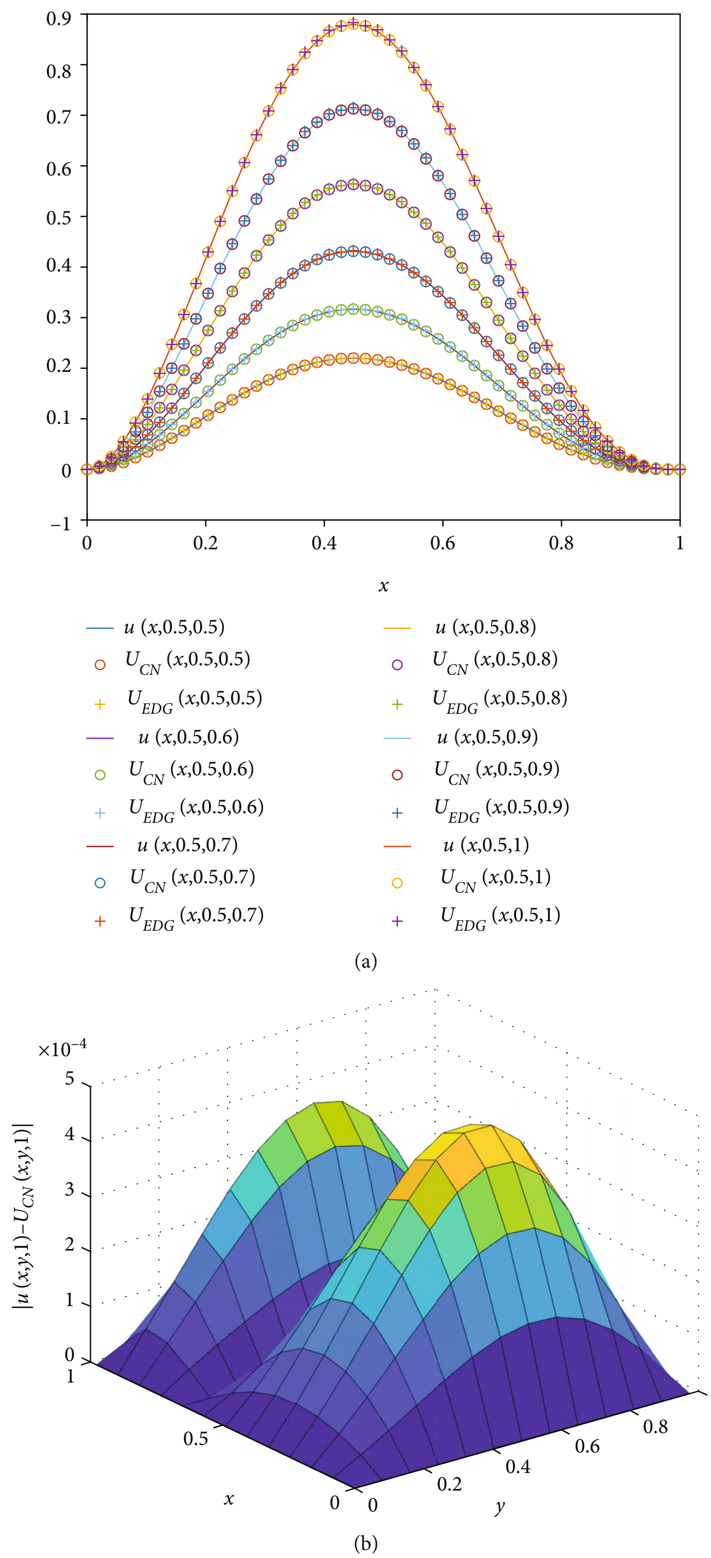


Figure 5: The exact, Crank–Nicolson and EDG solutions in different values of t and $y = 0.5$ (a) and the absolute error of Crank–Nicolson method at $T = 1$ (b) of Example 15 for $\Delta t = 0.01$, $\mathrm{Re} = 20$, $N = 49$, and $\alpha = 0.5$.

Table 7: Absolute errors of Example 16 at $T = 1$ for Re = 2, $\Delta t = 0.01$, $\alpha = 0.2$, and $\alpha = 0.4$.

N	$\alpha = 0.2$				$\alpha = 0.4$			
	CN	Time	EDG	Time	CN	Time	EDG	Time
9	$2.5900e-03$	0.21	$2.1227e-02$	0.19	$2.5752e-03$	0.20	$2.1177e-02$	0.18
19	$5.9920e-04$	0.83	$5.3687e-03$	0.44	$5.9600e-04$	0.84	$5.3597e-03$	0.59
49	$9.3436e-05$	9.41	$8.2492e-04$	5.19	$9.2993e-05$	9.70	$8.2383e-04$	5.12
99	$2.5225e-05$	117.52	$2.0397e-04$	39.94	$2.5145e-05$	119.21	$2.0376e-04$	39.86

Table 8: Absolute errors of Example 16 at $T = 1$ for Re = 2, $\Delta t = 0.01$, $\alpha = 0.6$, and $\alpha = 0.8$.

N	$\alpha = 0.6$				$\alpha = 0.8$			
	CN	Time	EDG	Time	CN	Time	EDG	Time
9	$2.5546e-03$	0.22	$2.1112e-02$	0.20	$2.5253e-03$	0.24	$2.1039e-02$	0.22
19	$5.9150e-04$	0.84	$5.3476e-03$	0.60	$5.8469e-04$	0.88	$5.3318e-03$	0.62
49	$9.2340e-05$	9.88	$8.2234e-04$	5.05	$9.0870e-05$	10.19	$8.2010e-04$	5.51
99	$2.5001e-05$	119.14	$2.0348e-04$	41.28	$2.4238e-05$	123.96	$2.0280e-04$	41.67

$$D_0\|e^{n+1}\|^2 \le C\Delta t^2 \sum_{k=1}^{n-1} \left\|e^k\right\|^2 + \Delta t\left(\lambda_1\left\|e^{n+1}\right\|^2 + \lambda_2\|e^n\|^2\right) + \alpha\Delta t\left(\varepsilon_{11} + \frac{1}{4\varepsilon_{12}}\right)\left\|R^{n+1/2}\right\|^2, 1 \le n \le N, \tag{81}$$

where

$$\lambda_1 = \frac{\alpha c_0}{2}(2 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + \varepsilon_8 + \varepsilon_9 + \varepsilon_{10}) + \alpha\varepsilon_{12} + C\varepsilon_{13} + C,$$

$$\lambda_2 = \frac{\alpha c_0}{2}\left(2 + \frac{1}{4\varepsilon_1} + \frac{1}{4\varepsilon_2} + \frac{1}{4\varepsilon_3} + \frac{1}{4\varepsilon_4} + \frac{1}{4\varepsilon_5} + \frac{1}{4\varepsilon_6} + \frac{1}{4\varepsilon_7} + \frac{1}{4\varepsilon_8} + \frac{1}{4\varepsilon_9} + \frac{1}{4\varepsilon_{10}}\right) + \frac{C}{4\varepsilon_{13}} + \frac{\alpha}{4\varepsilon_{11}} + 2C. \tag{82}$$

Therefore, we get

$$(C - \lambda_1)\left\|e^{n+1}\right\|^2 \le C\Delta t \sum_{k=1}^{n-1} \left\|e^k\right\|^2 + \lambda_2\|e^n\|^2 + \alpha\left(\varepsilon_{11} + \frac{1}{4\varepsilon_{12}}\right)\left\|R^{n+1/2}\right\|^2, 1 \le n \le N. \tag{83}$$

In the definition of λ_1, we choose epsilons which $C - \lambda_1$ be positive, so we have

$$\left\|e^{n+1}\right\|^2 \le C\Delta t \sum_{k=1}^{n-1} \left\|e^k\right\|^2 + C\|e^n\|^2 + \alpha C\left\|R^{n+1/2}\right\|^2, 1 \le n \le N. \tag{84}$$

Using Lemma 10 and $n\Delta t = T$, we get

$$\left\|e^{n+1}\right\|^2 \le C\left\|R^{n+1/2}\right\|^2 \le O\left(\Delta t + \Delta x^2 + \Delta y^2\right), 1 \le n \le N, \tag{85}$$

and this completes the proof.

Theorem 12. *The EDG method Equation (22) is convergent and the order of convergence is* $O(\Delta t + \Delta x^2 + \Delta y^2)$.

Proof. The proof is similar to Theorem 11.

6. Numerical Results

In this section, some numerical examples are considered to demonstrate the efficiency and accuracy of the proposed methods.

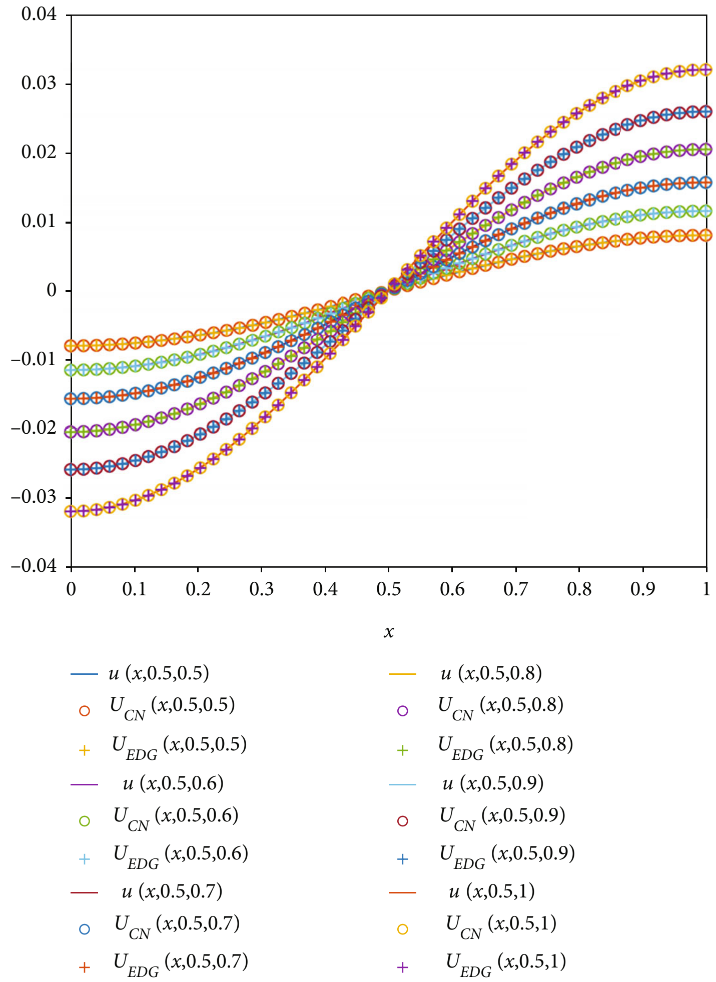
In numerical examples, we suppose that $u(x, t)$, $U_{CN}(x, t)$, and $U_{EDG}(x, t)$ denote the exact, Crank–Nicolson, and the EDG solution, respectively. Also in all Tables, the CN is an abbreviation for Crank–Nicolson Method.

The results obtained in this study show that the suggested methods have excellent stability, and they have verified the validity and effectiveness of the presented methods. Notably, we perform all of the computations by MATLAB R2019a software on a 64-bit PC with 2.30 GHz processor and 8 GB memory.

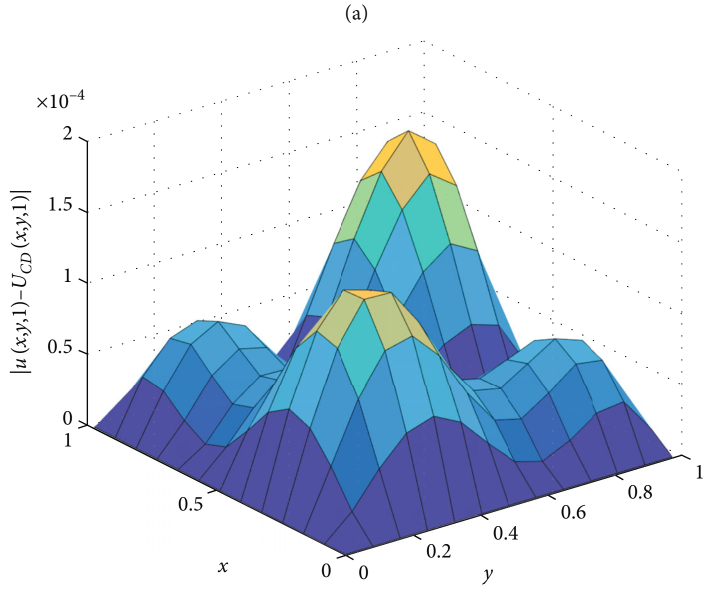
Example 13. Consider Equation (1) with the exact solution

$$u(x, y, t) = t^3\left(1 - x^2\right)^2\left(1 - y^2\right)^2, 0 \le x, y, t \le 1. \tag{86}$$

In Tables 1 and 2, the maximum of absolute errors and CPU Times for Crank–Nicolson and EDG methods for $\alpha = 0.1$, 0.3, and $\alpha = 0.7$, 0.9 with $T = 1$, $\Delta t = 0.01$, Re = 10, and different values of N are tabulated, respectively. These results confirm the convergent results. In Figure 3(a), the exact, Crank–Nicolson, and EDG solutions for $y = 0.5$, Re = 100,



(a)



(b)

Figure 6: The exact, Crank–Nicolson, and EDG solutions in different values of t (a) and the absolute error of Crank–Nicolson method at $T = 1$ (b) of Example 16 for $\Delta t = 0.01$, $\mathrm{Re} = 10$, $N = 49$, and $\alpha = 0.5$.

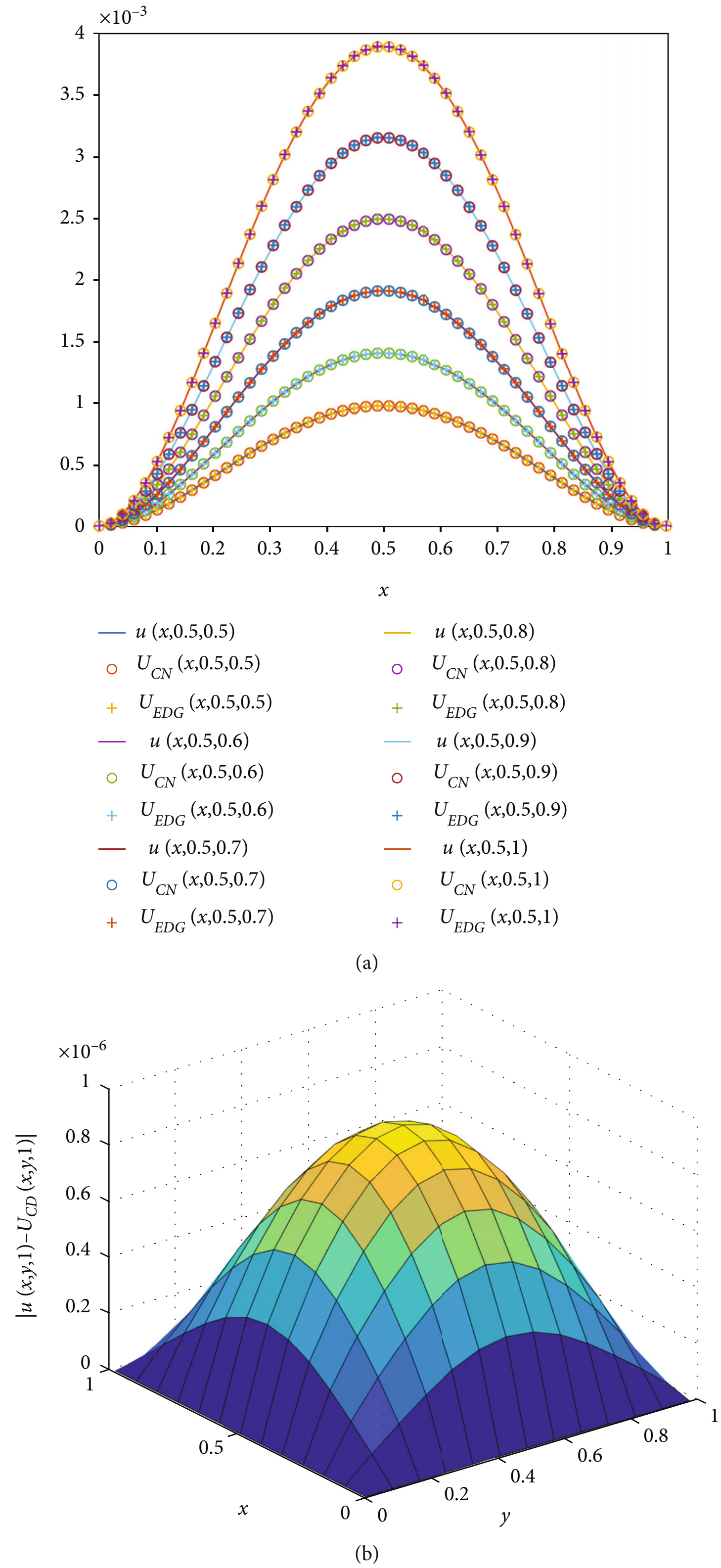


Figure 7: The exact, Crank–Nicolson and EDG solutions in different values of t (a) and the absolute error of Crank–Nicolson method at $T = 1$ (b) of Example 16 for $\Delta t = 0.01$, $\mathrm{Re} = 60$, $N = 49$, and $\alpha = 0.5$.

Table 9: Absolute errors of Example 17 at $T = 1$ for Re = 100, $\Delta t = 0.01$, $\alpha = 0.2$, and $\alpha = 0.4$.

N	$\alpha = 0.2$				$\alpha = 0.4$			
	CN	Time	EDG	Time	CN	Time	EDG	Time
9	2.3544e − 05	0.37	1.2078e − 04	0.15	2.0308e − 05	0.32	1.1028e − 04	0.14
19	5.3089e − 06	0.89	3.7501e − 05	0.55	4.5547e − 06	0.90	3.5462e − 05	0.73
49	7.8572e − 07	9.70	6.1862e − 06	4.84	6.5077e − 07	9.72	5.9056e − 06	5.23
99	1.9520e − 07	117.73	1.5238e − 06	39.65	1.9513e − 07	118.38	1.4377e − 06	40.22

Table 10: Absolute errors of Example 17 at $T = 1$ for Re = 100, $\Delta t = 0.01$, $\alpha = 0.6$, and $\alpha = 0.8$.

N	$\alpha = 0.6$				$\alpha = 0.8$			
	CN	Time	EDG	Time	CN	Time	EDG	Time
9	1.6919e − 05	0.32	9.8582e − 05	0.26	1.3331e − 05	0.37	8.5506e − 05	0.27
19	3.7515e − 06	0.89	3.3183e − 05	0.82	2.8606e − 06	0.94	3.0637e − 05	0.79
49	4.9131e − 07	9.91	5.5845e − 06	5.41	2.7513e − 07	10.06	5.1849e − 06	5.50
99	1.9501e − 07	118.22	1.3268e − 06	41.16	1.9462e − 07	119	1.1529e − 06	41.91

$N = 49$, and $\Delta t = 0.01$ in different values of t are illustrated. Furthermore in Figure 3(b), the absolute error at $T = 1$ for Re = 100, $N = 49$, $\Delta t = 0.01$, and $\alpha = 0.5$ is portrayed. According to the Figures, we can see that our numerical solutions correspond to the exact solutions.

Example 14. In this example, we assume that the exact solution of Equation (1) is

$$u(x, y, t) = x^2 y^2 \sin(\pi x) \sin(\pi y) \sin(t), 0 \le x, y, t \le 1. \tag{87}$$

The values of initial and boundary conditions and f can be achieved using the exact solution. The exact, Crank–Nicolson, and EDG solutions for $y = 0.5$, Re = 50, $N = 49$, and $\Delta t = 0.01$ in different values of t are shown in Figure 4(a) and in Figure 4(b), the absolute error of Crank–Nicolson method at $T = 1$ for Re = 50, $N = 49$, $\Delta t = 0.01$, and $\alpha = 0.5$ is depicted. Based on the Figures, we can see that the numerical solutions are a good approximation of the exact solutions. In Tables 3 and 4, the maximum of absolute errors and CPU Times for Crank–Nicolson and EDG methods for $\alpha = 0.1$, 0.3, and $\alpha = 0.7$, 0.9 with $T = 1$, $\Delta t = 0.01$, Re = 100, and different values of N are tabulated, respectively. The EDG method generates the numerical solution with almost the same accuracy as the Crank–Nicolson method, but uses less time-consuming in comparison to the Crank–Nicolson method.

Example 15. Assume that the exact solution of Equation (1) is as follows:

$$u(x, y, t) = t^2 x^2 (1 - x)^3 \sin(\pi y) \exp(x + y), 0 \le x, y, t \le 1. \tag{88}$$

The maximum of absolute errors and CPU Times for Crank–Nicolson and EDG methods for $\alpha = 0.1$, 0.3, 0.7, 0.9 with $T = 1$, $\Delta t = 0.01$, Re = 5, and different values of N are shown in Tables 5 and 6, respectively. In Figure 5(a), the exact, Crank–Nicolson, and EDG solutions for $y = 0.5$, Re = 20, $N = 49$, and $\Delta t = 0.01$ in different values of t are illustrated. Also, the absolute error of Crank–Nicolson method at $T = 1$ for Re = 20, $N = 49$, $\Delta t = 0.01$, and $\alpha = 0.5$ is portrayed in Figure 5(b). The numerical experiments verified our theoretical results once again.

Example 16. Consider Equation (1) with the exact solution

$$u(x, y, t) = t^2 \cos(\pi x) \cos(\pi y), 0 \le x, y, t \le 1. \tag{89}$$

This example does not apply to the initial and boundary conditions of the article, but the results of this example are as good as other examples.

In Tables 7 and 8, the maximum of absolute errors and the CPU time consumed by our proposed methods for $\alpha =$ 0.2, 0.4, and 0.6, 0.8 at $T = 1$, $\Delta t = 0.01$, Re = 2 with different mesh sizes are presented, respectively. Similar to other examples, the EDG method is faster than the Crank–Nicolson method. The exact, Crank–Nicolson, and EDG solutions for $y = 0.5$, Re = 10, $N = 49$, $\alpha = 0.5$, and $\Delta t = 0.01$ in different values of t are displayed in Figure 6(a). In addition, the absolute error of the Crank–Nicolson method at $T = 1$, $\alpha = 0.5$, Re = 10, $N = 49$, and $\Delta t = 0.01$ is displayed in Figure 6(b).

Example 17. In this example, we assume that the exact solution of Equation (1) is

$$u(x, y, t) = t^2 \left(x - x^2\right)^2 \left(y - y^2\right)^2, 0 \le x, y, t \le 1. \tag{90}$$

In Figure 7(a), the exact, Crank–Nicolson, and EDG solutions for $y = 0.5$, Re = 60, $N = 49$, and $\Delta t = 0.01$ in different values of t are shown. Besides, the absolute error of Crank–Nicolson is demonstrated in Figure 7(b). Obviously, our schemes are very accurate and quickly converge to the exact solution. The maximum of absolute errors and CPU Times for Crank–Nicolson and EDG methods for $\alpha = 0.2$, 0.4, and

0.6, 0.8 at $T = 1$, $\Delta t = 0.01$, and $\mathrm{Re} = 100$ with various mesh sizes are expressed in Tables 9 and 10, respectively.

7. Conclusion

In this paper, we introduced the Crank–Nicolson method and the EDG method which derived from 45° rotation of the Crank–Nicolson approximation point and Taylor expansion to solve the 2D time-fractional Burgers' equation with Caputo-Fabrizio derivative. The error analysis and local truncation error of these methods gave in detail. The stability of the proposed numerical methods is analyzed by the Von-Neumann method. The convergence analysis of the CN and EDG methods proved. Some test problems chose to investigate the applicability and practical efficiency. From Tables 1–10, the results showed a good agreement with the exact solution, and the EDG method was faster than the CN method. Numerical experiments showed the efficiency of the proposed methods in terms of CPU time and accuracy.

Data Availability

All results have been obtained by conducting the numerical procedure and the ideas can be shared for the researchers.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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