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Research Article

Painlevé Analysis, Soliton Molecule, and Lump Solution of the Higher-Order Boussinesq Equation

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The Painlevé integrability of the higher-order Boussinesq equation is proved by using the standard Weiss-Tabor-Carnevale (WTC) method. The multisoliton solutions of the higher-order Boussinesq equation are obtained by introducing dependent variable transformation. The soliton molecule and asymmetric soliton of the higher-order Boussinesq equation can be constructed by the velocity resonance mechanism. Lump solution can be derived by solving the bilinear form of the higher-order Boussinesq equation. By some detailed calculations, the lump wave of the higher-order Boussinesq equation is just the bright form. These types of the localized excitations are exhibited by selecting suitable parameters.

1. Introduction

The soliton molecule as a boundary state is comprised by a balance of repulsive and attractive forces between solitons caused by the nonlinear and dispersive effects [1]. The characteristics of the soliton molecule in both experiment and simulation have attracted considerable attention [2–4]. The soliton molecules are first discovered by theoretical analysis of the nonlinear Schröinger equation [5] and the complex Ginzburg-Landau equation [6]. The theoretical frameworks to address the soliton molecule have been proposed [7, 8]. Recently, Lou proposed the velocity resonance mechanism to form the soliton molecule [9]. The high-order dispersive terms play a key role in the velocity resonance mechanism [10]. The velocity resonance mechanism is developed to some integrable systems, the (2+1)-dimensional fifth-order Korteweg-de Vries (KdV) equation [11], the complex modified KdV equation [12], the (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation [13], and so on [14–16]. Combining the Darboux transformation and the variable separation approach, some interactions between soliton molecules and breather solutions and between soliton molecules and dromions are explored [11–15, 17]. In addition to the soliton molecule, lump solutions are a kind of rational function solutions which have become a hot field in nonlinear systems [18–22]. Lump solutions will decay polynomially in all directions of space [23]. The Hirota bilinear method is a useful method to find lump solutions. Lump waves of the nonlinear systems have been validated by the Hirota bilinear method [24–29]. In this paper, our objective is to explore a higher-order Boussinesq equation which is considered as the combination between the fourth-order and sixth-order Boussinesq forms. We get lots of interesting results for the higher-order Boussinesq equation, such as the multisoliton, the soliton molecule, and lump solution. Particularly, the soliton molecule of the higher-order Boussinesq equation is not valid for just the fourth-order Boussinesq equation or the sixth-order Boussinesq equation.

The higher-order Boussinesq equation reads

$$u_{tt} + \gamma u_{xx} - \alpha\left(u_{xxxx} + 6u_x^2 + 6uu_{xx}\right) - \beta(15uu_{xxxx} + 30u_x u_{xxx} + 15u_{xx}^2 + 45u^2 u_{xx} + 90uu_x^2 + u_{xxxxxx}) = 0, \quad (1)$$

where α, β, and γ are arbitrary constants. The higher-order Boussinesq equation (1) will become the fourth-order Boussinesq equation and the sixth-order Boussinesq equation with $\beta = 0$ and $\alpha = 0$, respectively. The higher-order Boussinesq equation (1) can describe the propagation of long waves in shallow water. The integrability of the fourth-order Boussinesq equation was handled by the inverse scattering transform method [30]. The group-invariant solutions of the fourth-order Boussinesq equation were obtained by similarity reductions [31]. The high-dimensional Boussinesq equations can be constructed by means of the fourth-order Boussinesq equation [32]. The N-soliton solutions for the fourth-order [33] and the sixth-order Boussinesq equations [34] were obtained by the Hirota bilinear method.

This paper is organized as follows. In Section 2, the Painlevé integrability of the higher-order Boussinesq equation is proved by the standard Weiss-Tabor-Carnevale (WTC) approach. In Section 3, the higher-order Boussinesq equation can be transformed to the bilinear form by the dependent variable transformation. The Hirota's bilinear method is employed to derive the multisoliton by handling the bilinear form of the higher-order Boussinesq equation. The soliton molecule of the higher-order Boussinesq equation is constructed by a new resonance condition. In Section 4, lump solution of the higher-order Boussinesq equation is obtained by solving the corresponding Hirota bilinear form. Finally, the conclusions of this paper follow in Section 5.

2. Painlevé Analysis for the Higher-Order Boussinesq Equation

According to the WTC approach [35], the solution of the higher-order Boussinesq equation can be written as

$$u = \sum_{j=0}^{\infty} u_j f^{j-\delta}, \tag{2}$$

with δ being a positive integer. The solution of the model is single valued about the arbitrary movable singularity manifold f. From the leading order analysis, we get

$$\delta = 2,\ u_0 = -2f_x^2. \tag{3}$$

By inserting the expression

$$u = -\frac{2f_x^2}{f^2} + \frac{u_j}{f^{j-2}}, \tag{4}$$

into (1) and vanishing the coefficient term of f^{j-8}, the polynomial equation in j is derived as

$$(j+1)(j-2)(j-3)(j-6)(j-7)(j-10) = 0. \tag{5}$$

The corresponding resonances occur at

$$j = -1, 2, 3, 6, 7, 10. \tag{6}$$

The resonance at $j=-1$ corresponds to the fact that the singularity manifold $f(x,t)=0$ is an arbitrary function. By vanishing subsequent coefficient terms of f, we infer that the number of arbitrary functions $(f, u_2, u_3, u_6, u_7, u_{10})$ is the same as the number of resonances $(-1, 2, 3, 6, 7, 10)$. The higher-order Boussinesq equation is thus the Painlevé integrability in the sense of the WTC test.

3. Multisoliton and Soliton Molecule for the Higher-Order Boussinesq Equation

To determine the multisoliton solutions of the higher-order Boussinesq equation (1), the dependent variable transformation reads

$$u = 2(\ln f)_{xx}. \tag{7}$$

By substituting (7) into (1), the bilinear form of the higher-order Boussinesq equation reads

$$\left(D_t^2 + \gamma D_x^2 - \alpha D_x^4 - \beta D_x^6\right) f \cdot f = 0, \tag{8}$$

where D is bilinear derivative operator [36].

$$D_x^l D_t^m (f\cdot g) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^l \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m f(x,t)\cdot g\left(x',t'\right)\Big|_{x'=x,t'=t}. \tag{9}$$

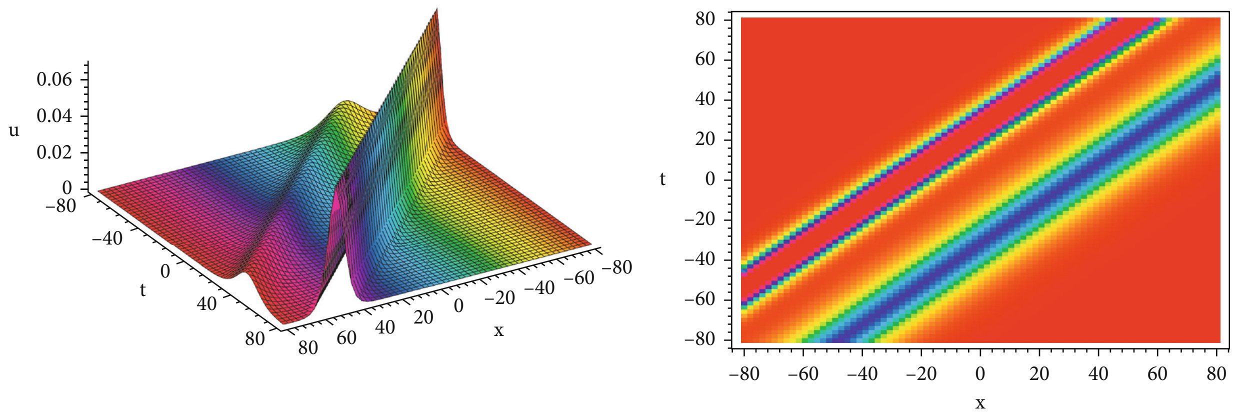
The N-soliton solutions of the higher-order Boussinesq equation can be calculated as

$$f = \sum_{\mu=0,1} \exp\left(\sum_{1\le i<j}^{N} \mu_i\mu_j \ln\left(A_{ij}\right) + \sum_{i=1}^{N} \mu_i\eta_i\right), \tag{10}$$

where $\sum_{\mu=0,1}$ is the summation with possible combinations of $\mu_i = 0, 1 (i = 1, 2, \cdots, N)$ and $\eta_i = k_i x + \omega_i t + c_i$. By substituting (7) and (10) into (1), the phases shift A_{ij} and the dispersion relation read as

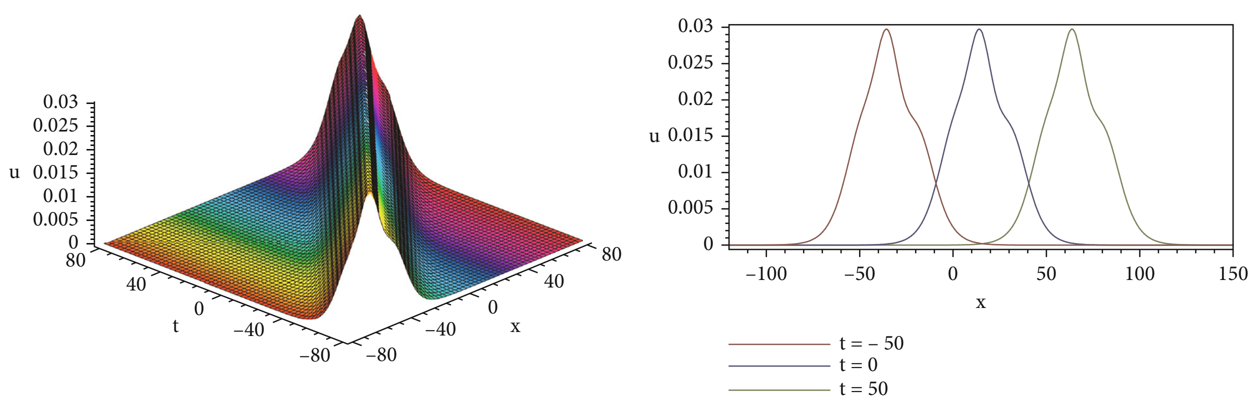
$$A_{ij} = \frac{2\alpha\left(3k_ik_j - 2M\right) + \beta\left(15k_ik_jM - 6M^2 - 8k_i^2k_j^2\right) + 2\sqrt{\alpha k_i^2 + \beta k_i^4 - \gamma}\sqrt{\alpha k_j^2 + \beta k_j^4 - \gamma} + 2\gamma}{-2\alpha\left(3k_ik_j + 2M\right) - \beta\left(15k_ik_jM + 6M^2 + 8k_i^2k_j^2\right) + 2\sqrt{\alpha k_i^2 + \beta k_i^4 - \gamma}\sqrt{\alpha k_i^2 + \beta k_i^4 - \gamma} + 2\gamma},$$
$$M = k_i^2 + k_j^2, \tag{11}$$

$$\omega_i^2 + \gamma k_i^2 - \alpha k_i^4 - \beta k_i^6 = 0. \tag{12}$$



(a) (b)

Figure 1: (a) Soliton molecule of the higher-order Boussinesq equation with the parameters (15). (b) Density plot of the corresponding soliton molecule.



(a) (b)

Figure 2: (a) Asymmetric soliton of the higher-order Boussinesq equation with the parameters (16). (b) The wave propagation pattern along x-axis by selecting different time $t = -50$, $t = 0$, $t = 50$.

We shall study the soliton molecule with a new resonance condition. The new resonance condition $(k_i \neq k_j)$ of velocity resonance reads

$$\frac{k_i}{k_j} = \frac{\omega_i}{\omega_j} = \frac{\sqrt{\alpha k_i^2 + \beta k_i^4 - \gamma}}{\sqrt{\alpha k_j^2 + \beta k_j^4 - \gamma}}. \quad (13)$$

By solving the above condition (13), the velocity resonant condition becomes

$$k_j = \pm \frac{\sqrt{-\beta\left(\alpha + \beta k_i^2\right)}}{\beta}. \quad (14)$$

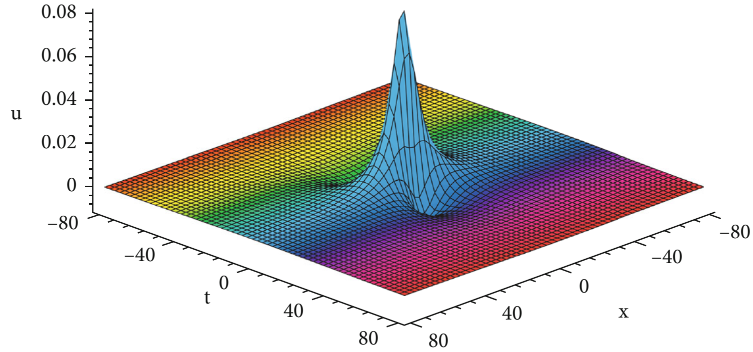
A soliton molecule and an asymmetric soliton can be constructed by selecting appropriate parameters in (13) or (14). We take two-soliton ($N = 2$) in (10) and the new resonance condition to describe these phenomena. For the Figure 1, the parameters are selected as

$$k_1 = \frac{1}{6}, k_2 = \frac{\sqrt{5}}{6}, \alpha = -\frac{1}{2}, \beta = 3, \gamma = -1, c_1 = 0, c_2 = 10. \quad (15)$$

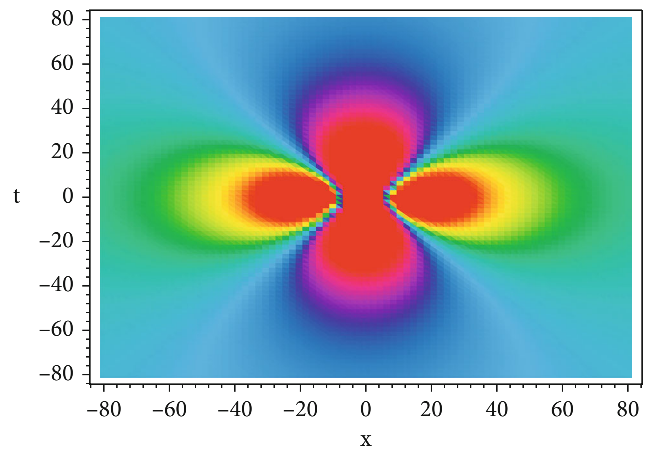
For the Figure 2, the parameters are selected as

$$k_1 = \frac{1}{6}, k_2 = \frac{\sqrt{5}}{6}, \alpha = -\frac{1}{2}, \beta = 3, \gamma = -1, c_1 = 0, c_2 = -3. \quad (16)$$

The soliton molecule and the asymmetric soliton are described in Figures 1 and 2, respectively. Two solitons in molecule are different amplitude, while two solitons in molecule possess the same velocity simultaneously.



(a)



(b)

FIGURE 3: (a) The three-dimensional plot of a lump wave of the higher-order Boussinesq equation with the parameters (21). (b) The corresponding density plot.

4. Lump Solution of the Higher-Order Boussinesq Equation

To obtain lump solution of the higher-order Boussinesq equation (1), we take a quadratic function f as

$$f=(a_2x+a_3t+a_4)^2+(a_5x+a_6t+a_7)^2+a_1. \tag{17}$$

By substituting (17) into the Hirota bilinear form (8) and balancing the different powers of x and t, the parameters are constrained as

$$a_1=\frac{3\alpha\left(a_2^2+a_5^2\right)}{\gamma},\ a_3=\sqrt{\gamma}a_5.a_6=-\sqrt{\gamma}a_2. \tag{18}$$

The function of u can be localized in (x,t)-plane with the parameters satisfying

$$\alpha>0,\gamma>0. \tag{19}$$

By substituting (17) into (7), a lump wave of (1) is generated

$$u=\frac{2A}{f}-\frac{4(Ax+B)^2}{f^2}, \tag{20}$$

with $a_2^2+a_5^2=A$, $a_2a_4+a_5a_7=B$. To describe the lump wave of the higher-order Boussinesq equation (1), the parameters are selected as

$$\alpha=16,\gamma=1,a_2=1,a_4=2,a_5=3,a_7=2. \tag{21}$$

The spatiotemporal structure and the corresponding density of a lump wave are described in Figures 3(a) and 3(b), respectively. The critical point of the lump wave can be calculated by solving

$$\frac{\partial u(x,t)}{\partial x}=0,\ \frac{\partial u(x,t)}{\partial t}=0. \tag{22}$$

By solving the above condition (22), we find that the function u reaches the maximum value at the point $(a_2a_7-a_4a_5/\sqrt{\gamma}A,-B/A)$ and the minimum values at two points $((a_2a_7-a_4a_5/\sqrt{\gamma}A),\sqrt{\gamma}B\pm 3\sqrt{\alpha}A)$. By substituting the above three points values into (20), the maximum and minimum values of the function u are $4\gamma/3\alpha$ and $-4\gamma^2(A^4B^2\gamma^3-3\gamma\alpha A^4+2A^3B\gamma^{5/2}(3\sqrt{\alpha}A^2+B)+\gamma^2A^2(3\sqrt{\alpha}A^2+B)^2)/(A^2B^2\gamma^3+3\gamma\alpha A^2+2AB\gamma^{5/2}(3\sqrt{\alpha}A^2+B)+\gamma^2(3\sqrt{\alpha}A^2+B)^2)^2$, respectively. We can only get the bright lump form of the higher-order Boussinesq equation by the above detail analysis.

5. Conclusion

In summary, the Painlevé property, the soliton molecule, and the lump solution of the higher-order Boussinesq equation (1) are studied by the standard WTC and the Hirota bilinear methods. The multisoliton solutions of (1) are obtained by introducing dependent variable transformation. The soliton molecule and the asymmetric soliton are constructed by the velocity resonance mechanism. The lump solution can be derived by using a positive quadratic function. By detail calculations of the maximum and minimum values of the function u, lump wave of the higher-order Boussinesq equation is just the bright form.

In this paper, the higher-order Boussinesq equation which possesses Painlevé integrability is constructed by introducing the Hirota bilinear operator D_x^6 based on the fourth-order Boussinesq equation. Similar introducing high-order Hirota bilinear operator procedure, we propose one equation

$$\left(D_t^2 + \gamma D_x^2 - \sum_{i=1}^{n} \left(\alpha_i D_x^{2+2i}\right)\right) f \cdot f = 0, \tag{23}$$

with α_i being arbitrary constants. The integrability-properties and nonlinear excitations of (23) are worthy to study further.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

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