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Research Article

The Existence of Strong Solution for Generalized Navier-Stokes Equations with $p(x)$-Power Law under Dirichlet Boundary Conditions

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In this note, in 2D and 3D smooth bounded domain, we show the existence of strong solution for generalized Navier-Stokes equation modeling by $p(x)$-power law with Dirichlet boundary condition under the restriction $(3n/(n+2)n+2) < p(x) < (2(n+1))/(n-1)$. In particular, if we neglect the convective term, we get a unique strong solution of the problem under the restriction $(2(n+1))/(n+3) < p(x) < (2(n+1))/(n-1)$, which arises from the nonflatness of domain.

1. Introduction

In this note, we consider the steady flows of non-Newtonian fluids in $\mathbb{R}^n$, $n = 2, 3$, which is modeled by the following system:

$$\begin{cases} -\operatorname{div}\left(\left(1+|\mathscr{D}u|^2\right)^{(p(x)-2)/2}\mathscr{D}u\right) + (u\cdot\nabla)u + \nabla\pi = f, & \text{in } \Omega, \\ \operatorname{div}\ u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where u is the velocity, π the pressure, f the external force, $\mathscr{D}u := (\nabla u + \nabla u^T)/2$, Ω a bounded domain, and $p(x) > 1$ a prescribed function.

This system arises from flows of electrorheological [1], thermorheological [2], chemically reacting non-Newtonian fluids [3].

For the existence of weak solutions to the problem (1), we refer to [1, 4].

Local higher differentiability for weak solutions to the problem (1) with $p(x) = \text{const}$ has first shown in [5] for $1.8 < p < 2$ in 3D. This result was improved to $p = 1.8$ in [6] by Wolf. For$p > 2$in 3D,$W^{2,2}_{\text{loc}}$-regularity is proved (see [7]).

Global higher differentiability for weak solutions to the problem (1) with $p(x) = \text{const}$ have been studied by several authors; for example, see [8–19] under the condition $f \in L^{p\wedge'}(\Omega)$ with $p\wedge' = \widehat{p}/(\widehat{p}-1)$ and $\widehat{p} := \min\{p, 2\}$. It was first established in [8] by Beirao da Veiga. He developed a crucial device which was to denote the second-order derivatives of the velocity in the normal direction through ones (and the first-order derivatives of the pressure) in the tangential directions by using the very explicit form of the main equations. But in contrast to interior regularity, the interaction between pressure and nonlinearity of leading term results in the lower regularity for the second-order derivatives of the velocity (and for the first-order derivatives of the pressure) in the normal direction, in comparison to the tangential directions. His idea reveals to be quite fruitful in many subsequent papers. He [10, 11] studied global higher differentiability of weak solution to the problem with the boundary condition (3) for $p > 15/8$ in 3D cubic domain.

With the help of the anisotropic Sobolev embedding theorem, Berselli [15] obtained an improved integrability of velocity gradient than in [11] in 3D cubic domain. His idea is that it is possible to apply the anisotropic Sobolev embedding theorem because of the difference in the regularity levels between the second-order derivatives of the velocity in the normal direction and the tangential ones. Beirao da

Veiga [9] showed global $W^{2,(4p-2)/(p+1)} \cap W^{1,4p-2}$-regularity for $(20/11) < p < 2$ by combining the idea from [15] with a delicate estimate on the convective term in 3D cubic domain and then in [12] extended it to nonflat boundaries. In [19], Crispo proved the same type of results in cylindrical domains. In [14], the authors showed global $W^{2,q}$-regularity for shear-thickening flows, i.e., $p > 2$ in n-D bounded smooth domain.

Recently, global higher differentiability of weak solution to the problem (1) in 3D smooth domain is studied by us in [20] by using a global higher integrability condition, which holds under the condition $f \in L^{\alpha p \wedge'(x)}(\Omega)$, where $p\wedge'(x) = (\widehat{p}(x))/(\widehat{p}(x)-1)$, $\widehat{p}(x) \coloneqq \min\{p(x), 2\}$, and $\alpha > 1$. This is slightly stronger rather than the standard condition $f \in L^{p\wedge'}(\Omega)$ for the case $p = \text{const}$. On the other hand, local higher differentiability of local weak solution to the problem (1) in 3D has been obtained in [21, 22] by relying on the local higher integrability result from [23].

In [24], the existence and uniqueness of $C^{1,\gamma}(\bar{\Omega}) \cap W^{2,2}(\Omega)$-solution corresponding to small data are proved, without further restrictions on the bounds on $p(x)$.

For interior or boundary partial regularity, we refer to [23, 25–27].

If one assumes the condition $f \in L^{p\wedge'(x)}(\Omega)$ for the problem (1), then when applying difference quotient, due to the $p(x)$-dependence of leading term, the additional term will appear:

$$\int \left(1 + |\mathscr{D}u|^2\right)^{(p(x)-2)/2} |\mathscr{D}u|^2 \log^2\left(1 + |\mathscr{D}u|^2\right) dx, \tag{2}$$

which cannot be estimated in terms of a priori estimate on weak solutions. So for the system (1) with $p \neq \text{const}$, the existence of strong solutions has been studied. In 3D, the existence of local strong solutions to the system (1) is first shown for $1.8 < p(x) < 6$ in [1] (chapter 3) by Ruzicka. In [28], Ettwein and Růžička showed the existence of $W^{2,(3\widehat{p}(x))/(\widehat{p}(x)+1)}_{\text{loc}}$-solutions without the artificial upper bound $p(x) < 6$. For 2D bounded domains, we refer to [29, 30].

In [31, 32], we gain the existence of strong solution for the system (1) under the standard assumption $f \in L^{p\wedge'(x)}(\Omega)$. But in that case, we consider the following the boundary condition: for $\Omega = (0,1)^n$, $n = 2, 3$, and $\Gamma \coloneqq \{x \in \partial\Omega : |x_1|,|x_{n-1}| < 1, x_n = 0 \text{ or } x_n = 1\}$,

$$u|_\Gamma = 0, u \text{ is } x' - \text{periodic}, \tag{3}$$

where by x'-periodic, we mean periodic of period 1 both in x_1 and x_{n-1}. This allows us to consider a bounded domain and simultaneously a flat boundary. Thus, it is natural to ask whether the sharp results proved in [31, 32] are valid for smooth domain. This is the aim of this note.

It seems to be possible to obtain the existence of $C^{1,\alpha}(\bar{\Omega})$-strong solution to the problem (1) in 2D by the result of this note and the same argument as in [32]. Very recently, we [32] show the result in the case of the boundary condition (3). For $C^{1,\alpha}(\Omega)$-regularity in 2D, we refer to [18, 29, 30, 33–36].

Set

$$\Omega_1 \coloneqq \{x \in \Omega \mid p(x) < 2\}, \Omega_2 \coloneqq \Omega \setminus \Omega_1. \tag{4}$$

For $n = 3$, we define

$$\bar{p}(x) \coloneqq 3p(x) - |p(x) - 2| - \mu, \ \bar{r}(x) \coloneqq \begin{cases} \dfrac{2\bar{p}(x)}{2(2-p(x)) + \bar{p}(x)}, & \text{if } x \in \Omega_1, \\ \dfrac{2\bar{p}(x)}{p(x) + \bar{p}(x) - 2}, & \text{if } x \in \Omega_2, \end{cases} \tag{5}$$

where μ is arbitrarily close to 0 for $p \neq \text{const}$, and for $n = 2$,

$$\bar{p}(x) \equiv q, \bar{r}(x) \coloneqq \begin{cases} 2 - \mu_0, & \text{if } p(x) \neq 2, \\ 2, & \text{if } p(x) = 2, \end{cases} \tag{6}$$

where q is arbitrary real number such that $1 < q < \infty$ and $\mu_0 > 0$ arbitrary close to 0.

The main results are as follows.

Theorem 1. *Let $n = 2, 3$. Assume that $\partial\Omega \in C^{2,1}$, $p(x) \in C^{0,1}(\bar{\Omega})$, $p \neq const$, and*

$$\frac{3n}{n+2} < p(x) < \frac{2(n+1)}{n-1}, \tag{7}$$

and $f \in L^{p\wedge'(x)}(\Omega)$ for

$$\widehat{p}(x) \coloneqq \min\{p(x), 2\}. \tag{8}$$

Then, for $\bar{p}(x), \bar{r}(x)$ from (5) and (6), there is a strong solution (u, π) to the problem (1) satisfying

$$\|\pi\|_{\bar{p}(x),\Omega} + \|u\|_{1,\bar{p}(x),\Omega} \leq C, \tag{9}$$

$$\|\nabla\pi\|_{\bar{r}(x),\Omega} + \left\|\nabla^2 u\right\|_{\bar{r}(x),\Omega} \leq C, \tag{10}$$

where the constants C depend on $p_-, p_+, \|f\|_{\widehat{p}}{}'^{(x)}$.

Moreover, for the problem (1) without the convective term, there is a unique strong solution satisfying (9) and (10) provided that

$$\frac{2(n+1)}{n+3} < p(x) < \frac{2(n+1)}{n-1}. \tag{11}$$

Remark 2. We note that if $p = \text{const}$, then the condition (11) will be no longer needed (see [12–14, 16, 20]).

But due to $p(x)$-dependence of the leading term, we cannot obtain $W^{2,q}$-regularity of weak solution to (1) provided $f \in L^{p\wedge'(x)}(\Omega)$. So it is customary to consider an approximate problem. Fortunately, in [31, 32], the condition (11) does not appear but we consider the boundary condition (3). The condition (11) arises from the nonflatness of Ω and

Dirichlet boundary condition. More precisely, due to them, there appears the term

$$\left\| \left(\frac{1+|\mathscr{D}u_\lambda|^2}{1+\lambda|\mathscr{D}u_\lambda|^2} \right)^{(p(x)-2)/2} \mathscr{D}u_\lambda \right\|_{\widehat{p}'(x)}, \quad (12)$$

in deriving of $W^{2,q}$-regularity of the approximate solutions independent of parameter (see (41) and (45)). The term disappears in the case (3). For the estimate of the term, we need an additional condition (11). It is open whether the condition can be removed when one considers the approximate problem with Dirichlet boundary condition over nonflat domain. In fact, Kaplicky ([37], Lemma 4.2) showed (9) and (10) for the approximate problem to (1) with $p = \text{const} \in (2, 4)$ in 2D.

Due to $p(x)$-dependence of the extra stress tensor, we consider an approximate problem with $((1+|\mathscr{D}u_\lambda|^2)/(1+\lambda|\mathscr{D}u_\lambda|^2))^{(p(x)-2)/2}\mathscr{D}u_\lambda$ for $\lambda \in (0,1)$ instead of $(1+|\mathscr{D}u|^2)^{(p(x)-2)/2}\mathscr{D}u$. It is easy to see that the approximate solution belongs to $\mathscr{V}_2(\Omega) \cap W^{2,2}(\Omega)$ owing to the trivial inequality $1 \le (1+a^2)/(1+\lambda a^2) \le 1/\lambda$. The main point is to derive the estimates about all derivatives of the approximation solutions in suitable Sobolev spaces, which are independent of parameter. This allows us to show convergence of approximation solutions to the one to problem (1).

The paper is organized as follows. In Section 2, we give preliminaries. Section 3 is devoted to prove the main results.

2. Preliminaries

2.1. Notations. By $p'(x)$, we denote the conjugate function of $p(x)$. For $p \in L^\infty(\Omega)$, $p \ge 1$, define $p_- := \operatorname{essinf}\, p(x)$, $p_+ := \operatorname{esssup}\, p(x)$. Let p^* be a Sobolev conjugate exponent, i.e., $p^* = np/(n-p)$ when $p < n$ and $p^* = q$ for all $q \in (1,\infty)$ when $p = n$ and $p^* = \infty$ otherwise.

For $n \times n$-matrices F, H denote $F : H = \sum_{i,j=1}^n F_{ij}H_{ij}$, $|F| \equiv (F : F)^{1/2}$. For two vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} \otimes \mathbf{b} = \{\mathbf{a}_i \mathbf{b}_j\}_{ij}$ and $\mathbf{a}$ e$\mathbf{b} := 1/2(\mathbf{a} \otimes \mathbf{b} + (\mathbf{a} \otimes \mathbf{b})^T)$.

For $p \in L^\infty(\Omega)$, $p \ge 1$, the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined by

$$L^{p(x)}(\Omega) := \left\{ u : \Omega \longrightarrow \mathbb{R} \mid u \text{ is measurable and } \rho_{p(x)}(u) := \int_\Omega |u|^{p(x)} dx < \infty \right\}, \quad (13)$$

endowed with the norm $\|u\|_{p(x),\Omega} := \inf \{ \lambda > 0 \mid \rho_{p(x)}(u/\lambda) \le 1 \}$. Then, we define the variable exponent Sobolev space by

$$W^{k,p(x)}(\Omega) := \left\{ u \mid \nabla^\alpha u \in L^{p(x)}(\Omega), \forall |\alpha| \le k \right\}, \quad (14)$$

with the norm $\|u\|_{k,p(x),\Omega} := \sum_{|\alpha|\le k} \|\nabla^\alpha u\|_{p(x),\Omega}$. We define $W_0^{k,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$. Let $W^{-1,p'(x)}(\Omega)$ be dual of $W_0^{1,p(x)}(\Omega)$.

We do not distinguish between scalar, vector-valued, and tensor-valued function spaces in the notations. Define

$$\mathscr{V}_{p(x)} := \left\{ u \in W_0^{1,p(x)}(\Omega) \mid \operatorname{div}\ u = 0 \right\}. \quad (15)$$

Definition 3. We say that function u is a weak solution to the problem (1) if $u \in \mathscr{V}_{p(x)}$ and it satisfies

$$\begin{aligned} &\int_\Omega \left(1+|\mathscr{D}u|^2\right)^{(p(x)-2)/2} \mathscr{D}u : \mathscr{D}\phi dx + \int_\Omega (u\cdot\nabla)u\cdot\phi dx \\ &\quad = \int_\Omega f\cdot\phi dx, \quad \forall \phi \in \mathscr{V}_{p(x)}. \end{aligned} \quad (16)$$

We refer to the term strong solution as a weak solution which additionally satisfies $u \in W^{2,q}(\Omega)$ for some $1 \le q < \infty$.

2.2. Some Problems Related to Flattening of the Boundary. As before, our problem is reduced to a problem involving a flat boundary by a suitable change of variables. Here, we follow the arguments and notations in [14]. Since $\partial\Omega \in C^{2,1}$, for each point $P \in \partial\Omega$, there are local coordinates such that in these coordinates, we have $P = 0$ and $\partial\Omega$ is locally described by a $C^{2,1}$-function $\theta_P : Q_d^{n-1}(0) \longrightarrow Q_d^1(0)$, where $Q_d^k(0)$ is the k-dimensional cubic with center 0 and length $2d$ (which is small enough and will be fixed later), with the following properties:

$$\begin{aligned} &x \in \Omega_P := \Omega \cap \left(Q_d^{n-1}(0) \times Q_d^1(0)\right) \Longleftrightarrow x_n > \theta_P(x_1, \cdots x_{n-1}), \\ &\nabla\theta_P(0) = 0, |\nabla\theta_P(y)| < cd, \quad y \in Q_d^{n-1}(0). \end{aligned} \quad (17)$$

As $\partial\Omega$ is a compact, there exist a finite set of points $\Gamma \subset \partial\Omega$ and an open set $\Omega_0 \subset\subset \Omega$ such that $\Omega \subset \Omega_0 \cup \bigcup_{P\in\Gamma} \Omega_P$. We construct a partition of unity $\{\zeta_0, \zeta_P, P \in \Gamma\}$, corresponding to this covering, such that $\operatorname{dist}(\operatorname{supp} \zeta_P, \partial\Omega_P \setminus \partial\Omega) \ge h_0$ for all $P \in \Gamma$ and some suitable small $h_0 > 0$. Let us fix some $P \in \Gamma$.

Set $x' := (x_1, \cdots x_{n-1})$. For $h < h_0, i \in \{1, 2\}$, and a function φ with $\operatorname{supp}\varphi \subset \operatorname{supp}\zeta$, we define tangential translation through

$$\varphi_\tau\left(x', x_n\right) := \varphi\left(x' + he^i, x_n + \theta_P\left(x' + he^i\right) - \theta_P\left(x'\right)\right) \quad (18)$$

and tangential derivative through

$$\partial_\tau\varphi := \lim_{h\longrightarrow 0} \frac{\varphi_\tau - \varphi}{h}. \quad (19)$$

Now, we give the two propositions below related to the tangential derivatives.

Proposition 4 (see [12, 14]). *Let* supp $\varphi \subset$ supp ζ_P *and* $\varphi \in W_0^{1,1}(\Omega_P)$. *Then,*

$$\begin{aligned} \nabla \partial_\tau \varphi &= \partial_\tau \nabla \varphi + (\partial_n \varphi)_\tau \otimes \partial_\tau \nabla \theta_P, \\ \mathscr{D} \partial_\tau \varphi &= \partial_\tau \mathscr{D} \varphi + (\partial_n \varphi)_\tau \odot \partial_\tau \nabla \theta_P, \\ div\, \partial_\tau \varphi &= \partial_\tau div\, \varphi + (\partial_n \varphi)_\tau \cdot \partial_\tau \nabla \theta_P. \end{aligned} \tag{20}$$

Proposition 5 (see [20]). *Let* $1 < p(x) < \infty$, $p(x) \in C^{0,1}(\Omega_P)$. *Then, there holds*

$$\|\zeta_P \partial_\tau \nabla u\|_{p(x)} \le c\|\zeta_P \partial_\tau \mathscr{D} u\|_{p(x)} + c|\nabla \zeta_P| \|\nabla u\|_{p(x)}, \quad \forall u \in W_0^{1,p(x)}(\Omega). \tag{21}$$

3. The Proof of Main Results

We use universal constants $c, C > 0$, which may vary in different occurrences. In particular, C depends on p_-, p_+, Ω, n, $\|\nabla p(x)\|_\infty$, $\|f\|_{\widehat{p}}{}'^{(x)}$, while c on $p_-, p_+, \Omega, n, \|\nabla p(x)\|_\infty$.

To begin with, let us define

$$\begin{aligned} Y_\lambda &:= \frac{1 + |\mathscr{D} u_\lambda|^2}{1 + \lambda |\mathscr{D} u_\lambda|^2}, \qquad M_\lambda := Y_\lambda^{(p(x)-2)/2}, \\ S_\lambda(x, \mathscr{D} u_\lambda) &:= M_\lambda \mathscr{D} u_\lambda, \quad V_{p(x)}^\lambda(\mathscr{D} u_\lambda) := M_\lambda^{1/2} \mathscr{D} u_\lambda. \end{aligned} \tag{22}$$

As before, in order to prove the main result, i.e., Theorem 1, we will consider the following approximate problem:

$$\begin{cases} -\text{div } S_\lambda(x, \mathscr{D} u_\lambda) + (u_\lambda \cdot \nabla) u_\lambda + \nabla \pi_\lambda = f, & \text{in } \Omega, \\ \text{div } u_\lambda = 0, & \text{in } \Omega, \\ u_\lambda = 0, & \text{on } \partial\Omega, \end{cases} \tag{23}$$

where $\lambda \in (0, 1]$. Let us denote

$$L_0^2(\Omega) := \left\{ f \in L^2(\Omega), \int_\Omega f dx = 0 \right\}. \tag{24}$$

It is known that there exists a weak solution $(u_\lambda, \pi_\lambda) \in \mathscr{V}_2(\Omega) \times L_0^2(\Omega)$ to the problem (23) by compactness method if $f \in L^2(\Omega)$. Furthermore, the following facts are valid.

Proposition 6. *Let* u_λ *be a weak solution to the problem (23). Assume that* $f \in L^{p\wedge'(x)}(\Omega)$ *and* $3/2 \le p(x) < \infty$. *Then, the following hold:*

$$\int_\Omega M_\lambda |\mathscr{D} u_\lambda|^2 dx + \int_\Omega |\mathscr{D} u_\lambda|^{p\wedge(x)} dx \le c \int_\Omega |f|^{p\wedge'(x)} dx + c|\Omega|, \tag{25}$$

$$\left\| M_\lambda^{1/2} \right\|_{(2p(x))/(p(x)-2), \Omega_2} + \left\| M_\lambda^{-(1/2)} \right\|_{(2p(x))/(2-p(x)), \Omega_1} \le C, \tag{26}$$

where $\widehat{p}(x)$ *is from (8). For the problem (23) without convective term, these are valid for all* $1 < p(x) < \infty$.

Remark 7. In fact, the proposition above was proved in ([31], Lemma 4.1) and [32] for the boundary condition (3). But it is easy to see that these inequalities hold also for Dirichlet boundary condition.

Moreover, noting that

$$1 \le \frac{1 + a^2}{1 + \lambda a^2} \le \frac{1}{\lambda}, \quad \text{for } \lambda \in (0, 1], \tag{27}$$

we can prove by the same line in [20] that if $f \in L^2(\Omega)$, $p(x) \in C^{0,1}(\bar{\Omega})$, then a weak solution u_λ, π_λ of the problem (23) satisfies

$$\begin{aligned} u_\lambda &\in \mathscr{V}_2(\Omega) \cap W^{2,2}(\Omega), \\ \pi_\lambda &\in L_0^2(\Omega) \cap W^{1,2}(\Omega). \end{aligned} \tag{28}$$

However, the norms of u_λ in $\mathscr{V}_2(\Omega) \cap W^{2,2}(\Omega)$ and π_λ in $L_0^2(\Omega) \cap W^{1,2}(\Omega)$ are dependent of λ.

Thus, from now on, we focus on the estimates about the derivatives of the approximation solutions in suitable Sobolev spaces, which are independent of parameter. This allows us to show convergence of the approximation solutions to the one to problem (1) in the spaces.

Hereafter, all constants are independent of parameter λ and introduce a shorthand notation $S_\lambda := S_\lambda(x, \mathscr{D} u_\lambda)$ if it will be clear from the context.

Let the assumptions of Theorem 1 hold.

To prove Theorem 1, we will combine the methods in [20, 31, 32].

The proof is divided into two cases: with and without the convective term.

3.1. The Proof of Theorem 1 without the Convective Term

Step 1. Estimates of the approximate solutions independent of parameter in tangential directions.

In this step, our aim is to prove that

$$\int M_\lambda |\partial_\tau \mathscr{D} u_\lambda|^2 \zeta^2 dx \le \left(C + c\|f\|_{p\wedge'(x)}^2 \right) + cT_\lambda + c\|S_\lambda\|_{\widehat{p}'(x)}^2, \tag{29}$$

$$\|\partial_\tau \pi_\lambda \zeta\|_{\check{p}'(x)} \le \left(C + c\|f\|_{p\wedge'(x)} \right) + C\left(A_\lambda^{1/2} + T_\lambda^{1/2} + c\|S_\lambda\|_{p\wedge'(x)} \right), \tag{30}$$

where $\widehat{p}(x)$ is from (8), $\check{p}(x) := \max\{2, p(x)\}$, and

$$A_\lambda := \int M_\lambda |\partial_\tau \mathscr{D} u_\lambda|^2 \zeta^2 dx, \; T_\lambda := \int M_\lambda |\mathscr{D} u_\lambda|^2 \log^2 Y_\lambda dx. \tag{31}$$

Fix $P \in \Gamma$, and let $\zeta = \zeta_P$, $\Omega = \Omega_P$, and $\theta = \theta_P$ be as in Subsection 2.2. For simplicity we will omit the symbol "Ω_P" in integration ones and denote the symbols "$\Omega_P \cap \Omega_i$," $i = 1, 2$, by "Ω_i."

By ([20], Proposition 14.3.15) and Propositions 4 and 5, there exists a solution $\psi \in W_0^{1,\hat{p}(x)}(\Omega)$ to the problem

$$\operatorname{div} \psi = -\operatorname{div} \partial_\tau \left(\zeta^2 \partial_\tau u_\lambda\right), \text{ in } \Omega ; \psi = 0, \text{ on } \partial\Omega, \tag{32}$$

satisfying

$$\|\mathscr{D}\psi\|_{\hat{p}(x)} \le c\|\zeta \partial_\tau \mathscr{D} u_\lambda\|_{\hat{p}(x)} + c\|\nabla u_\lambda\|_{\hat{p}(x)}, \tag{33}$$

where the constant c depends on p_-, p_+, Ω, n.

Multiplying the first equations in (23) by $\phi = \partial_\tau(\zeta^2 \partial_\tau u_\lambda) + \psi$, integrating by parts and using Proposition 4, we get

$$\begin{aligned}\int \zeta \partial_\tau S_\lambda : \mathscr{D}(\zeta \partial_\tau u_\lambda) dx = &- \int \partial_\tau S_\lambda \\ &: (\nabla\zeta \odot \zeta \partial_\tau u_\lambda) dx - \int S_\lambda \\ &: \left(\left(\partial_n \left(\zeta^2 \partial_\tau u_\lambda\right)\right) \odot \partial_\tau \nabla\theta\right) dx + \int f \cdot \left[\partial_\tau \left(\zeta^2 \partial_\tau u_\lambda\right) + \psi\right] dx - \int_\Omega S_\lambda \\ &: \mathscr{D}\psi dx =: \sum_{i=1}^{4} I_i.\end{aligned} \tag{34}$$

We apply Proposition 4 to the left hand side of (34) to get

$$\begin{aligned}I_0 := \int \zeta \partial_\tau S_\lambda : \mathscr{D}(\zeta \partial_\tau u_\lambda) dx = &\int \zeta^2 \partial_\tau S_\lambda \\ &: \partial_\tau \mathscr{D} u_\lambda dx + \int \zeta^2 \partial_\tau S_\lambda \\ &: \left[(\partial_n u_\lambda)_\tau \odot \partial_\tau \nabla\theta\right] dx + \int \partial_\tau S_\lambda \\ &: (\partial_\tau u_\lambda \odot \nabla\zeta)\zeta dx =: J_1 + J_2 + J_3.\end{aligned} \tag{35}$$

It is clear that

$$\partial_\tau S_\lambda = \sum_{i,j=1}^{n} \frac{\partial S_\lambda}{\partial \mathscr{D}_{ij}} \partial_\tau \mathscr{D}_{ij}(u_\lambda) + \frac{\partial S_\lambda}{\partial p(x)} \partial_\tau p(x) =: J_{1,1} + J_{1,2}. \tag{36}$$

From ([31], (3.6)), we have

$$\int J_{1,1} : \partial_\tau \mathscr{D} u_\lambda \zeta^2 dx \ge c(p_-) A_\lambda. \tag{37}$$

where $c(p_-) = \hat{p}_- - 1$. It is shown in ([31], (3.7)) that

$$\int J_{1,2} : \partial_\tau \mathscr{D} u_\lambda \zeta^2 dx \le \frac{c(p_-)}{16} A_\lambda + c T_\lambda. \tag{38}$$

Combining (37) with (38) yields that

$$\frac{15 c(p_-)}{16} A_\lambda \le J_1 + c T_\lambda. \tag{39}$$

The terms J_2, J_3 can be rewritten as

$$\begin{aligned}J_2 + J_3 = -\int S_\lambda : \partial_\tau \left[(\partial_n u_\lambda)_\tau \odot \partial_\tau \nabla\theta \zeta^2\right] dx - \int S_\lambda \\ : \partial_\tau((\partial_\tau u_\lambda \odot \nabla\zeta)\zeta) dx.\end{aligned} \tag{40}$$

Hence these terms can be estimated as follows: by Korn's inequality

$$\begin{aligned}|J_2| + |J_3| &\le c\|S_\lambda\|_{\hat{p}}{}'^{(x)} \left(\|\nabla u_\lambda\|_{\hat{p}(x)} + \|\partial_\tau \nabla u_\lambda \zeta\|_{\hat{p}(x)}\right) \\ &\overset{(13)}{\le} C\|S_\lambda\|_{\hat{p}}{}'^{(x)} \left(1 + \|\partial_\tau \mathscr{D} u_\lambda \zeta\|_{\hat{p}(x)}\right) \\ &\le C\|S_\lambda\|_{\hat{p}}{}'^{(x)} \left(1 + \left\|M_\lambda^{-(1/2)}\right\|_{(2p(x))/(2-p(x)),\Omega_1} A_\lambda^{1/2} + A_\lambda^{1/2}\right) \\ &\overset{(14)}{\le} c\|S_\lambda\|_{\hat{p}'(x)}^2 + \frac{c(p_-)}{16} A_\lambda + C.\end{aligned} \tag{41}$$

Combining (39), (41) with (35), we arrive at

$$\frac{7 c(p_-)}{8} A_\lambda \le c I_0 + c T_\lambda + c\|S_\lambda\|_{\hat{p}'(x)}^2 + C. \tag{42}$$

Since $I_1 = -J_3$, the term I_1 can be estimated as J_3 by

$$I_1 \quad \le c\|S_\lambda\|_{\hat{p}'(x)}^2 + \frac{c(p_-)}{8} A_\lambda + C. \tag{43}$$

Note that by Proposition 4

$$\partial_n \left(\zeta^2 \partial_\tau u_\lambda\right) = \zeta^2 \partial_\tau \partial_n u_\lambda + 2\zeta \partial_n \zeta \partial_\tau u_\lambda. \tag{44}$$

Hence the term I_2 also can be estimated as $J_2 + J_3$:

$$\begin{aligned}I_2 &= -\int S_\lambda : \left(\zeta^2 \partial_\tau \partial_n u_\lambda + 2\zeta \partial_n \zeta \partial_\tau u_\lambda\right) dx \\ &\le c\|S_\lambda\|_{\hat{p}'(x)}^2 + \frac{c(p_-)}{8} A_\lambda + C.\end{aligned} \tag{45}$$

We use Hölder's inequality to get

$$\begin{aligned}I_3 &\le c \int |f||\partial_\tau u_\lambda| dx + c \int |f|[|\zeta \partial_\tau \nabla u_\lambda| + |\psi|] dx \\ &\overset{(13)}{\le} C\|f\|_{p\wedge'(x)} + \|f\|_{p\wedge'(x)} \left[\|\psi\|_{\hat{p}(x)} + \|\zeta \partial_\tau \nabla u_\lambda\|_{\hat{p}(x)}\right] \\ &\overset{(13),(17)}{\le} C\left(\|f\|_{p\wedge'(x)} + 1\right) + c\|f\|_{p\wedge'(x)} \|\zeta \partial_\tau \mathscr{D} u_\lambda\|_{\hat{p}(x)} \\ &\le C\left(\|f\|_{p\wedge'(x)} + 1\right) + c\|f\|_{p\wedge'(x)} \left(\left\|M_\lambda^{-(1/2)}\right\|_{(2p(x))/(2-p(x)),\Omega_1} + 1\right) A_\lambda^{1/2} \\ &\overset{(14)}{\le} C\left(1 + \|f\|_{p\wedge'(x)}^2\right) + \frac{c(p_-)}{8} A_\lambda.\end{aligned} \tag{46}$$

Using (33) and (25) yields that

$$I_4 \le \int |S_\lambda||\mathscr{D}\psi|dx \le c\|\mathscr{D}\psi\|_{\widehat{p}(x)}\|S_\lambda\|_{\widehat{p}}{}'^{(x)} \le \frac{c(p_-)}{8}A_\lambda + C + c\|S_\lambda\|^2_{\widehat{p}'(x)}. \tag{47}$$

Equation (34) together with (42)-(47) yields the desired estimate (29).

Next, let us prove (30).

In [32] (4.20), it is proved that

$$\|\pi_\lambda\|_{\check{p}}{}'^{(x),\Omega} \le C. \tag{48}$$

Indeed, though the formula (4.20) from [32] is proved in 2D for $p(x) > 3/2$, it also holds for $p(x) > 3n/(n+2)$, $n = 2, 3$. Moreover if neglecting the convective term, this holds for $p(x) > 1$.

Let $\phi \in C_0^2(\Omega)$. Multiplying the first equations of (23) by $\partial_\tau(\zeta\phi)$ and integrating by parts, we have

$$\int \partial_\tau \pi_\lambda \operatorname{div}(\zeta\phi)dx = \int \partial_\tau S_\lambda : \mathscr{D}(\zeta\phi)dx + \int \pi_\lambda(\partial_n(\zeta\phi))_\tau \cdot \partial_\tau\nabla\theta dx + \int f\cdot\partial_\tau(\zeta\phi)dx - \int S_\lambda : \left((\partial_n(\zeta\phi))_\tau \odot \partial_\tau\nabla\theta\right)dx. \tag{49}$$

Since $\check{p}'(x) \le p\wedge'(x)$, we have

$$\int S_\lambda : \left((\partial_n(\zeta\phi))_\tau \odot \partial_\tau\nabla\theta\right)dx \le c\|S_\lambda\|_{\check{p}}{}'^{(x)}\|\nabla(\zeta\phi)\|_{\check{p}(x)} \le c\|S_\lambda\|_{\widehat{p}}{}'^{(x)}\|\phi\|_{1,\check{p}(x)}, \tag{50}$$

$$\int \pi_\lambda(\partial_n(\zeta\phi))_\tau \cdot \partial_\tau\nabla\theta dx \le c\|\pi_\lambda\|_{\check{p}}{}'^{(x)}\|\nabla(\zeta\phi)\|_{\check{p}(x)} \overset{(30)}{\le} C\|\phi\|_{1,\check{p}(x)}, \tag{51}$$

$$\int f\cdot\partial_\tau(\zeta\phi)dx \le c\|f\|_{\widehat{p}}{}'^{(x)}\|\nabla(\zeta\phi)\|_{\widehat{p}(x)} \le c\|f\|_{\widehat{p}}{}'^{(x)}\|\phi\|_{1,\check{p}(x)}. \tag{52}$$

It follows that

$$\begin{aligned}\int \partial_\tau S_\lambda : \mathscr{D}(\zeta\phi)dx &\le \int (|J_{1,1} + J_{1,2}|)|\mathscr{D}(\zeta\phi)|dx \\ &\le c\int [M_\lambda|\partial_\tau\mathscr{D}u_\lambda| + M_\lambda|\mathscr{D}u_\lambda| \log Y_\lambda]|\mathscr{D}(\zeta\phi)|dx \\ &\le cA_\lambda^{1/2}\left[\|\mathscr{D}\phi\|_{2,\Omega_1} + \|M_\lambda^{1/2}\|_{(2p(x))/(p(x)-2),\Omega_2}\|\mathscr{D}\phi\|_{p(x),\Omega_1}\right] + cT_\lambda^{1/2}\|\mathscr{D}\phi\|_{\check{p}(x)} \\ &\overset{(14)}{\le} C\left(A_\lambda^{1/2} + T_\lambda^{1/2}\right)\|\phi\|_{1,\check{p}(x)}.\end{aligned} \tag{53}$$

The left hand of (49) can be rewritten as

$$\int \partial_\tau\pi_\lambda \operatorname{div}\phi dx = -\int \nabla(\partial_\tau\pi_\lambda)\cdot\phi dx. \tag{54}$$

Equation (49) together with (50)-(54) implies that

$$\left|\int \nabla(\partial_\tau\pi_\lambda\zeta)\cdot\phi dx\right| \le \left[\left(C + c\|f\|_{p\wedge'(x)}\right) + C\left(A_\lambda^{1/2} + T_\lambda^{1/2}\right)\right]\|\phi\|_{1,\check{p}(x)} + c\|S_\lambda\|_{\widehat{p}}{}'^{(x)}. \tag{55}$$

Thus, by the negative norm theorem ([38], Theorem 14.3.18) and (55), we have

$$\begin{aligned}\|\partial_\tau\pi_\lambda\zeta\|_{\check{p}}{}'^{(x)} &\le c\|\nabla(\partial_\tau\pi_\lambda\zeta)\|_{-1,\check{p}'(x)} + c\|\partial_\tau\pi_\lambda\zeta\|_{-1,\check{p}'(x)} \\ &= c\sup_{\phi\in W_0^{1,\check{p}(x)}(\Omega)}\frac{\langle\nabla(\partial_\tau\pi_\lambda\zeta),\phi\rangle}{\|\phi\|_{1,\check{p}(x)}} + c\sup_{\phi\in W_0^{1,\check{p}(x)}(\Omega)}\frac{\langle\partial_\tau\pi_\lambda\zeta,\phi\rangle}{\|\phi\|_{1,\check{p}(x)}} \\ &\le \left(C + c\|f\|_{p\wedge'(x)}\right) + C\left(A_\lambda^{1/2} + T_\lambda^{1/2}\right) + c\|S_\lambda\|_{p_1'} + c\sup_{\phi\in W_0^{1,\check{p}(x)}(\Omega)}\frac{\|\pi_\lambda\|_{\check{p}'(x)}\|\partial_\tau\phi\|_{\check{p}(x)}}{\|\phi\|_{1,\check{p}(x)}} \\ &\le \left(C + c\|f\|_{p\wedge'(x)}\right) + C\left(A_\lambda^{1/2} + T_\lambda^{1/2}\right) + c\|S_\lambda\|_{\widehat{p}}{}'^{(x)},\end{aligned} \tag{56}$$

which is the just (30).

Step 2. Estimates of the approximate solutions independent of parameter in all directions.

Our aim in this step is to show (79) and (80).

Here, we follow the notations from [20]. By mimicking the derivation of [20], (4.51), we can get

$$M_\lambda|\nabla\mathscr{D}u_\lambda| \le cM_\lambda(|\partial_\tau\nabla u_\lambda| + |\mathscr{D}u_\lambda| \log Y_\lambda) + c(|\partial_\tau\pi_\lambda| + |f|). \tag{57}$$

Now, we want to derive some estimates on the first derivatives of $V_{p(x)}^\lambda(\mathscr{D}u_\lambda)$ from (22), independent of parameter λ. This allows us to prove boundedness of T_λ and $\|S_\lambda\|_{\widehat{p}}{}'^{(x)}$ and to improve regularity of solutions to problem (1).

Since $1/3|\nabla^2u_\lambda| \le |\mathscr{D}(\nabla u_\lambda)| \le 3|\nabla^2u_\lambda|$ (see, [1]), it follows from (57) that

$$\begin{aligned}\left|\nabla V_{p(x)}^\lambda(\mathscr{D}u_\lambda)\right| &\le cM_\lambda^{1/2}\left|\nabla^2u_\lambda\right| + cM_\lambda^{1/2}\log Y_\lambda|\mathscr{D}u_\lambda| \\ &\le cM_\lambda^{1/2}(|\partial_\tau\nabla u_\lambda| + |\mathscr{D}u_\lambda| \log Y_\lambda) + cM_\lambda^{-(1/2)}(|\partial_\tau\pi_\lambda| + |f|).\end{aligned} \tag{58}$$

Similarly, we have

$$\left|\partial_\tau V^\lambda_{p(x)}(\mathscr{D}u_\lambda)\right| \le cM_\lambda^{1/2}|\partial_\tau(\mathscr{D}u_\lambda)| + cM_\lambda^{1/2}\log Y_\lambda|\mathscr{D}u_\lambda|. \tag{59}$$

Now, define

$$r^1(x) := \frac{2p(x)}{p(x)+|p(x)-2|} = \min\left\{p(x), p'(x)\right\}. \tag{60}$$

It follows from (58) that

$$\begin{aligned}\left\|\nabla\left(V^\lambda_{p(x)}(\mathscr{D}u_\lambda)\right)\zeta\right\|_{r^1(x)} \le{}& c\left\|M_\lambda^{1/2}\partial_\tau(\nabla u_\lambda)\zeta\right\|_{r^1(x)} \\ &+ c\left\|M_\lambda^{1/2}\log Y_\lambda\mathscr{D}u_\lambda\zeta\right\|_{r^1(x)} \\ &+ c\left\|M_\lambda^{-(1/2)}(|\partial_\tau\pi_\lambda|+|f|)\zeta\right\|_{r^1(x)}.\end{aligned} \tag{61}$$

Using Hölder's inequality and the fact that $r^1(x)\le 2$, we have by Korn's inequality

$$\begin{aligned}\left\|M_\lambda^{1/2}\partial_\tau\nabla u_\lambda\zeta\right\|_{r^1(x)} &\le \|\partial_\tau\nabla u_\lambda\zeta\|_{p(x),\Omega_1} + \left\|M_\lambda^{1/2}\partial_\tau\nabla u_\lambda\zeta\right\|_{p'(x),\Omega_2} \\ &\le c\|\partial_\tau\mathscr{D}u_\lambda\zeta\|_{p(x),\Omega_1} + \left\|M_\lambda^{1/2}\right\|_{(2p(x))/(p(x)-2),\Omega_2}|_{2,\Omega_2} + C \\ &\le c\left[\left\|M_\lambda^{-(1/2)}\right\|_{(2p(x))/(2-p(x)),\Omega_1}\right. \\ &\quad\left. + \left\|M_\lambda^{1/2}\right\|_{(2p(x))/(p(x)-2),\Omega_2}\right]A_\lambda^{1/2} + C \\ &\overset{(14)}{\le} CA_\lambda^{1/2} + C.\end{aligned} \tag{62}$$

Taking into account the fact that $r^1(x)\le 2$, we have

$$c\left\|M_\lambda^{1/2}\log Y_\lambda\mathscr{D}u_\lambda\right\|_{r^1(x)} \le cT_\lambda^{1/2}. \tag{63}$$

From (30) and the fact that $M_\lambda^{-(1/2)}\le 1$ and $r^1(x)=\check{p}'(x)$ in Ω_2, it follows that

$$\begin{aligned}\left\|M_\lambda^{-(1/2)}(|\partial_\tau\pi_\lambda|+|f|)\zeta\right\|_{r^1(x)} \le{}& \left\|M_\lambda^{-(1/2)}\right\|_{(2p(x))/(2-p(x)),\Omega_1}\|\partial_\tau\pi_\lambda\zeta+f\|_{2,\Omega_1} \\ &+ \|\partial_\tau\pi_\lambda\zeta+f\|_{p'(x),\Omega_2} \\ \overset{(14)}{\le}{}& C\left(1+\|f\|_{p\wedge'(x)} + A_\lambda^{1/2} + T_\lambda^{1/2} + \|S_\lambda\|_{p\wedge'(x)}\right).\end{aligned} \tag{64}$$

Gathering (61)-(64) and using (29), we obtain

$$\left\|\nabla\left(V^\lambda_{p(x)}(\mathscr{D}u_\lambda)\right)\zeta\right\|_{r^1(x)} \le C\left(1+\|f\|_{p\wedge'(x)} + T_\lambda^{1/2} + c\|S_\lambda\|_{p\wedge'(x)}\right). \tag{65}$$

On the other hand, it follows from (59) and (29) that

$$\begin{aligned}\left\|\partial_\tau\left(V^\lambda_{p(x)}(\mathscr{D}u_\lambda)\right)\zeta\right\|_2 \le{}& c\left\|M_\lambda^{1/2}\partial_\tau(\mathscr{D}u_\lambda)\zeta\right\|_2 \\ &+ c\left\|M_\lambda^{1/2}\log Y_\lambda\mathscr{D}u_\lambda\zeta\right\|_2 \\ \le{}& c\left(1+\|f\|_{p\wedge'(x)} + T_\lambda^{1/2} + \|S_\lambda\|_{p\wedge'(x)}\right).\end{aligned} \tag{66}$$

Define

$$\begin{aligned}m^1(x) :={}& \left(\frac{1}{n}\left[\frac{p(x)+|p(x)-2|}{2p(x)} + \frac{n-1}{2} - 1\right]\right)^{-1} - \mu \\ ={}& \frac{2np(x)}{p(x)(n-2)+|p(x)-2|} - \mu,\end{aligned} \tag{67}$$

where μ is an arbitrary small positive real number. Applying the anisotropic embedding theorem (see [39], Theorem 2.5), we obtain that by (65) and (66),

$$\left\|V^\lambda_{p(x)}(\mathscr{D}u_\lambda)\right\|_{m^1(x)} \le c\left(1+\|f\|_{p\wedge'(x)} + T_\lambda^{1/2} + \|S_\lambda\|_{p\wedge'(x)}\right). \tag{68}$$

It is shown in [31, 32] that

$$T_\lambda \le \varepsilon_0\left\|V^\lambda_{p(x)}(\mathscr{D}u_\lambda)\right\|^2_{m^1(x)} + C. \tag{69}$$

Now, let us estimate the term $\|S_\lambda\|_{\widehat{p}}{}'^{(x)}$. Note that

$$\frac{p(x)}{p(x)-1} < m^1(x), \tag{70}$$

provided that

$$p(x)<2, p(x) > \frac{2(n+1)}{n+3}. \tag{71}$$

This implies immediately that

$$\|S_\lambda\|_{p}{}'^{(x),\Omega_1} \le \left\|V^\lambda_{p(x)}(\mathscr{D}u_\lambda)\right\|_{p}{}'^{(x),\Omega_1} \le \varepsilon_0\left\|V^\lambda_{p(x)}(\mathscr{D}u_\lambda)\right\|_{m^1(x)} + C. \tag{72}$$

Before we begin with the estimate on $\|S_\lambda\|_{2,\Omega_2}$, we claim that for $p\ge 2$,

$$\left(\frac{1+a^2}{1+\lambda a^2}\right)^{(p-2)/2} a \le c\left|\left(\frac{1+a^2}{1+\lambda a^2}\right)^{(p-2)/4} a\right|^{(2(p-1))/p} + c. \tag{73}$$

Indeed, setting $Y := (1+a^2)/(1+\lambda a^2)$ and using Young's inequality with a pair $(p/(p-2), p/2)$, we get

$$\begin{aligned} Y^{(p-2)/4} &= Y^{((p-2)p-2/4)((p-2)p-2/p)} Y^{(p-2)/4(1-((p-2)p-2/p))} \\ &= \left(Y^{(p-2)/4} Y^{1/2}\right)^{(p-2)/p} \le c\left|Y^{(p-2)/4}(1+a)\right|^{(p-2)/p} \\ &\le c\left|Y^{(p-2)/4} a\right|^{(p-2)/p} + cY^{((p-2)p-2/4)((p-2)p-2/p)} \\ &\le c\left|Y^{(p-2)/4} a\right|^{(p-2)/p} + \frac{1}{2} Y^{(p-2)/4} + c, \end{aligned} \tag{74}$$

which implies (73).

Now, let us estimate the term $\|S_\lambda\|_{2,\Omega_2}$. Using the interpolation and Young's inequalities yields that

$$\begin{aligned} \|S_\lambda\|_{2,\Omega_2} &\overset{(52)}{\le} c + c\left\|V^\lambda_{p(x)}(\mathscr{D}u_\lambda)\right\|^{(2(p_2-1))/p_2}_{(4(p_2-1))/p_2,\Omega_2} \\ &\le c + c\left(\left\|V^\lambda_{p(x)}(\mathscr{D}u_\lambda)\right\|^{1-\beta}_{2,\Omega_2}\left\|V^\lambda_{p(x)}(\mathscr{D}u_\lambda)\right\|^{\beta}_{m^1_-,\Omega_2}\right)^{(2(p_2-1))/p_2}, \end{aligned} \tag{75}$$

where

$$\begin{gathered} p_2 := \sup_{\Omega_p\cap\Omega_2} p(x), m^1_- := \inf_{\Omega_p\cap\Omega_2} m^1(x) = \frac{2np_2}{p_2(n-2)+|p_2-2|} - \mu, \\ \beta := \frac{np_2(p_2-2)}{2(p_2-1)(p_2+2)}. \end{gathered} \tag{76}$$

Hence, noting that $m^1(x)$ is decreased in p over Ω_2, we have

$$\|S_\lambda\|_{2,\Omega_2} \overset{(52)}{\le} c + c\left\|V^\lambda_{p(x)}(\mathscr{D}u_\lambda)\right\|^{(n(p_2-2))/(p_2+2)}_{m^1_-,\Omega_2} \le \varepsilon_0\left\|V^\lambda_{p(x)}(\mathscr{D}u_\lambda)\right\|_{m^1(x)} + C, \tag{77}$$

provided that $(n(p_2-2))/(p_2+2) < 1$, i.e.,

$$p(x) > 2, p(x) < \frac{2(n+1)}{n-1}. \tag{78}$$

Inserting (69), (72), and (77) into (68) and taking (71) and (78) into account, we conclude that

$$\left\|V^\lambda_{p(x)}(\mathscr{D}u_\lambda)\right\|_{m^1(x)} \le c\left(1+\|f\|_{p\wedge'(x)}\right) \tag{79}$$

and furthermore from (65) that

$$\left\|\nabla\left(V^\lambda_{p(x)}(\mathscr{D}u_\lambda)\right)\zeta\right\|_{r^1(x)} \le C\left(1+\|f\|_{p\wedge'(x)}\right), \tag{80}$$

provided that

$$\frac{2(n+1)}{n+3} < p(x) < \frac{2(n+1)}{n-1}, \tag{81}$$

which is the just (11).

Step 3. Since $m^1(x) > 2$, the estimate (79) gives us the better information on $V^\lambda_{p(x)}(\mathscr{D}u_\lambda)$ than an a priori estimate (25). By the same argument, it is possible to gradually improve the integrability of $V^\lambda_{p(x)}(\mathscr{D}u_\lambda)$ with the estimates independent of parameter and to get (7) and (8) with u_λ instead of u. This bootstrap argument is the same as in [20, 31, 32], and so we omit them. In particular, since the estimates (9) and (10) with u_λ are independent of parameter λ, it is possible to apply limiting process $\lambda \longrightarrow 0$ and as a result, we have that there exists a strong solution to (1) without the convective term satisfying (9) and (10).

Thus, Theorem 1 is proved without the convective term.

3.2. The Proof of Theorem 1 with the Convective Term

Step 1. Estimates of the approximate solutions independent of parameter in tangential directions.

Our aim in this step is to show that the estimates (29) and (30) are valid under the additional assumption $p(x) > 3n/(n+2)$.

To begin with, let us show the validity of (29).

In this case, on (34), there will be added the term

$$\begin{aligned} I_5 &= \int\left[(\partial_\tau u_\lambda\cdot\nabla)u_\lambda + \left((u_\lambda)_\tau\cdot\nabla\right)\partial_\tau u_\lambda - \left(u_\tau\cdot(\partial_n u_\lambda)_\tau\otimes\partial_\tau\nabla\theta\right)\right] \\ &\quad\cdot\zeta^2\partial_\tau u_\lambda dx - \int(u_\lambda\cdot\nabla)u_\lambda\cdot\psi dx \\ &= \int\left[|\partial_\tau u_\lambda|^2(\nabla\zeta\cdot u_\lambda)\zeta + \left((u_\lambda)_\tau\cdot\left[(\partial_n u_\lambda)_\tau\otimes\partial_\tau\nabla\theta\right]\right)\cdot\partial_\tau u_\lambda\zeta^2\right]dx \\ &\quad - \int\zeta^2(\partial_\tau u_\lambda\cdot\nabla)u_\lambda\cdot\partial_\tau u_\lambda dx - \int(u_\lambda\cdot\nabla)u_\lambda\cdot\psi dx \\ &=: I_{51} + I_{52} + I_{53}, \end{aligned} \tag{82}$$

see [20], (5.1).

Let $p_1 := \inf_{\Omega_p} p(x)$. Taking into account $2\widehat{p}'_1 < (p\wedge_1)^*$ for $\widehat{p}_1 > 3n/(n+2)$ and using Hölder's, Sobolev's, and Korn's inequalities, we have

$$\begin{aligned} I_{52} &\le c\|\partial_\tau u_\lambda\|_{\widehat{p}_1}\|u_\lambda\|_{2\widehat{p}'_1}\|\zeta\partial_\tau u_\lambda\|_{2\widehat{p}'_1} \\ &\le c\|\nabla u_\lambda\|_{\widehat{p}_1}\|u_\lambda\|_{\widehat{p}_1^*}\|\zeta\partial_\tau u_\lambda\|_{\widehat{p}_1^*} \\ &\le c\|\nabla u_\lambda\|^2_{\widehat{p}(x)}\|\mathscr{D}(\zeta\partial_\tau u_\lambda)\|_{\widehat{p}(x)} \\ &\overset{(13)}{\le} C + C\left(\left\|M_\lambda^{-(1/2)}\right\|_{(2p(x))/(2-p(x)),\Omega_1} + 1\right)A_\lambda^{1/2} \\ &\overset{(14)}{\le} C + \frac{c(p_-)}{24}A_\lambda. \end{aligned} \tag{83}$$

On the other hand, by Hölder's inequality, we have

$$I_{51} \quad \le \|\zeta \partial_\tau u_\lambda\|^2_{2\widehat{p}'_1} \|\nabla u_\lambda\|_{\widehat{p}_1}. \tag{84}$$

Note that $3n/(n+2) < \widehat{p}_1 \le 2$. Since $\widehat{p}_1 < 2\widehat{p}'_1 < (p\wedge_1)^*$, we can interpolate $\|\zeta\partial_\tau u\|_{2\widehat{p}'_1}$ between $L^{p\wedge_1}$ and $L^{(p\wedge_1)^*}$. Using this interpolation and Korn's, Hölder's, and Sobolev's inequalities, it follows that for $\alpha = ((n+2)\widehat{p}_1 - 3n)/2\widehat{p}_1$,

$$\begin{aligned} I_{51} &\overset{(13)}{\le} C\|\zeta\partial_\tau u_\lambda\|^2_{2(\widehat{p}_1)'} \le C\|\zeta\partial_\tau u_\lambda\|^{2\alpha}_{\widehat{p}_1}\|\zeta\partial_\tau u_\lambda\|^{2(1-\alpha)}_{(p\wedge_1)^*} \\ &\le C\|\zeta\partial_\tau u_\lambda\|^2_{\widehat{p}_1} + C\|\zeta\partial_\tau u_\lambda\|^{2\alpha}_{\widehat{p}_1}\|\zeta\partial_\tau \mathscr{D} u_\lambda\|^{2(1-\alpha)}_{\widehat{p}(x)} \\ &\le C + C\|\zeta\partial_\tau u_\lambda\|^{2\alpha}_{\widehat{p}_1}\left(\left\|M_\lambda^{-(1/2)}\right\|^{2(1-\alpha)}_{(2p(x))/(2-p(x)),\Omega_1} + 1\right)A_\lambda^{1-\alpha} \\ &\overset{(14)}{\le} C + \frac{c(p_-)}{24}A_\lambda, \end{aligned} \tag{85}$$

where, in the last estimate, we use Young's inequality with a pair $(1/(1-\alpha), 1/\alpha)$. From (33), (25), and (26), it follows that

$$\begin{aligned} I_{53} &\le \|u_\lambda\|^2_{2\widehat{p}'_1}\|\nabla\psi\|_{\widehat{p}_1} \le C + C\|\mathscr{D}(\zeta\partial_\tau u_\lambda)\|_{\widehat{p}(x)} \\ &\le C + C\left(\left\|M_\lambda^{-(1/2)}\right\|_{(2p(x))/(2-p(x)),\Omega_1} + 1\right)A_\lambda^{(1/2)} \\ &\le C + \frac{c(p_-)}{24}A_\lambda. \end{aligned} \tag{86}$$

Thus, identity (82) together with (83)-(86) yields

$$I_5 \le C + \frac{c(p_-)}{8}A_\lambda. \tag{87}$$

Thus, the estimate (29) continues to hold for the full problem (23).

Next, let us prove that there also holds (30). Indeed, for the full problem (23), there will be additionally added the term

$$-\int (u_\lambda\cdot\nabla)u_\lambda\cdot\partial_\tau(\zeta\phi)dx, \tag{88}$$

on the right hand side of (49). It is clear that

$$\begin{aligned} \frac{np(x)}{(n+1)p(x)-2n} &\le 2^*, \quad x\in\Omega_1, \\ 4 &\le p^*(x), \quad x\in\Omega_2. \end{aligned} \tag{89}$$

Recalling that $W^{1,\check{p}(x)}(\Omega)^\circ L^q(\Omega)$ for any $q<\infty$ and using Hölder's inequality, we obtain

$$\begin{aligned} \int_\Omega (u_\lambda\cdot\nabla)u_\lambda\cdot\partial_\tau(\zeta\phi)dx &= \int_\Omega(\partial_\tau u_\lambda\cdot\nabla)u_\lambda\cdot\zeta\phi dx \\ &\quad + \int_\Omega (u_\lambda\cdot\nabla)\partial_\tau u_\lambda\cdot\zeta\phi dx \\ &\le c\|\partial_\tau u_\lambda\zeta\|_{p^*(x),\Omega_1}\|\nabla u_\lambda\|_{p(x),\Omega_1}\|\phi\|_{(np(x))/((n+1)p(x)-2n),\Omega_1} \\ &\quad + \|\partial_\tau u_\lambda\zeta\|_{4,\Omega_2}\|\nabla u_\lambda\|_{2,\Omega_2}\|\phi\|_{4,\Omega_2} \\ &\quad + c\|u_\lambda\|_{p^*(x),\Omega_1}\|\nabla\partial_\tau u_\lambda\zeta\|_{p(x),\Omega_1}\|\phi\|_{(np(x))/((n+1)p(x)-2n),\Omega_1} \\ &\quad + \|u_\lambda\|_{4,\Omega_2}\|\nabla\partial_\tau u_\lambda\zeta\|_{2,\Omega_2}\|\phi\|_{4,\Omega_2} \overset{(13),(14)}{\le} CA_\lambda^{1/2}\|\phi\|_{1,\check{p}(x)}. \end{aligned} \tag{90}$$

Thus, there also holds (30) for the problem (23).

Step 2. Estimates of the approximate solutions independent of parameter in all directions.

Here, we prove the validity of (79).

For the problem (23), there will be additionally added the term

$$c\left\|M_\lambda^{-(1/2)}(u_\lambda\cdot\nabla)u_\lambda\zeta\right\|_{r^1(x)} \tag{91}$$

on the right hand side of (61) and hence,

$$\begin{aligned} \left\|V^\lambda_{p(x)}(\mathscr{D}u_\lambda)\right\|_{m^1(x)} &\le c\Big(1 + \|f\|_{p\wedge'(x)} + T_\lambda^{1/2} + \|S_\lambda\|_{p\wedge'(x)} \\ &\quad + \left\|M_\lambda^{-(1/2)}(u_\lambda\cdot\nabla)u_\lambda\zeta\right\|_{r^1(x)}\Big), \end{aligned} \tag{92}$$

where $m^1(x)$ is from (67).

By (25) and (26) and Korn's, Hölder's, and Young's inequalities, we have

$$\begin{aligned} &\left\|M_\lambda^{-(1/2)}(u_\lambda\cdot\nabla)u_\lambda\zeta\right\|_{r^1(x),\Omega_1} \\ &\quad\le \left\|M_\lambda^{-(1/2)}\right\|_{(2p(x))/(2-p(x)),\Omega_1}\|u_\lambda\|_{(np(x))/(n-p(x)),\Omega_1}\|\nabla u_\lambda\zeta\|_{(2np(x))/((n+2)p(x)-2n),\Omega_1} \\ &\quad\le C\|\mathscr{D}u_\lambda\zeta\|_{(2np(x))/((n+2)p(x)-2n),\Omega_1} + C \\ &\quad\le C\left\|M_\lambda^{-(1/2)}\right\|_{(2p(x))/(2-p(x)),\Omega_1}\left\|V^\lambda_{p(x)}(\mathscr{D}u_\lambda)\zeta\right\|_{(np(x))/((n+1)p(x)-2n),\Omega_1} + C \\ &\quad\le \varepsilon_0\left\|V^\lambda_{p(x)}(\mathscr{D}u_\lambda)\right\|_{m^1(x)} + C, \end{aligned} \tag{93}$$

where, to perform the last estimate, we use the fact

$$\frac{np(x)}{(n+1)p(x)-2n} < \frac{2np(x)}{p(x)(n-3)+2} = m^1(x), \quad \text{for } n=2,3, x\in\Omega_1, \tag{94}$$

which holds by the condition $p(x) > 3n/(n+2)$.

On the other hand, by the same line above, we obtain that

$$\begin{aligned}\left\|M_\lambda^{-(1/2)}(u_\lambda\cdot\nabla)u_\lambda\zeta\right\|_{r^1(x),\Omega_2} &\le \|u_\lambda\|_{2n/(n-2),\Omega_2}\|\nabla u_\lambda\zeta\|_{(2np(x))/((n+2)p(x)-2n),\Omega_2}\\ &\le C\|\mathscr{D}u_\lambda\zeta\|_{(2np(x))/((n+2)p(x)-2n),\Omega_2}+C\\ &\overset{M_\lambda^{1/2}\ge 1}{\le} C\left\|V_{p(x)}^\lambda(\mathscr{D}u_\lambda)\zeta\right\|_{(2np(x))/((n+2)p(x)-2n),\Omega_2}+C\\ &\le \varepsilon_0\left\|V_{p(x)}^\lambda(\mathscr{D}u_\lambda)\right\|_{m^1(x)}+C,\end{aligned} \tag{95}$$

where, to perform the last estimate, we use the fact

$$\frac{2np(x)}{(n+2)p(x)-2n}<\frac{2np(x)}{p(x)(n-1)-2}=m^1(x),\quad \text{for } n=2,3,x\in\Omega_2, \tag{96}$$

which also holds by the condition $p(x)>3n/(n+2)$.

Inserting (93) and (95) into (92) and noting (69), (72), and (77), we conclude that

$$\left\|V_{p(x)}^\lambda(\mathscr{D}u_\lambda)\right\|_{m^1(x)}\le C\left(1+\|f\|_{p\wedge'(x)}\right). \tag{97}$$

This is just the estimate (79).

Step 3. The rest is the same as the previous subsection. Thus, Theorem 1 is completely proved.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there is no conflict of interest.

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