Hindawi
Advances in Mathematical Physics
Volume 2021, Article ID 6678150, 5 pages
https://doi.org/10.1155/2021/6678150

Research Article

An Interpolation Theorem for Quasimartingales in Noncommutative Symmetric Spaces

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Received 25 November 2020; Revised 7 January 2021; Accepted 18 January 2021; Published 31 January 2021

Academic Editor: Francesco Toppan

Let E be a separable symmetric space on $(0,\infty)$ and $E(\mathcal{M})$ the corresponding noncommutative space. In this paper, we introduce a kind of quasimartingale spaces which is like but bigger than $E(\mathcal{M})$ and obtain the following interpolation result: let $\widehat{E}(\mathcal{M})$ be the space of all bounded $E(\mathcal{M})$-quasimartingales and $1 < p < p_E < q_E < q < \infty$. Then, there exists a symmetric space F on $(0,\infty)$ with nontrivial Boyd indices such that $\widehat{E}(\mathcal{M}) = (\widehat{L}_p(\mathcal{M}), \widehat{L}_q(\mathcal{M})_{F,K}$.

1. Introduction

Let E be a symmetric space on $(0,\infty)$ with the Fatou property and $1 < p < p_E < q_E < q < \infty$. Kalton and Montgomery-Smith [1] proved that there exists a symmetric space F with nontrivial Boyd indices such that $E = (L_p((0,\infty)), L_q((0,\infty))_{F,K}$. As is now well-known, the preceding interpolation result can automatically lift to the noncommutative setting (see [[2], Theorem 3.4]): let E be a symmetric space on $(0,\infty)$ with the Fatou property and $(\mathcal{M},\tau)$ a semifinite von Neumann algebra. Then, the following are equivalent:

(i) $1 < p < p_E < q_E < q < \infty$.

(ii) There exists a symmetric space F on $(0,\infty)$ with nontrivial Boyd indices such that

$$E(\mathcal{M}) = \left(L_p(\mathcal{M}), L_q(\mathcal{M})\right)_{F,K}. \qquad (1)$$

In this paper, we replace the space $E(\mathcal{M})$ in (2) with a bigger and more complex space $\widehat{E}(\mathcal{M})$ (see Definition 3) and obtain a generalized interpolation result. Our main result can be stated as follows (see Section 2 for the unexplained notations).

Theorem 1. *Let E be a separable symmetric space on $(0,\infty)$ with $1 < p < p_E < q_E < q < \infty$. Then, there exists a symmetric Banach function space F on $(0,\infty)$ SSS with nontrivial Boyd indices such that*

$$\widehat{E}(\mathcal{M}) = \left(\widehat{L}_p(\mathcal{M}), \widehat{L}_q(\mathcal{M}\right)_{F,K}. \qquad (2)$$

2. Preliminaries

2.1. Noncommutative Spaces. Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace τ. We denote by $L_0(\mathcal{M})$ the family of all τ-measurable operators. Note that $\chi(\lambda,\infty)(|x|)$ is the spectral projection of $x \in L_0(\mathcal{M})$ associated with the interval (λ,∞). For $x \in L_0(\mathcal{M})$, define its generalized singular number by

$$\mu_t(x) = \inf\left\{\lambda > 0 : \tau\left(\chi_{(\lambda,\infty)}(|x|)\right) \le t\right\}, t > 0. \qquad (3)$$

Note that the function $t \mapsto \mu_t(x)$ from $(0,\infty)$ into $[0,\infty)$ is right continuous and nonincreasing. For the case that $\mathcal{M}$ is the abelian von Neumann algebra $L(0,\infty)$ with the trace given by integration with respect to the Lebesgue measure, $L_0(\mathcal{M})$ is the space of all measurable functions, and $\mu(f)$ is the decreasing rearrangement of the measurable function f (see [3, 4]).

Recall that a Banach function space $(E, \|\cdot\|_E)$ on $(0,\infty)$ is called symmetric if for any $g \in E$ and any measurable function f with $\mu(f) \le \mu(g)$, we have $f \in E$ and $\|f\|_E \le \|g\|_E$. The Köthe dual of E is the function space defined by setting

$$E^{\times} = \left\{ f \in L_0(0,\infty) : \int_0^{\infty} |f(t)g(t)|dt < \infty : \forall g \in E \right\}. \quad (4)$$

When equipped with the norm $\|f\|_{E^{\times}} := \sup \{\int_0^{\infty}|f(t) g(t)|dt : \|g\|_E \le 1\}$, $E^{\times}$ is a symmetric Banach function space. For any $s > 0$, we define the dilation operator D_s on E by

$$(D_s f)(t) = f(t/s), t > 0, f \in E. \quad (5)$$

Define the lower and upper Boyd indices of E by

$$p_E := \lim_{s\to\infty} \frac{\log s}{\log \|D_s\|} \text{ and } q_E := \lim_{s\to 0^+} \frac{\log s}{\log \|D_s\|}, \quad (6)$$

respectively. It is well-known that $1 \le p_E \le q_E \le \infty$, and we shall say that E has nontrivial Boyd indices, whenever $1 < p_E \le q_E < \infty$. We refer to [1, 5] for unexplained terminology from function space theory.

For a given symmetric Banach function space $(E, \|\cdot\|_E)$ on $(0,\infty)$, we define the corresponding noncommutative space by setting

$$E(\mathscr{M}, \tau) = \{x \in L_0(\mathscr{M}) : \mu_t(x) \in E\}, \quad (7)$$

equipped with the norm

$$\|x\|_{E(\mathscr{M},\tau)} := \|\mu_t(x)\|_E. \quad (8)$$

It is well-known that $E(\mathscr{M}, \tau)$ is a Banach space and is referred to as the noncommutative symmetric space associated with $(\mathscr{M}, \tau)$ corresponding to the function space $(E, \|\cdot\|_E)$. Note that if $1 \le p < \infty$ and $E = L_p(0,\infty)$, then $E(\mathscr{M}, \tau) = L_p(\mathscr{M}, \tau)$ is the usual noncommutative L_p-space associated with $(\mathscr{M}, \tau)$.

Recall that $l\infty$ is a von Neumann algebra equipped with the trace: $\gamma(a) = \sum a_n$, $a = (a_n) \in l_{\infty}^{+}$ (see [6]). Now, let $\mathscr{N} = \mathscr{M} \bar{\otimes} l_{\infty}$ be the von Neumann algebra tensor product and $\nu = \tau \otimes \gamma$ the tensor trace. This gives rise to noncommutative spaces $E(\mathscr{M} \bar{\otimes} l_{\infty})$. Note that $L_p(\mathscr{M} \bar{\otimes} l_{\infty})$ coincides with the space $l_p(L_p(\mathscr{M}))$.

2.2. Noncommutative Martingales. A noncommutative probability space is a couple $(\mathscr{M}, \tau)$, where $\mathscr{M}$ is a finite von Neumann algebra and τ is a normal faithful trace with $\tau(1) = 1$. Let $(\mathscr{M}_n)_{n\ge1}$ be an increasing sequence of von Neumann subalgebras of $\mathscr{M}$ such that the union of the $\mathscr{M}_n{}'s$ is weak* dense in $\mathscr{M}$. Let ε_n be the conditional expectation with respect to $\mathscr{M}_n$.

Definition 2. A sequence $x = (x_n)_{n\ge1}$ in $L_1(\mathscr{M})$ is called a sequence of martingale differences if $x_n \in E(\mathscr{M}_n)$ for $n \ge 1$ and if $\varepsilon_n(x_{n+1}) = 0$ for all $n \ge 0$.

In this paper, we always consider noncommutative martingales associated with a noncommutative probability space unless explicit explanation.

2.3. Interpolation. Let (X_0, X_1) be a compatible couple of quasi-Banach space. Its K-functional is defined by

$$K_t(x; X_0, X_1) = \inf \left\{ \|x_0\|_{X_O} + t\|x_1\|_{x_1} : x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1 \right\}, \quad (9)$$

for $x \in X_0 + X_1$ and $t > 0$. Let E be a symmetric Banach space on $(0,\infty)$. Set

$$\|x\|_{(X_0,X_1)_{E,K}} = \left\| \frac{K_t(x; X_0, X_1)}{t} \right\|_E. \quad (10)$$

Then, the interpolation space $_{(X_0,X_1)E,K}$ is defined as $_{(X_0,X_1)E,K} = \{x \in X_0 + X_1 : \|x\|_{(X_0,X_1)_{E,K}} < \infty\}$ equipped with the norm $\|\cdot\|_{(X_0,X_1)_{E,K}}$.

3. Main Result

The main result in this section is Theorem 1, which extends the result of Kalton and Montgomery-Smith [1] to a $E(\mathscr{M})$ -quasimartingale spaces. We first introduce the quasimartingale spaces.

Definition 3. Let E be a symmetric Banach function space on $(0,\infty)$. A sequence $x = (x_n)_{n\ge1}$ is called a $E(\mathscr{M})$-quasimartingale with respect to $(\mathscr{M}_n)_{n\ge1}$ if $x_n \in E(\mathscr{M}_n)$ for $n \ge 1$ and (with $\varepsilon_0 = 0, x_0 = 0$)

$$\left\| \sum_{n=1}^{\infty} \varepsilon_{n-1}(dx_n) \otimes e_n \right\| E(\mathscr{M} \bar{\otimes} l_{\infty}) < \infty. \quad (11)$$

We set

$$\|x\|_{\widehat{E}(\mathscr{M})} := \sup_n \|y_n\|_{E(\mathscr{M})} + \left\| \sum_{n=1}^{\infty} \varepsilon_{n-1}(dx_n) \otimes e_n \right\| E(\mathscr{M} \bar{\otimes} l_{\infty}), \quad (12)$$

where $y_n = \sum_{k=1}^{n}(dx_k - \varepsilon_{k-1}(dx_k))$ and $(e_n)n \ge 1$ denote the standard basic sequence of l_{∞}.

If $\|x\|_{\widehat{E}(\mathscr{M})} < \infty$, x is called a bounded $E(\mathscr{M})$-quasimartingale. The quasimartingale space $\widehat{E}(\mathscr{M})$ is defined as the space of all bounded $\widehat{E}(\mathscr{M})$ -quasimartingales, equipped with the norm $\|\cdot\|_{\widehat{E}(\mathscr{M})}$.

Remark 4.

(i) In the case $E = L_p$ for $1 \le p < \infty$, we have that $\widehat{E}(\mathcal{M}) = \widehat{L_p}(\mathcal{M})$ and

$$\|x\|_{\widehat{L_p}(\mathcal{M})} := \sup_n \|y_n\|_{L_p(\mathcal{M})} + \left(\sum_{n=1}^{\infty} \|\varepsilon_{n-1}(dx_n)\|_{Lp(\mathcal{M})}^p \right)^{1/p} \quad (13)$$

(ii) Let $x = (x_n)_{n\ge1}$ be a bounded $E(\mathcal{M})$-quasimartingale. Set $z_n = \sum_{k=1}^n (\mathscr{E}_{k-1}(dx_k))$ and $y_n = x_n - z_n (n \ge 1)$. Then, $z = (z_n)_{n\ge1}$ is a predicable $E(\mathcal{M})$-quasimartingale with $z_1 = 0$, and $y = (y_n)_{n\ge1}$ is a bounded $E\ (\mathcal{M})$ -martingale. Moreover, the decomposition,

$$x = y + z, \quad (14)$$

is unique (see Lemma 2.5, [7]).

The following lemma is the key ingredient of our proof of Theorem 7.

Lemma 5. *Let E be a separable symmetric space on $(0, \infty)$ with $1 < p_E \le q_E < \infty$. Then,*

$$(E\wedge(\mathcal{M}))^* = E\wedge^{\times}(\mathcal{M}), \quad (15)$$

with equivalent norms.

Proof. Let $\mu = (\mu_n)_{n\ge1} \in E\wedge^{\times}(\mathcal{M})$ and $x = (x_n)_{n\ge1} \in \widehat{E}(\mathcal{M})$. Let $\mu_n = \nu_n + \omega_n$ and $x_n = y_n + z_{n(n\ge1)}$ be the decomposition of μ and x as in (14). Then, $y = (y_n)_{n\ge1}$ is a bounded $E(\mathcal{M})$-martingale and $\nu = (\nu_n)_{n\ge1}$ is a bounded $\widehat{E}(\mathcal{M})$ -martingale. Thus, there exist $y_\infty \in E(\mathcal{M})$ and $\nu_\infty \in E^{\times}(\mathcal{M})$ such that

$$y_n \xrightarrow{E(\mathcal{M})} y\infty, \nu_n \xrightarrow{E^{\times}(\mathcal{M})} \nu_\infty. \quad (16)$$

Now, we define a linear functional on $\widehat{E}(\mathcal{M})$ by

$$l_\mu(x) = \tau(\nu_\infty y_\infty) + \sum_{n=1}^{\infty} \tau(d\omega_n dz_n). \quad (17)$$

Then, by Hölder's inequality,

$$\begin{aligned} |l_\mu(x)| &\le \|\nu_\infty\|_{E^{\times}(\mathcal{M})} \|y_\infty\|_{E(\mathcal{M})} \\ &\quad + \tau \otimes \gamma \left(\sum_{n=1}^{\infty} d\omega_n \otimes e_n \sum_{n=1}^{\infty} dz_n \otimes e_n \right) \\ &\le \|\nu_\infty\|_{E^{\times}(\mathcal{M})} \|y_\infty\|_{E(\mathcal{M})} \\ &\quad + \left\| \sum_{n=1}^{\infty} d\omega_n \otimes e_n \right\|_{E(\mathcal{M}\bar{\otimes} l_\infty)} \left\| \sum_{n=1}^{\infty} dz_n \otimes e_n \right\| E^{\times}(\mathcal{M} \bar{\otimes} l_\infty) \\ &\le \|\mu\|_{E\wedge^{\times}(\mathcal{M})} \|x\|_{\widehat{E}(\mathcal{M})}. \end{aligned} \quad (18)$$

Thus, $l_\mu(x)$ is continuous on $\widehat{E}(\mathcal{M})$ and $\|l_\mu\| \le \|\mu\|_{E\wedge^{\times}(\mathcal{M})}$.

We pass to the converse inclusion. Let $l \in (E\wedge(\mathcal{M}))^*$. Let l_1 be the restriction of l on $E(\mathcal{M})$. Noting that $(E(\mathcal{M}))^* = E^{\times}(\mathcal{M})$, there exists an operator $\nu \in E^{\times}(\mathcal{M})$ and $\|\nu\|_{E^{\times}(\mathcal{M})} \le \|l\|$ such that

$$l_1(a) = \tau(a\nu), a \in E\ (\mathcal{M}). \quad (19)$$

On the other hand, let $F_E(M)$ be the space of all sequences $db = (db_n)_{n\ge1}$ such that $b = (b_n)_{n\ge1}$ is a predictable $E(\mathcal{M})$ -quasimartingale with $b_1 = 0$ equipped with the norm $\|db\|_{F_E(\mathcal{M})} = \|\sum_{n=1}^{\infty} db_n \otimes e_n\| E(\mathcal{M} \bar{\otimes} l\infty)$. Define a functional on $F_E(\mathcal{M})$ by

$$l_2(db) = l(b),\ db = (db_n)_{n\ge1} \in F_E(M). \quad (20)$$

Then, by the inequality

$$|l_2(db)| \le \|l\| \|b\|_{\widehat{E}(\mathcal{M})} = \|l\| \|db\|_{F_E(\mathcal{M})}, \quad (21)$$

we have l_2 is a continuous linear functional on $F_E(M)$ and $\|l_2\| \le \|l\|$. Note that $F_E(M)$ is isometric to the subspace of $E(M \bar{\otimes} l\infty)$. By the Hahn-Banach theorem, l_2 extends to a functional on $E(M \bar{\otimes} l\infty)$. Since $(E(M \bar{\otimes} l\infty)) * \ = E \times (M \bar{\otimes} l_\infty)$, the representation theorem allows us to find a sequence $(W_n')_{n\ge1} \in E^{\times}(M \bar{\otimes} l_\infty)$ such that

$$\begin{aligned} l_2(s) &= \sum_{n=1}^{\infty} \nu\left(\left(w_n' \otimes e_n \right) (s_n \otimes e_n) \right) \\ &= \sum_{n=1}^{\infty} \tau\left(w_n' s_n \right), (s_n)_{n\ge1} \in E(\mathcal{M} \bar{\otimes} l_\infty), \end{aligned} \quad (22)$$

and

$$\left\| \sum_{n=1}^{\infty} w_n' \otimes e_n \right\|_{E^{\times}(\mathcal{M}\bar{\otimes} l_\infty)} \le \|l_2\|. \quad (23)$$

Set $\omega_1 = 0$ and $w_n = \sum_{k=1}^n \varepsilon_{k-1}(w_k')(n \ge 2)$. For any $db = (db_n)_{n\ge1} \in F_E(M)$, noting that $db = (db_n)_{n\ge1}$ is predicable, it follows from (22) that

$$l_2(db) = \sum_{n=1}^{\infty} \tau\left(\varepsilon_{n-1}\left(w_n' db_n\right)\right) = \sum_{n=1}^{\infty} \tau\left(db_n \varepsilon_{n-1}\left(w_n'\right)\right) = \sum_{n=1}^{\infty} \tau(dw_n db_n). \tag{24}$$

It is easy to see that $\omega = (\omega_n)_{n\geq 1}$ is predictable with $\omega_1 = 0$ and

$$\left\|\sum_{n=1}^{\infty} dw_n \otimes e_n\right\|_{E^\times(\mathscr{M}\bar{\otimes} l_\infty)} = \left\|\sum_{n=1}^{\infty} \varepsilon_{n-1}\left(w_n'\right) \otimes e_n\right\|_{E^\times(\mathscr{M}\bar{\otimes} l_\infty)} \leq \left\|\sum_{n=1}^{\infty} w_n' \otimes e_n\right\|_{E^\times(\mathscr{M}\bar{\otimes} l_\infty)} \leq \|l_2\|. \tag{25}$$

Set $\mu_n = \nu_n + \omega_n (n \geq 1)$, where $\nu_n = \mathcal{E}n(\nu)(n \geq 1)$. Then, $\mu = (\mu_n)_{n\geq 1} \in E\wedge^\times(M)$ and

$$\|\mu\|_{E\wedge^\times(\mathscr{M})} = \|\nu\|_{E^\times(\mathscr{M})} + \left\|\sum_{n=1}^{\infty} dw_n \otimes e_n\right\|_{E^\times(\mathscr{M}\bar{\otimes} l_\infty)} \leq 2\|l\|. \tag{26}$$

For any $x = (x_n)_{n\geq 1} \in \widehat{E}(\mathscr{M})$, let $x_n = y_n + z_n (n \geq 1)$ be its decomposition as in (14).

Noting that $y = (y_n)_{n\geq 1}$ is a bounded $E(\mathrm{M})$ martingale and $dz = (dz_n)_{n\geq 1} \in F_E(M)$, it follows from (19) and (24) that

$$l(x) = l(y) + l(z) = \tau(y_\infty \nu_\infty) + \sum_{n=1}^{\infty} \tau(dw_n dz_n). \tag{27}$$

The proof is completed.

The following lemma is about the duality theorem of interpolation spaces.

Lemma 6 (see [2]). *Let E be separable and (X_1, X_2) be a couple of Banach spaces such that $X_1 \cap X_2$ is dense in both X_1 and X_2. Then,*

$$(X_1, X_2)^*_{E,K} = (X_2^*, X_1^*)_{E^\times,K}. \tag{28}$$

Proof of Theorem 7. Let $x \in (\widehat{L}_p(\mathscr{M}), \widehat{L}_q(\mathscr{M}))_{F,K}$ and $x = x^0 + x^1$ be a decomposition of x, where $x^0 \in L_p(\mathscr{M}), x^1 \in L_q(\mathscr{M})$. Let $x_n^k = y_n^k + z_n^k (n \geq 1)$ be the decomposition of x^k $(k = 0, 1)$ as in (14). Then, y^0 is a bounded $L_p(M)$-martingale, and y^1 is a bounded $L_q(M)$-martingale. Thus, there exist $y_\infty^0 \in L_p(\mathscr{M})$ and $y_\infty^1 \in L_q(\mathscr{M})$ such that

$$y_n^0 \xrightarrow{L_p(\mathscr{M})} y_\infty^0, \quad y_n^1 \xrightarrow{L_p(\mathscr{M})} y_\infty^1, \quad \|y_\infty^0\|_{L_p(\mathscr{M})} = \sup_n \|y_n^0\|_{L_p(\mathscr{M})}, \quad \|y_\infty^1\|_{L_p(\mathscr{M})} = \sup_n \|y_n^1\|_{L_p(\mathscr{M})}. \tag{29}$$

Using Definition 3, we get that

$$\|x^0\|_{\widehat{L}_p(\mathscr{M})} = \|y_\infty^0\|_{L_p(\mathscr{M})} + \|dz^0\|_{L_p(\mathscr{M}\bar{\otimes} l_\infty)}, \quad \|x^1\|_{\widehat{L}_p(\mathscr{M})} = \|y_\infty^1\|_{L_q(\mathscr{M})} + \|dz^1\|_{L_q(\mathscr{M}\bar{\otimes} l_\infty)}. \tag{30}$$

Set $y = y_\infty^0 + y_\infty^1, z = z^0 + z^1$.Then, by the definition of K -functionals and (30),

$$\begin{aligned} &K_t\left(y\,;L_p(\mathscr{M}), L_q(\mathscr{M})\right) + K_t\left(dz\,;L_p(\mathscr{M}\bar{\otimes} l_\infty), L_q(\mathscr{M}\bar{\otimes} l_\infty)\right) \\ &\quad\leq \|y_\infty^0\|_{L_p(\mathscr{M})} + t\|y_\infty^1\|_{L_q(\mathscr{M})} + \|dz^0\|_{L_p(\mathscr{M}\bar{\otimes} l_\infty)} \\ &\qquad + t\|dz^1\|_{L_q(\mathscr{M}\bar{\otimes} l_\infty)} = \|x^0\|_{\widehat{L}_p(\mathscr{M})} + t\|x^1\|_{\widehat{L}_q(\mathscr{M})}. \end{aligned} \tag{31}$$

Thus, taking infimum over all decomposition of x, we obtain

$$\begin{aligned} &K_t\left(y\,;L_p(M), L_q(M)\right) + K_t\left(dz\,;L_p(M\bar{\otimes} l_\infty), L_q(M\bar{\otimes} l_\infty)\right) \\ &\quad\leq K_t\left(x\,;\widehat{L}_p(M), \widehat{L}_q(M)\right). \end{aligned} \tag{32}$$

Therefore, using the equality $\|x\|_{(X_0,X_1)F,K} = \|K_t(x\,;X_0,X_1)/t\|_F$, we have

$$\begin{aligned} &\|y\|_{\left(L_p(M),L_q(M)\right)F,K} + \|dz\|_{L_p\left(M\bar{\otimes} l_\infty\right),Lq\left(M\bar{\otimes} l_\infty\right))F,K} \\ &\quad\leq 2\|x\|_{\left(\widehat{L}_p(M),\widehat{L}_q(M)\right)F,K}. \end{aligned} \tag{33}$$

It follows from the equality $E(M) = (L_p(M), L_q(M))_{F,K}$ that

$$\|x\|_{\widehat{E}(M)} = \|y\|_{E(M)} + \|dz\|_{E\left(M\bar{\otimes} l_\infty\right)} \leq 2\|x\|_{\left(\widehat{L}_p(M),\widehat{L}_q(M)\right)_{F,K}}. \tag{34}$$

For any $x \in (\widehat{L}_p(M), \widehat{L}_q(M))_{F,K}$, we obtain $x \in \widehat{E}(M)$ which implies that

$$\widehat{E}(\mathscr{M}) \supset \left(\widehat{L}_p(\mathscr{M}), \widehat{L}_q(\mathscr{M})\right)_{F,K}. \tag{35}$$

Similarly, we have

$$E\wedge^\times(\mathscr{M}) \supset \left(\widehat{L}_{q'}(\mathscr{M}), \widehat{L}_{p'}(\mathscr{M})\right)_{F^\times,K}, \tag{36}$$

where p' and q' denote the conjugate index of p and q. By Lemma 5 and Lemma 6, we obtain that

$$\begin{aligned}\widehat{E}(\mathscr{M}) &= (E\wedge^{\times}(\mathscr{M}))^{*} \subset \left(\widehat{L}_{q'}(\mathscr{M}), \widehat{L}_{p'}(\mathscr{M})\right)^{*}_{F^{\times},K} \\ &= \left(\widehat{L}_{p}(\mathscr{M}), \widehat{L}_{q}(\mathscr{M})\right)_{F,K}.\end{aligned} \tag{37}$$

Thus,

$$\widehat{E}(\mathscr{M}) = \left(\widehat{L}_{p}(\mathscr{M}), \widehat{L}_{q}(\mathscr{M})\right)_{F,K}. \tag{38}$$

The proof is completed.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (11801489, 11671308).

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