Hindawi
Advances in Mathematical Physics
Volume 2021, Article ID 6760596, 21 pages
https://doi.org/10.1155/2021/6760596

Research Article

Dynamic Characteristics of Four Systems of Difference Equations with Higher Order

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Received 17 August 2021; Revised 18 October 2021; Accepted 13 November 2021; Published 13 December 2021

Academic Editor: Mohammad Mirzazadeh

In this paper, we explore the global dynamical characteristics, boundedness, and rate of convergence of certain higher-order discrete systems of difference equations. More precisely, it is proved that for all involved respective parameters, discrete systems have a trivial fixed point. We have studied local and global dynamical characteristics at trivial fixed point and proved that trivial fixed point of the discrete systems is globally stable under respective definite parametric conditions. We have also studied boundedness and rate of convergence for under consideration discrete systems. Finally, theoretical results are confirmed numerically. Our findings in this paper are considerably extended and improve existing results in the literature.

1. Introduction

1.1. Motivation and Literature Review. No one can deny the significance of difference equations these days. These equation models not only are the discrete physical phenomenon but also are the integral part of numerical schemes used to solve differential equations. These equations are widely applied in many branches of scientific field. Difference equations describe the phenomenon of discrete dynamical systems and have applications in various branches of science such as statistical problems, resource management, neural networks, ecology, economics, queuing problems, number theory, sociology, physics, engineering, psychology, quanta in radiation, genetics in biology, combinatorial analysis, probability theory, geometry, population dynamics, electrical networks, stochastic time series, and queuing problems [1, 2]. Studying the global characteristics of higher-order nonlinear difference equations or systems of difference equations is a difficult but rewarding task. These findings pave the way for the construction of a basic theory of higher-order difference equations. Recently, many mathematicians have explored the dynamics of difference equation along their system. For instance, Oğul et al. [3] investigated the dynamics of the following higher-order system:

$$x_{n+1} = \frac{x_{n-15}}{\pm 1 \pm x_{n-3} x_{n-7} x_{n-11} x_{n-15}}. \tag{1}$$

Kulenović et al. [4] investigated the dynamic behavior of the difference equation:

$$x_{n+1} = \frac{\alpha x_n + \beta x_{n-1}}{A + x_{n-1}}. \tag{2}$$

Zhang et al. [5] investigated the dynamic behavior of the difference equation system:

$$\begin{aligned} x_{n+1} &= A + \frac{x_n}{\sum_{i=1}^{k} y_{n-i}}, \\ y_{n+1} &= B + \frac{y_n}{\sum_{i=1}^{k} x_{n-i}}. \end{aligned} \tag{3}$$

Kalabušić et al. [6] investigated the dynamic behavior of the difference equation system:

$$\begin{aligned} x_{n+1} &= \frac{\alpha_1 + \beta_1 x_n}{y_n}, \\ y_{n+1} &= \frac{\alpha_2 + \gamma_2 y_n}{A_2 + x_n}. \end{aligned} \tag{4}$$

Kalabušić et al. [7] investigated the dynamic behavior of the difference equation system:

$$x_{n+1} = \frac{\alpha_1 + \beta_1 x_n}{A_1 + y_n}, \quad y_{n+1} = \frac{\gamma_2 y_n}{A_2 + B_2 x_n + y_n}. \tag{5}$$

Kalabušić et al. [8] investigated the dynamic behavior of the difference equation system:

$$x_{n+1} = \frac{\beta_1 x_n}{B_1 x_n + y_n}, \quad y_{n+1} = \frac{\alpha_2 + \gamma_2 y_n}{A_2 + x_n}. \tag{6}$$

Garić-Demirović et al. [9] investigated the dynamic behavior of four distinct difference equation systems. Elsayed [10, 11] studied solutions form of difference equations and their systems. Further, Khan and Qureshi [12] studied the dynamic behavior of a competitive system. DeVault et al. [13] investigated the dynamic behavior of the difference equation:

$$x_{n+1} = \frac{A}{x_n} + \frac{1}{x_{n-2}}. \tag{7}$$

Abu-Saris and DeVault [14] studied global attractivity of the difference equation:

$$x_{n+1} = A + \frac{x_n}{x_{n-k}}. \tag{8}$$

On the other hand, in recent years, many works have been published that discussed dynamic behavior of difference equation along their systems [15–20]. In continuation of existing study, it is important to mention that dynamical characteristics of the following difference equation have been investigated by Bajo and Liz [21]:

$$x_{n+1} = \frac{x_{n-1}}{\alpha_1 + \alpha_2 \prod_{i=0}^{1} x_{n-i}}, \tag{9}$$

where $\alpha_\ell (\ell = 1, 2)$ and $x_{-\ell} (\ell = 1, 0)$ are real constants. By extending the work of [21], Zhang et al. [22] have explored the dynamical properties of the system:

$$x_{n+1} = \frac{x_{n-2}}{\alpha_2 + \prod_{i=0}^{2} y_{n-i}}, \quad y_{n+1} = \frac{y_{n-2}}{\alpha_1 + \prod_{i=0}^{2} x_{n-i}}, \tag{10}$$

where $\alpha_\ell (\ell = 1, 2)$ and $x_{-\ell}, y_{-\ell} (\ell = 2, 1, 0)$ are real constants.

1.2. Objective, Contributions, and Novelties. Motivated from the aforementioned studies, the objective of the present study is to investigate the behavior of certain rational systems by extending the work done by [21, 22]:

$$x_{n+1} = \frac{\alpha_{10} y_{n-k}}{\alpha_{11} + \alpha_{12} \prod_{i=0}^{k} z_{n-i}}, \quad y_{n+1} = \frac{\alpha_{13} z_{n-k}}{\alpha_{14} + \alpha_{15} \prod_{i=0}^{k} x_{n-i}}, \quad z_{n+1} = \frac{\alpha_{16} x_{n-k}}{\alpha_{17} + \alpha_{18} \prod_{i=0}^{k} y_{n-i}}, \tag{11}$$

$$x_{n+1} = \frac{\alpha_{19} z_{n-k}}{\alpha_{20} + \alpha_{21} \prod_{i=0}^{k} x_{n-i}}, \quad y_{n+1} = \frac{\alpha_{22} x_{n-k}}{\alpha_{23} + \alpha_{24} \prod_{i=0}^{k} y_{n-i}}, \quad z_{n+1} = \frac{\alpha_{25} y_{n-k}}{\alpha_{26} + \alpha_{27} \prod_{i=0}^{k} z_{n-i}}, \tag{12}$$

$$x_{n+1} = \frac{\alpha_{28} y_{n-k}}{\alpha_{29} + \alpha_{30} \prod_{i=0}^{k} y_{n-i}}, \quad y_{n+1} = \frac{\alpha_{31} z_{n-k}}{\alpha_{32} + \alpha_{33} \prod_{i=0}^{k} z_{n-i}}, \quad z_{n+1} = \frac{\alpha_{34} x_{n-k}}{\alpha_{35} + \alpha_{36} \prod_{i=0}^{k} x_{n-i}}, \tag{13}$$

$$x_{n+1} = \frac{\alpha_{37} z_{n-k}}{\alpha_{38} + \alpha_{39} \prod_{i=0}^{k} z_{n-i}}, \quad y_{n+1} = \frac{\alpha_{40} x_{n-k}}{\alpha_{41} + \alpha_{42} \prod_{i=0}^{k} x_{n-i}}, \quad z_{n+1} = \frac{\alpha_{43} y_{n-k}}{\alpha_{44} + \alpha_{45} \prod_{i=0}^{k} y_{n-i}}, \tag{14}$$

where $\alpha_\ell (\ell = 10, \cdots, 45)$ and $x_{-\ell}, y_{-\ell}, z_{-\ell} (\ell = -k, -k+1, \cdots, 1, 0)$ are real constants. More precisely, our main finding in this paper includes the following:

(i) Exploration of trivial fixed point of discrete systems (11)–(14)

(ii) Construction of the corresponding linearized system

(iii) Investigation of global dynamics by stability theory

(iv) Study of boundedness of positive solution and convergence rate of discrete systems (11)–(14)

(v) Validation of obtained results numerically

1.3. Paper Structure. The rest of the paper is structured as follows: In the subsequent section, we explore trivial fixed point and linearized form of discrete systems (11)–(14). Local dynamical characteristics of systems (11)–(14) are investigated in Section 3 while the boundedness for (11)–(14) is explored in Section 4. In Section 5, global dynamics is investigated. Section 6 includes the investigation of rate of converges for said discrete systems. Obtained results are numerically confirmed in Section 7. The conclusion and future work are given in Section 8.

2. Linearized Form and Trivial Fixed Point of Systems (11)–(14)

Linearized form and trivial fixed points of discrete systems (11)–(14) are studied in this section.

2.1. Fixed Point. Obviously, $P_0 = (0, 0, 0)$ is the trivial fixed point of discrete systems (11)–(14). Now, in the rest of the section, linearized form for discrete systems (11)–(14) is explored.

2.2. Linearized Form of Discrete System (11). The map for the linearized form of discrete system (11) is

$$\left(\beta^{(11)}, \beta_n^{(11)}, \cdots, \beta_{n-k}^{(11)}, \gamma^{(11)}, \gamma_n^{(11)}, \cdots, \gamma_{n-k}^{(11)}, \delta^{(11)}, \delta_n^{(11)}, \cdots, \delta_{n-k}^{(11)}\right) \mapsto \widehat{\Gamma} = \left(x_{n+1}, x_n, \cdots, x_{n-k+1}, y_{n+1}, y_n, \cdots, y_{n-k+1}, z_{n+1}, z_n, \cdots, z_{n-k+1}\right), \tag{15}$$

where

$$\begin{aligned}
\beta^{(11)} &= \frac{\alpha_{10} y_{n-k}}{\alpha_{11} + \alpha_{12} \prod_{i=0}^{k} z_{n-i}}, \beta_n^{(11)} = x_n, \cdots, \beta_{n-k}^{(11)} = x_{n-k+1}, \\
\gamma^{(11)} &= \frac{\alpha_{13} z_{n-k}}{\alpha_{14} + \alpha_{15} \prod_{i=0}^{k} x_{n-i}}, \gamma_n^{(11)} = y_n, \cdots, \gamma_{n-k}^{(11)} = y_{n-k+1}, \\
\delta^{(11)} &= \frac{\alpha_{16} x_{n-k}}{\alpha_{17} + \alpha_{18} \prod_{i=0}^{k} y_{n-i}}, \delta_n^{(11)} = z_n, \cdots, \delta_{n-k}^{(11)} = z_{n-k+1}.
\end{aligned} \tag{16}$$

The linearized form of (11) at Λ under (15) is

$$\Gamma_{n+1} \wedge^{(11)} = J\Big|_{\Lambda} \Gamma_n \wedge^{(11)}, \tag{17}$$

where $\Gamma_n \wedge^{(11)} = \left(x_n, x_{n-1}, \cdots, x_{n-k}, y_n, y_{n-1}, \cdots, y_{n-k}, z_n, z_{n-1}, \cdots, z_{n-k}\right)^T$, is

$$J|_{\Lambda} = \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{\alpha_{10}}{\alpha_{11}+\alpha_{12}z^{k+1}} & -\frac{\alpha_{10}\alpha_{12}yz^k}{(\alpha_{11}+\alpha_{12}z^{k+1})^2} & -\frac{\alpha_{10}\alpha_{12}yz^k}{(\alpha_{11}+\alpha_{12}z^{k+1})^2} & \cdots & -\frac{\alpha_{10}\alpha_{12}yz^k}{(\alpha_{11}+\alpha_{12}z^{k+1})^2} & -\frac{\alpha_{10}\alpha_{12}yz^k}{(\alpha_{11}+\alpha_{12}z^{k+1})^2} \\
1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
-\frac{\alpha_{13}\alpha_{15}zx^k}{(\alpha_{14}+\alpha_{15}x^{k+1})^2} & -\frac{\alpha_{13}\alpha_{15}zx^k}{(\alpha_{14}+\alpha_{15}x^{k+1})^2} & \cdots & -\frac{\alpha_{13}\alpha_{15}zx^k}{(\alpha_{14}+\alpha_{15}x^{k+1})^2} & -\frac{\alpha_{13}\alpha_{15}zx^k}{(\alpha_{14}+\alpha_{15}x^{k+1})^2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{\alpha_{13}}{\alpha_{14}+\alpha_{15}x^{k+1}} \\
0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & \frac{\alpha_{16}}{\alpha_{17}+\alpha_{18}y^{k+1}} & -\frac{\alpha_{16}\alpha_{18}xy^k}{(\alpha_{17}+\alpha_{18}y^{k+1})^2} & -\frac{\alpha_{16}\alpha_{18}xy^k}{(\alpha_{17}+\alpha_{18}y^{k+1})^2} & \cdots & -\frac{\alpha_{16}\alpha_{18}xy^k}{(\alpha_{17}+\alpha_{18}y^{k+1})^2} & -\frac{\alpha_{16}\alpha_{18}xy^k}{(\alpha_{17}+\alpha_{18}y^{k+1})^2} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0
\end{pmatrix}, \tag{18}$$

$$\Lambda = (x, y, z). \tag{19}$$

2.3. Linearized Form of Discrete System (12). Linearized form of discrete system (12) at Λ under the map:

$$\left(\beta^{(12)}, \beta_n^{(12)}, \cdots, \beta_{n-k}^{(12)}, \gamma^{(12)}, \gamma_n^{(12)}, \cdots, \gamma_{n-k}^{(12)}, \delta^{(12)}, \delta_n^{(12)}, \cdots, \delta_{n-k}^{(12)}\right) \mapsto \widehat{\Gamma}, \tag{20}$$

$$\Gamma_{n+1} \wedge^{(12)} = J\Big|_{\Lambda} \Gamma_n \wedge^{(12)}, \tag{21}$$

where

$$J|_{\Lambda} = \begin{pmatrix}
-\frac{\alpha_{19}\alpha_{21}zx^k}{(\alpha_{20}+\alpha_{21}x^{k+1})^2} & -\frac{\alpha_{19}\alpha_{21}zx^k}{(\alpha_{20}+\alpha_{21}x^{k+1})^2} & \cdots & -\frac{\alpha_{19}\alpha_{21}zx^k}{(\alpha_{20}+\alpha_{21}x^{k+1})^2} & -\frac{\alpha_{19}\alpha_{21}zx^k}{(\alpha_{20}+\alpha_{21}x^{k+1})^2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{\alpha_{19}}{\alpha_{20}+\alpha_{21}z^{k+1}} \\
1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & \frac{\alpha_{22}}{\alpha_{23}+\alpha_{24}y^{k+1}} & -\frac{\alpha_{22}\alpha_{24}xy^k}{(\alpha_{23}+\alpha_{24}y^{k+1})^2} & -\frac{\alpha_{22}\alpha_{24}xy^k}{(\alpha_{23}+\alpha_{24}y^{k+1})^2} & \cdots & -\frac{\alpha_{22}\alpha_{24}xy^k}{(\alpha_{23}+\alpha_{24}y^{k+1})^2} & -\frac{\alpha_{22}\alpha_{24}xy^k}{(\alpha_{23}+\alpha_{24}y^{k+1})^2} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{\alpha_{25}}{\alpha_{26}+\alpha_{27}z^{k+1}} & -\frac{\alpha_{25}\alpha_{27}yz^k}{(\alpha_{26}+\alpha_{27}z^{k+1})^2} & -\frac{\alpha_{25}\alpha_{27}yz^k}{(\alpha_{26}+\alpha_{27}z^{k+1})^2} & \cdots & -\frac{\alpha_{25}\alpha_{27}yz^k}{(\alpha_{26}+\alpha_{27}z^{k+1})^2} & -\frac{\alpha_{25}\alpha_{27}yz^k}{(\alpha_{26}+\alpha_{27}z^{k+1})^2} \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0
\end{pmatrix}. \tag{22}$$

and

$$\beta^{(12)}=\frac{\alpha_{19}z_{n-k}}{\alpha_{20}+\alpha_{21}\prod_{i=0}^{k}x_{n-i}},\beta_n^{(12)}=x_n,\cdots,\beta_{n-k}^{(12)}=x_{n-k+1},$$
$$\gamma^{(12)}=\frac{\alpha_{22}x_{n-k}}{\alpha_{23}+\alpha_{24}\prod_{i=0}^{k}y_{n-i}},\gamma_n^{(12)}=y_n,\cdots,\gamma_{n-k}^{(12)}=y_{n-k+1},$$
$$\delta^{(12)}=\frac{\alpha_{25}y_{n-k}}{\alpha_{26}+\alpha_{27}\prod_{i=0}^{k}z_{n-i}},\delta_n^{(12)}=z_n,\cdots,\delta_{n-k}^{(12)}=z_{n-k+1}. \tag{23}$$

2.4. Linearized Form of Discrete System (13). Linearized form of discrete system (13) at Λ under the map:

$$\left(\beta^{(13)},\beta_n^{(13)},\cdots,\beta_{n-k}^{(13)},\gamma^{(13)},\gamma_n^{(13)},\cdots,\gamma_{n-k}^{(13)},\delta^{(13)},\delta_n^{(13)},\cdots,\delta_{n-k}^{(13)}\right)\mapsto\widehat{\Gamma}, \tag{24}$$

is

$$\Gamma_{n+1}\wedge^{(13)}=J\Big|_{\Lambda}\Gamma_n\wedge^{(13)}, \tag{25}$$

where

$$J|_{\Lambda}=\begin{pmatrix}
0 & 0 & \cdots & 0 & 0\\
1 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & \cdots & 1 & 0\\
0 & 0 & \cdots & 0 & 0\\
0 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & \cdots & 0 & 0\\
-\frac{\alpha_{34}\alpha_{36}x^{k+1}}{\left(\alpha_{35}+\alpha_{36}x^{k+1}\right)^2} & -\frac{\alpha_{34}\alpha_{36}x^{k+1}}{\left(\alpha_{35}+\alpha_{36}x^{k+1}\right)^2} & \cdots & -\frac{\alpha_{34}\alpha_{36}x^{k+1}}{\left(\alpha_{35}+\alpha_{36}x^{k+1}\right)^2} & \frac{\alpha_{34}\alpha_{35}}{\left(\alpha_{35}+\alpha_{36}x^{k+1}\right)^2}\\
0 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & \cdots & 0 & 0\\
\\
-\frac{\alpha_{28}\alpha_{30}y^{k+1}}{\left(\alpha_{29}+\alpha_{30}y^{k+1}\right)^2} & -\frac{\alpha_{28}\alpha_{30}y^{k+1}}{\left(\alpha_{29}+\alpha_{30}y^{k+1}\right)^2} & \cdots & -\frac{\alpha_{28}\alpha_{30}y^{k+1}}{\left(\alpha_{29}+\alpha_{30}y^{k+1}\right)^2} & \frac{\alpha_{28}\alpha_{29}}{\left(\alpha_{29}+\alpha_{30}y^{k+1}\right)^2}\\
0 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & \cdots & 0 & 0\\
0 & 0 & \cdots & 0 & 0\\
1 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & \cdots & 1 & 0\\
0 & 0 & \cdots & 0 & 0\\
0 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & \cdots & 0 & 0\\
\\
0 & 0 & \cdots & 0 & 0\\
0 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & \cdots & 0 & 0\\
-\frac{\alpha_{31}\alpha_{33}z^{k+1}}{\left(\alpha_{32}+\alpha_{33}z^{k+1}\right)^2} & -\frac{\alpha_{31}\alpha_{33}z^{k+1}}{\left(\alpha_{32}+\alpha_{33}z^{k+1}\right)^2} & \cdots & -\frac{\alpha_{31}\alpha_{33}z^{k+1}}{\left(\alpha_{32}+\alpha_{33}z^{k+1}\right)^2} & \frac{\alpha_{31}\alpha_{32}}{\left(\alpha_{32}+\alpha_{33}z^{k+1}\right)^2}\\
0 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & \cdots & 0 & 0\\
\\
0 & 0 & \cdots & 0 & 0\\
1 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & \cdots & 1 & 0
\end{pmatrix}
\begin{array}{l}
\beta^{(13)}=\frac{\alpha_{28}y_{n-k}}{\alpha_{29}+\alpha_{30}\prod_{i=0}^{k}y_{n-i}},\beta_n^{(13)}=x_n,\cdots,\beta_{n-k}^{(13)}=x_{n-k+1},\\
,\gamma^{(13)}=\frac{\alpha_{31}z_{n-k}}{\alpha_{32}+\alpha_{33}\prod_{i=0}^{k}z_{n-i}},\gamma_n^{(13)}=y_n,\cdots,\gamma_{n-k}^{(13)}=y_{n-k+1},\\
\delta^{(13)}=\frac{\alpha_{34}x_{n-k}}{\alpha_{35}+\alpha_{36}\prod_{i=0}^{k}x_{n-i}},\delta_n^{(13)}=z_n,\cdots,\delta_{n-k}^{(13)}=z_{n-k+1}.
\end{array} \tag{26}$$

2.5. Linearized Form of Discrete System (14). Linearized form of discrete system (14) at Λ under the map:

$$\left(\beta^{(14)}, \beta_n^{(14)}, \cdots, \beta_{n-k}^{(14)}, \gamma^{(14)}, \gamma_n^{(14)}, \cdots, \gamma_{n-k}^{(14)}, \delta^{(14)}, \delta_n^{(14)}, \cdots, \delta_{n-k}^{(14)}\right) \mapsto \widehat{\Gamma}, \tag{27}$$

is

$$\Gamma_{n+1}\wedge^{(14)} = J\Big|_{\Lambda} \Gamma_n \wedge^{(14)}, \tag{28}$$

where

$$J|_{\Lambda} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ -\frac{\alpha_{40}\alpha_{42}x^{k+1}}{\left(\alpha_{41}+\alpha_{42}x^{k+1}\right)^2} & -\frac{\alpha_{40}\alpha_{42}x^{k+1}}{\left(\alpha_{41}+\alpha_{42}x^{k+1}\right)^2} & \cdots & -\frac{\alpha_{40}\alpha_{42}x^{k+1}}{\left(\alpha_{41}+\alpha_{42}x^{k+1}\right)^2} & \frac{\alpha_{40}\alpha_{41}}{\left(\alpha_{41}+\alpha_{42}x^{k+1}\right)^2} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ -\frac{\alpha_{43}\alpha_{45}y^{k+1}}{\left(\alpha_{44}+\alpha_{45}y^{k+1}\right)^2} & -\frac{\alpha_{43}\alpha_{45}y^{k+1}}{\left(\alpha_{44}+\alpha_{45}y^{k+1}\right)^2} & \cdots & -\frac{\alpha_{43}\alpha_{45}y^{k+1}}{\left(\alpha_{44}+\alpha_{45}y^{k+1}\right)^2} & \frac{\alpha_{43}\alpha_{44}}{\left(\alpha_{44}+\alpha_{45}y^{k+1}\right)^2} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ \\ -\frac{\alpha_{37}\alpha_{39}z^{k+1}}{\left(\alpha_{38}+\alpha_{39}z^{k+1}\right)^2} & -\frac{\alpha_{37}\alpha_{39}z^{k+1}}{\left(\alpha_{38}+\alpha_{39}z^{k+1}\right)^2} & \cdots & -\frac{\alpha_{37}\alpha_{39}z^{k+1}}{\left(\alpha_{38}+\alpha_{39}z^{k+1}\right)^2} & \frac{\alpha_{37}\alpha_{38}}{\left(\alpha_{38}+\alpha_{39}z^{k+1}\right)^2} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ \\ 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

$$\begin{aligned} \beta^{(14)} &= \frac{\alpha_{37}z_{n-k}}{\alpha_{38}+\alpha_{39}\prod_{i=0}^{k} z_{n-i}}, \beta_n^{(14)} = x_n, \cdots, \beta_{n-k}^{(14)} = x_{n-k+1}, \\ \gamma^{(14)} &= \frac{\alpha_{40}x_{n-k}}{\alpha_{41}+\alpha_{42}\prod_{i=0}^{k} x_{n-i}}, \gamma_n^{(14)} = y_n, \cdots, \gamma_{n-k}^{(14)} = y_{n-k+1}, \\ \delta^{(14)} &= \frac{\alpha_{43}y_{n-k}}{\alpha_{43}+\alpha_{45}\prod_{i=0}^{k} y_{n-i}}, \delta_n^{(14)} = z_n, \cdots, \delta_{n-k}^{(14)} = z_{n-k+1}. \end{aligned} \tag{29}$$

3. Local Dynamical Characteristics of Discrete Systems (11)–(14)

Hereafter by Theorem 1.1 of [16], local dynamical characteristics about P_0 of discrete systems (11)–(14) is explored.

3.1. Local Dynamical Characteristics of Discrete System (11)

Theorem 1. *P_0 of discrete system (11) is a sink if*

$$\begin{aligned} &\frac{\alpha_{10}}{\alpha_{11}} < 1, \\ &\frac{\alpha_{13}}{\alpha_{14}} < 1, \\ &\frac{\alpha_{16}}{\alpha_{17}} < 1. \end{aligned} \tag{30}$$

Proof. At P_0, (17) becomes

$$\Gamma_{n+1}\wedge^{(11)} = J\Big|_{P_0} \Gamma_n \wedge^{(11)}. \tag{31}$$

From (18), one has

$$B_1 = J|_{P_0} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{\alpha_{10}}{\alpha_{11}} & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{\alpha_{13}}{\alpha_{14}} \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \frac{\alpha_{16}}{\alpha_{17}} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{32}$$

Now, if $\sigma_1, \sigma_2, \sigma_3, \cdots, \sigma_{k+1}, \cdots, \sigma_{2k+1}, \cdots, \sigma_{3k+3}$ represent characteristic roots of P_0 and diagonal matrix: $\widehat{\Omega} = \operatorname{diag}\left(\varrho_1, \varrho_2, \cdots, \varrho_{k+1}, \cdots, \varrho_{2k+1}, \cdots, \varrho_{3k+3}\right)$ where

$$\begin{aligned} \varrho_1 &= \varrho_{k+2} = \varrho_{2k+3} = 1, \\ \varrho_{1+\varsigma} &= \varrho_{k+2+\varsigma} = \varrho_{2k+3+\varsigma} = 1 - \varsigma\varepsilon, \ 1 \le \varsigma \le k \text{ where } 0 < \varepsilon < 1, \end{aligned} \tag{33}$$

$$0<\varepsilon<\min\left\{\frac{1}{k}\left(1-\frac{\alpha_{10}}{\alpha_{11}}\right),\frac{1}{k}\left(1-\frac{\alpha_{13}}{\alpha_{14}}\right),\frac{1}{k}\left(1-\frac{\alpha_{16}}{\alpha_{17}}\right)\right\}. \tag{34}$$

Now,

$$\widehat{\Omega}B_1\Omega\wedge^{-1}=\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \varrho_1\varrho_{2k+2}^{-1}\frac{\alpha_{10}}{\alpha_{11}} & 0 & 0 & \cdots & 0 & 0\\
\varrho_2\varrho_1^{-1} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & \cdots & \varrho_{k+1}\varrho_k^{-1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \varrho_{k+2}\varrho_{3k+3}^{-1}\frac{\alpha_{13}}{\alpha_{14}}\\
0 & 0 & \cdots & 0 & 0 & \varrho_{k+3}\varrho_{k+2}^{-1} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \varrho_{2k+2}\varrho_{2k+1}^{-1} & 0 & 0 & 0 & \cdots & 0 & 0\\
0 & 0 & \cdots & 0 & \varrho_{2k+3}\varrho_{k+1}^{-1}\frac{\alpha_{16}}{\alpha_{17}} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0\\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \varrho_{3k+2}\varrho_{3k+1}^{-1} & 0 & \cdots & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \varrho_{3k+3}\varrho_{3k+2}^{-1} & 0
\end{pmatrix}. \tag{35}$$

So,

$$\begin{aligned}
&0<\varrho_{k+1}<\cdots<\varrho_2<\varrho_1,\\
&0<\varrho_{2k+2}<\cdots<\varrho_{k+2},\\
&0<\varrho_{3k+3}<\cdots<\varrho_{2k+3}.
\end{aligned} \tag{36}$$

From (36), one obtains

$$\begin{aligned}
&\varrho_2\varrho_1^{-1}<1,\cdots,\varrho_{k+1}\varrho_k^{-1}<1,\\
&\varrho_{k+3}\varrho_{k+2}^{-1}<1,\cdots,\varrho_{2k+2}\varrho_{2k+1}^{-1}<1,\\
&\varrho_{2k+4}\varrho_{2k+3}^{-1}<1,\cdots,\varrho_{3k+3}\varrho_{3k+2}^{-1}<1.
\end{aligned} \tag{37}$$

From (33) and (34), one gets

$$\begin{aligned}
\varrho_1\varrho_{2k+2}^{-1}\frac{\alpha_{10}}{\alpha_{11}}&=\varrho_{2k+2}^{-1}\frac{\alpha_{10}}{\alpha_{11}}=\frac{\alpha_{10}}{\alpha_{11}}\frac{1}{1-k\varepsilon}<1,\\
\varrho_{k+2}\varrho_{3k+3}^{-1}\frac{\alpha_{13}}{\alpha_{14}}&=\varrho_{3k+3}^{-1}\frac{\alpha_{13}}{\alpha_{14}}=\frac{\alpha_{13}}{\alpha_{14}}\frac{1}{1-k\varepsilon}<1,\\
\varrho_{2k+3}\varrho_{k+1}^{-1}\frac{\alpha_{16}}{\alpha_{17}}&=\varrho_{k+1}^{-1}\frac{\alpha_{16}}{\alpha_{17}}=\frac{\alpha_{16}}{\alpha_{17}}\frac{1}{1-k\varepsilon}<1.
\end{aligned} \tag{38}$$

Finally, from (37) and (38), one gets

$$\begin{aligned}
\max_{1\le v\le 3k+3}|\rho_v|=\left\|\widehat{\Omega}B_1\Omega\wedge^{-1}\right\|=\max\Big\{&\varrho_1\varrho_{2k+2}^{-1}\frac{\alpha_{10}}{\alpha_{11}},\varrho_2\varrho_1^{-1},\cdots,\\
&\varrho_{k+1}\varrho_k^{-1},\varrho_{k+3}\varrho_{k+2}^{-1},\cdots,\varrho_{2k+2}\varrho_{2k+1}^{-1},\varrho_{3k+2}\varrho_{3k+1}^{-1},\cdots,\\
&\varrho_{3k+3}\varrho_{3k+2}^{-1},\varrho_{k+2}\varrho_{3k+3}^{-1}\frac{\alpha_{13}}{\alpha_{14}},\varrho_{2k+3}\varrho_{k+1}^{-1}\frac{\alpha_{16}}{\alpha_{17}}\Big\}<1.
\end{aligned} \tag{39}$$

From (39), we get the required result. □

In a similar way, dynamical characteristics of discrete systems (12)–(14) around P_0 can be investigated.

3.2. Local Dynamical Characteristics of Discrete Systems (12)–(14)

Theorem 2. *For local dynamical characteristics about P_0 of discrete systems (12)–(14), the following statements hold:*

(i) P_0 of (12) is a sink if

$$\begin{aligned}
&\frac{\alpha_{19}}{\alpha_{20}}<1,\\
&\frac{\alpha_{22}}{\alpha_{23}}<1,\\
&\frac{\alpha_{25}}{\alpha_{26}}<1
\end{aligned} \tag{40}$$

(ii) P_0 *of (13) is a sink if*

$$\begin{aligned} \frac{\alpha_{28}}{\alpha_{29}} &< 1, \\ \frac{\alpha_{31}}{\alpha_{32}} &< 1, \\ \frac{\alpha_{34}}{\alpha_{35}} &< 1 \end{aligned} \tag{41}$$

(iii) P_0 *of (14) is a sink if*

$$\begin{aligned} \frac{\alpha_{37}}{\alpha_{38}} &< 1, \\ \frac{\alpha_{40}}{\alpha_{41}} &< 1, \\ \frac{\alpha_{43}}{\alpha_{44}} &< 1 \end{aligned} \tag{42}$$

Proof. Its proof is the same as the proof of Theorem 1. □

4. Boundedness of Discrete Systems (11)–(14)

The boundedness of discrete systems (11)–(14) is explored in this section.

4.1. Boundedness of Discrete System (11)

Theorem 3. *If* $X_n = \{(x_n, y_n, z_n)\}$ *is the positive solution of discrete system (11), then for* $\mu \geq 0$*, the following holds:*

$$0 < x_n \leq \begin{cases} \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\mu+1}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\mu}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\mu} y_{-k}, & \text{if } n = (3+3k)\mu + 1, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\mu+1}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\mu}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\mu} y_{1-k}, & \text{if } n = (3+3k)\mu + 2, \\ \vdots & \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\mu+1}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\mu}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\mu} y_{-1}, & \text{if } n = (3+3k)\mu + k, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\mu+1}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\mu}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\mu} y_{0}, & \text{if } n = (3+3k)\mu + k + 1, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\mu+1}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\mu+1}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\mu} z_{-k}, & \text{if } n = (3+3k)\mu + k + 2, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\mu+1}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\mu+1}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\mu} z_{1-k}, & \text{if } n = (3+3k)\mu + k + 3, \\ \vdots & \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\mu+1}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\mu+1}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\mu} z_{-1}, & \text{if } n = (3+3k)\mu + 2k + 1, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\mu+1}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\mu+1}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\mu} z_{0}, & \text{if } n = (3+3k)\mu + 2k + 2, \\ \left(\frac{\alpha_{10}\alpha_{13}\alpha_{16}}{\alpha_{11}\alpha_{14}\alpha_{17}}\right)^{\mu+1} x_{-k}, & \text{if } n = (3+3k)\mu + 2k + 3, \\ \left(\frac{\alpha_{10}\alpha_{13}\alpha_{16}}{\alpha_{11}\alpha_{14}\alpha_{17}}\right)^{\mu+1} x_{1-k}, & \text{if } n = (3+3k)\mu + 2k + 4, \\ \vdots & \\ \left(\frac{\alpha_{10}\alpha_{13}\alpha_{16}}{\alpha_{11}\alpha_{14}\alpha_{17}}\right)^{\mu+1} x_{-1}, & \text{if } n = (3+3k)\mu + 3k + 2, \\ \left(\frac{\alpha_{10}\alpha_{13}\alpha_{16}}{\alpha_{11}\alpha_{14}\alpha_{17}}\right)^{\mu+1} x_{0}, & \text{if } n = (3+3k)\mu + 3k + 3. \end{cases} \tag{43}$$

$$0 < y_n \leq \begin{cases} \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\mu}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\mu+1}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\mu} z_{-k}, & \text{if } n = (3+3k)\mu + 1, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\mu}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\mu+1}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\mu} z_{1-k}, & \text{if } n = (3+3k)\mu + 2, \\ \vdots & \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\mu}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\mu+1}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\mu} z_{-1} & \text{if } n = (3+3k)\mu + k, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\mu}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\mu+1}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\mu} z_{0}, & \text{if } n = (3+3k)\mu + k + 1, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\mu}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\mu+1}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\mu+1} x_{-k}, & \text{if } n = (3+3k)\mu + k + 2, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\mu+1}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\mu+1} x_{1-k}, & \text{if } n = (3+3k)\mu + k + 3, \\ \vdots & \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\mu}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\mu+1}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\mu} x_{-1}, & \text{if } n = (3+3k)\mu + 2k + 1, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\mu}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\mu+1}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\mu} x_{0}, & \text{if } n = (3+3k)\mu + 2k + 2, \\ \left(\frac{\alpha_{10}\alpha_{13}\alpha_{16}}{\alpha_{11}\alpha_{14}\alpha_{17}}\right)^{\mu+1} y_{-k}, & \text{if } n = (3+3k)\mu + 2k + 3, \\ \left(\frac{\alpha_{10}\alpha_{13}\alpha_{16}}{\alpha_{11}\alpha_{14}\alpha_{17}}\right)^{\mu+1} y_{1-k}, & \text{if } n = (3+3k)\mu + 2k + 4, \\ \vdots & \\ \left(\frac{\alpha_{10}\alpha_{13}\alpha_{16}}{\alpha_{11}\alpha_{14}\alpha_{17}}\right)^{\mu+1} y_{-1}, & \text{if } n = (3+3k)\mu + 3k + 2, \\ \left(\frac{\alpha_{10}\alpha_{13}\alpha_{16}}{\alpha_{11}\alpha_{14}\alpha_{17}}\right)^{\mu+1} y_{0}, & \text{if } n = (3+3k)\mu + 3k + 3. \end{cases} \tag{44}$$

Finally,

$$0 < z_n \leq \begin{cases} \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\mu}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\mu}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\mu+1} x_{-k}, & \text{if } n = (3+3k)\mu + 1, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\mu}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\mu}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\mu+1} x_{1-k}, & \text{if } n = (3+3k)\mu + 2, \\ \vdots & \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\mu}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\mu}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\mu+1} x_{-1}, & \text{if } n = (3+3k)\mu + k, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\mu}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\mu}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\mu+1} x_{0}, & \text{if } n = (3+3k)\mu + k + 1, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\mu+1}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\mu}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\mu+1} y_{-k}, & \text{if } n = (3+3k)\mu + k + 2, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\mu+1}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\mu}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\mu+1} y_{1-k}, & \text{if } n = (3+3k)\mu + k + 3, \\ \vdots & \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\mu+1}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\mu}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\mu+1} y_{-1}, & \text{if } n = (3+3k)\mu + 2k + 1, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\mu+1}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\mu}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\mu+1} y_{0}, & \text{if } n = (3+3k)\mu + 2k + 2, \\ \left(\frac{\alpha_{10}\alpha_{13}\alpha_{16}}{\alpha_{11}\alpha_{14}\alpha_{17}}\right)^{\mu+1} z_{-k}, & \text{if } n = (3+3k)\mu + 2k + 3, \\ \left(\frac{\alpha_{10}\alpha_{13}\alpha_{16}}{\alpha_{11}\alpha_{14}\alpha_{17}}\right)^{\mu+1} z_{1-k}, & \text{if } n = (3+3k)\mu + 2k + 4, \\ \vdots & \\ \left(\frac{\alpha_{10}\alpha_{13}\alpha_{16}}{\alpha_{11}\alpha_{14}\alpha_{17}}\right)^{\mu+1} z_{-1}, & \text{if } n = (3+3k)\mu + 3k + 2, \\ \left(\frac{\alpha_{10}\alpha_{13}\alpha_{16}}{\alpha_{11}\alpha_{14}\alpha_{17}}\right)^{\mu+1} z_{0}, & \text{if } n = (3+3k)\mu + 3k + 3. \end{cases} \tag{45}$$

Proof. Noticeably, (43), (44), and (45) hold if $\mu = 0$. If (43), (44), and (45) are true for $\mu = \sigma$, i.e.,

$$0 < x_n \le \begin{cases} \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+1}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma} y_{-k}, & \text{if } n = (3+3k)\sigma + 1, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+1}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma} y_{1-k}, & \text{if } n = (3+3k)\sigma + 2, \\ \vdots & \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+1}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma} y_{-1}, & \text{if } n = (3+3k)\sigma + k, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+1}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma} y_{0}, & \text{if } n = (3+3k)\sigma + k + 1, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+1}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+1}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma} z_{-k}, & \text{if } n = (3+3k)\sigma + k + 2, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+1}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+1}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma} z_{1-k}, & \text{if } n = (3+3k)\sigma + k + 3, \\ \vdots & \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+1}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+1}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma} z_{-1}, & \text{if } n = (3+3k)\sigma + 2k + 1, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+1}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+1}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma} z_{0}, & \text{if } n = (3+3k)\sigma + 2k + 2, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\frac{\alpha_{13}}{\alpha_{14}}\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} x_{-k}, & \text{if } n = (3+3k)\sigma + 2k + 3, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\frac{\alpha_{13}}{\alpha_{14}}\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} x_{1-k}, & \text{if } n = (3+3k)\sigma + 2k + 4, \\ \vdots & \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\frac{\alpha_{13}}{\alpha_{14}}\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} x_{-1}, & \text{if } n = (3+3k)\sigma + 3k + 2, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\frac{\alpha_{13}}{\alpha_{14}}\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} x_{0}, & \text{if } n = (3+3k)\sigma + 3k + 3. \end{cases}$$

$$0 < y_n \le \begin{cases} \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+1}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma} z_{-k}, & \text{if } n = (3+3k)\sigma + 1, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+1}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma} z_{1-k}, & \text{if } n = (3+3k)\sigma + 2, \\ \vdots & \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+1}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma} z_{-1}, & \text{if } n = (3+3k)\sigma + k, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+1}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma} z_{0}, & \text{if } n = (3+3k)\sigma + k + 1, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+1}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} x_{-k}, & \text{if } n = (3+3k)\sigma + k + 2, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+1}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} x_{1-k}, & \text{if } n = (3+3k)\sigma + k + 3, \\ \vdots & \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+1}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma} x_{-1}, & \text{if } n = (3+3k)\sigma + 2k + 1, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+1}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma} x_{0}, & \text{if } n = (3+3k)\sigma + 2k + 2, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\frac{\alpha_{13}}{\alpha_{14}}\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} y_{-k}, & \text{if } n = (3+3k)\sigma + 2k + 3, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\frac{\alpha_{13}}{\alpha_{14}}\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} y_{1-k}, & \text{if } n = (3+3k)\sigma + 2k + 4, \\ \vdots & \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\frac{\alpha_{13}}{\alpha_{14}}\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} y_{-1}, & \text{if } n = (3+3k)\sigma + 3k + 2, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\frac{\alpha_{13}}{\alpha_{14}}\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} y_{0}, & \text{if } n = (3+3k)\sigma + 3k + 3. \end{cases} \tag{46}$$

Finally,

$$0 < z_n \le \begin{cases} \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} x_{-k}, & \text{if } n = (3+3k)\sigma + 1, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} x_{1-k}, & \text{if } n = (3+3k)\sigma + 2, \\ \vdots & \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} x_{-1}, & \text{if } n = (3+3k)\sigma + k, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} x_{0}, & \text{if } n = (3+3k)\sigma + k + 1, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+1}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} y_{-k}, & \text{if } n = (3+3k)\sigma + k + 2, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+1}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} y_{1-k}, & \text{if } n = (3+3k)\sigma + k + 3, \\ \vdots & \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+1}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} y_{-1}, & \text{if } n = (3+3k)\sigma + 2k + 1, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+1}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} y_{0}, & \text{if } n = (3+3k)\sigma + 2k + 2, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\frac{\alpha_{13}}{\alpha_{14}}\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} z_{-k}, & \text{if } n = (3+3k)\sigma + 2k + 3, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\frac{\alpha_{13}}{\alpha_{14}}\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} z_{1-k}, & \text{if } n = (3+3k)\sigma + 2k + 4, \\ \vdots & \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\frac{\alpha_{13}}{\alpha_{14}}\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} z_{-1}, & \text{if } n = (3+3k)\sigma + 3k + 2, \\ \left(\frac{\alpha_{10}}{\alpha_{11}}\frac{\alpha_{13}}{\alpha_{14}}\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} z_{0}, & \text{if } n = (3+3k)\sigma + 3k + 3. \end{cases} \tag{47}$$

Than for $\mu = \sigma + 1$, we have

$$x_{(3k+3)(\sigma+1)+1} = \frac{\alpha_{10} y_{(3k+3)\sigma+2k+3}}{\alpha_{11} + \alpha_{12}\prod_{i=0}^{k} z_{(3k+3)\sigma+3k+3-i}}, \le \frac{\alpha_{10}}{\alpha_{11}} y_{(3k+3)\sigma+2k+3} \le \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+2}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+1}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} y_{-k},$$

$$x_{(3k+3)(\sigma+1)+2} = \frac{\alpha_{10} y_{(3k+3)\sigma+2k+4}}{\alpha_{11} + \alpha_{12}\prod_{i=0}^{k} z_{(3k+3)\sigma+3k+4-i}}, \le \frac{\alpha_{10}}{\alpha_{11}} y_{(3k+3)\sigma+2k+4} \le \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+2}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+1}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} y_{-k+1},$$

$$\vdots$$

$$x_{(3k+3)(\sigma+1)+k} = \frac{\alpha_{10} y_{(3k+3)\sigma+3k+2}}{\alpha_{11} + \alpha_{12}\prod_{i=0}^{k} z_{(3k+3)\sigma+4k+2-i}}, \le \frac{\alpha_{10}}{\alpha_{11}} y_{(3k+3)\sigma+3k+2} \le \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+2}\left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+1}\left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} y_{-1},$$

$$x_{(3k+3)(\sigma+1)+k+1} = \frac{\alpha_{10} y_{(3k+3)\sigma+3k+3}}{\alpha_{11} + \alpha_{12} \prod_{i=0}^{k} z_{(3k+3)\sigma+4k+3-i}}, \leq \frac{\alpha_{10}}{\alpha_{11}} y_{(3k+3)\sigma+3k+3} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+2} \left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+1} \left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} y_0,$$

$$y_{(3k+3)(\sigma+1)+1} = \frac{\alpha_{13} z_{(3k+3)\sigma+2k+3}}{\alpha_{14} + \alpha_{15} \prod_{i=0}^{k} x_{(3k+3)\sigma+3k+3-i}}, \leq \frac{\alpha_{13}}{\alpha_{14}} z_{(3k+3)\sigma+2k+3} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+1} \left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+2} \left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} z_{-k},$$

$$y_{(3k+3)(\sigma+1)+2} = \frac{\alpha_{13} z_{(3k+3)\sigma+2k+4}}{\alpha_{14} + \alpha_{15} \prod_{i=0}^{k} x_{(3k+3)\sigma+3k+4-i}}, \leq \frac{\alpha_{13}}{\alpha_{14}} z_{(3k+3)\sigma+2k+4} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+1} \left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+2} \left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} z_{-k+1},$$

$$\vdots$$

$$y_{(3k+3)(\sigma+1)+k} = \frac{\alpha_{13} z_{(3k+3)\sigma+3k+2}}{\alpha_{14} + \alpha_{15} \prod_{i=0}^{k} x_{(3k+3)\sigma+4k+2-i}}, \leq \frac{\alpha_{13}}{\alpha_{14}} z_{(3k+3)\sigma+3k+2} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+1} \left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+2} \left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} z_{-1},$$

$$y_{(3k+3)(\sigma+1)+k+1} = \frac{\alpha_{13} z_{(3k+3)\sigma+3k+3}}{\alpha_{14} + \alpha_{15} \prod_{i=0}^{k} x_{(3k+3)\sigma+4k+3-i}}, \leq \frac{\alpha_{13}}{\alpha_{14}} z_{(3k+3)\sigma+3k+3} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+1} \left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+2} \left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} z_0,$$

$$z_{(3k+3)(\sigma+1)+1} = \frac{\alpha_{16} x_{(3k+3)\sigma+2k+3}}{\alpha_{17} + \alpha_{18} \prod_{i=0}^{k} y_{(3k+3)\sigma+3k+3-i}}, \leq \frac{\alpha_{16}}{\alpha_{17}} x_{(3k+3)\sigma+2k+3} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+1} \left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+1} \left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+2} x_{-k},$$

$$z_{(3k+3)(\sigma+1)+2} = \frac{\alpha_{16} x_{(3k+3)\sigma+2k+4}}{\alpha_{17} + \alpha_{18} \prod_{i=0}^{k} y_{(3k+3)\sigma+3k+4-i}}, \leq \frac{\alpha_{16}}{\alpha_{17}} x_{(3k+3)\sigma+2k+4} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+1} \left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+1} \left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+2} x_{-k+1},$$

$$\vdots$$

$$z_{(3k+3)(\sigma+1)+k} = \frac{\alpha_{16} x_{(3k+3)\sigma+3k+2}}{\alpha_{17} + \alpha_{18} \prod_{i=0}^{k} y_{(3k+3)\sigma+4k+2-i}}, \leq \frac{\alpha_{16}}{\alpha_{17}} x_{(3k+3)\sigma+3k+2} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+1} \left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+1} \left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+2} x_{-1},$$

$$z_{(3k+3)(\sigma+1)+k+1} = \frac{\alpha_{16} x_{(3k+3)\sigma+3k+3}}{\alpha_{17} + \alpha_{18} \prod_{i=0}^{k} y_{(3k+3)\sigma+4k+3-i}}, \leq \frac{\alpha_{16}}{\alpha_{17}} x_{(3k+3)\sigma+3k+3} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+1} \left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+1} \left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+2} x_0. \tag{48}$$

And

$$x_{(3k+3)(\sigma+1)+k+2} = \frac{\alpha_{10} y_{(3k+3)(\sigma+1)+1}}{\alpha_{11} + \alpha_{12} \prod_{i=0}^{k} z_{(3k+3)\sigma+4k+4-i}}, \leq \frac{\alpha_{10}}{\alpha_{11}} y_{(3k+3)(\sigma+1)+1} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+2} \left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+2} \left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} z_{-k},$$

$$x_{(3k+3)(\sigma+1)+k+3} = \frac{\alpha_{10} y_{(3k+3)(\sigma+1)+2}}{\alpha_{11} + \alpha_{12} \prod_{i=0}^{k} z_{(3k+3)\sigma+4k+5-i}}, \leq \frac{\alpha_{10}}{\alpha_{11}} y_{(3k+3)(\sigma+1)+2} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+2} \left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+2} \left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} z_{-k+1},$$

$$\vdots$$

$$x_{(3k+3)(\sigma+1)+2k+1} = \frac{\alpha_{10} y_{(3k+3)(\sigma+1)+k}}{\alpha_{11} + \alpha_{12} \prod_{i=0}^{k} z_{(3k+3)\sigma+5k+3-i}}, \leq \frac{\alpha_{10}}{\alpha_{11}} y_{(3k+3)(\sigma+1)+k} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+2} \left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+2} \left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} z_{-1},$$

$$x_{(3k+3)(\sigma+1)+2k+2} = \frac{\alpha_{10} y_{(3k+3)(\sigma+1)+k+1}}{\alpha_{11} + \alpha_{12} \prod_{i=0}^{k} z_{(3k+3)\sigma+5k+4-i}}, \leq \frac{\alpha_{10}}{\alpha_{11}} y_{(3k+3)(\sigma+1)+2} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+2} \left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+2} \left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+1} z_0,$$

$$y_{(3k+3)(\sigma+1)+k+2} = \frac{\alpha_{13} z_{(3k+3)(\sigma+1)+1}}{\alpha_{14} + \alpha_{15} \prod_{i=0}^{k} x_{(3k+3)\sigma+4k+4-i}}, \leq \frac{\alpha_{13}}{\alpha_{14}} z_{(3k+3)(\sigma+1)+1} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+1} \left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+2} \left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+2} x_{-k},$$

$$y_{(3k+3)(\sigma+1)+k+3} = \frac{\alpha_{13} z_{(3k+3)(\sigma+1)+2}}{\alpha_{14} + \alpha_{15} \prod_{i=0}^{k} x_{(3k+3)\sigma+4k+5-i}}, \leq \frac{\alpha_{13}}{\alpha_{14}} z_{(3k+3)(\sigma+1)+2} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+1} \left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+2} \left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+2} x_{-k+1},$$

$$\vdots$$

$$y_{(3k+3)(\sigma+1)+2k+1} = \frac{\alpha_{13} z_{(3k+3)(\sigma+1)+k}}{\alpha_{14} + \alpha_{15} \prod_{i=0}^{k} x_{(3k+3)\sigma+5k+3-i}}, \leq \frac{\alpha_{13}}{\alpha_{14}} z_{(3k+3)(\sigma+1)+k} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+1} \left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+2} \left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+2} x_{-1},$$

$$y_{(3k+3)(\sigma+1)+2k+2} = \frac{\alpha_{13} z_{(3k+3)(\sigma+1)+k+1}}{\alpha_{14} + \alpha_{15} \prod_{i=0}^{k} x_{(3k+3)\sigma+5k+4-i}}, \leq \frac{\alpha_{13}}{\alpha_{14}} z_{(3k+3)(\sigma+1)+k+1} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+1} \left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+2} \left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+2} x_0,$$

$$z_{(3k+3)(\sigma+1)+k+2} = \frac{\alpha_{16} x_{(3k+3)(\sigma+1)+1}}{\alpha_{17} + \alpha_{18} \prod_{i=0}^{k} y_{(3k+3)\sigma+4k+4-i}}, \leq \frac{\alpha_{16}}{\alpha_{17}} x_{(3k+3)(\sigma+1)+1} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+2} \left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+1} \left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+2} y_{-k},$$

$$z_{(3k+3)(\sigma+1)+k+3} = \frac{\alpha_{16}x_{(3k+3)(\sigma+1)+2}}{\alpha_{17} + \alpha_{18}\prod_{i=0}^{k} y_{(3k+3)\sigma+4k+5-i}}, \leq \frac{\alpha_{16}}{\alpha_{17}} x_{(3k+3)(\sigma+1)+2} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+2} \left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+1} \left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+2} y_{-k+1},$$

$$\vdots$$

$$z_{(3k+3)(\sigma+1)+2k+1} = \frac{\alpha_{16}x_{(3k+3)(\sigma+1)+k}}{\alpha_{17} + \alpha_{18}\prod_{i=0}^{k} y_{(3k+3)\sigma+5k+3-i}}, \leq \frac{\alpha_{16}}{\alpha_{17}} x_{(3k+3)(\sigma+1)+k} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+2} \left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+1} \left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+2} y_{-1},$$

$$z_{(3k+3)(\sigma+1)+2k+2} = \frac{\alpha_{16}x_{(3k+3)(\sigma+1)+k+1}}{\alpha_{17} + \alpha_{18}\prod_{i=0}^{k} y_{(3k+3)\sigma+5k+4-i}}, \leq \frac{\alpha_{16}}{\alpha_{17}} x_{(3k+3)(\sigma+1)+k+1} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\right)^{\sigma+2} \left(\frac{\alpha_{13}}{\alpha_{14}}\right)^{\sigma+1} \left(\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+2} y_{0}. \quad (49)$$

Finally,

$$x_{(3k+3)(\sigma+1)+2k+3} = \frac{\alpha_{10}y_{(3k+3)(\sigma+1)+k+2}}{\alpha_{11} + \alpha_{12}\prod_{i=0}^{k} z_{(3k+3)\sigma+5k+5-i}}, \leq \frac{\alpha_{10}}{\alpha_{11}} y_{(3k+3)(\sigma+1)+k+2} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\frac{\alpha_{13}}{\alpha_{14}}\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+2} x_{-k},$$

$$x_{(3k+3)(\sigma+1)+2k+4} = \frac{\alpha_{10}y_{(3k+3)(\sigma+1)+k+3}}{\alpha_{11} + \alpha_{12}\prod_{i=0}^{k} z_{(3k+3)\sigma+5k+6-i}}, \leq \frac{\alpha_{10}}{\alpha_{11}} y_{(3k+3)(\sigma+1)+k+3} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\frac{\alpha_{13}}{\alpha_{14}}\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+2} x_{-k+1},$$

$$\vdots$$

$$x_{(3k+3)(\sigma+1)+3k+2} = \frac{\alpha_{10}y_{(3k+3)(\sigma+1)+2k+1}}{\alpha_{11} + \alpha_{12}\prod_{i=0}^{k} z_{(3k+3)\sigma+6k+4-i}}, \leq \frac{\alpha_{10}}{\alpha_{11}} y_{(3k+3)(\sigma+1)+2k+1} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\frac{\alpha_{13}}{\alpha_{14}}\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+2} x_{-1},$$

$$x_{(3k+3)(\sigma+1)+3k+3} = \frac{\alpha_{10}y_{(3k+3)(\sigma+1)+2k+2}}{\alpha_{11} + \alpha_{12}\prod_{i=0}^{k} z_{(3k+3)\sigma+6k+5-i}}, \leq \frac{\alpha_{10}}{\alpha_{11}} y_{(3k+3)(\sigma+1)+2k+2} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\frac{\alpha_{13}}{\alpha_{14}}\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+2} x_{0},$$

$$y_{(3k+3)(\sigma+1)+2k+3} = \frac{\alpha_{13}z_{(3k+3)(\sigma+1)+k+2}}{\alpha_{14} + \alpha_{15}\prod_{i=0}^{k} x_{(3k+3)\sigma+5k+5-i}}, \leq \frac{\alpha_{13}}{\alpha_{14}} z_{(3k+3)(\sigma+1)+k+2} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\frac{\alpha_{13}}{\alpha_{14}}\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+2} y_{-k},$$

$$y_{(3k+3)(\sigma+1)+2k+4} = \frac{\alpha_{13}z_{(3k+3)(\sigma+1)+k+3}}{\alpha_{14} + \alpha_{15}\prod_{i=0}^{k} x_{(3k+3)\sigma+5k+6-i}}, \leq \frac{\alpha_{13}}{\alpha_{14}} z_{(3k+3)(\sigma+1)+k+3} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\frac{\alpha_{13}}{\alpha_{14}}\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+2} y_{-k+1},$$

$$\vdots$$

$$y_{(3k+3)(\sigma+1)+3k+2} = \frac{\alpha_{13}z_{(3k+3)(\sigma+1)+2k+1}}{\alpha_{14} + \alpha_{15}\prod_{i=0}^{k} x_{(3k+3)\sigma+6k+4-i}}, \leq \frac{\alpha_{13}}{\alpha_{14}} z_{(3k+3)(\sigma+1)+2k+1} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\frac{\alpha_{13}}{\alpha_{14}}\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+2} y_{-1},$$

$$y_{(3k+3)(\sigma+1)+3k+3} = \frac{\alpha_{13}z_{(3k+3)(\sigma+1)+2k+2}}{\alpha_{14} + \alpha_{15}\prod_{i=0}^{k} x_{(3k+3)\sigma+6k+5-i}}, \leq \frac{\alpha_{13}}{\alpha_{14}} z_{(3k+3)(\sigma+1)+2k+2} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\frac{\alpha_{13}}{\alpha_{14}}\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+2} y_{0},$$

$$z_{(3k+3)(\sigma+1)+2k+3} = \frac{\alpha_{16}x_{(3k+3)(\sigma+1)+k+2}}{\alpha_{17} + \alpha_{18}\prod_{i=0}^{k} y_{(3k+3)\sigma+5k+5-i}}, \leq \frac{\alpha_{16}}{\alpha_{17}} x_{(3k+3)(\sigma+1)+k+2} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\frac{\alpha_{13}}{\alpha_{14}}\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+2} z_{-k},$$

$$z_{(3k+3)(\sigma+1)+2k+4} = \frac{\alpha_{16}x_{(3k+3)(\sigma+1)+k+3}}{\alpha_{17} + \alpha_{18}\prod_{i=0}^{k} y_{(3k+3)\sigma+5k+6-i}}, \leq \frac{\alpha_{16}}{\alpha_{17}} x_{(3k+3)(\sigma+1)+k+3} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\frac{\alpha_{13}}{\alpha_{14}}\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+2} z_{-k+1},$$

$$\vdots$$

$$z_{(3k+3)(\sigma+1)+3k+2} = \frac{\alpha_{16}x_{(3k+3)(\sigma+1)+2k+1}}{\alpha_{17} + \alpha_{18}\prod_{i=0}^{k} y_{(3k+3)\sigma+6k+4-i}}, \leq \frac{\alpha_{16}}{\alpha_{17}} x_{(3k+3)(\sigma+1)+2k+1} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\frac{\alpha_{13}}{\alpha_{14}}\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+2} z_{-1},$$

$$z_{(3k+3)(\sigma+1)+3k+3} = \frac{\alpha_{16}x_{(3k+3)(\sigma+1)+2k+2}}{\alpha_{17} + \alpha_{18}\prod_{i=0}^{k} y_{(3k+3)\sigma+6k+5-i}}, \leq \frac{\alpha_{16}}{\alpha_{17}} x_{(3k+3)(\sigma+1)+2k+2} \leq \left(\frac{\alpha_{10}}{\alpha_{11}}\frac{\alpha_{13}}{\alpha_{14}}\frac{\alpha_{16}}{\alpha_{17}}\right)^{\sigma+2} z_{0}. \quad (50)$$

□

Corollary 4. *$\{X_n\}$ of (11) is bounded if (30) holds.*

Proof. It is a direct result of Theorem 3.

Hereafter, we will present boundedness for discrete systems (12)–(14). □

Theorem 5. *(i) If $\{X_n\}$ is the positive solution of (12), then for $\mu \geq 0$, the following holds:*

$$0 < x_n \leq \begin{cases} \left(\frac{\alpha_{19}}{\alpha_{20}}\right)^{\mu+1}\left(\frac{\alpha_{22}}{\alpha_{23}}\right)^{\mu}\left(\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu} z_{-k}, & \text{if } n=(3+3k)\mu+1, \\ \left(\frac{\alpha_{19}}{\alpha_{20}}\right)^{\mu+1}\left(\frac{\alpha_{22}}{\alpha_{23}}\right)^{\mu}\left(\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu} z_{-k+1}, & \text{if } n=(3+3k)\mu+2, \\ \vdots & \\ \left(\frac{\alpha_{19}}{\alpha_{20}}\right)^{\mu+1}\left(\frac{\alpha_{22}}{\alpha_{23}}\right)^{\mu}\left(\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu} z_{-1}, & \text{if } n=(3+3k)\mu+k, \\ \left(\frac{\alpha_{19}}{\alpha_{20}}\right)^{\mu+1}\left(\frac{\alpha_{22}}{\alpha_{23}}\right)^{\mu}\left(\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu} z_{0}, & \text{if } n=(3+3k)\mu+k+1, \\ \left(\frac{\alpha_{19}}{\alpha_{20}}\right)^{\mu+1}\left(\frac{\alpha_{22}}{\alpha_{23}}\right)^{\mu+1}\left(\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu} y_{-k}, & \text{if } n=(3+3k)\mu+k+2, \\ \left(\frac{\alpha_{19}}{\alpha_{20}}\right)^{\mu+1}\left(\frac{\alpha_{22}}{\alpha_{23}}\right)^{\mu+1}\left(\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu} y_{-k+1}, & \text{if } n=(3+3k)\mu+k+3, \\ \vdots & \\ \left(\frac{\alpha_{19}}{\alpha_{20}}\right)^{\mu+1}\left(\frac{\alpha_{22}}{\alpha_{23}}\right)^{\mu+1}\left(\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu} y_{-1}, & \text{if } n=(3+3k)\mu+2k+1, \\ \left(\left(\frac{\alpha_{19}}{\alpha_{20}}\right)^{\mu+1}\left(\frac{\alpha_{22}}{\alpha_{23}}\right)^{\mu+1}\left(\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu}\right) y_{0}, & \text{if } n=(3+3k)\mu+2k+2, \\ \left(\frac{\alpha_{19}}{\alpha_{20}}\frac{\alpha_{22}}{\alpha_{23}}\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu+1} x_{-k}, & \text{if } n=(3+3k)\mu+2k+3, \\ \left(\frac{\alpha_{19}}{\alpha_{20}}\frac{\alpha_{22}}{\alpha_{23}}\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu+1} x_{-k+1}, & \text{if } n=(3+3k)\mu+2k+3, \\ \vdots & \\ \left(\frac{\alpha_{19}}{\alpha_{20}}\frac{\alpha_{22}}{\alpha_{23}}\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu+1} x_{-1}, & \text{if } n=(3+3k)\mu+3k+2, \\ \left(\frac{\alpha_{19}}{\alpha_{20}}\frac{\alpha_{22}}{\alpha_{23}}\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu+1} x_{0}, & \text{if } n=(3+3k)\mu+3k+3. \end{cases} \quad (51)$$

$$0 < y_n \leq \begin{cases} \left(\frac{\alpha_{19}}{\alpha_{20}}\right)^{\mu}\left(\frac{\alpha_{22}}{\alpha_{23}}\right)^{\mu+1}\left(\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu} x_{-k}, & \text{if } n=(3+3k)\mu+1, \\ \left(\frac{\alpha_{19}}{\alpha_{20}}\right)^{\mu}\left(\frac{\alpha_{22}}{\alpha_{23}}\right)^{\mu+1}\left(\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu} x_{-k+1}, & \text{if } n=(3+3k)\mu+2, \\ \vdots & \\ \left(\frac{\alpha_{19}}{\alpha_{20}}\right)^{\mu}\left(\frac{\alpha_{22}}{\alpha_{23}}\right)^{\mu+1}\left(\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu} x_{-1}, & \text{if } n=(3+3k)\mu+k, \\ \left(\frac{\alpha_{19}}{\alpha_{20}}\right)^{\mu}\left(\frac{\alpha_{22}}{\alpha_{23}}\right)^{\mu+1}\left(\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu} x_{0}, & \text{if } n=(3+3k)\mu+k+1, \\ \left(\frac{\alpha_{19}}{\alpha_{20}}\right)^{\mu}\left(\frac{\alpha_{22}}{\alpha_{23}}\right)^{\mu+1}\left(\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu} z_{-k}, & \text{if } n=(3+3k)\mu+k+2, \\ \left(\frac{\alpha_{19}}{\alpha_{20}}\right)^{\mu}\left(\frac{\alpha_{22}}{\alpha_{23}}\right)^{\mu+1}\left(\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu} z_{-k+1}, & \text{if } n=(3+3k)\mu+k+3, \\ \vdots & \\ \left(\frac{\alpha_{19}}{\alpha_{20}}\right)^{\mu}\left(\frac{\alpha_{22}}{\alpha_{23}}\right)^{\mu+1}\left(\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu} z_{-1}, & \text{if } n=(3+3k)\mu+2k+1, \\ \left(\frac{\alpha_{19}}{\alpha_{20}}\right)^{\mu}\left(\frac{\alpha_{22}}{\alpha_{23}}\right)^{\mu+1}\left(\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu} z_{0}, & \text{if } n=(3+3k)\mu+2k+2, \\ \left(\frac{\alpha_{19}}{\alpha_{20}}\frac{\alpha_{22}}{\alpha_{23}}\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu+1} y_{-k}, & \text{if } n=(3k+3)\mu+2k+3, \\ \left(\frac{\alpha_{19}}{\alpha_{20}}\frac{\alpha_{22}}{\alpha_{23}}\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu+1} y_{-k+1}, & \text{if } n=(3+3k)\mu+2k+3, \\ \vdots & \\ \left(\frac{\alpha_{19}}{\alpha_{20}}\frac{\alpha_{22}}{\alpha_{23}}\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu+1} y_{-1}, & \text{if } n=(3+3k)\mu+3k+2, \\ \left(\frac{\alpha_{19}}{\alpha_{20}}\frac{\alpha_{22}}{\alpha_{23}}\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu+1} y_{0}, & \text{if } n=(3+3k)\mu+3k+3. \end{cases}$$

Finally,

$$0 < z_n \leq \begin{cases} \left(\frac{\alpha_{19}}{\alpha_{20}}\right)^{\mu}\left(\frac{\alpha_{22}}{\alpha_{23}}\right)^{\mu}\left(\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu+1} y_{-k}, & \text{if } n=(3+3k)\mu+1, \\ \left(\frac{\alpha_{19}}{\alpha_{20}}\right)^{\mu}\left(\frac{\alpha_{22}}{\alpha_{23}}\right)^{\mu}\left(\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu+1} y_{-k+1}, & \text{if } n=(3+3k)\mu+2, \\ \vdots & \\ \left(\frac{\alpha_{19}}{\alpha_{20}}\right)^{\mu}\left(\frac{\alpha_{22}}{\alpha_{23}}\right)^{\mu}\left(\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu+1} y_{-1}, & \text{if } n=(3+3k)\mu+k, \\ \left(\frac{\alpha_{19}}{\alpha_{20}}\right)^{\mu}\left(\frac{\alpha_{22}}{\alpha_{23}}\right)^{\mu}\left(\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu+1} y_{0}, & \text{if } n=(3+3k)\mu+k+1, \\ \left(\frac{\alpha_{19}}{\alpha_{20}}\right)^{\mu+1}\left(\frac{\alpha_{22}}{\alpha_{23}}\right)^{\mu}\left(\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu+1} x_{-k}, & \text{if } n=(3+3k)\mu+k+2, \\ \left(\frac{\alpha_{19}}{\alpha_{20}}\right)^{\mu+1}\left(\frac{\alpha_{22}}{\alpha_{23}}\right)^{\mu}\left(\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu+1} x_{-k+1}, & \text{if } n=(3+3k)\mu+k+3, \\ \vdots & \\ \left(\frac{\alpha_{19}}{\alpha_{20}}\right)^{\mu+1}\left(\frac{\alpha_{22}}{\alpha_{23}}\right)^{\mu}\left(\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu+1} x_{-1}, & \text{if } n=(3+3k)\mu+2k+1, \\ \left(\frac{\alpha_{19}}{\alpha_{20}}\right)^{\mu+1}\left(\frac{\alpha_{22}}{\alpha_{23}}\right)^{\mu}\left(\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu+1} x_{0}, & \text{if } n=(3+3k)\mu+2k+2, \\ \left(\frac{\alpha_{19}}{\alpha_{20}}\frac{\alpha_{22}}{\alpha_{23}}\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu+1} z_{-k}, & \text{if } n=(3+3k)\mu+2k+3, \\ \left(\frac{\alpha_{19}}{\alpha_{20}}\frac{\alpha_{22}}{\alpha_{23}}\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu+1} z_{-k+1}, & \text{if } n=(3+3k)\mu+2k+3, \\ \vdots & \\ \left(\frac{\alpha_{19}}{\alpha_{20}}\frac{\alpha_{22}}{\alpha_{23}}\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu+1} z_{-1}, & \text{if } n=(3+3k)\mu+3k+2, \\ \left(\frac{\alpha_{19}}{\alpha_{20}}\frac{\alpha_{22}}{\alpha_{23}}\frac{\alpha_{25}}{\alpha_{26}}\right)^{\mu+1} z_{0}, & \text{if } n=(3+3k)\mu+3k+3. \end{cases} \quad (52)$$

(ii) If $\{X_n\}$ is the positive solution of (13), then for $\mu \geq 0$, the following holds:

$$0 < x_n \leq \begin{cases} \left(\frac{\alpha_{28}}{\alpha_{29}}\right)^{\mu+1}\left(\frac{\alpha_{31}}{\alpha_{32}}\right)^{\mu}\left(\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu} y_{-k}, & \text{if } n=(3+3k)\mu+1, \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\right)^{\mu+1}\left(\frac{\alpha_{31}}{\alpha_{32}}\right)^{\mu}\left(\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu} y_{-k+1}, & \text{if } n=(3+3k)\mu+2, \\ \vdots & \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\right)^{\mu+1}\left(\frac{\alpha_{31}}{\alpha_{32}}\right)^{\mu}\left(\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu} y_{-1}, & \text{if } n=(3+3k)\mu+k, \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\right)^{\mu+1}\left(\frac{\alpha_{31}}{\alpha_{32}}\right)^{\mu}\left(\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu} y_{0}, & \text{if } n=(3+3k)\mu+k+1, \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\right)^{\mu+1}\left(\frac{\alpha_{31}}{\alpha_{32}}\right)^{\mu+1}\left(\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu} z_{-k}, & \text{if } n=(3+3k)\mu+k+2, \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\right)^{\mu+1}\left(\frac{\alpha_{31}}{\alpha_{32}}\right)^{\mu+1}\left(\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu} z_{-k+1}, & \text{if } n=(3+3k)\mu+k+3, \\ \vdots & \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\right)^{\mu+1}\left(\frac{\alpha_{31}}{\alpha_{32}}\right)^{\mu+1}\left(\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu} z_{-1}, & \text{if } n=(3+3k)\mu+2k+1, \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\right)^{\mu+1}\left(\frac{\alpha_{31}}{\alpha_{32}}\right)^{\mu+1}\left(\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu} z_{0}, & \text{if } n=(3+3k)\mu+2k+2, \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\frac{\alpha_{31}}{\alpha_{32}}\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu+1} x_{-k}, & \text{if } n=(3+3k)\mu+2k+3, \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\frac{\alpha_{31}}{\alpha_{32}}\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu+1} x_{-k+1}, & \text{if } n=(3+3k)\mu+2k+4, \\ \vdots & \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\frac{\alpha_{31}}{\alpha_{32}}\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu+1} x_{-1}, & \text{if } n=(3+3k)\mu+3k+2, \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\frac{\alpha_{31}}{\alpha_{32}}\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu+1} x_{0}, & \text{if } n=(3+3k)\mu+3k+3. \end{cases} \quad (53)$$

And

$$0 < y_n \le \begin{cases} \left(\frac{\alpha_{28}}{\alpha_{29}}\right)^{\mu}\left(\frac{\alpha_{31}}{\alpha_{32}}\right)^{\mu+1}\left(\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu} z_{-k}, & \text{if } n=(3+3k)\mu+1, \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\right)^{\mu}\left(\frac{\alpha_{31}}{\alpha_{32}}\right)^{\mu+1}\left(\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu} z_{-k+1}, & \text{if } n=(3+3k)\mu+2 \\ \vdots & \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\right)^{\mu}\left(\frac{\alpha_{31}}{\alpha_{32}}\right)^{\mu+1}\left(\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu} z_{-1}, & \text{if } n=(3+3k)\mu+k, \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\right)^{\mu}\left(\frac{\alpha_{31}}{\alpha_{32}}\right)^{\mu+1}\left(\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu} z_{0}, & \text{if } n=(3+3k)\mu+k+1, \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\right)^{\mu}\left(\frac{\alpha_{31}}{\alpha_{32}}\right)^{\mu+1}\left(\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu} x_{-k}, & \text{if } n=(3+3k)\mu+k+2, \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\right)^{\mu}\left(\frac{\alpha_{31}}{\alpha_{32}}\right)^{\mu+1}\left(\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu} x_{-k+1}, & \text{if } n=(3+3k)\mu+k+3, \\ \vdots & \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\right)^{\mu}\left(\frac{\alpha_{31}}{\alpha_{32}}\right)^{\mu+1}\left(\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu} x_{-1}, & \text{if } n=(3+3k)\mu+2k+1, \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\right)^{\mu}\left(\frac{\alpha_{31}}{\alpha_{32}}\right)^{\mu+1}\left(\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu} x_{0}, & \text{if } n=(3+3k)\mu+2k+2, \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\frac{\alpha_{31}}{\alpha_{32}}\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu+1} y_{-k}, & \text{if } n=(3+3k)\mu+2k+3, \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\frac{\alpha_{31}}{\alpha_{32}}\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu+1} y_{-k+1}, & \text{if } n=(3+3k)\mu+2k+4, \\ \vdots & \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\frac{\alpha_{31}}{\alpha_{32}}\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu+1} y_{-1}, & \text{if } n=(3+3k)\mu+3k+2, \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\frac{\alpha_{31}}{\alpha_{32}}\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu+1} y_{0}, & \text{if } n=(3+3k)\mu+3k+3. \end{cases} \quad (54)$$

Finally,

$$0 < z_n \le \begin{cases} \left(\frac{\alpha_{28}}{\alpha_{29}}\right)^{\mu}\left(\frac{\alpha_{31}}{\alpha_{32}}\right)^{\mu}\left(\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu+1} x_{-k}, & \text{if } n=(3+3k)\mu+1, \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\right)^{\mu}\left(\frac{\alpha_{31}}{\alpha_{32}}\right)^{\mu}\left(\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu+1} x_{-k+1}, & \text{if } n=(3+3k)\mu+2, \\ \vdots & \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\right)^{\mu}\left(\frac{\alpha_{31}}{\alpha_{32}}\right)^{\mu}\left(\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu+1} x_{-1}, & \text{if } n=(3+3k)\mu+k, \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\right)^{\mu}\left(\frac{\alpha_{31}}{\alpha_{32}}\right)^{\mu}\left(\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu+1} x_{0}, & \text{if } n=(3+3k)\mu+k+1, \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\right)^{\mu+1}\left(\frac{\alpha_{31}}{\alpha_{32}}\right)^{\mu}\left(\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu+1} y_{-k}, & \text{if } n=(3+3k)\mu+k+2, \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\right)^{\mu+1}\left(\frac{\alpha_{31}}{\alpha_{32}}\right)^{\mu}\left(\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu+1} y_{-k+1}, & \text{if } n=(3+3k)\mu+k+3, \\ \vdots & \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\right)^{\mu+1}\left(\frac{\alpha_{31}}{\alpha_{32}}\right)^{\mu}\left(\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu+1} y_{-1}, & \text{if } n=(3+3k)\mu+2k+1, \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\right)^{\mu+1}\left(\frac{\alpha_{31}}{\alpha_{32}}\right)^{\mu}\left(\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu+1} y_{0}, & \text{if } n=(3+3k)\mu+2k+2, \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\frac{\alpha_{31}}{\alpha_{32}}\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu+1} z_{-k}, & \text{if } n=(3+3k)\mu+2k+3, \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\frac{\alpha_{31}}{\alpha_{32}}\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu+1} z_{-k+1}, & \text{if } n=(3+3k)\mu+2k+4, \\ \vdots & \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\frac{\alpha_{31}}{\alpha_{32}}\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu+1} z_{-1}, & \text{if } n=(3+3k)\mu+3k+2, \\ \left(\frac{\alpha_{28}}{\alpha_{29}}\frac{\alpha_{31}}{\alpha_{32}}\frac{\alpha_{34}}{\alpha_{35}}\right)^{\mu+1} z_{0}, & \text{if } n=(3+3k)\mu+3k+3. \end{cases} \quad (55)$$

(iii) If $\{X_n\}$ is the positive solution of (14), then for $\mu \ge 0$, the following holds:

$$0 < x_n \le \begin{cases} \left(\frac{\alpha_{37}}{\alpha_{38}}\right)^{\mu+1}\left(\frac{\alpha_{40}}{\alpha_{41}}\right)^{\mu}\left(\frac{\alpha_{43}}{\alpha_{44}}\right)^{\mu} z_{-k}, & \text{if } n=(3+3k)\mu+1, \\ \left(\frac{\alpha_{37}}{\alpha_{38}}\right)^{\mu+1}\left(\frac{\alpha_{40}}{\alpha_{41}}\right)^{\mu}\left(\frac{\alpha_{43}}{\alpha_{44}}\right)^{\mu} z_{-k+1}, & \text{if } n=(3+3k)\mu+2, \\ \vdots & \\ \left(\frac{\alpha_{37}}{\alpha_{38}}\right)^{\mu+1}\left(\frac{\alpha_{40}}{\alpha_{41}}\right)^{\mu}\left(\frac{\alpha_{43}}{\alpha_{44}}\right)^{\mu} z_{-1}, & \text{if } n=(3+3k)\mu+k, \\ \left(\frac{\alpha_{37}}{\alpha_{38}}\right)^{\mu+1}\left(\frac{\alpha_{40}}{\alpha_{41}}\right)^{\mu}\left(\frac{\alpha_{43}}{\alpha_{44}}\right)^{\mu} z_{0}, & \text{if } n=(3+3k)\mu+k+1, \\ \left(\frac{\alpha_{37}}{\alpha_{38}}\right)^{\mu+1}\left(\frac{\alpha_{40}}{\alpha_{41}}\right)^{\mu+1}\left(\frac{\alpha_{43}}{\alpha_{44}}\right)^{\mu} y_{-k}, & \text{if } n=(3+3k)\mu+k+2, \\ \left(\frac{\alpha_{37}}{\alpha_{38}}\right)^{\mu+1}\left(\frac{\alpha_{40}}{\alpha_{41}}\right)^{\mu+1}\left(\frac{\alpha_{43}}{\alpha_{44}}\right)^{\mu} y_{-k+1}, & \text{if } n=(3+3k)\mu+k+3, \\ \vdots & \\ \left(\frac{\alpha_{37}}{\alpha_{38}}\right)^{\mu+1}\left(\frac{\alpha_{40}}{\alpha_{41}}\right)^{\mu+1}\left(\frac{\alpha_{43}}{\alpha_{44}}\right)^{\mu} y_{-1}, & \text{if } n=(3+3k)\mu+2k+1, \\ \left(\frac{\alpha_{37}}{\alpha_{38}}\right)^{\mu+1}\left(\frac{\alpha_{40}}{\alpha_{41}}\right)^{\mu+1}\left(\frac{\alpha_{43}}{\alpha_{44}}\right)^{\mu} y_{0}, & \text{if } n=(3+3k)\mu+2k+2, \\ \left(\frac{\alpha_{37}}{\alpha_{38}}\frac{\alpha_{40}}{\alpha_{41}}\frac{\alpha_{43}}{\alpha_{44}}\right)^{\mu+1} x_{-k}, & \text{if } n=(3+3k)\mu+2k+3, \\ \left(\frac{\alpha_{37}}{\alpha_{38}}\frac{\alpha_{40}}{\alpha_{41}}\frac{\alpha_{43}}{\alpha_{44}}\right)^{\mu+1} x_{-k+1}, & \text{if } n=(3+3k)\mu+2k+4, \\ \vdots & \\ \left(\frac{\alpha_{37}}{\alpha_{38}}\frac{\alpha_{40}}{\alpha_{41}}\frac{\alpha_{43}}{\alpha_{44}}\right)^{\mu+1} x_{-1}, & \text{if } n=(3+3k)\mu+3k+2, \\ \left(\frac{\alpha_{37}}{\alpha_{38}}\frac{\alpha_{40}}{\alpha_{41}}\frac{\alpha_{43}}{\alpha_{44}}\right)^{\mu+1} x_{0}, & \text{if } n=(3+3k)\mu+3k+3. \end{cases} \quad (56)$$

And

$$0 < y_n \le \begin{cases} \left(\frac{\alpha_{37}}{\alpha_{38}}\right)^{\mu}\left(\frac{\alpha_{40}}{\alpha_{41}}\right)^{\mu+1}\left(\frac{\alpha_{43}}{\alpha_{44}}\right)^{\mu} x_{-k}, & \text{if } n=(3+3k)\mu+1, \\ \left(\frac{\alpha_{37}}{\alpha_{38}}\right)^{\mu}\left(\frac{\alpha_{40}}{\alpha_{41}}\right)^{\mu+1}\left(\frac{\alpha_{43}}{\alpha_{44}}\right)^{\mu} x_{-k+1}, & \text{if } n=(3+3k)\mu+2, \\ \vdots & \\ \left(\frac{\alpha_{37}}{\alpha_{38}}\right)^{\mu}\left(\frac{\alpha_{40}}{\alpha_{41}}\right)^{\mu+1}\left(\frac{\alpha_{43}}{\alpha_{44}}\right)^{\mu} x_{-1}, & \text{if } n=(3+3k)\mu+k, \\ \left(\frac{\alpha_{37}}{\alpha_{38}}\right)^{\mu}\left(\frac{\alpha_{40}}{\alpha_{41}}\right)^{\mu+1}\left(\frac{\alpha_{43}}{\alpha_{44}}\right)^{\mu} x_{0}, & \text{if } n=(3+3k)\mu+k+1, \\ \left(\frac{\alpha_{37}}{\alpha_{38}}\right)^{\mu}\left(\frac{\alpha_{40}}{\alpha_{41}}\right)^{\mu+1}\left(\frac{\alpha_{43}}{\alpha_{44}}\right)^{\mu} z_{-k}, & \text{if } n=(3+3k)\mu+k+2, \\ \left(\frac{\alpha_{37}}{\alpha_{38}}\right)^{\mu}\left(\frac{\alpha_{40}}{\alpha_{41}}\right)^{\mu+1}\left(\frac{\alpha_{43}}{\alpha_{44}}\right)^{\mu} z_{-k+1}, & \text{if } n=(3+3k)\mu+k+3, \\ \vdots & \\ \left(\frac{\alpha_{37}}{\alpha_{38}}\right)^{\mu}\left(\frac{\alpha_{40}}{\alpha_{41}}\right)^{\mu+1}\left(\frac{\alpha_{43}}{\alpha_{44}}\right)^{\mu} z_{-1}, & \text{if } n=(3+3k)\mu+2k+1, \\ \left(\frac{\alpha_{37}}{\alpha_{38}}\right)^{\mu}\left(\frac{\alpha_{40}}{\alpha_{41}}\right)^{\mu+1}\left(\frac{\alpha_{43}}{\alpha_{44}}\right)^{\mu} z_{0}, & \text{if } n=(3+3k)\mu+2k+2, \\ \left(\frac{\alpha_{37}}{\alpha_{38}}\frac{\alpha_{40}}{\alpha_{41}}\frac{\alpha_{43}}{\alpha_{44}}\right)^{\mu+1} y_{-k}, & \text{if } n=(3+3k)\mu+2k+3, \\ \left(\frac{\alpha_{37}}{\alpha_{38}}\frac{\alpha_{40}}{\alpha_{41}}\frac{\alpha_{43}}{\alpha_{44}}\right)^{\mu+1} y_{-k+1}, & \text{if } n=(3+3k)\mu+2k+4, \\ \vdots & \\ \left(\frac{\alpha_{37}}{\alpha_{38}}\frac{\alpha_{40}}{\alpha_{41}}\frac{\alpha_{43}}{\alpha_{44}}\right)^{\mu+1} y_{-1}, & \text{if } n=(3+3k)\mu+3k+2, \\ \left(\frac{\alpha_{37}}{\alpha_{38}}\frac{\alpha_{40}}{\alpha_{41}}\frac{\alpha_{43}}{\alpha_{44}}\right)^{m+1} y_{0}, & \text{if } n=(3+3k)\mu+3k+3. \end{cases} \quad (57)$$

Finally,

$$0<z_n\le\begin{cases}\left(\frac{\alpha_{37}}{\alpha_{38}}\right)^{\mu}\left(\frac{\alpha_{40}}{\alpha_{41}}\right)^{\mu}\left(\frac{\alpha_{43}}{\alpha_{44}}\right)^{\mu+1}y_{-k}, & \text{if } n=(3+3k)\mu+1,\\ \left(\frac{\alpha_{37}}{\alpha_{38}}\right)^{\mu}\left(\frac{\alpha_{40}}{\alpha_{41}}\right)^{\mu}\left(\frac{\alpha_{43}}{\alpha_{44}}\right)^{\mu+1}y_{-k+1}, & \text{if } n=(3+3k)\mu+2,\\ \vdots & \\ \left(\frac{\alpha_{37}}{\alpha_{38}}\right)^{\mu}\left(\frac{\alpha_{40}}{\alpha_{41}}\right)^{\mu}\left(\frac{\alpha_{43}}{\alpha_{44}}\right)^{\mu+1}y_{-1}, & \text{if } n=(3+3k)\mu+k,\\ \left(\frac{\alpha_{37}}{\alpha_{38}}\right)^{\mu}\left(\frac{\alpha_{40}}{\alpha_{41}}\right)^{\mu}\left(\frac{\alpha_{43}}{\alpha_{44}}\right)^{\mu+1}y_{0}, & \text{if } n=(3+3k)\mu+k+1,\\ \left(\frac{\alpha_{37}}{\alpha_{38}}\right)^{\mu+1}\left(\frac{\alpha_{40}}{\alpha_{41}}\right)^{\mu}\left(\frac{\alpha_{43}}{\alpha_{44}}\right)^{\mu+1}x_{-k}, & \text{if } n=(3+3k)\mu+k+2,\\ \left(\frac{\alpha_{37}}{\alpha_{38}}\right)^{\mu+1}\left(\frac{\alpha_{40}}{\alpha_{41}}\right)^{\mu}\left(\frac{\alpha_{43}}{\alpha_{44}}\right)^{\mu+1}x_{-k+1}, & \text{if } n=(3+3k)\mu+k+3,\\ \vdots & \\ \left(\frac{\alpha_{37}}{\alpha_{38}}\right)^{\mu+1}\left(\frac{\alpha_{40}}{\alpha_{41}}\right)^{\mu}\left(\frac{\alpha_{43}}{\alpha_{44}}\right)^{\mu+1}x_{-1}, & \text{if } n=(3+3k)\mu+2k+1,\\ \left(\frac{\alpha_{37}}{\alpha_{38}}\right)^{\mu+1}\left(\frac{\alpha_{40}}{\alpha_{41}}\right)^{\mu}\left(\frac{\alpha_{43}}{\alpha_{44}}\right)^{\mu+1}x_{0}, & \text{if } n=(3+3k)\mu+2k+2\\ \left(\frac{\alpha_{37}\alpha_{40}\alpha_{43}}{\alpha_{38}\alpha_{41}\alpha_{44}}\right)^{\mu+1}z_{-k}, & \text{if } n=(3+3k)\mu+2k+3,\\ \left(\frac{\alpha_{37}\alpha_{40}\alpha_{43}}{\alpha_{38}\alpha_{41}\alpha_{44}}\right)^{\mu+1}z_{1-k}, & \text{if } n=(3+3k)\mu+2k+4,\\ \vdots & \\ \left(\frac{\alpha_{37}\alpha_{40}\alpha_{43}}{\alpha_{38}\alpha_{41}\alpha_{44}}\right)^{\mu+1}z_{-1}, & \text{if } n=(3+3k)\mu+3k+2,\\ \left(\frac{\alpha_{37}\alpha_{40}\alpha_{43}}{\alpha_{38}\alpha_{41}\alpha_{44}}\right)^{\mu+1}z_{0}, & \text{if } n=(3+3k)\mu+3k+3.\end{cases}\tag{58}$$

Proof. Same as the proof of Theorem 3. □

Corollary 6. *The following statements are true for (12)–(14):*

(i) *If (40) holds, then $\{(X_n)\}$ of discrete system (12) is bounded*

(ii) *If (41) holds, then $\{(X_n)\}$ of discrete system (13) is bounded*

(iii) *If (42) holds, then $\{(X_n)\}$ of discrete system (14) is bounded*

Proof. Its consequence of Theorem 5. □

5. Global Dynamics about P_0 of (11)–(14)

Theorem 7. *If (30) holds, then P_0 of (11) is globally stable.*

Proof. If (30) is true, then from discrete system (11), one has

$$\begin{aligned}x_{n+1}&=\frac{\alpha_{10}y_{n-k}}{\alpha_{11}+\alpha_{12}\prod_{i=0}^{k}z_{n-i}}\le\frac{\alpha_{10}}{\alpha_{11}}y_{n-k}<y_{n-k},\\ y_{n+1}&=\frac{\alpha_{13}z_{n-k}}{\alpha_{14}+\alpha_{15}\prod_{i=0}^{k}x_{n-i}}\le\frac{\alpha_{13}}{\alpha_{14}}z_{n-k}<z_{n-k},\\ z_{n+1}&=\frac{\alpha_{16}x_{n-k}}{\alpha_{17}+\alpha_{18}\prod_{i=0}^{k}y_{n-i}}\le\frac{\alpha_{16}}{\alpha_{17}}x_{n-1}<x_{n-k}.\end{aligned}\tag{59}$$

From (59), one gets

$$\begin{aligned}&x_{(3k+3)n+1}<y_{(3k+3)n-k}\text{ and }x_{(3k+3)n+3k+4}<y_{(3k+3)n+2k+3},\\ &y_{(3k+3)n+1}<z_{(3k+3)-k}\text{ and }y_{(3k+3)n+3k+4}<z_{(3k+3)n+2k+3},\\ &z_{(3k+3)n+1}<x_{(3k+3)n-k}\text{ and }z_{(3k+3)n+3k+4}<x_{(3k+3)n+2k+3}.\end{aligned}\tag{60}$$

Moreover, from (60), one gets

$$\begin{aligned}&x_{(3k+3)n+3k+4}<y_{(3k+3)n+2k+3}<z_{(3k+3)n+k+2}<x_{(3k+3)n+1},\\ &y_{(3k+3)n+3k+4}<z_{(3k+3)n+2k+3}<x_{(3k+3)n+k+2}<y_{(3k+3)n+1},\\ &z_{(3k+3)n+3k+4}<x_{(3k+3)n+2k+3}<y_{(3k+3)n+k+2}<z_{(3k+3)n+1}.\end{aligned}\tag{61}$$

From (61), one can conclude that $\{x_{(3k+3)n+1}\},\cdots,\{x_{(3k+3)n+3k+3}\}$; $\{y_{(3k+3)n+1}\},\cdots,\{y_{(3k+3)n+3k+3}\}$; and $\{z_{(3k+3)n+1}\},\cdots,\{z_{(3k+3)n+3k+3}\}$ are decreasing. Thus, $\{x_n\},\{y_n\}$, and $\{z_n\}$ are decreasing, and hence, one can conclude that $\lim_{n\longrightarrow\infty}(x_n,y_n,z_n)=P_0$. □

Theorem 8. *If respective parametric conditions (40), (41), and (42) hold, then P_0 of discrete systems (12)–(14) is globally stable.*

Proof. Like as the proof of Theorem 7. □

6. Convergence Rate

Theorem 9. *If respective parametric conditions (30), (40), (41), and (42) hold, then error vector*

$$\widetilde{\varrho}_n=\begin{pmatrix}\varrho_n^1\\ \vdots\\ \varrho_{n-k}^1\\ \varrho_n^2\\ \vdots\\ \varrho_{n-k}^2\\ \varrho_n^3\\ \vdots\\ \varrho_{n-k}^3\end{pmatrix},\tag{62}$$

of positive solution: $\{X_n\}$ of corresponding discrete systems (11)–(14) satisfies the following relations:

$$\lim_{n\longrightarrow\infty}\sqrt[n]{\|\varrho_n\|}=\left|\nu J|_{P_0}\right|,\qquad \lim_{n\longrightarrow\infty}\frac{\|\varrho_{n+1}\|}{\|\varrho_n\|}=\left|\nu J|_{P_0}\right|, \tag{63}$$

where $|\nu J|_{P_0}|$ is equivalent to the modulus of one of the characteristic roots of $J|_{P_0}$ calculated at trivial fixed point P_0.

Proof. Let $\{(X_n)\}$ be a positive solution of (11) for which $\lim_{n\longrightarrow\infty}(x_n, y_n, z_n)=\Lambda$. In order for error terms, we have

$$\begin{aligned} x_{n+1}-x &= \frac{\alpha_{10}y_{n-k}}{\alpha_{11}+\alpha_{12}\prod_{i=0}^{k}z_{n-i}}-\frac{\alpha_{10}y}{\alpha_{11}+\alpha_{12}z^{k+1}},\\ &= \frac{\alpha_{10}}{\alpha_{11}+\alpha_{12}\prod_{i=0}^{k}z_{n-i}}(y_{n-k}-y)\\ &- \frac{\alpha_{10}\alpha_{12}y\prod_{i=1}^{k}z_{n-i}}{\left(\alpha_{11}+\alpha_{12}\prod_{i=0}^{k}z_{n-i}\right)\left(\alpha_{11}+\alpha_{12}z^{k+1}\right)}(z_n-z)\\ &- \frac{\alpha_{10}\alpha_{12}yz\prod_{i=2}^{k}z_{n-i}}{\left(\alpha_{11}+\alpha_{12}\prod_{i=0}^{k}z_{n-i}\right)\left(\alpha_{11}+\alpha_{12}z^{k+1}\right)}(z_{n-1}-z)\\ &-\cdots- \frac{\alpha_{10}\alpha_{12}yz^{k-1}z_{n-k}}{\left(\alpha_{11}+\alpha_{12}\prod_{i=0}^{k}z_{n-i}\right)\left(\alpha_{11}+\alpha_{12}z^{k+1}\right)}(z_{n-k+1}-z)\\ &- \frac{\alpha_{10}\alpha_{12}yz^{k}}{\left(\alpha_{11}+\alpha_{12}\prod_{i=0}^{k}z_{n-i}\right)\left(\alpha_{11}+\alpha_{12}z^{k+1}\right)}(z_{n-k}-z), \end{aligned} \tag{64}$$

$$\begin{aligned} y_{n+1}-y &= \frac{\alpha_{13}z_{n-k}}{\alpha_{14}+\alpha_{15}\prod_{i=0}^{k}x_{n-i}}-\frac{\alpha_{13}z}{\alpha_{14}+\alpha_{15}x^{k+1}},\\ &= -\frac{\alpha_{13}\alpha_{15}z\prod_{i=1}^{k}x_{n-i}}{\left(\alpha_{14}+\alpha_{15}\prod_{i=0}^{k}x_{n-i}\right)\left(\alpha_{14}+\alpha_{15}x^{k+1}\right)}(x_n-x)\\ &- \frac{\alpha_{13}\alpha_{15}zx\prod_{i=2}^{k}x_{n-i}}{\left(\alpha_{14}+\alpha_{15}\prod_{i=0}^{k}x_{n-i}\right)\left(\alpha_{14}+\alpha_{15}x^{k+1}\right)}(x_{n-1}-x)\\ &-\cdots- \frac{\alpha_{13}\alpha_{15}zx^{k-1}x_{n-k}}{\left(\alpha_{14}+\alpha_{15}\prod_{i=0}^{k}x_{n-i}\right)\left(\alpha_{14}+\alpha_{15}x^{k+1}\right)}(x_{n-k+1}-x)\\ &- \frac{\alpha_{13}\alpha_{15}zx^{k}}{\left(\alpha_{14}+\alpha_{15}\prod_{i=0}^{k}x_{n-i}\right)\left(\alpha_{14}+\alpha_{15}x^{k+1}\right)}(x_{n-k}-x)\\ &+ \frac{\alpha_{13}}{\alpha_{14}+\alpha_{15}\prod_{i=0}^{1}x_{n-k}}(z_{n-k}-z), \end{aligned} \tag{65}$$

$$\begin{aligned} z_{n+1}-z &= \frac{\alpha_{16}x_{n-k}}{\alpha_{17}+\alpha_{18}\prod_{i=0}^{k}y_{n-i}}-\frac{\alpha_{16}x}{\alpha_{17}+\alpha_{18}y^{k+1}},\\ &= \frac{\alpha_{16}}{\alpha_{17}+\alpha_{18}\prod_{i=0}^{k}y_{n-i}}(x_{n-k}-x)\\ &- \frac{\alpha_{16}\alpha_{18}x\prod_{i=1}^{k}y_{n-i}}{\left(\alpha_{17}+\alpha_{18}\prod_{i=0}^{k}y_{n-i}\right)\left(\alpha_{17}+\alpha_{18}y^{k+1}\right)}(y_n-y)\\ &- \frac{\alpha_{16}\alpha_{18}xy\prod_{i=2}^{k}y_{n-i}}{\left(\alpha_{17}+\alpha_{18}\prod_{i=0}^{k}y_{n-i}\right)\left(\alpha_{17}+\alpha_{18}y^{k+1}\right)}(y_{n-1}-y)\\ &-\cdots- \frac{\alpha_{16}\alpha_{18}xy^{k-1}y_{n-k}}{\left(\alpha_{17}+\alpha_{18}\prod_{i=0}^{k}y_{n-i}\right)\left(\alpha_{17}+\alpha_{18}y^{k+1}\right)}(y_{n-k+1}-y)\\ &\cdot \frac{\alpha_{16}\alpha_{18}xy^{k}}{\left(\alpha_{17}+\alpha_{18}\prod_{i=0}^{k}y_{n-i}\right)\left(\alpha_{17}+\alpha_{18}y^{k+1}\right)}(y_{n-k}-y). \end{aligned} \tag{66}$$

Now, set

$$\varrho_n^1=x_n-x,\qquad \varrho_n^2=y_n-y,\qquad \varrho_n^3=z_n-z. \tag{67}$$

Utilizing (67) in (64) and (66), one gets

$$\begin{aligned} \varrho_{n+1}^1 &= A_n^{n-k}\varrho_{n-k}^2+B_n^n\varrho_n^3+B_n^{n-1}\varrho_{n-1}^3+\cdots+B_n^{n-k+1}\varrho_{n-k+1}^3+B_n^{n-k}\varrho_{n-k}^3,\\ \varrho_{n+1}^2 &= C_n^n\varrho_n^1+C_n^{n-1}\varrho_{n-1}^1+\cdots+C_n^{n-k+1}\varrho_{n-k+1}^1+C_n^{n-k}\varrho_{n-k}^1+D_n^{n-k}\varrho_{n-k}^3,\\ \varrho_{n+1}^3 &= E_n^{n-k}\varrho_{n-k}^1+F_n^n\varrho_n^2+F_n^{n-1}\varrho_{n-1}^2+\cdots+F_n^{n-k+1}\varrho_{n-k+1}^2+F_n^{n-k}\varrho_{n-k}^2, \end{aligned} \tag{68}$$

where

$$\begin{aligned} A_n^{n-k} &= \frac{\alpha_{10}}{\alpha_{11}+\alpha_{12}z^{k+1}},\\ B_n^n &= -\frac{\alpha_{10}\alpha_{12}y\prod_{i=1}^{k}z_{n-i}}{\left(\alpha_{11}+\alpha_{12}\prod_{i=0}^{k}z_{n-i}\right)\left(\alpha_{11}+\alpha_{12}z^{k+1}\right)},\\ B_n^{n-1} &= -\frac{\alpha_{10}\alpha_{12}yz\prod_{i=2}^{k}z_{n-i}}{\left(\alpha_{11}+\alpha_{12}\prod_{i=0}^{k}z_{n-i}\right)\left(\alpha_{11}+\alpha_{12}z^{k+1}\right)},\\ &\vdots\\ B_n^{n-k+1} &= -\frac{\alpha_{10}\alpha_{12}yz^{k-1}z_{n-k}}{\left(\alpha_{11}+\alpha_{12}\prod_{i=0}^{k}z_{n-i}\right)\left(\alpha_{11}+\alpha_{12}z^{k+1}\right)},\\ B_n^{n-k} &= -\frac{\alpha_{10}\alpha_{12}yz^{k}}{\left(\alpha_{11}+\alpha_{12}\prod_{i=0}^{k}z_{n-i}\right)\left(\alpha_{11}+\alpha_{12}z^{k+1}\right)},\\ C_n^n &= -\frac{\alpha_{13}\alpha_{15}z\prod_{i=1}^{k}x_{n-i}}{\left(\alpha_{14}+\alpha_{15}\prod_{i=0}^{k}x_{n-i}\right)\left(\alpha_{14}+\alpha_{15}x^{k+1}\right)},\\ C_n^{n-1} &= -\frac{\alpha_{13}\alpha_{15}zx\prod_{i=2}^{k}x_{n-i}}{\left(\alpha_{14}+\alpha_{15}\prod_{i=0}^{k}x_{n-i}\right)\left(\alpha_{14}+\alpha_{15}x^{k+1}\right)},\\ &\vdots\\ C_n^{n-k+1} &= -\frac{\alpha_{13}\alpha_{15}zx^{k-1}x_{n-k}}{\left(\alpha_{14}+\alpha_{15}\prod_{i=0}^{k}x_{n-i}\right)\left(\alpha_{14}+\alpha_{15}x^{k+1}\right)},\\ C_n^{n-k} &= -\frac{\alpha_{13}\alpha_{15}zx^{k}}{\left(\alpha_{14}+\alpha_{15}\prod_{i=0}^{k}x_{n-i}\right)\left(\alpha_{14}+\alpha_{15}x^{k+1}\right)}, \end{aligned} \tag{69}$$

$$\begin{aligned} D_n^{n-k} &= \frac{\alpha_{13}}{\alpha_{14}+\alpha_{15}\prod_{i=0}^{1}x_{n-k}},\\ E_n^{n-k} &= \frac{\alpha_{16}}{\alpha_{17}+\alpha_{18}\prod_{i=0}^{k}y_{n-i}},\\ F_n^n &= -\frac{\alpha_{16}\alpha_{18}x\prod_{i=1}^{k}y_{n-i}}{\left(\alpha_{17}+\alpha_{18}\prod_{i=0}^{k}y_{n-i}\right)\left(\alpha_{17}+\alpha_{18}y^{k+1}\right)},\\ F_n^{n-1} &= -\frac{\alpha_{16}\alpha_{18}xy\prod_{i=2}^{k}y_{n-i}}{\left(\alpha_{17}+\alpha_{18}\prod_{i=0}^{k}y_{n-i}\right)\left(\alpha_{17}+\alpha_{18}y^{k+1}\right)},\\ &\vdots\\ F_n^{n-k+1} &= -\frac{\alpha_{16}\alpha_{18}xy^{k-1}y_{n-k}}{\left(\alpha_{17}+\alpha_{18}\prod_{i=0}^{k}y_{n-i}\right)\left(\alpha_{17}+\alpha_{18}y^{k+1}\right)},\\ F_n^{n-k} &= \frac{\alpha_{16}\alpha_{18}xy^{k}}{\left(\alpha_{17}+\alpha_{18}\prod_{i=0}^{k}y_{n-i}\right)\left(\alpha_{17}+\alpha_{18}y^{k+1}\right)}. \end{aligned} \tag{70}$$

From (69) and (70), we get

$$\begin{aligned}
&\lim_{n\longrightarrow\infty} A_n^{n-k} = \frac{\alpha_{10}}{\alpha_{11}+\alpha_{12}z^{k+1}},\\
&\lim_{n\longrightarrow\infty} B_n^{n} = \lim_{n\longrightarrow\infty} B_n^{n-1} = \cdots = \lim_{n\longrightarrow\infty} B_n^{n-k+1} = \lim_{n\longrightarrow\infty} B_n^{n-k} = -\frac{\alpha_{10}\alpha_{12}yz^k}{\left(\alpha_{11}+\alpha_{12}z^{k+1}\right)^2},\\
&\lim_{n\longrightarrow\infty} C_n^{n} = \lim_{n\longrightarrow\infty} C_n^{n-1} = \cdots = \lim_{n\longrightarrow\infty} C_n^{n-k+1} = \lim_{n\longrightarrow\infty} C_n^{n-k} = -\frac{\alpha_{13}\alpha_{15}zx^k}{\left(\alpha_{14}+\alpha_{15}x^{k+1}\right)^2},\\
&\lim_{n\longrightarrow\infty} D_n^{n-k} = \frac{\alpha_{13}}{\alpha_{14}+\alpha_{15}x^{k+1}},\\
&\lim_{n\longrightarrow\infty} E_n^{n-k} = \frac{\alpha_{16}}{\alpha_{17}+\alpha_{18}y^{k+1}},\\
&\lim_{n\longrightarrow\infty} F_n^{n} = \lim_{n\longrightarrow\infty} F_n^{n-1} = \cdots = \lim_{n\longrightarrow\infty} F_n^{n-k+1} = \lim_{n\longrightarrow\infty} F_n^{n-k} = -\frac{\alpha_{16}\alpha_{18}xy^k}{\left(\alpha_{17}+\alpha_{18}y^{k+1}\right)^2}.
\end{aligned} \tag{71}$$

So the limiting system [23]:

$$\widetilde{\mathrm{Q}_{n+1}} = \tilde{K}\widetilde{\mathrm{Q}_n}, \tag{72}$$

where $\widetilde{\mathrm{Q}_n}$ is depicted in (62) and $\tilde{K}$ is same as $J|_{\wedge}$ about $\wedge$. In particular about P_0, it becomes

$$\begin{pmatrix} \mathrm{Q}_{n+1}^1 \\ \vdots \\ \mathrm{Q}_{n-k+1}^1 \\ \mathrm{Q}_{n+1}^2 \\ \vdots \\ \mathrm{Q}_{n-k+1}^2 \\ \mathrm{Q}_{n+1}^3 \\ \vdots \\ \mathrm{Q}_{n-k+1}^3 \end{pmatrix} = \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{\alpha_{10}}{\alpha_{11}} & 0 & 0 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{\alpha_{13}}{\alpha_{14}} \\
0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & \frac{\alpha_{16}}{\alpha_{17}} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 & 0
\end{pmatrix} \begin{pmatrix} \mathrm{Q}_{n}^1 \\ \vdots \\ \mathrm{Q}_{n-k}^1 \\ \mathrm{Q}_{n}^2 \\ \vdots \\ \mathrm{Q}_{n-k}^2 \\ \mathrm{Q}_{n}^3 \\ \vdots \\ \mathrm{Q}_{n-k}^3 \end{pmatrix}, \tag{73}$$

which is the same as the linearized system of (11) about P_0. Adopting a similar procedure, one can prove the rest of the results for discrete systems (12)–(14). □

7. Numerical Simulations

We give some simulations in this section to verify not only the main finding but also show different behavior of discrete systems (11)–(14). For the systems under consideration, these numerical simulations provide time plots for x_n, y_n, z_n, as well as a global attractor. The following scenarios should be considered in this regard:

Example 1. Figure 1 indicates the behavior of discrete system (11) about P_0 if $k = 2$, α_ℓ ($\ell = 1, 2, \cdots, 9$), respectively, are 4, 5, 3, 25, 26, 1, 32, 33, and 4 and initial values $x_{-\ell}$, $\ell_{-\ell}$, $z_{-\ell}$ ($\ell = 2, 1, 0$), respectively, are 0.7, 0.9, 0.4, 1.1, 0.9, 0.4, 0.9, 0.9, and 0.8. Moreover, Figures 1(a)–1(c) show that P_0 of discrete system (11) is a sink, and its attractor is shown in Figure 1(d).

Example 2. Figure 2 indicates the behavior of discrete system (11) about P_0 if $k = 2$, α_ℓ ($\ell = 1, 2, \cdots, 9$), respectively, are 6, 5, 0.1, 25, 22, 0.3, 32, 30, and 0.4 and initial values $x_{-\ell}$, $\ell_{-\ell}$, and

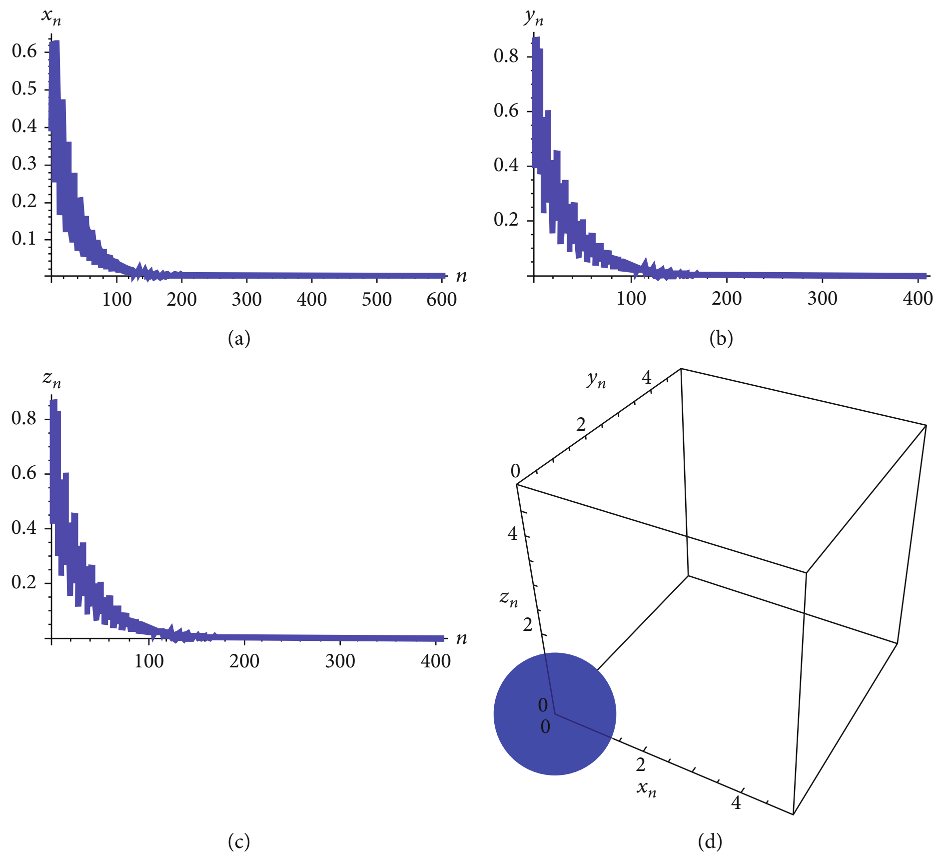

Figure 1: Dynamical characteristics of (11).

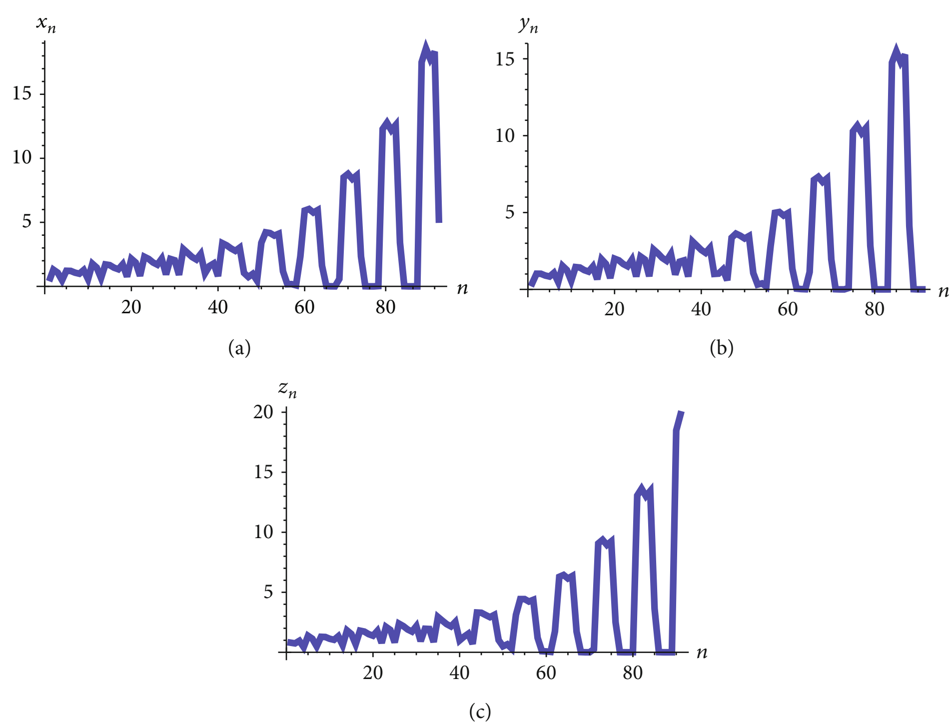

Figure 2: Dynamical characteristics of (11).

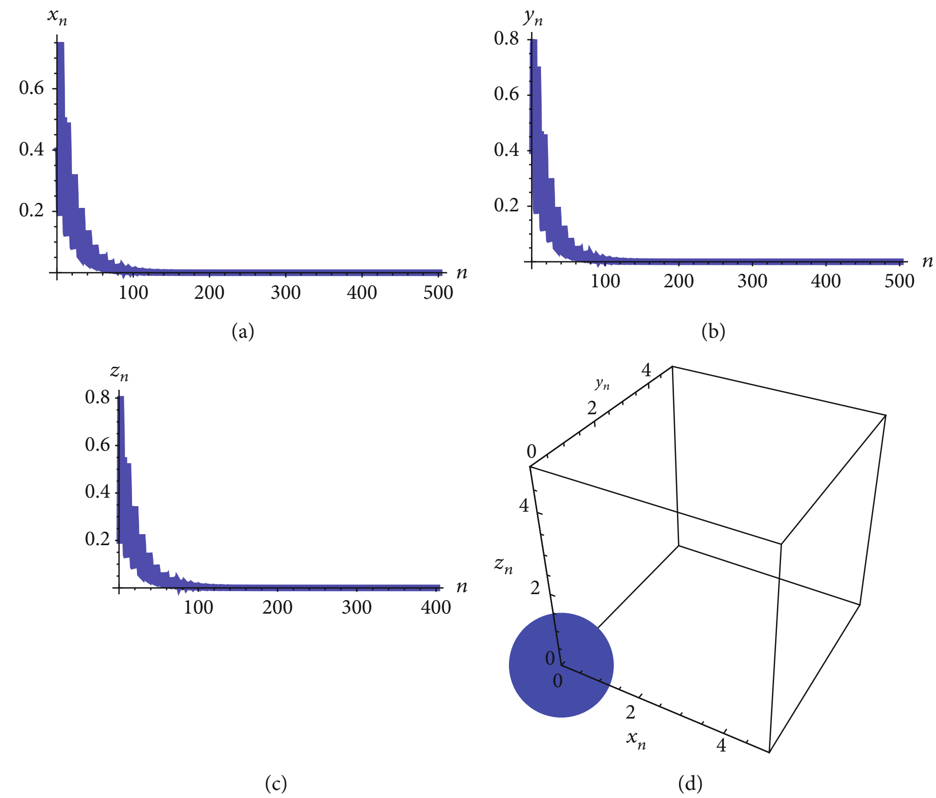

Figure 3: Dynamical characteristics of (12).

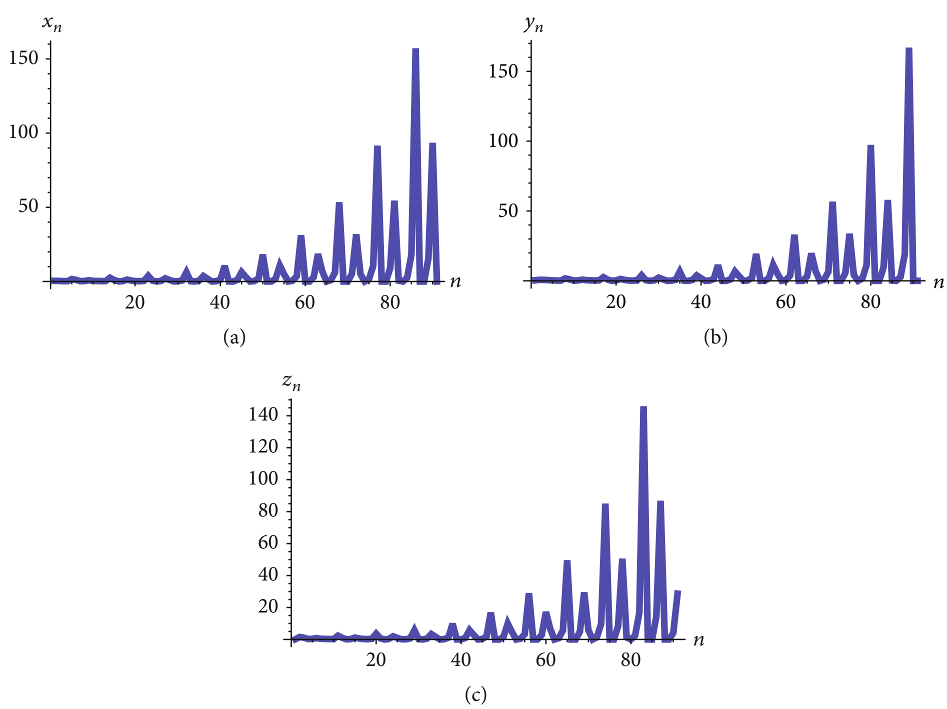

Figure 4: Dynamical characteristics of (12).

$z_{-\ell}(\ell = 2, 1, 0)$, respectively, are 0.7, 0.9, 0.4, 1.1, 0.9, 0.4, 0.9, 0.9, and 0.8. Moreover, Figures 2(a)–2(c) show that P_0 of discrete system (11) is unstable.

Example 3. Figure 3 indicates the behavior of discrete system (12) about P_0 if $k = 2$, α_ℓ $(\ell = 1, 2, \cdots, 9)$, respectively, are 14, 15, 3, 15, 16, 3, 3, 4, and 4 and $x_{-\ell}$, $\ell_{-\ell}$, and $z_{-\ell}(\ell = 2, 1, 0)$,

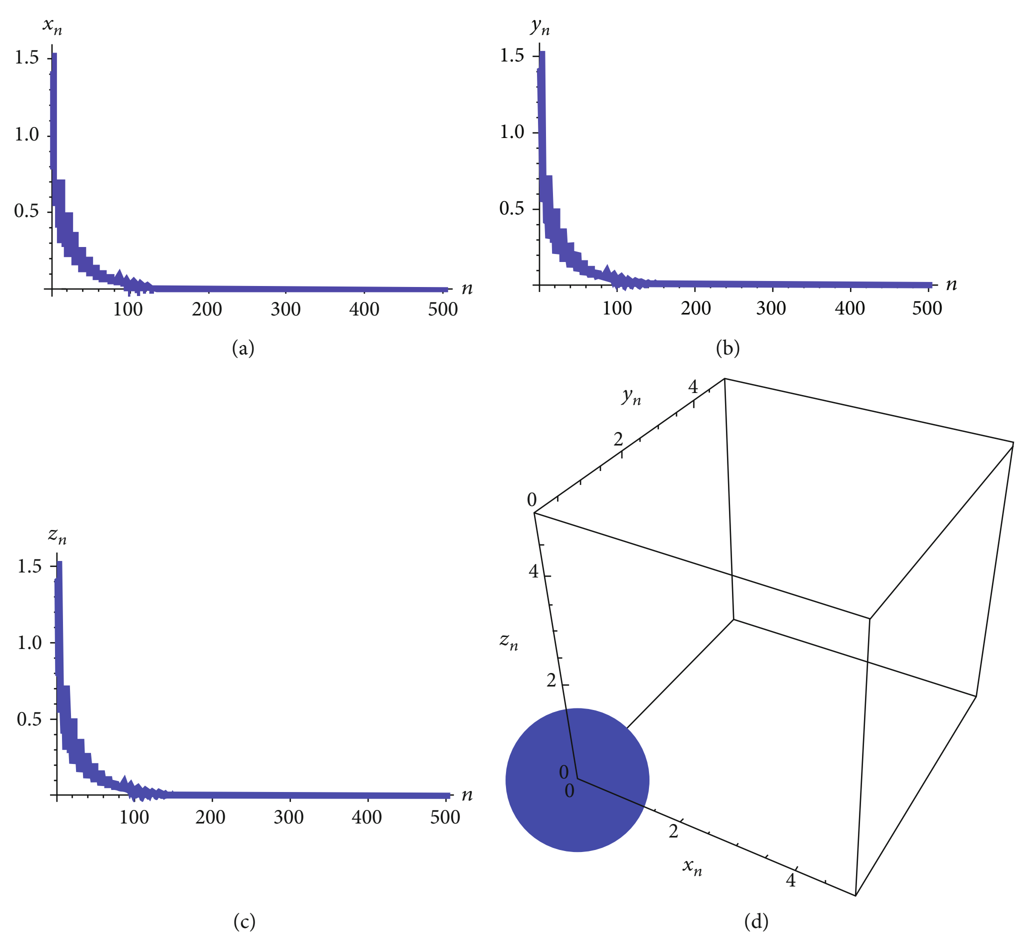

Figure 5: Dynamical characteristics of (13).

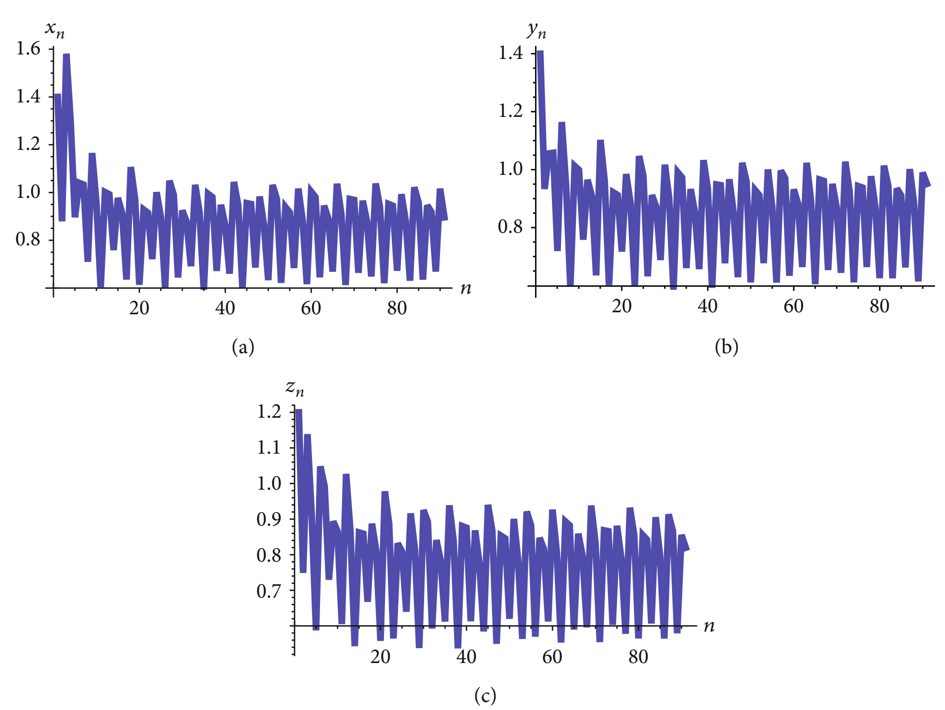

Figure 6: Dynamical characteristics of (13).

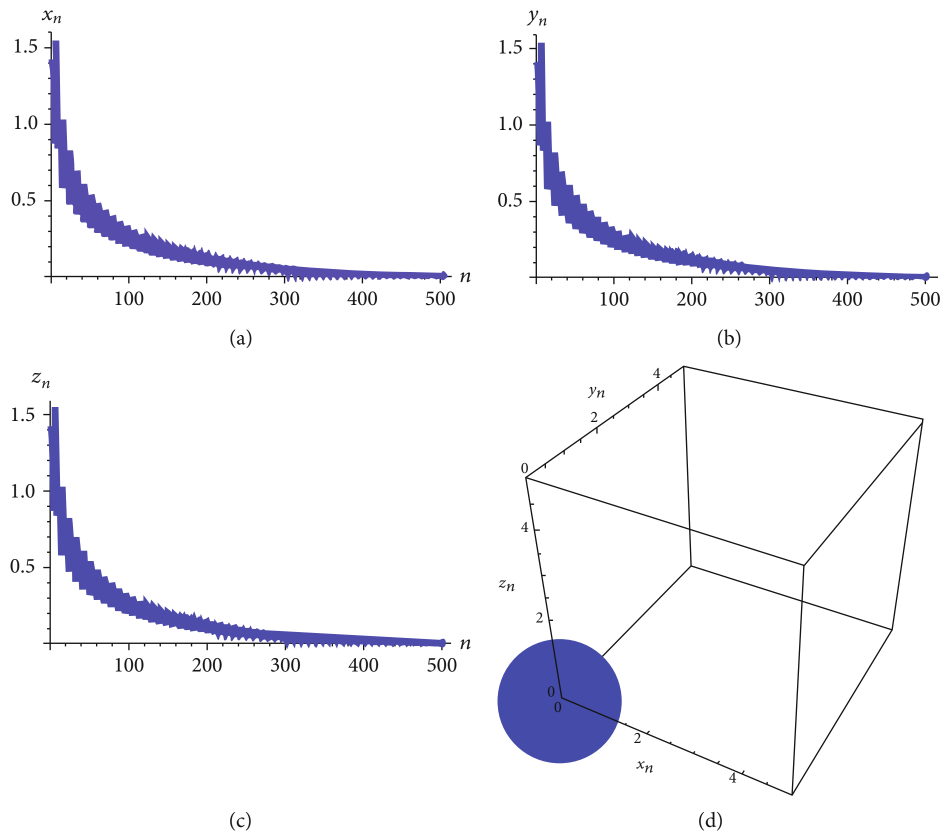

Figure 7: Dynamical characteristics of (14).

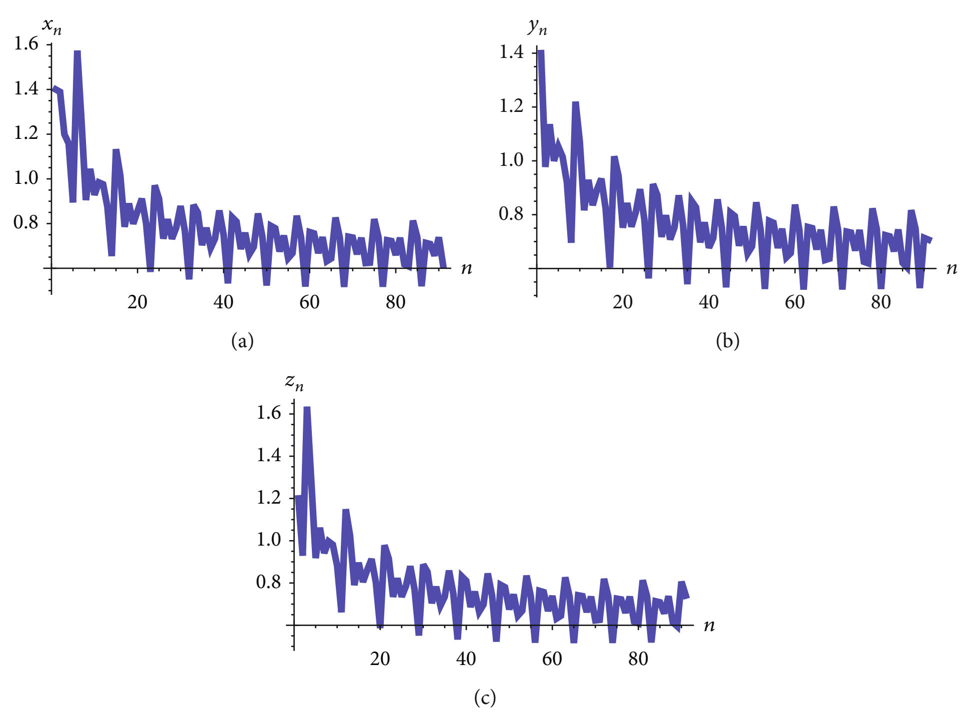

Figure 8: Dynamical characteristics of (14).

respectively, are 0.9, 0.9, 0.4, 1.1, 0.9, 0.4, 0.45, 0.23, and 0.2. Moreover, Figures 3(a)–3(c) show that P_0 of discrete system (12) is a sink, and its attractor is shown in Figure 3(d).

Example 4. Figure 4 indicates the behavior of discrete system (12) about P_0 if $k=2$, $\alpha_\ell(\ell=1,2,\cdots,9)$, respectively, are 14, 13, 13, 17, 16, 13, 6, 4, and 14 and $x_{-\ell}$, $\ell_{-\ell}$, and $z_{-\ell}$ $(\ell=2,1,0)$, respectively, are 0.9, 0.9, 0.4, 1.1, 0.9, 0.4, 0.45, 0.23, and 0.2. Moreover, Figures 4(a)–4(c) show that P_0 of discrete system (12) is unstable.

Example 5. Figure 5 indicates the behavior of discrete system (13) about P_0 if $k=2$, α_ℓ $(\ell=1,2,\cdots,9)$, respectively, are 24, 25, 3, 5, 6, 3, 22, 23, and 8 and $x_{-\ell}$, $\ell_{-\ell}$, and $z_{-\ell}(\ell=2,1,0)$, respectively, are 1.9, 1.9, 1.4, 1.1, 1.9, 1.4, 1.45, 1.23, and 1.2. Moreover, Figures 5(a)–5(c) show that P_0 of discrete system (13) is a sink, and its attractor is shown in Figure 5(d).

Example 6. Figure 6 indicates the behavior of discrete system (13) about P_0 if $k=2$, α_ℓ $(\ell=1,2,\cdots,9)$, respectively, are 27, 25, 3, 8, 6, 3, 25, 23, and 8 and $x_{-\ell}$, $\ell_{-\ell}$, and $z_{-\ell}(\ell=2,1,0)$, respectively, are 1.9, 1.9, 1.4, 1.1, 1.9, 1.4, 1.45, 1.23, and 1.2. Moreover, Figures 6(a)–6(c) show that P_0 of discrete system (13) is unstable.

Example 7. Figure 7 indicates the behavior of discrete system (14) about P_0 if $k=2$, α_ℓ $(\ell=1,2,\cdots,9)$, respectively, are 124, 125, 3, 5, 6, 3, 122, 123, and 8 and $x_{-\ell}$, $\ell_{-\ell}$, and $z_{-\ell}(\ell=2,1,0)$, respectively, are 1.9, 1.9, 1.4, 1.1, 1.9, 1.4, 1.45, 1.23, and 1.2. Moreover, Figures 7(a)–7(c) show that P_0 of discrete system (14) is a sink, and its attractor is shown in Figure 7(d).

Example 8. Figure 8 indicates the behavior of discrete system (14) about P_0 if $k=2$, α_ℓ $(\ell=1,2,\cdots,9)$, respectively, are 124, 123, 3, 15, 14, 3, 122, 121, and 8 and $x_{-\ell}$, $\ell_{-\ell}$, and $z_{-\ell}$ $(\ell=2,1,0)$, respectively, are 1.9, 1.9, 1.4, 1.1, 1.9, 1.4, 1.45, 1.23, and 1.2. Moreover, Figures 8(a)–8(c) show that P_0 of discrete system (14) is unstable.

8. Conclusion and Future Work

In this paper, we explored global dynamical characteristics, boundedness, and convergence rate of certain higher-order discrete systems of difference equations which is a natural extension of [21, 22]. We have proved that trivial fixed point P_0 of discrete systems (11)–(14) is globally stable if, respectively, parametric conditions (i) $\alpha_{10}/\alpha_{11}<1$, $\alpha_{13}/\alpha_{14}<1$, and $\alpha_{16}/\alpha_{17}<1$; (ii) $\alpha_{19}/\alpha_{20}<1$, $\alpha_{22}/\alpha_{23}<1$, and $\alpha_{25}/\alpha_{26}<1$; (iii) $\alpha_{28}/\alpha_{29}<1$, $\alpha_{31}/\alpha_{32}<1$, and $\alpha_{34}/\alpha_{35}<1$; and (iv) $\alpha_{37}/\alpha_{38}<1$, $\alpha_{40}/\alpha_{41}<1$, and $\alpha_{43}/\alpha_{44}<1$ hold. Further, we have proved that every positive solution of discrete systems (11)–(14) is bounded if (i) $\alpha_{10}/\alpha_{11}<1$, $\alpha_{13}/\alpha_{14}<1$, and $\alpha_{16}/\alpha_{17}<1$; (ii) $\alpha_{19}/\alpha_{20}<1$, $\alpha_{22}/\alpha_{23}<1$, and $\alpha_{25}/\alpha_{26}<1$; (iii) $\alpha_{28}/\alpha_{29}<1$, $\alpha_{31}/\alpha_{32}<1$, and $\alpha_{34}/\alpha_{35}<1$; and (iv) $\alpha_{37}/\alpha_{38}<1$, $\alpha_{40}/\alpha_{41}<1$, and $\alpha_{43}/\alpha_{44}<1$ hold. It is also examined that positive solution of discrete systems (11)–(14) converges to P_0. Finally, numerical verification of theoretical results is performed. Closed-form solution and calculation of forbidden set for the discrete systems (11)–(14) are our next aim to study.

Data Availability

All the data utilized in this article have been included, and the sources from where they were adopted were cited accordingly.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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