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Research Article

Periodic Peakon and Smooth Periodic Solutions for KP-MEW(3,2) Equation

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In this paper, we consider the KP-MEW(3,2) equation by the bifurcation theory of dynamical systems when integral constant is considered. The corresponding traveling wave system is a singular planar dynamical system with one singular straight line. The phase portrait for $c < 0$, $0 < c < 1$, and $c > 1$ is drawn. Exact parametric representations of periodic peakon solutions and smooth periodic solution are presented.

1. Introduction

In 2012, Saha [1] considered KP-MEW equation:

$$\left(q_t \pm (q^m)_x \pm (q^n)_{xxt}\right)_x - q_{yy} = 0. \quad (1)$$

It is constructed by combining the MEW equation with the sense of the KP equation. He obtained the smooth and nonsmooth traveling wave solutions of KP-MEW Eq. (1) by using bifurcation theory of dynamical system. In his paper, he neglected integral constants when transferred (1) to an ordinary differential equation. However, if integral constant does not equal to zero, what is the traveling wave solution? For the arised question, we investigate the following KP-MEW(3,2) equation:

$$\left(q_t + (q^3)_x + (q^2)_{xxt}\right)_x - q_{yy} = 0, \quad (2)$$

by using bifurcation theory of dynamical systems [2–10], when integral constant is considered.

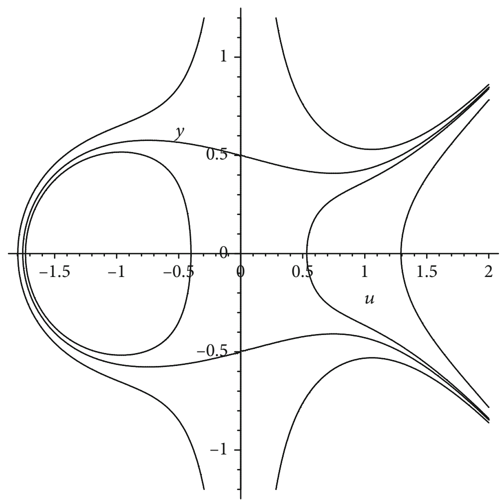
About the topic, Wei et al. [11] constructed the single peak solitary wave solutions of generalized KP-MEW (2,2) equation with boundary condition by using the differential equation qualitative theory. Li and Song [12] found the kink-type wave and compaction-type wave solutions of generalized KP-MEW (2,2) equation by using bifurcation method and a numerical simulation approach of dynamical systems. Zhong et al. [13] obtained the cuspons, peakons, compacton, smooth, and loop soliton solutions of generalized KP-MEW equation by applying the bifurcation theory technique. In [14], lie symmetry analysis was performed on generalized KP-MEW equation. The authors derived symmetries and adjoint representations for KP-MEW equation. Seadawy et al. [15] investigated the solitary wave solutions of generalized KP-MEW-Burgers equation by applying modification form of extended auxiliary equation mapping method.

For finding the traveling wave solutions of nonlinear partial differential equations, there are more plentiful methods to be adopted, such as mapping method and the extended mapping method [16], tanh-coth expansion method [17, 18], Darboux transformation [19], first integral method [20], and exp-expansion method [21]. Practically, there is no unified technique that can be employed to handle all types of nonlinear differential equations. It is important to discuss the dynamical behavior of a nonlinear traveling system. Therefore, in our paper, we try to find some new traveling wave solutions for Eq. (2) by bifurcation theory of dynamical system.

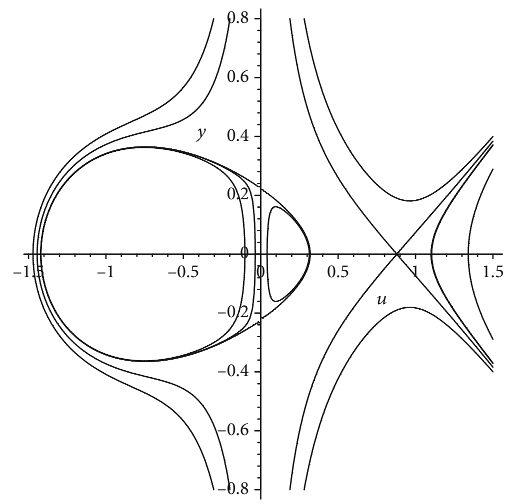
2. Phase Portraits

Making the transformations $q(x,t) = q(x-ct) = u(\xi)$, integrating it twice, (2) arrives to

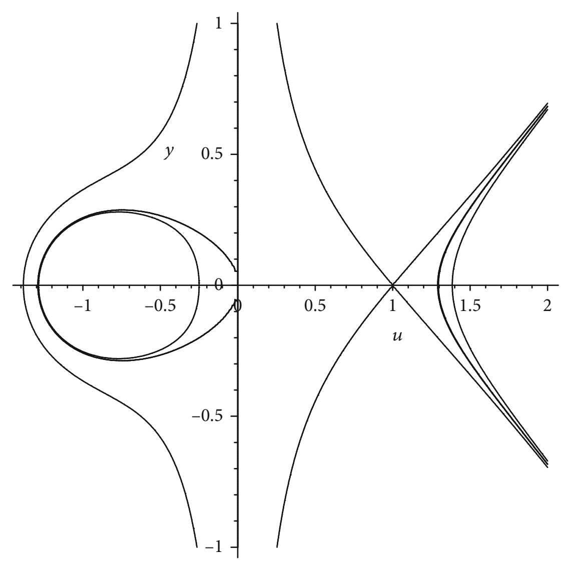
$$(1-c)u + u^3 - c\left(u^2\right)'' = g, \quad (3)$$



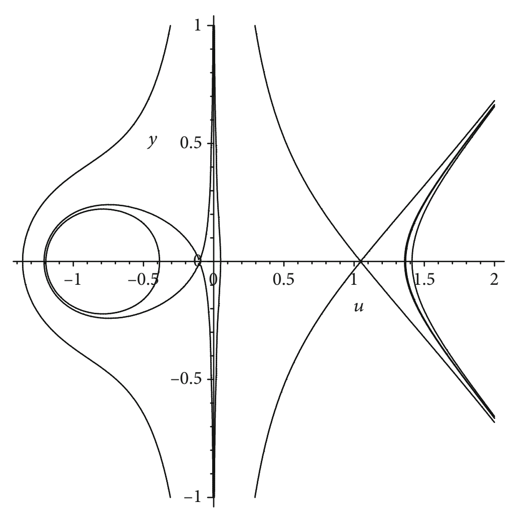
(a) $g \leq -g_1$



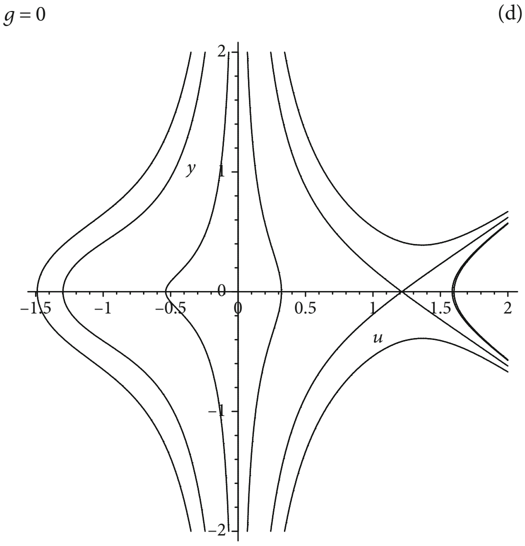
(b) $-g_1 < g < 0$



(c) $g = 0$



(d) $0 < g < g_1$



(e) $g \geq g_1$

Figure 1: Phase portraits of system (4) when $c > 1$.

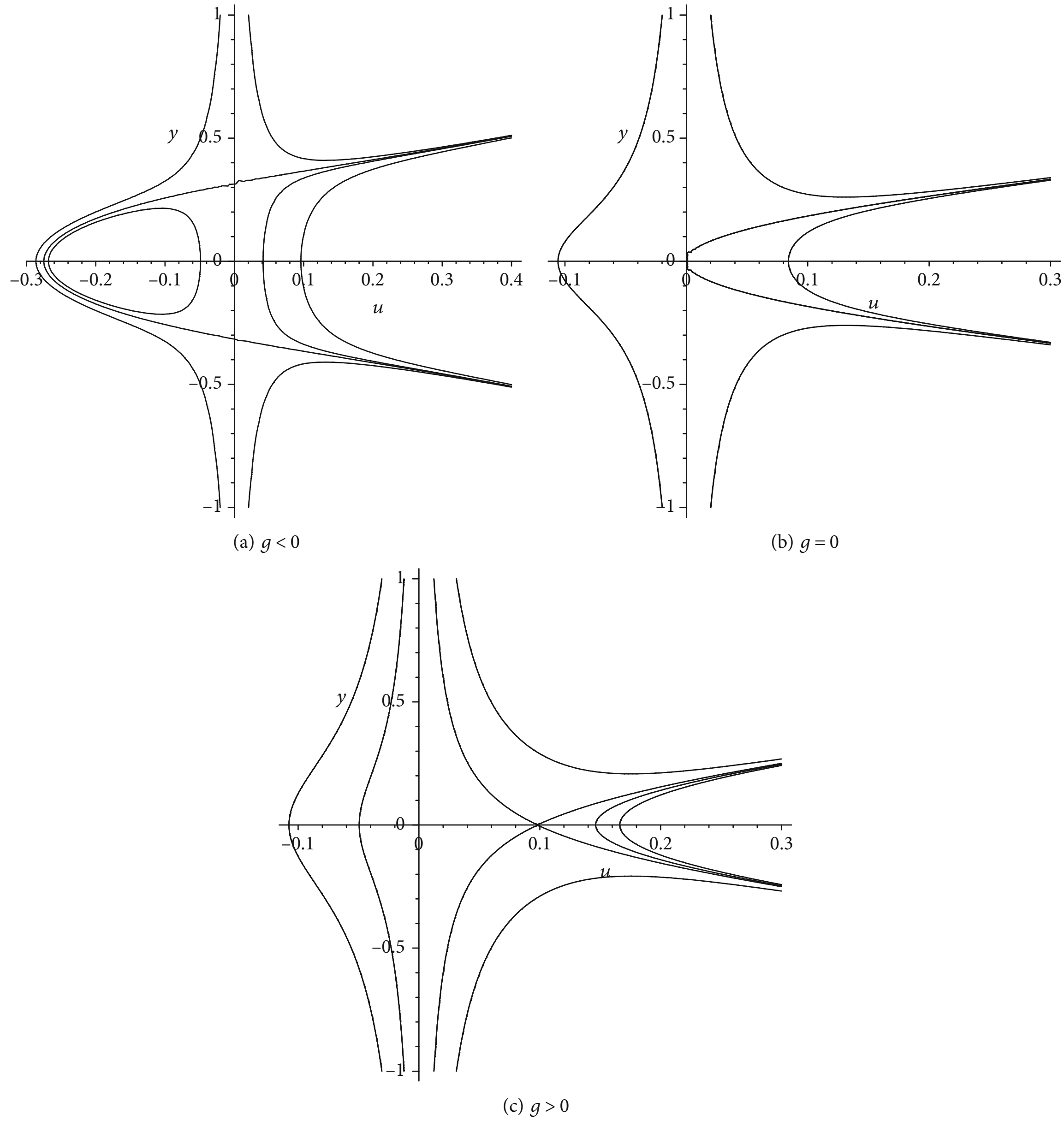


Figure 2: Phase portraits of system (4) when $0<c<1$.

where c is the wave speed, g is the integral constant, and $'$ is the derivative with respect to ξ.

Equation (3) is equivalent to the planar dynamical system

$$\begin{cases} \dfrac{du}{d\xi} = y, \\ \dfrac{dy}{d\xi} = \dfrac{-g+(1-c)u+u^3-2cy^2}{2cu}. \end{cases} \quad (4)$$

Using the transformation $d\xi = 2cud\tau$, (4) changes to

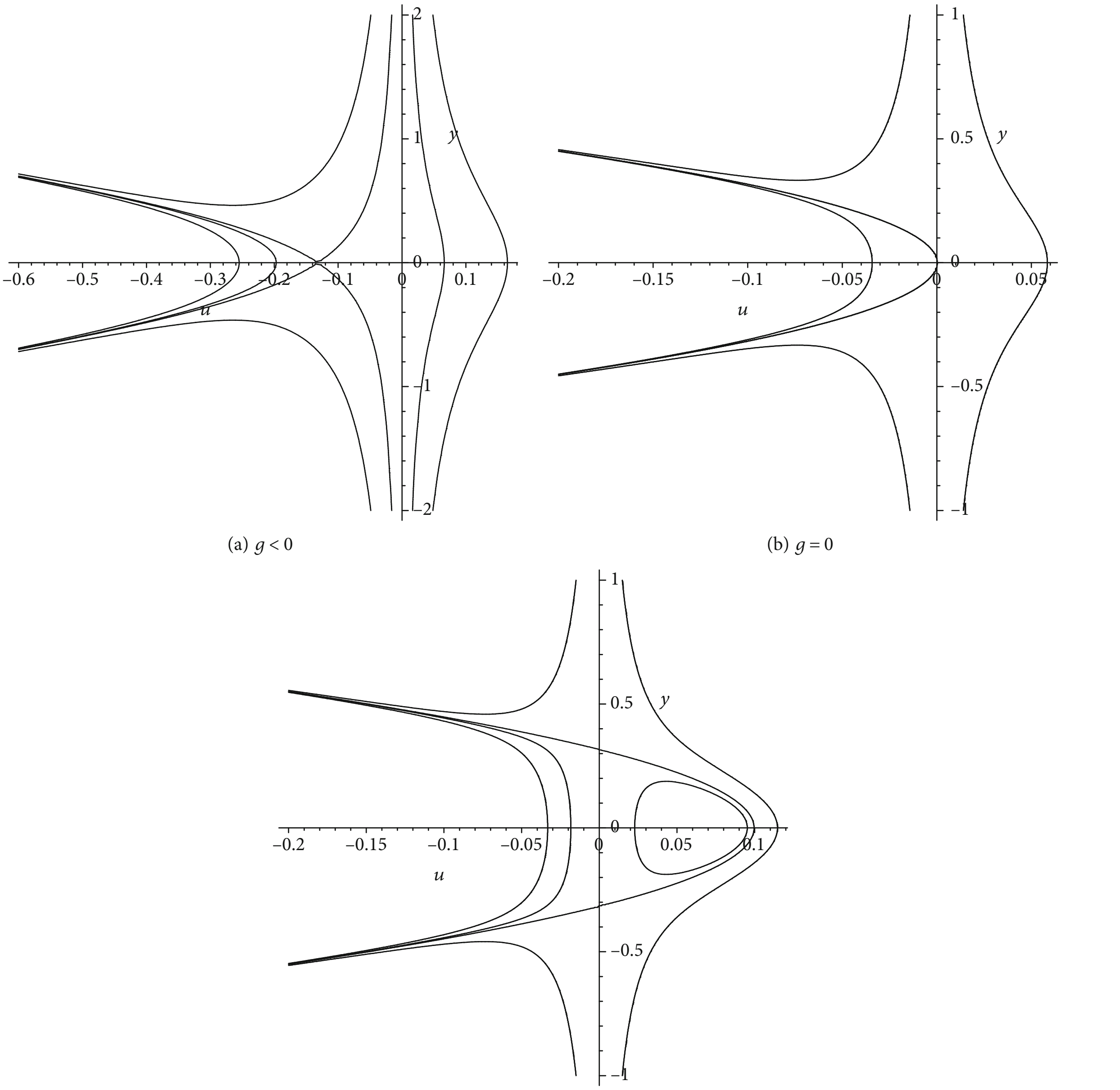
$$\begin{cases} \dfrac{du}{d\tau} = 2cuy, \\ \dfrac{dy}{d\tau} = -g+(1-c)u+u^3-2cy^2, \end{cases} \quad (5)$$

with the first integral

$$H(u,y) = cu^2y^2 + \left(\frac{g}{2}u^2 - \frac{1-c}{3}u^3 - \frac{1}{5}u^5\right) = h, \quad (6)$$

where h is an integral constant. Consequently, systems (4) and (5) have the same topological phase portraits except for the straight line $u=0$.

Setting $f(u) = -g+(1-c)u+u^3$, $\Delta = g^2/4+(1-c)^3/27$. Denote that $u_1 = (g/2+\sqrt{\Delta})^{1/3} + (g/2-\sqrt{\Delta})^{1/3}$, $u_2 = \rho(g/2+\sqrt{\Delta})^{1/3} + \rho^2(g/2-\sqrt{\Delta})^{1/3}$, and $u_3 = \rho^2(g/2+\sqrt{\Delta})^{1/3} + \rho(g/2-\sqrt{\Delta})^{1/3}$, where $\rho = -1+\sqrt{3}i/2$ is an imaginary number. Based on the formula of finding roots for cubic equations, we have the following conclusions: (i) if $\Delta>0$, $f(u)$ has unique equilibrium point $E_1(u_1,0)$; (ii) if $\Delta=0$,



(a) $g<0$

(b) $g=0$

(c) $g>0$

Figure 3: Phase portraits of system (4) when $c<0$.

$f(u)$ has two equilibrium points $E_1(u_1,0)$ and $E_2(u_2,0)$; (iii) if $\Delta<0, f(u)$ has three equilibrium points $E_1(u_1,0)$, $E_2(u_2,0)$, and $E_3(u_3,0)$. Especially, on the straight line $u=0$, there are two equilibrium points $S_{\pm}(0,\pm\sqrt{g/2c})$ for $gc>0$.

Let $M(u_e,y_e)$ be the coefficient matrix of the linearized system of (5) at an equilibrium point (u_e,y_e) and $J(u_e,y_e) = detM(u_e,y_e)$. Hence, it holds

$$J(u_e,y_e) = -8c^2y_e^2 - 2cu_e\left(1-c+3u_e^2\right). \tag{7}$$

It has the following proposition.

Proposition 1. *For an equilibrium point of a planar integrable system, if $J<0$, then the equilibrium point is a saddle point; if $J>0$, then it is a center point; if $J=0$ and the Poincare index of the equilibrium point is zero, then it is cusped.*

For convenience, if $c>1$, noted that $g_1=\sqrt{-4(1-c)^3/27}$ and $g_2=\sqrt{-80(1-c)^3/243}$. We will discuss the phase portrait for $c>1$, $0<c<1$, and $c<0$ (see on Figures 1–3).

3. Some Exact Parametric Representations of Solutions for System (4)

In this section, some traveling wave solutions of system (4) are given, with the exact parametric representations of its

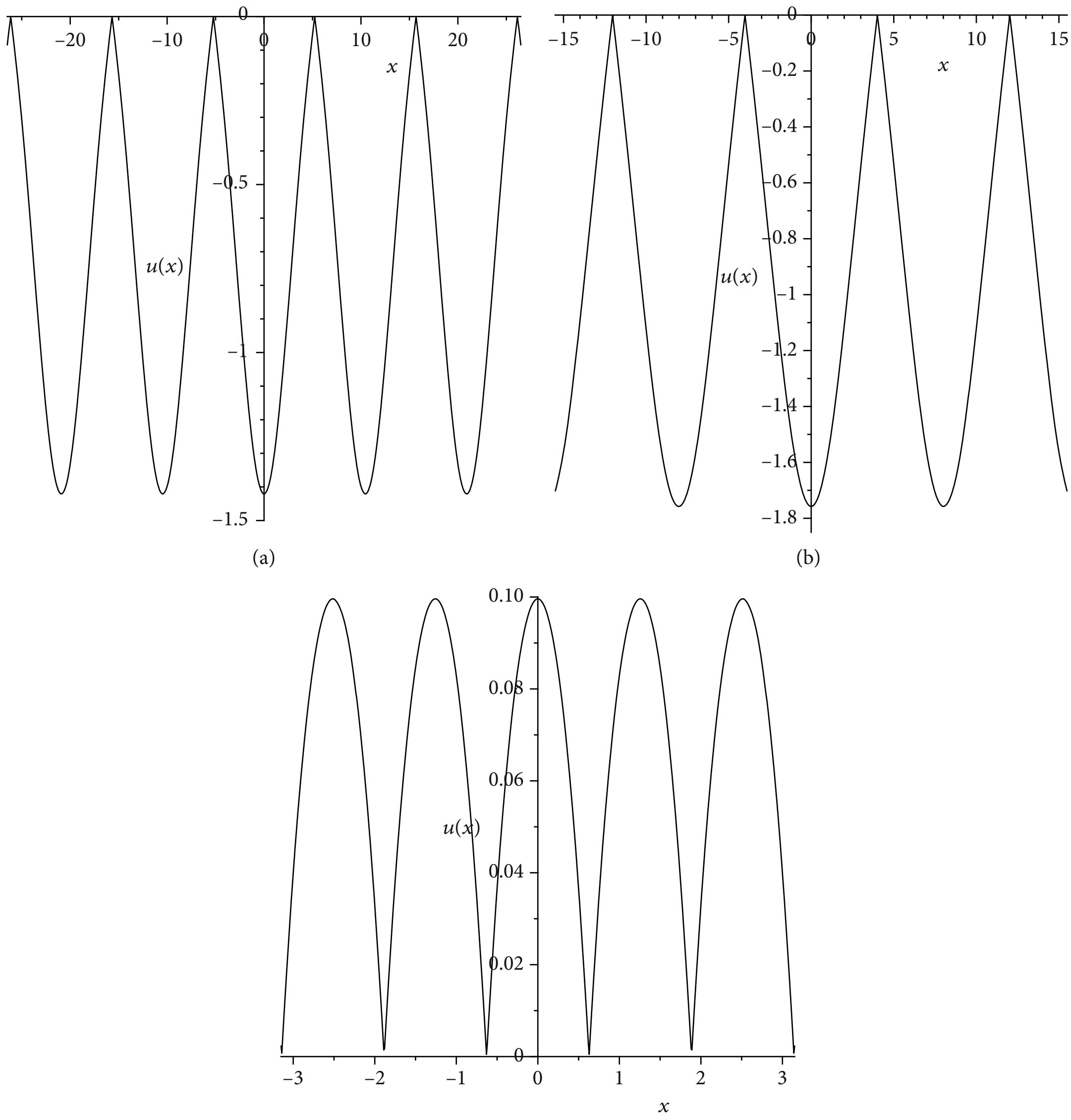


Figure 4: Periodic peakon solutions of (2).

solutions presented. With the aid of [22], we will obtain the exact traveling wave solutions of KP-MEW (3,2) equation.

3.1. Periodic Peakon Solutions

(i) In this subsection, the periodic peakon solution of system (4) is discussed; for the cases $c > 1, -g_2 < g < 0$, the phase portrait is shown as Figure 1(b); for $h = 0$, there are two arch orbits connecting $S_{\pm}(0,\pm\sqrt{g/2c})$ on both sides of the straight line $u = 0$. At the same time,

$$y^2 = \frac{1}{5c}\left(u^3 + \frac{5(1+c)}{3}u - \frac{5}{2}g\right) = \frac{1}{5c}(u-u_1)(u-u_2)(u-u_3), \tag{8}$$

with $u_1 < u_2 < u_3$. Consequently, from the first equation of (6), the parametric representations are as follows:

$$u(\xi) = u_1 + (u_2 - u_1)sn^2(\omega_0\xi, k_0), \tag{9}$$

where $sn(\omega_0\xi, k_0)$ is the Jacobin elliptic function with the modulo $k_0 = \sqrt{u_2 - u_1/u_3 - u_1}$, $\omega_0 = \sqrt{5c/4(u_3 - u_1)}$. The profile of periodic peakon solution (9) is shown in Figure 4(a).

(ii) For the cases $c > 1$, $g \le g_1$, the phase portrait is shown as Figure 1(a); for $h = 0$, there is an arch orbit connecting $S_{\pm}(0,\pm\sqrt{g/2c})$ on the left side of the straight line $u = 0$. (6) can be reduced to

$$y^2 = \frac{1}{5c}(u-u_4)(u-z_1)(u-z_2) = \frac{1}{5c}(u-u_4)\left[(u-b_1)^2 + a_1^2\right], \tag{10}$$

where $u_4 < u$, z_1, and z_2 are conjugate complex numbers. Thus, we can obtain the parametric representations of the periodic peakon solution (see Figure 4(b)).

$$u(\xi) = u_4 - \frac{Acn(\omega_1\xi, k_1) - A}{1 + cn(\omega_1\xi, k_1)}, \tag{11}$$

where $cn(x, k)$ is the Jacobin elliptic function, $A^2 = (b_1 - u_4)^2 + a_1^2$, $\omega_1 = \sqrt{A/5c}$, $k_1^2 = (A + b_1 - u_4)/2A$.

(iii) For the cases $0 < c < 1$, $g < 0$, the phase portrait is shown as Figure 2(a); for $h = 0$, there is an arch orbit connecting $S_\pm(0, \pm\sqrt{g/2c})$ on the left side of the straight line $u = 0$. From (6), we have

$$y^2 = \frac{1}{5c}(u-u_5)(u-z_3)(u-z_4), \tag{12}$$

where $u_5 < u$, z_3, and z_4 are conjugate complex numbers, so it has the parametric representations of periodic peakon solution as (11).

(iv) For the cases $c < 0$, $g < 0$, the phase portrait is shown as Figure 3(c); for $h = 0$, there is an arch orbit connecting $S_\pm(0, \pm\sqrt{g/2c})$ on the right side of the straight line $u = 0$. It holds

$$y^2 = \frac{1}{-5c}(u_6-u)(u-z_5)(u-z_6) = \frac{1}{-5c}(u_6-u)\left[(u-b_2)^2 + a_2^2\right], \tag{13}$$

where $u < u_6$, z_5, and z_6 are conjugate complex numbers, so it has the parametric representations of periodic peakon solution:

$$u(\xi) = u_6 + \frac{A - Acn(\omega_2\xi, k_2)}{1 + cn(\omega_2\xi, k_2)}, \tag{14}$$

where $cn(x, k)$ is the Jacobin elliptic function, $A^2 = (b_2 - u_6)^2 + a_2^2$, $\omega_2 = \sqrt{A/-5c}$, $k_2^2 = (A - b_2 + u_6)/2A$. The profile of periodic peakon solution (14) is shown in Figure 4(c).

3.2. Smooth Periodic Solution. The smooth periodic solution of system (4) is discussed; for the cases $c > 1$, $g = 0$, the phase portrait is shown as Figure 1(c); for $h = 0$, there is a homoclinic orbit enclosing the equilibrium point $E_1(u_1, 0)$ and connecting the straight line $u = 0$. At the same time,

$$y^2 = \frac{1}{5c}\left(u^3 + \frac{5(1+c)}{3}u - \frac{5}{2}g\right) = \frac{1}{5c}u\left(u^2 - u_7^2\right), \tag{15}$$

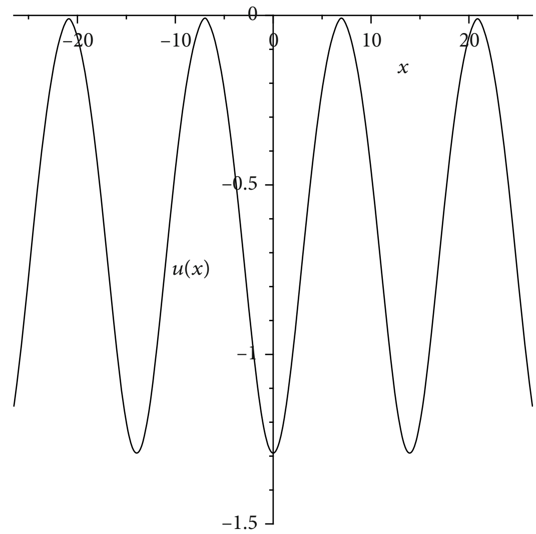


Figure 5: Smooth periodic solution of (2).

with $0 < u_7 < u$. Consequently, from the first equation of (6), the parametric representations are as follows:

$$u(\xi) = -u_7 + u_7 sn^2\left(\sqrt{\frac{u_7}{10c}}\xi, \sqrt{\frac{1}{2}}\right), \tag{16}$$

where $sn(x, k)$ is the Jacobin elliptic function. The profile of smooth periodic solution (16) is shown in Figure 5.

Remark 2. All the phase portrait bifurcations and the traveling wave solutions obtained of KP-MEW (3,2) equation in presented paper were not mentioned in [1].

4. Conclusion

In present paper, the method of bifurcation theory of dynamical systems is used to investigate KP-MEW (3,2) equation. We obtain the parametric representations of periodic peakon and smooth periodic wave solutions. The phase portrait bifurcation of the traveling wave system corresponding to the equation is shown.

Data Availability

All the underlying data supporting the results of your study can be found.

Conflicts of Interest

The authors declare that there is no conflict of interests.

Acknowledgments

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