On the Exact Solitary Wave Solutions to the New \((2 + 1)\) and \((3 + 1)\)-Dimensional Extensions of the Benjamin-Ono Equations

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In this paper, the Kudryashov method to construct the new exact solitary wave solutions for the newly developed \((2 + 1)\)-dimensional Benjamin-Ono equation is successfully employed. In the same vein, also the new \((2 + 1)\)-dimensional Benjamin-Ono equation to \((3 + 1)\)-dimensional spaces is extended and then analyzed and investigated. Different forms of exact solitary wave solutions to this new equation were also determined. Graphical illustrations for certain solutions in both equations are provided. We alternatively offer that the determining method is general, impressive, outspoken, and powerful and can be exerted to create exact solutions of various kinds of nonlinear models originated in mathematical physics and engineering.

1. Introduction

Nonlinear evolution equations have been known for their vital roles in many fields of engineering and nonlinear sciences for long. A lot of these equations are famous in fluid flow problems and shallow water waves applications. A very good example for such equations is the Benjamin-Ono equation [1] that describes inner waves of deep-stratified fluids that reads

\[
 u_{tt} + \alpha (u^2)_{xx} + \beta u_{xxxx} = 0, \quad (1)
\]

where \(\alpha\) and \(\beta\) are nonzero constants for monitoring the nonlinear term and depth of the fluid, respectively. Further, different studies have been carried out on this important model ranging from analytical solution, numerical solution, stability, and well-posedness among others. For instance, the multisoliton solution and time-periodic solutions of the Benjamin-Ono equation were presented by Matsuno [2] and Ambrose and Wilkening [3], respectively (see also Angulo et al. [4] for the stability, Tutiya and Shiraishi [5] for discrete solutions, and [6–11] for other related studies).

Additionally, the \((2 + 1)\)-dimensional version of Benjamin-Ono equation Eq. (1) was recently introduced by Wazwaz [12]. The new equations has the form

\[
 u_{tt} + \alpha (u^2)_{xx} + \beta u_{xxxx} + \gamma u_{yyyy} = 0, \quad (2)
\]

where \(\alpha, \beta,\) and \(\gamma\) are nonzero constants. Note that \(\gamma\) should not be zero; otherwise, we recover Eq. (1). In [12], the Hirota bilinear method and certain ansatzs methods have been used to construct a variety of multiple and complex soliton solutions and also checked the Painlevé integrality condition.

However, in this paper, we further extend the new \((2 + 1)\)-dimensional Benjamin-Ono equation [12] given in Eq. (2) to \((3 + 1)\)-dimensional spaces and call it the \((3 + 1)\)-dimensional Benjamin-Ono equation given by

\[
 u_{tt} + \alpha (u^2)_{xx} + \beta u_{xxxx} + \gamma u_{yyyy} + \delta u_{zzzz} = 0, \quad (3)
\]

where \(\alpha, \beta, \gamma,\) and \(\delta\) are nonzero constants. Furthermore, to present more new solitary wave solutions for the \((2 + 1)\)-dimensional Benjamin-Ono equation in Eq. (2) and also to study the \((3 + 1)\)-dimensional Benjamin-Ono equation, we developed Eq. (3), to employ the Kudryashov method [13, 14] as a powerful integration method for
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2. Analysis of the Method

To illustrate the idea of the Kudryashov method [13, 14], we consider the following system of nonlinear differential equations:

\[ F(u, T_u, T_{xx}u, T_{tt}u, T_{xxxx}u, \ldots) = 0. \]  

(4)

Applying the transformation

\[ u(x, t) = f(\xi), \quad \xi = ax - ct - x_0, \]  

(5)

where \( a \) and \( c \) are nonzero constants and \( x_0 \) is arbitrary constant, converts Eq. (4) to a nonlinear ordinary differential equations as follows

\[ H(f', f'', f''', \ldots) = 0, \]  

(6)

where the derivatives are with respect to \( \xi \). It is assumed that the solutions of Eq. (6) are presented as a finite series, say

\[ f(\xi) = a_0 + \sum_{i=1}^{N} a_i \Phi^i(\xi), \]  

(7)

where \( a_i, i = 1, 2, \ldots, N \) (\( a_N \neq 0 \)), are constants to be computed, and \( \Phi(\xi) \) is given by the following function:

\[ \Phi(\xi) = \frac{1}{1 + we^{\xi}}, \]  

(8)

which satisfies the ordinary differential equation

\[ \Phi'(\xi) = \Phi(\xi)(\Phi(\xi) - 1). \]  

(9)

Also, the value of \( N \) is determined by homogenous balancing method (see [13, 14]). Substituting Eq. (7) and its necessary derivatives like

\[ f' = \sum_{i=1}^{N} a_i \Phi^i(\Phi - 1), \]  

(10)

\[ f'' = \sum_{i=1}^{N} a_i \Phi^i((1 + i)\Phi - i), \]  

\[ \vdots \]

into Eq. (6) gives

\[ P(\Phi(\xi)) = 0, \]  

(11)

where \( P(\Phi(\xi)) \) is a polynomial in \( \Phi(\xi) \). Equating the coefficient of each power of \( \Phi(\xi) \) in Eq. (11) to zero, a system of algebraic equations will be obtained whose solution yields the exact solutions of Eq. (4).

3. Applications

In this section, some new solitary wave solutions of the (2 + 1)-dimensional and (3 + 1)-dimensional Benjamin-Ono equations are constructed using the Kudryashov method presented above.

3.1. The (2 + 1)-Dimensional Benjamin-Ono Equation. In this section, we will study the (2 + 1)-dimensional Benjamin-Ono equation given by Eq. (2)

\[ u_{tt} + \alpha(u^2)_{xx} + \beta u_{xxxx} + \gamma u_{yyyy} = 0, \]  

(12)

where \( u(x, y, t) \) is a sufficiently often a differentiable function and \( \alpha, \beta, \gamma \) are nonzero parameters.

To determine certain solitary wave solutions, we first substitute

\[ u(x, y, t) = f(\xi), \quad \xi = ax + by - ct - x_0, \]  

(13)

into Eq. (12) where

\[ T_u u = c^2 f' (\xi), \quad T_{xx} u = a f' (\xi), \quad T_{tt} u = a^2 f'' (\xi), \]  

\[ T_{xxxx} u = a^4 f''''(\xi), \quad T_{yyyy} u = b^4 f''''(\xi), \]  

(14)

and convert Eq. (12) to a nonlinear ordinary differential equation given below:

\[ c^2 f'''' + aa^2 (f^2)'' + \beta a^4 f'''' + \gamma b^4 f'''' = 0. \]  

(15)

Integrating Eq. (15) twice with respect to \( \xi \), yields

\[ c^2 f + aa^2 (f^2) + \beta a^4 f'' + \gamma b^4 f'' = 0, \]  

(16)

where the integrating constant is considered zero. Balancing
Eq. (15) with Case 3.

Substituting Eq. (17) into Eq. (16) and equating the coefficient of each power of $\Phi(\xi)$ to zero, we get a system of algebraic equations given below:

$$c^2a_0 + a^2aa_0 = 0,$$

$$c^2a_1 + a^4b_1 + b^4g_1 + 2a^2aa_1 = 0,$$

$$-3a^4b_1 - 3b^4g_1 + a^2aa_1^2 + c^2a_2 + 4a^4b_2 + 4b^4g_2 + 2a^2aa_2 = 0,$$

$$2a^4b_1 + 2b^4g_1 - 10a^4b_2 - 10b^4g_2 + 2a^2aa_1 = 0,$$

$$6a^4b_2 + 6b^4g_2 + a^2aa_2 = 0.$$

Solving the above nonlinear algebraic system, the following results will be concluded as follows.

**Case 1.**

$$a_0 = 0, a_1 = -\frac{6c^2}{a^2}, a_2 = \frac{6c^2}{a^2}b = \mp \frac{(c^2 - a^4)^{1/4}}{y^{1/4}}.$$

Hence, the solution is formed as

$$u_{1,2}(x, y, t) = -\frac{6c^2}{a^2} + \frac{6c^2}{a^2}b = \mp \frac{(c^2 - a^4)^{1/4}}{y^{1/4}}, \xi = ax + by - ct - x_0.$$  

(20)

**Case 2.**

$$a_0 = -\frac{c^2}{a^2}, a_1 = \frac{6c^2}{a^2}, a_2 = -\frac{6c^2}{a^2}b = \mp \frac{(c^2 - a^4)^{1/4}}{y^{1/4}}.$$

Hence, the solution is formed as

$$u_{3,4}(x, y, t) = -\frac{c^2}{a^2} + \frac{6c^2}{a^2}b = \mp \frac{(c^2 - a^4)^{1/4}}{y^{1/4}}, \xi = ax + by - ct - x_0.$$  

(22)

**Case 3.**

$$a_0 = 0, a_1 = -\frac{6c^2}{a^2}, a_2 = \frac{6c^2}{a^2}b = \mp \frac{(c^2 - a^4)^{1/4}}{y^{1/4}}.$$  

Hence, the solution is formed as

$$u_{3,4}(x, y, t) = -\frac{6c^2}{a^2} + \frac{6c^2}{a^2}b = \mp \frac{(c^2 - a^4)^{1/4}}{y^{1/4}}, \xi = ax + by + dt - x_0.$$  

(23)

**Case 4.**

$$a_0 = -\frac{c^2}{a^2}, a_1 = \frac{6c^2}{a^2}, a_2 = -\frac{6c^2}{a^2}b = \mp \frac{i(c^2 - a^4)^{1/4}}{y^{1/4}}.$$  

Hence, the solution is formed as

$$u_{7,8}(x, y, t) = -\frac{c^2}{a^2} + \frac{6c^2}{a^2}b = \mp \frac{(c^2 - a^4)^{1/4}}{y^{1/4}}, \xi = ax + by + dt - x_0.$$  

(26)

The corresponding dynamic characteristics of the periodic wave solution are plotted in Figures 1 and 2 and arise at spaces $y = -1, y = 0, y = 1$, in Figure 3, they arise at spaces $y = -10, y = -7$, and $y = 1$, and also in Figure 4, they arise at spaces $y = -10, y = 0$, and $y = 1$ with the following special parameters:

$$a = 2, \alpha = 2, \beta = -3, c = 2, y = 2, r = 1, w = 0.3, t = 20.$$  

(27)

3.2 The (3 + 1)-Dimensional Benjamin-Ono Equation. In this section, we will study the (3 + 1)-dimensional Benjamin-Ono equation which we give as

$$u_t + \alpha(u^2)_{xx} + \beta u_{xxxx} + \gamma u_{yyyy} + \delta u_{zzzz} = 0,$$  

(28)

where $u(x, y, z, t)$ is a sufficiently often differentiable function and $\alpha, \beta, \gamma$ and $\delta$ are nonzero parameters. Also to determine some soliton solutions, we first substitute the transformation

$$u(x, y, z, t) = f(\xi), \xi = ax + by + dz - ct - x_0.$$  

(29)

into Eq. (28) where

$$T_{tt} = c^2 f'(\xi), T_{tx} = af'(\xi), T_{xxx} = a^3 f'''(\xi),$$

$$T_{xxxx} = a^4 f''''(\xi), T_{yyyy} = b^4 f''''(\xi), T_{zzzz} = a^4 f''''(\xi),$$

(30)

which converts Eq. (28) into a nonlinear ordinary differential equation as follows:

$$c^2 f'' + aa^2 f'''' + \beta a^4 f'''' + \gamma b^4 f'''' + \delta d^4 f'''' = 0.$$  

(31)
Integrating (31) once with respect to $\xi$ and setting the integrating constant zero yield

$$c^2 f + a^2 (f^2) + \beta a^4 f'' + \gamma b^4 f'' + \delta d^4 f'' = 0.$$ (32)

Balancing $f^2$ and $f''$ in Eq. (32) results to $2N = N + 2$, so $N = 2$. This offers a truncated series as the following form:

$$f(\xi) = a_0 + a_1 \Phi(\xi) + a_2 \Phi^2(\xi).$$ (33)

Substituting Eq. (33) into Eq. (32) and equating the coefficient of each power of $\Phi(\xi)$ to zero, we get the following system of algebraic equations:

$$c^2 a_0 + a^2 a a_0^2 = 0,$$

$$c^2 a_1 + a^4 a^4 + b^4 a_1^4 + d^4 a_1 + 2 a^4 a_0 a_1 = 0,$$

$$-3 a^4 a_1 + 3 b^4 a_0 a_1 - 3 d^4 a_1 + a^2 a a_2^2 + c^2 a_2 + 4 a^4 a_2$$

$$+ 4 b^4 a_0 + 4 d^4 a_2 + 2 a^2 a a_0 a_2 = 0,$$

$$2 a^4 a_1 + 2 b^4 a_0 a_1 + 2 d^4 a_1 - 10 a^4 a_2 - 10 b^4 a_2$$

$$- 10 d^4 a_2 + 2 a^2 a a_0 a_2 = 0,$$

$$6 a^4 a_2 + 6 b^4 a_0 a_2 + 6 d^4 a_2 + a^2 a a_2^2 = 0.$$ (34)

Solving the above system, yields the following.
Case 1.

\[ \begin{align*}
 a_0 &= 0, \\
 a_1 &= \frac{-6c^2}{a^2\alpha}, \\
 a_2 &= \frac{6c^2}{a^2\alpha}, \\
 b &= \pm \left( \frac{c^2 - a^4\beta - d^4\delta}{y^{1/4}} \right)^{1/4}.
\end{align*} \]

\[ (35) \]

Hence, the solution is formed as

\[ u_{1,2}(x, y, z, t) = -\frac{6c^2/a^2\alpha}{1 + we^5} + \frac{6c^2/a^2\alpha}{(1 + we^5)^2} \xi, \]

\[ = ax + by + dz - ct - x_0. \]

\[ (36) \]

Case 2.

\[ \begin{align*}
 a_0 &= -\frac{c^2}{a^2\alpha}, \\
 a_1 &= \frac{6c^2}{a^2\alpha}, \\
 a_2 &= -\frac{6c^2}{a^2\alpha}, \\
 b &= \pm \left( \frac{c^2 - a^4\beta - d^4\delta}{y^{1/4}} \right)^{1/4}.
\end{align*} \]

\[ (37) \]

Hence, the solution is formed as

\[ u_{3,4}(x, y, t) = -\frac{c^2}{a^2\alpha} + \frac{6c^2/a^2\alpha}{1 + we^5} - \frac{6c^2/a^2\alpha}{(1 + we^5)^2} \xi, \]

\[ = ax + by + dz - ct - x_0. \]

\[ (38) \]
Case 3.

\[ a_0 = 0, a_1 = - \frac{6c^2}{a^2\alpha}, a_2 = \frac{6c^2}{a^2\alpha}, b = \mp \frac{i(-c^2 - a^4\beta - d^4\delta)^{1/4}}{\gamma^{1/4}}. \]  

Hence, the solution is formed as

\[ u_{5,6}(x, y, t) = -\frac{6c^2/a^2\alpha}{1 + we^2} + \frac{6c^2/a^2\alpha}{(1 + we^2)^2} \xi, \]  

where \( \xi = ax + by + dz - ct - x_0. \)  

Case 4.

\[ a_0 = -\frac{c^2}{a^2\alpha}, a_1 = \frac{6c^2}{a^2\alpha}, a_2 = \frac{6c^2}{a^2\alpha}, b = \mp \frac{i(c^2 - a^4\beta - d^4\delta)^{1/4}}{\gamma^{1/4}}. \]  

Hence, the solution is formed as

\[ u_{7,8}(x, y, t) = -\frac{c^2/a^2\alpha}{1 + we^2} + \frac{6c^2/a^2\alpha}{1 + we^2} - \frac{6c^2/a^2\alpha}{(1 + we^2)^2} \xi, \]  

where \( \xi = ax + by + dz - ct - x_0. \)  

The corresponding dynamic characteristics of the periodic wave solution are plotted in Figures 5 and 6 and arise at spaces \( y = -1, y = 0, \) and \( y = 1, \) in Figure 7, they arise at
and three dimensional plots for the solutions are plotted. Figures 1 and 4 show the graph of the solutions (20)–(26) for the (2 + 1)-dimensional Benjamin-Ono equation, respectively. Figures 5 and 8 show the behavior of the solutions (36)–(42) for the (3 + 1)-dimensional Benjamin-Ono equation, respectively.

5. Conclusion

In conclusion, we have presented new solitary wave solutions for the (2 + 1)-dimensional Benjamin-Ono equation introduced recently by Wazwaz and also extended it to (3 + 1)-dimensional spaces called the (3 + 1)-dimensional Benjamin-Ono equation. While constructing the solitary wave solutions, we make use of the Kudryashov method.

4. Some Graphical Illustrations

We depict in this section some graphical illustrations of the obtained solutions for the (2 + 1)- and (3 + 1)-dimensional extensions of the Benjamin-Ono equations, both the two

Figure 7: Graph of the absolute value of Eq. (40) for the (2 + 1)-dimensional Benjamin-Ono equation at $a = 2$, $\alpha = 2$, $\beta = -3$, $c = 2$, $y = 2$, $r = 1$, $w = 0.3$, $d = 2$, $\delta = -2$, $z = 1$, $t = 20$ and for 2 plot spaces $y = -10, -7, 1$.

Figure 8: Graph of the absolute value of Eq. (42) for the (2 + 1)-dimensional Benjamin-Ono equation at $a = 2$, $\alpha = 2$, $\beta = -3$, $c = 2$, $y = 2$, $r = 1$, $w = 0.3$, $d = 2$, $\delta = -2$, $z = 1$, $t = 20$ and for 2 plot spaces $y = -10, -3, 1$.
being one of the powerful integration methods for treating various nonlinear evolution equations and construct various exponential solutions to both equations. The development of offered method may allow the extensions of the Benjamin-Ono equations to be used in more general configurations. The solutions are all verified by putting them back into the original equations with the aid of the Maple symbolic computation package 18.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflict of interest.

Acknowledgments

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