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Research Article

Remarks on the Systems of Semilinear Fractional Rayleigh-Stokes Equation

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In this paper, we study the Cauchy problem for a system of Rayleigh-Stokes equations. In this system of equations, we use derivatives in the classical Riemann-Liouville sense. This system has many applications in some non-Newtonian fluids. We obtained results for the existence, uniqueness, and frequency of the solution. We discuss the stability of the solutions and find the solution spaces. Our main technique is to use the Banach mapping theorem combined with some techniques in Fourier analysis.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a smooth domain with the boundary $\partial\Omega$, and $T > 0$ is a given time. In this paper, we study the initial value problem for systems of Rayleigh- Stokes problem as follows

$$\begin{cases} \partial_t u + (-\Delta)^\beta u - d\partial_t^\alpha \Delta u = \mathcal{G}(u, v), & (x, t) \in \Omega \times (0, T), \\ \partial_t v + (-\Delta)^\beta v - d\partial_t^\alpha \Delta v = \mathcal{H}(u, v), & (x, t) \in \Omega \times (0, T), \\ u(x, t) = 0, & x \in \partial\Omega, \\ u(x, 0) = \varphi(x), & x \in \Omega, \\ v(x, 0) = \theta(x), & x \in \Omega. \end{cases} \quad (1)$$

where (φ, θ) is Cauchy input data. Some functions $\mathcal{G}, \mathcal{H}$ called the source data which are defined later. Here, $\partial_t = \partial/\partial t$, and ∂_t^α is the Riemann-Liouville fractional derivative of order $0 < \alpha < 1$ given by [1, 2]:

$$\partial_t^\alpha w(x, t) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \left(\int_0^t (t-s)^{-\alpha} w(x, s) ds \right), \quad (2)$$

where $\Gamma(.)$ is the Gamma function. As far as we know, there are currently several definitions for fraction derivatives and fraction integrals, such as Riemann-Liouville, Caputo, Hadamard, Riesz, and Griinwald-Letnikov. Some works are attracting the attention of the community, like Debbouche and his group [3–5], Karapinar et al. [6–13], Benchohra et al. [14–16], Inc et al. [17–20].

And some very interesting results about unchanged level derivatives are surveyed by.

In fluid dynamics, the Rayleigh problem is the first Stokes problem, which determines the flow generated by the sudden motion of an infinitely long plate from the resting state, named after Lord Rayleigh and Sir George Stokes. This problem is considered to be one of the simplest problems with the correct solution for the Navier-Stokes equation. In recent times, with the development of fractions, a number of authors such as Shen et al. [21] investigated Rayleigh-Stokes, which is a more general form than the classical model. The fractional Rayleigh-Stokes equation (1) has applications in non-Newtonian behavior of fluids [21], and other applications of this equation can be given in [21, 22]. We list some papers on fractional Rayleigh stokes in the following.

(i) The initial and boundary values for the Rayleigh-Stokes problem in the case of homogeneity have been explored in a number of interesting papers; see for example [23–27] and its references

(ii) The authors in [21, 28, 29] used the Fourier transform and the fractional Laplace transform to obtain the exact solution

(iii) Numerical solutions for Problem (1) has been studied by many authors in [4, 23, 24, 30, 31]

(iv) In [32], the authors concerned with the following problem for a following stochastic Rayleigh-Stokes equation

$$\begin{cases} \partial_t X(t) + \left(1+\alpha\partial_t^\beta\right)\mathscr{A}X(t) = f(t, X(t)) + \gamma(t)\dot{B}_Q^h(t), & t \in (0, T), \\ X(t)|_{\partial\mathscr{D}} = 0, X(0) = x_0, & t \in (0, T). \end{cases} \tag{3}$$

The existence and uniqueness of mild solution in each case are established separately by applying a standard method that is Banach fixed point theorem. In [22], Caraballo et al. investigated the following time-fractional Rayleigh-Stokes stochastic equation

$$\left(\frac{\partial}{\partial t} - \Delta - \mathrm{u}\frac{\partial^\alpha}{\partial t^\alpha}\Delta\right)u = F(t, u) + \sigma(t, u)\dot{\mathscr{W}}(t), \text{on } J \times \mathscr{X}, \tag{4}$$

where $\{\mathscr{W}(t,.)\}_{t\in\bar{J}}$ represents a standard Wiener process.

To the best of the author's knowledge, the problem of the system of equations for a fractional Rayleigh-Stokes with a nonlinear source, i.e., Problem (1), has yet to be studied. The goal of this paper is to develop a theory of the existence and regularity estimate for the mild solution to the Problem (1). Our main technique is to use the Banach mapping theorem combined with some techniques in Fourier analysis.

2. The Existence and Regularity of the Solution

In this section, we consider the existence and mild solution of Problem (1). Before going into the main theorem of this section, we briefly discuss spectral, eigenvalues, and related functional spaces on the Laplacian operator.

Let $\mathscr{A} = -\Delta$. The domain $D(\mathscr{A}^s)$ for $s \geq 0$ is a Banach space equipped with the norm

$$\|h\|_{D(\mathscr{A}^s)} := \left(\sum_{n=1}^{\infty} |\langle h, \varphi_n\rangle|^2 \lambda_n^{2s}\right)^{1/2}, h \in D(\mathscr{A}^s). \tag{5}$$

The definition of the negative fractional power $\mathscr{A}^{-s}$ can be found in [33]. Its domain $D(\mathscr{A}^{-s})$ is a Hilbert space endowed with the dual inner product $\langle .,.\rangle_{-s,s}$ taken between $D(\mathscr{A}^{-s})$ and $D(\mathscr{A}^s)$. This generates the norm

$$\|h\|_{D(\mathscr{A}^{-s})} = \left(\sum_{n=1}^{\infty} \left||h, \varphi_n|_{-s,s}\right|^2 \lambda_n^{-2s}\right)^{1/2}. \tag{6}$$

A couple (u, v) of functions $u(x, t), v(x, t)$: $\bar{Q}_T \longrightarrow \mathbb{R}$, $(\bar{Q}_T = \bar{\Omega} \times [0, T])$ are called a function of two variables x, t

$$\begin{aligned} (u, v)&: \bar{Q}_T \longrightarrow \mathbb{R}^2, \\ (u, v)(x, t) &= (u(x, t), v(x, t)). \end{aligned} \tag{7}$$

Here, the norm of $(u, v) \in \mathbb{X} \times \mathbb{X}$ (for any space $\mathbb{X}$) is defined

$$\|(u, v)\|_{(\mathbb{X})^2} = \|u\|_{\mathbb{X}} + \|v\|_{\mathbb{X}}. \tag{8}$$

Theorem 1. *Let $(\varphi, \theta) \in H^m(\Omega) \times H^m(\Omega)$. Let $\mathscr{G}, \mathscr{H}$ satisfies $\mathscr{G}(0, 0) = \mathscr{H}(0, 0) = 0$ and the globally Lipschitz function*

$$\begin{aligned} \|\mathscr{G}(v_1, v_2) - \mathscr{G}(\bar{v}_1, \bar{v}_2)\| &\leq L_g\left(\|v_1 - \bar{v}_1\|_{L^2(\Omega)} + \|v_2 - \bar{v}_2\|_{L^2(\Omega)}\right), \\ \|\mathscr{H}(v_1, v_2) - \mathscr{H}(\bar{v}_1, \bar{v}_2)\| &\leq L_h\left(\|v_1 - \bar{v}_1\|_{L^2(\Omega)} + \|v_2 - \bar{v}_2\|_{L^2(\Omega)}\right), \end{aligned} \tag{9}$$

for any $v_1, \bar{v}_1, v_2, \bar{v}_2 \in L^2(\Omega)$. Then, problem (1) has a unique solution (u, v) belongs to $L^\infty(0, T; L^2(\Omega)) \times L^\infty(0, T; L^2(\Omega))$ and regularity estimates hold

$$\|(u, v)\|_{L^\infty\left(0,T;L^2(\Omega)\right)\times L^\infty\left(0,T;L^2(\Omega)\right)} \leq \mathscr{D}_1\left(\|\varphi\|_{H^m(\Omega)} + \|\theta\|_{H^m(\Omega)}\right). \tag{10}$$

Proof. Assume that the mild solution u is described by a Fourier series

$$u(x, t) = \sum_j \langle u(x, t), e_j(x)\rangle e_j(x), v(x, t) = \sum_j \langle v(x, t), e_j(x)\rangle e_j(x). \tag{11}$$

Thanks to the results of [24], we deduce that the solution of Problem (1) with the initial condition $u(x, 0) = \varphi(x)$ is given by

$$\begin{aligned} u_j(t) = &\left(\int_0^\infty e^{-vt}\mathscr{K}(j, \alpha, \beta, v)dv\right)\varphi_j \\ &+ \int_0^t \left(\int_0^\infty e^{-v(t-s)}\mathscr{K}(j, \alpha, \beta, v)dv\right) \\ &\cdot \langle \mathscr{G}(u(x, s), v(x, s))), e_j(x)\rangle ds, \end{aligned} \tag{12}$$

where $\mathscr{K}$ is represented as follows

$$\mathscr{K}(j, \alpha, \beta, v) = \frac{d}{\pi}\frac{\lambda_j^\beta \sin(\alpha\pi)v^\alpha}{\left(-v + \lambda_j^\beta dv^\alpha \cos\alpha\pi + \lambda_j^\beta\right)^2 + \left(\lambda_j^\beta dv^\alpha \sin\alpha\pi\right)^2}. \tag{13}$$

Hence, the mild solution of Problem (1) is given by

$$u(x,t)=\sum_j\left(\int_0^\infty e^{-\nu t}\mathcal{K}(j,\alpha,\beta,\nu)d\nu\right)\varphi_j e_j(x)+\sum_j\left(\int_0^t\left(\int_0^\infty e^{-\nu(t-s)}\mathcal{K}(j,\alpha,\beta,\nu)d\nu\right)\cdot\left\langle\mathcal{G}(u(x,s),v(x,s))),e_j(x)\right\rangle ds\right)e_j(x), \tag{14}$$

$$v(x,t)=\sum_j\left(\int_0^\infty e^{-\nu t}\mathcal{K}(j,\alpha,\beta,\nu)d\nu\right)\theta_j e_j(x)+\sum_j\left(\int_0^t\left(\int_0^\infty e^{-\nu(t-s)}\mathcal{K}(j,\alpha,\beta,\nu)d\nu\right)\cdot\left\langle\mathcal{H}(u(x,s),v(x,s))),e_j(x)\right\rangle ds\right)e_j(x). \tag{15}$$

We get the following estimate

$$\int_0^\infty e^{-\nu t}\mathcal{K}(j,\alpha,\beta,\nu)d\nu\le\frac{\mathcal{C}(\alpha,\beta,d)}{1+\lambda_j^\beta t^{1-\alpha}}\cdot \tag{16}$$

Now, we continue to show that (1) has a unique mild solution.

For any $a\ge 0$, denote by $(L_a^\infty(0,T;H^m(\Omega)))^2$ the function space $(L^\infty(0,T;H^m(\Omega)))^2$ associated with the norm

$$\|(\chi_1,\chi_2)\|_{a,m}:=\sup_{0\le t\le T}\|\exp(-at)\chi_1(\cdot,t)\|_{H^m(\Omega)}+\sup_{0\le t\le T}\|\exp(-at)\chi_2(\cdot,t)\|_{H^m(\Omega)}, \tag{17}$$

for any

$$(\chi_1,\chi_2)\in L^\infty\left(0,T;L^2(\Omega)\right)\times L^\infty\left(0,T;L^2(\Omega)\right). \tag{18}$$

Let us give the following operator

$$Q(\chi_1,\chi_2)(t)=(\mathcal{Q}_1(\chi_1,\chi_2)(t),\mathcal{Q}_2(\chi_1,\chi_2)(t)), \tag{19}$$

where $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are defined by the following

$$\mathcal{Q}_1(\chi_1,\chi_2)(t)=\sum_j\left(\int_0^\infty e^{-\nu t}\mathcal{K}(j,\alpha,\beta,\nu)d\nu\right)\varphi_j e_j(x)+\sum_j\left(\int_0^t\left(\int_0^\infty e^{-\nu(t-s)}\mathcal{K}(j,\alpha,\beta,\nu)d\nu\right)\cdot\left\langle\mathcal{G}(\chi_1(x,s),\chi_2(x,s))),e_j(x)\right\rangle ds\right)e_j(x),$$

$$\mathcal{Q}_2(\chi_1,\chi_2)(t)=\sum_j\left(\int_0^\infty e^{-\nu t}\mathcal{K}(j,\alpha,\beta,\nu)d\nu\right)\theta_j e_j(x)+\sum_j\left(\int_0^t\left(\int_0^\infty e^{-\nu(t-s)}\mathcal{K}(j,\alpha,\beta,\nu)d\nu\right)\cdot\left\langle\mathcal{H}(\chi_1(x,s),\chi_2(x,s))),e_j(x)\right\rangle ds\right)e_j(x). \tag{20}$$

From two equality as above, if $(\chi_1,\chi_2)=(0,0)$, we find that two following equality

$$\mathcal{Q}_1(\chi_1,\chi_2)(t)=\sum_j\left(\int_0^\infty e^{-\nu t}\mathcal{K}(j,\alpha,\beta,\nu)d\nu\right)\varphi_j e_j(x),\quad \mathcal{Q}_2(\chi_1,\chi_2)(t)=\sum_j\left(\int_0^\infty e^{-\nu t}\mathcal{K}(j,\alpha,\beta,\nu)d\nu\right)\theta_j e_j(x). \tag{21}$$

Hence, we get that for any $a>0$

$$\|\exp(-at)\mathcal{Q}_1(\chi_1,\chi_2)(t)\|_{H^m(\Omega)}+\|\exp(-at)\mathcal{Q}_2(\chi_1,\chi_2)(t)\|_{H^m(\Omega)}=e^{-at}\sqrt{\sum_j\lambda_j^{2m}\left(\int_0^\infty e^{-\nu t}\mathcal{K}(j,\alpha,\beta,\nu)d\nu\right)^2\varphi_j^2}+e^{-at}\sqrt{\sum_j\lambda_j^{2m}\left(\int_0^\infty e^{-\nu t}\mathcal{K}(j,\alpha,\beta,\nu)d\nu\right)^2\theta_j^2}. \tag{22}$$

Since the fact that

$$\int_0^\infty e^{-\nu t}\mathcal{K}(j,\alpha,\beta,\nu)d\nu\le\mathcal{C}(\alpha,\beta,d), \tag{23}$$

we know that

$$\sum_j\lambda_j^{2m}\left(\int_0^\infty e^{-\nu t}\mathcal{K}(j,\alpha,\beta,\nu)d\nu\right)^2\varphi_j^2\le|\mathcal{C}(\alpha,\beta,d)|^2\sum_j\lambda_j^{2m}\varphi_j^2, \tag{24}$$

$$\sum_j\lambda_j^{2m}\left(\int_0^\infty e^{-\nu t}\mathcal{K}(j,\alpha,\beta,\nu)d\nu\right)^2\theta_j^2\le|\mathcal{C}(\alpha,\beta,d)|^2\sum_j\lambda_j^{2m}\theta_j^2. \tag{25}$$

Combining (22), (24), and (25), we find that

$$\|\exp(-at)\mathcal{Q}_1(\chi_1,\chi_2)(t)\|_{H^m(\Omega)}+\|\exp(-at)\mathcal{Q}_2(\chi_1,\chi_2)(t)\|_{H^m(\Omega)}\le\mathcal{C}(\alpha,\beta,d)\left(\|\varphi\|_{H^m(\Omega)}+\|\theta\|_{H^m(\Omega)}\right), \tag{26}$$

which allows us to deduce that

$$Q(\chi_1,\chi_2)\in\left(L_a^\infty\left(0,T;L^2(\Omega)\right)\right)^2,\ \text{if}\,(\chi_1,\chi_2)=(0,0). \tag{27}$$

Let two functions (χ_1,χ_2) and $(\bar{\chi}_1,\bar{\chi}_2)$ in the space $(L_a^\infty(0,T;L^2(\Omega)))^2$. Then, using Parseval's equality, we get

$$\begin{aligned}&\|\mathcal{Q}_1(\chi_1,\chi_2)(t)-\mathcal{Q}_1(\bar{\chi}_1,\bar{\chi}_2)(t)\|_{H^m(\Omega)}\\&\quad\le\int_0^t\sqrt{\sum_j\lambda_j^{2m}\left(\int_0^\infty e^{-\nu(t-s)}\mathcal{K}(j,\alpha,\beta,\nu)d\nu\right)\left\langle\mathcal{G}(\chi_1(x,s),\chi_2(x,s)))-\mathcal{G}(\bar{\chi}_1(x,s),\bar{\chi}_2(x,s))),e_j(x)\right\rangle^2}ds.\end{aligned} \tag{28}$$

Noting that

$$\int_0^\infty e^{-\nu(t-s)}\mathcal{K}(j,\alpha,\beta,\nu)d\nu\le\mathcal{C}(\alpha,\beta,d)\lambda_j^{-\beta}(t-s)^{\alpha-1}. \tag{29}$$

Hence, and noting that $\lambda_j^{m-\beta}\le\lambda_1^{m-\beta}$, we find that

$$\begin{aligned}&e^{-at}\|\mathcal{Q}_1(\chi_1,\chi_2)(t)-\mathcal{Q}_1(\bar{\chi}_1,\bar{\chi}_2)(t)\|_{H^m(\Omega)}\\&\quad\le\mathcal{C}(\alpha,\beta,d)\lambda_1^{m-\beta}e^{-at}\int_0^t(t-s)^{\alpha-1}\|\mathcal{G}(\chi_1(x,s),\chi_2(x,s)))\\&\qquad-\mathcal{G}(\bar{\chi}_1(x,s),\bar{\chi}_2(x,s)))\|_{L^2(\Omega)}ds\\&\quad\le\mathcal{C}(\alpha,\beta,d)\lambda_1^{m-\beta}L_g\int_0^t(t-s)^{\alpha-1}e^{-a(t-s)}e^{-as}\Big(\|\chi_1(s)-\bar{\chi}_1(s)\|_{L^2(\Omega)}\\&\qquad+\|\chi_2(s)-\bar{\chi}_2(s)\|_{L^2(\Omega)}\Big)ds.\end{aligned} \tag{30}$$

Let us emphasize that for $0\le s\le T$ then

$$\begin{aligned}&e^{-as}\left(\|\chi_1(s)-\bar{\chi}_1(s)\|_{L^2(\Omega)}+\|\chi_2(s)-\bar{\chi}_2(s)\|_{L^2(\Omega)}\right)\\&\quad\le C_{1,m}\sup_{0\le t\le T}\|\exp(-at)\chi_1(\cdot,t)\|_{L^2(\Omega)}\\&\qquad+C_{1,m}\sup_{0\le t\le T}\|\exp(-at)\chi_2(\cdot,t)\|_{L^2(\Omega)}\\&\quad=C_{1,m}\|(\chi_1,\chi_2)\|_{a,m}.\end{aligned} \tag{31}$$

Combining (30) and (31), we find that

$$\begin{aligned}&e^{-at}\|\mathcal{Q}_1(\chi_1,\chi_2)(t)-\mathcal{Q}_1(\bar{\chi}_1,\bar{\chi}_2)(t)\|_{H^m(\Omega)}\\&\quad\le\mathcal{C}_1(\alpha,\beta,d,m)\left(\int_0^t(t-s)^{\alpha-1}e^{-a(t-s)}ds\right)\|(\chi_1,\chi_2)\|_{a,m},\end{aligned} \tag{32}$$

where we denote

$$\mathcal{C}_1(\alpha,\beta,d,m)=\mathcal{C}(\alpha,\beta,d)C_{1,m}\lambda_1^{m-\beta}L_g. \tag{33}$$

Next, we need to deal with the integral quantity $\text{Int1}=\int_0^t(t-s)^{\alpha-1}e^{-a(t-s)}ds$. By change variable $s=t\gamma$, we find that

$$\begin{aligned}\text{Int1}&=\int_0^1(t-t\gamma)^{\alpha-1}\exp(-at(1-\gamma))td\gamma\\&=\int_0^1t^\alpha(1-\gamma)^{\alpha-1}\exp(-at(1-\gamma))d\gamma\\&=(at(1-\gamma))^{\frac{\alpha}{2}}\exp(-at(1-\gamma))\left(\frac{t}{a}\right)^{\alpha/2}\int_0^1(1-\gamma)^{\alpha/2-1}d\gamma.\end{aligned} \tag{34}$$

Thank the inequality $y\le e^y$ for $y\ge0$, we know that

$$(at(1-\gamma))^{\alpha/2}\le\exp(at(1-\gamma)). \tag{35}$$

Combining with the fact that $\int_0^1(1-\gamma)^{\alpha/2-1}d\gamma=2/\alpha$, we deduce that the following inequality

$$\int_0^t(t-s)^{\alpha-1}e^{-a(t-s)}ds\le\frac{2}{\alpha}\left(\frac{T}{a}\right)^{\alpha/2}. \tag{36}$$

This inequality together with (32) leads to

$$\begin{aligned}&e^{-at}\|\mathcal{Q}_1(\chi_1,\chi_2)(t)-\mathcal{Q}_1(\bar{\chi}_1,\bar{\chi}_2)(t)\|_{H^m(\Omega)}\\&\quad\le\frac{2\mathcal{C}_1(\alpha,\beta,d,m)}{\alpha}\left(\frac{T}{a}\right)^{\alpha/2}\|(\chi_1,\chi_2)\|_{a,m}.\end{aligned} \tag{37}$$

By a similar way, we also get that

$$\begin{aligned}&e^{-at}\|\mathcal{Q}_2(\chi_1,\chi_2)(t)-\mathcal{Q}_2(\bar{\chi}_1,\bar{\chi}_2)(t)\|_{H^m(\Omega)}\\&\quad\le\frac{2\mathcal{C}_1(\alpha,\beta,d,m)}{\alpha}\left(\frac{T}{a}\right)^{\alpha/2}\|(\chi_1,\chi_2)\|_{a,m}.\end{aligned} \tag{38}$$

Therefore, we can deduce that

$$\|Q(\chi_1,\chi_2)-Q(\bar{\chi}_1,\bar{\chi}_2)\|_{a,m}\le\frac{4\mathcal{C}_1(\alpha,\beta,d,m)}{\alpha}\left(\frac{T}{a}\right)^{\alpha/2}\|(\chi_1,\chi_2)\|_{a,m}. \tag{39}$$

Since the limitation

$$\lim_{a\longrightarrow+\infty}\frac{4\mathscr{C}_1(\alpha,\beta,d,m)}{\alpha}\left(\frac{T}{a}\right)^{\alpha/2}=0, \tag{40}$$

we know that there exists a positive a^* such that $4\mathscr{C}_1(\alpha,\beta,d,m)/\alpha(T/a^*)^{\alpha/2}<1$. Thus, we can deduce that Q is contractive on $(L^\infty_{a^*}(0,T;H^m(\Omega)))^2$. Applying Banach fixed point theorem, we get that Q has a fixed point (u,v), so, the function (u,v) is also the unique solution of (1). Since (14), we find that

$$\begin{aligned}\|u(.,t)\|_{H^m(\Omega)} &\le \left\|\sum_j\left(\int_0^\infty e^{-\nu t}\mathscr{K}(j,\alpha,\beta,\nu)d\nu\right)\varphi_j e_j(x)\right\|_{H^m(\Omega)}\\ &+\left\|\sum_j\left(\int_0^t\left(\int_0^\infty e^{-\nu(t-s)}\mathscr{K}(j,\alpha,\beta,\nu)d\nu\right)\right.\right.\\ &\left.\cdot\left\langle\mathscr{G}(u(x,s),v(x,s))),e_j(x)\right\rangle ds\right)e_j(x)\|_{H^m(\Omega)}\\ &\le\sqrt{\sum_j\lambda_j^{2m}\left(\int_0^\infty e^{-\nu t}\mathscr{K}(j,\alpha,\beta,\nu)d\nu\right)^2\varphi_j^2}\\ &+\int_0^t\sqrt{\sum_j\lambda_j^{2m}\left(\int_0^\infty e^{-\nu(t-s)}\mathscr{K}(j,\alpha,\beta,\nu)d\nu\right)\left\langle\mathscr{G}(u(x,s),v(x,s))),e_j(x)\right\rangle^2}ds.\end{aligned} \tag{41}$$

By looking at (24) and (29), we follow from (41) that

$$\begin{aligned}\|u(.,t)\|_{H^m(\Omega)}&\le\mathscr{C}(\alpha,\beta,d)\|\varphi\|_{H^m(\Omega)}\\ &+\mathscr{C}(\alpha,\beta,d)\lambda_1^{m-\beta}\int_0^t(t-s)^{\alpha-1}\|\mathscr{G}(u(x,s),v(x,s)))\|_{L^2(\Omega)}ds,\end{aligned} \tag{42}$$

where we have used that $\mathscr{G}(0,0)=0$ and

$$\|\mathscr{G}(u(x,s),v(x,s)))\|_{L^2(\Omega)}\le\mathscr{G}(0,0)+L_g\left(\|u(.,s)\|_{L^2(\Omega)}+\|v(.,s)\|_{L^2(\Omega)}\right). \tag{43}$$

Hence, we derive that

$$\begin{aligned}\|u(.,t)\|_{H^m(\Omega)}&\le\mathscr{C}(\alpha,\beta,d)\|\varphi\|_{H^m(\Omega)}\\ &+\mathscr{C}(\alpha,\beta,d)\lambda_1^{m-\beta}L_g\int_0^t(t-s)^{\alpha-1}\\ &\cdot\left(\|u(.,s)\|_{L^2(\Omega)}+\|v(.,s)\|_{L^2(\Omega)}\right)ds.\end{aligned} \tag{44}$$

By a similar way, we also obtain that the following estimate

$$\begin{aligned}\|v(.,t)\|_{H^m(\Omega)}&\le\mathscr{C}(\alpha,\beta,d)\|\theta\|_{H^m(\Omega)}\\ &+\mathscr{C}(\alpha,\beta,d)\lambda_1^{m-\beta}L_g\int_0^t(t-s)^{\alpha-1}\\ &\cdot\left(\|u(.,s)\|_{L^2(\Omega)}+\|v(.,s)\|_{L^2(\Omega)}\right)ds.\end{aligned} \tag{45}$$

Combining (44) and (45), we get that

$$\begin{aligned}\|u(.,t)\|_{H^m(\Omega)}+\|v(.,t)\|_{H^m(\Omega)}&\le\mathscr{C}(\alpha,\beta,d)\\ &\cdot\left(\|\varphi\|_{H^m(\Omega)}+\|\theta\|_{H^m(\Omega)}\right)\\ &+2\mathscr{C}(\alpha,\beta,d)\lambda_1^{m-\beta}L_g\int_0^t(t-s)^{\alpha-1}\\ &\cdot\left(\|u(.,s)\|_{L^2(\Omega)}+\|v(.,s)\|_{L^2(\Omega)}\right)ds.\end{aligned} \tag{46}$$

By applying Gronwall's inequality as in [34], we obtain that

$$\begin{aligned}\|u(.,t)\|_{H^m(\Omega)}+\|v(.,t)\|_{H^m(\Omega)}&\le\mathscr{C}(\alpha,\beta,d)\\ &\cdot\left(\|\varphi\|_{H^m(\Omega)}+\|\theta\|_{H^m(\Omega)}\right)E_{\alpha,1}\left(2\mathscr{C}(\alpha,\beta,d)\lambda_1^{m-\beta}L_g\right).\end{aligned} \tag{47}$$

Hence, we can deduce that (u,v) belongs to $L^\infty(0,T;L^2(\Omega))\times L^\infty(0,T;L^2(\Omega))$, and furthermore, we also derive that

$$\|(u,v)\|_{L^\infty(0,T;L^2(\Omega))\times L^\infty(0,T;L^2(\Omega))}\le\mathscr{D}_1\left(\|\varphi\|_{H^m(\Omega)}+\|\theta\|_{H^m(\Omega)}\right), \tag{48}$$

where we set

$$\mathscr{D}_1 = \mathscr{C}(\alpha,\beta,d)\mathrm{E}_{\alpha,1}\left(2\mathscr{C}(\alpha,\beta,d)\lambda_1^{m-\beta}L_g\right). \tag{49}$$

□

Theorem 2. *Let $(\varphi,\theta)\in H^{m+2}(\Omega)\times H^{m+2}(\Omega)$. Then, Problem (1) has a unique solution $(u,v)\in C^{\alpha}([0,T];H^m(\Omega))\times C^{\alpha}([0,T];H^m(\Omega))$.*

Proof. For $k>0$, we set the following space

$$\begin{aligned}&C^k([0,T];H^m(\Omega))\times C^k([0,T];H^m(\Omega))\\&=\left\{(v_1,v_2)\in C([0,T];H^m(\Omega))\times C([0,T];H^m(\Omega))\sup_{0\le t<s\le T}\frac{\|v_1(.,t)-v_1(.s)\|_{H^m(\Omega)}}{|t-s|^k}+\sup_{0\le t<s\le T}\frac{\|v_2(.,t)-v_2(.s)\|_{H^m(\Omega)}}{|t-s|^k}<\infty\right\}.\end{aligned} \tag{50}$$

It is easy to see that

$$\begin{aligned}u\left(x,t'\right)-u(x,t)&=\sum_j\left[\int_0^\infty e^{-vt'}\mathscr{K}(j,\alpha,\beta,v)dv-\int_0^\infty e^{-vt}\mathscr{K}(j,\alpha,\beta,v)dv\right]\varphi_je_j(x)\\&\quad+\sum_j\left[\int_t^{\tilde{t}}\left(\int_0^\infty e^{-vs}\mathscr{K}(j,\alpha,\beta,v)dv\right)\left\langle\mathscr{G}\left(u\left(x,t'-s\right),v\left(x,t'-s\right)\right),e_j(x)\right\rangle ds\right]e_j(x)\\&\quad+\sum_j\left(\int_0^t\left(\int_0^\infty e^{-vs}\mathscr{K}(j,\alpha,\beta,v)dv\right)\left\langle\mathscr{G}\left(u\left(x,t'-s\right),v\left(x,t'-s\right)\right)-\mathscr{G}(u(x,t-s),v(x,t-s)),e_j(x)\right\rangle ds\right)e_j(x)\\&=J_1\left(x,t'-t\right)+J_2\left(x,t'-t\right)+J_3\left(x,t'-t\right).\end{aligned} \tag{51}$$

First, we look at the second term $J_2(x,t'-t)$. Using the inequality, we find that

$$\begin{aligned}\left\|J_2\left(.,t'-t\right)\right\|_{H^m(\Omega)}&\le\int_t^{\tilde{t}}\left\|\sum_j\left(\int_0^\infty e^{-vs}\mathscr{K}(j,\alpha,\beta,v)dv\right)\left\langle\mathscr{G}\left(u\left(x,t'-s\right),v\left(x,t'-s\right)\right),e_j(x)\right\rangle\right\|_{H^m(\Omega)}ds\\&=\int_t^{t'}\sqrt{\sum_j\lambda_j^{2m}\left(\int_0^\infty e^{-vs}\mathscr{K}(j,\alpha,\beta,v)dv\right)^2\left\langle\mathscr{G}\left(u\left(x,t'-s\right),v\left(x,t'-s\right)\right),e_j(x)\right\rangle^2}.\end{aligned} \tag{52}$$

Using (16), we obtain that

$$\begin{aligned}&\sqrt{\sum_j\lambda_j^{2m}\left(\int_0^\infty e^{-vs}\mathscr{K}(j,\alpha,\beta,v)dv\right)^2\left\langle\mathscr{G}\left(u\left(x,t'-s\right),v\left(x,t'-s\right)\right),e_j(x)\right\rangle^2}\\&\le\mathscr{C}(\alpha,\beta,d)\lambda_1^{m-\beta}s^{\alpha-1}\left\|\mathscr{G}\left(u\left(x,t'-s\right),v\left(x,t'-s\right)\right)\right\|_{L^2(\Omega)}\\&\le\mathscr{C}(\alpha,\beta,d)\lambda_1^{m-\beta}L_gs^{\alpha-1}\left(\left\|u\left(.,t'-s\right)\right\|_{L^2(\Omega)}+\left\|v\left(.,t'-s\right)\right\|_{L^2(\Omega)}\right)\\&\le\mathscr{C}(\alpha,\beta,d)\lambda_1^{m-\beta}L_gs^{\alpha-1}\|(u,v)\|_{L^\infty\left(0,T;L^2(\Omega)\right)\times L^\infty\left(0,T;L^2(\Omega)\right)}\le\bar{\mathscr{D}}_1s^{\alpha-1}.\end{aligned} \tag{53}$$

Combining (52) and (53) and using the inequality $(c+d)^\alpha\le c^\alpha+d^\alpha$ for any $0<\alpha<1$, we get that

$$\left\|J_2\left(.,t'-t\right)\right\|_{H^m(\Omega)}\le\bar{\mathscr{D}}_1\int_t^{t'}s^{\alpha-1}ds=\bar{\mathscr{D}}_1\frac{\left(t'\right)^\alpha-t^\alpha}{\alpha}\le\bar{\mathscr{D}}_1\frac{\left(t'-t\right)^\alpha}{\alpha}. \tag{54}$$

Next, we consider the second term $J_2(x,t'-t)$. It is easy to observe that

$$\begin{aligned}J_1\left(x,t'-t\right)&=\sum_j\left[\int_0^\infty e^{-vt'}\mathscr{K}(j,\alpha,\beta,v)dv\right.\\&\quad\left.-\int_0^\infty e^{-vt}\mathscr{K}(j,\alpha,\beta,v)dv\right]\varphi_je_j(x)\\&=\int_t^{t'}\left(\sum_j\bar{K}(s,j,\alpha,\beta,v)\varphi_je_j(x)\right)ds.\end{aligned} \tag{55}$$

Here, we set the following function

$$\bar{K}(t, j, \alpha, \beta, \nu) = \frac{d}{dt}\left(\int_0^\infty e^{-\nu t}\mathscr{K}(j, \alpha, \beta, \nu)d\nu\right). \tag{56}$$

In order to give the further process, we use one result in Theorem 1 [24]. If $f \in H^2(\Omega)$, then, we get the following estimate

$$\sum_j \left|\bar{K}(s, j, \alpha, \beta, \nu)f_j\right|^2 \le \mathscr{D}(\Omega, d, T, N)^2 s^{-2\alpha}\|f\|^2_{H^2(\Omega)} red \cdot . \tag{57}$$

Set the function $f^0(x) = \sum_j \lambda_j^m \varphi_j e_j(x)$. Since the fact that $\varphi \in H^{2+m}(\Omega)$, we know that f^0 belongs to the space $H^2(\Omega)$. Hence, using Parseval's equality, we get that

$$\begin{aligned}\left\|J_1\left(.,t'-t\right)\right\|_{H^m(\Omega)} &\le \int_t^{t'}\sqrt{\sum_j \lambda_j^{2m}\left|\bar{K}(s, j, \alpha, \beta, \nu)\right|^2\left|\varphi_j\right|^2}ds \\ &\le \int_t^{t'}\sqrt{\sum_j \left|\bar{K}(s, j, \alpha, \beta, \nu)f_j^0\right|^2}ds \\ &\le \mathscr{D}(\Omega, d, T, N)\left(\int_t^{t'} s^{\alpha-1}ds\right)\|\varphi\|_{H^{2+m}(\Omega)} \\ &= \mathscr{D}(\Omega, d, T, N)\frac{\left(t'\right)^\alpha - t^\alpha}{\alpha}\|\varphi\|_{H^{2+m}(\Omega)} \\ &\le \mathscr{D}(\Omega, d, T, N)\frac{\left(t'-t\right)^\alpha}{\alpha}\|\varphi\|_{H^{2+m}(\Omega)}.\end{aligned} \tag{58}$$

Next, we treat the third term $J_3(x, t'-t)$. Using the inequality, we find that

$$\begin{aligned}\left\|J_3\left(.,t'-t\right)\right\|_{H^m(\Omega)} &\le \mathscr{C}(\alpha, \beta, d)\lambda_1^{m-\beta}\int_0^t s^{\alpha-1} \\ &\quad\cdot \left\|\mathscr{G}\left(u\left(x, t'-s\right), v\left(x, t'-s\right)\right)\right) \\ &\quad - \mathscr{G}(u(x, t-s), v(x, t-s)))\Big\|_{L^2(\Omega)} ds \\ &\le \mathscr{C}(\alpha, \beta, d)\lambda_1^{m-\beta}\int_0^t (t-s)^{\alpha-1} \\ &\quad\cdot \left\|\mathscr{G}\left(u\left(x, s+t'-t\right), v\left(x, s+t'-t\right)\right)\right) \\ &\quad - \mathscr{G}(u(x, s), v(x, s)))\Big\|_{L^2(\Omega)} ds \\ &\le \mathscr{C}(\alpha, \beta, d)\lambda_1^{m-\beta}L_g\int_0^t (t-s)^{\alpha-1} \\ &\quad\cdot \left(\left\|u\left(.,s+t'-t\right) - u(.,s)\right\|_{L^2(\Omega)}\right. \\ &\quad \left.-\left\|v\left(.,s+t'-t\right) - v(.,s)\right\|_{L^2(\Omega)}\right)ds.\end{aligned} \tag{59}$$

Combining (51), (52), (58), and (59), we obtain that

$$\begin{aligned}\left\|u\left(.,t'\right) - u(.,t)\right\|_{H^m(\Omega)} &\le \left\|J_1\left(.,t'-t\right)\right\|_{H^m(\Omega)} \\ &\quad + \left\|J_2\left(.,t'-t\right)\right\|_{H^m(\Omega)} + \left\|J_3\left(.,t'-t\right)\right\|_{H^m(\Omega)} \\ &\le \mathscr{D}(\Omega, d, T, N)\frac{\left(t'-t\right)^\alpha}{\alpha}\|\varphi\|_{H^{2+m}(\Omega)} \\ &\quad + \bar{\mathscr{D}}_1\frac{\left(t'-t\right)^\alpha}{\alpha} + \mathscr{C}(\alpha, \beta, d)\lambda_1^{m-\beta}L_g \\ &\quad \cdot \int_0^t (t-s)^{\alpha-1}\left(\left\|u\left(.,s+t'-t\right) - u(.,s)\right\|_{L^2(\Omega)}\right. \\ &\quad \left.- \left\|v\left(.,s+t'-t\right) - v(.,s)\right\|_{L^2(\Omega)}\right)ds.\end{aligned} \tag{60}$$

By a similar way as above, we also obtain that

$$\begin{aligned}&\left\|v\left(.,t'\right) - v(.,t)\right\|_{H^m(\Omega)} \le \mathscr{D}(\Omega, d, T, N)\frac{\left(t'-t\right)^\alpha}{\alpha}\|\theta\|_{H^{2+m}(\Omega)} \\ &+ \bar{\mathscr{D}}_1\frac{\left(t'-t\right)^\alpha}{\alpha} + \mathscr{C}(\alpha, \beta, d)\lambda_1^{m-\beta}L_h \\ &\cdot \int_0^t (t-s)^{\alpha-1}\left(\left\|u\left(.,s+t'-t\right) - u(.,s)\right\|_{L^2(\Omega)}\right. \\ &\left.- \left\|v\left(.,s+t'-t\right) - v(.,s)\right\|_{L^2(\Omega)}\right)ds.\end{aligned} \tag{61}$$

Combining (60) and (61), we derive that

$$\begin{aligned}&\left\|u\left(.,t'\right) - u(.,t)\right\|_{H^m(\Omega)} + \left\|v\left(.,t'\right) - v(.,t)\right\|_{H^m(\Omega)} \\ &\le \mathscr{D}(\Omega, d, T, N)\frac{\left(t'-t\right)^\alpha}{\alpha}\left(\|\varphi\|_{H^{2+m}(\Omega)} + \|\theta\|_{H^{2+m}(\Omega)}\right) \\ &\quad + 2\bar{\mathscr{D}}_1\frac{\left(t'-t\right)^\alpha}{\alpha} + \bar{C}\int_0^t (t-s)^{\alpha-1}\left(\left\|u\left(.,s+t'-t\right) - u(.,s)\right\|_{H^m(\Omega)}\right. \\ &\quad \left.- \left\|v\left(.,s+t'-t\right) - v(.,s)\right\|_{H^m(\Omega)}\right)ds.\end{aligned} \tag{62}$$

Here, we set $\bar{C} = C_m 2\mathscr{C}(\alpha, \beta, d)\lambda_1^{m-\beta}(L_g + L_h)$. Let $h > 0$ fixed and let the following function

$$\begin{aligned}\mathscr{Y}_\rho(t) &= e^{-\rho t}\|u(.,t+h) - u(.,t)\|_{H^m(\Omega)} \\ &\quad + e^{-\rho t}\|v(.,t+h) - v(.,t)\|_{H^m(\Omega)}.\end{aligned} \tag{63}$$

From some above observations, we can deduce that

$$\begin{aligned} e^{\rho t}\mathcal{Y}_\rho(t) &\le \mathcal{D}(\Omega, d, T, N)\frac{h^\alpha}{\alpha}\left(\|\varphi\|_{H^{2+m}(\Omega)} + \|\theta\|_{H^{2+m}(\Omega)}\right) \\ &\quad + 2\bar{\mathcal{D}}_1\frac{h^\alpha}{\alpha} + \bar{C}e^{\rho t}\int_0^t e^{-\rho(t-s)}(t-s)^{\alpha-1}e^{-\rho s} \\ &\quad \cdot \left(\|u(.,s+h) - u(.,s)\|_{H^m(\Omega)} - \|v(.,s+h) - v(.,s)\|_{H^m(\Omega)}\right)ds \\ &\le \mathcal{D}(\Omega, d, T, N)\frac{h^\alpha}{\alpha}\left(\|\varphi\|_{H^{2+m}(\Omega)} + \|\theta\|_{H^{2+m}(\Omega)}\right) \\ &\quad + 2\bar{\mathcal{D}}_1\frac{h^\alpha}{\alpha} + \bar{C}e^{\rho t}\left(\int_0^t e^{-\rho(t-s)}(t-s)^{\alpha-1}ds\right)\max_{s\in[0,T]}\mathcal{Y}_\rho(s). \end{aligned} \tag{64}$$

From (36), we get that

$$\int_0^t (t-s)^{\alpha-1}e^{-\rho(t-s)}ds \le \frac{2}{\alpha}\left(\frac{T}{\rho}\right)^{\alpha/2}. \tag{65}$$

This together with (64) that

$$\begin{aligned} \mathcal{Y}_\rho(t) &\le \mathcal{D}(\Omega, d, T, N)\frac{h^\alpha}{\alpha}\left(\|\varphi\|_{H^{2+m}(\Omega)} + \|\theta\|_{H^{2+m}(\Omega)}\right) \\ &\quad + 2\bar{\mathcal{D}}_1\frac{h^\alpha}{\alpha} + \bar{C}\left(\frac{T}{\rho}\right)^{\alpha/2}\max_{s\in[0,T]}\mathcal{Y}_\rho(s) red \cdot . \end{aligned} \tag{66}$$

This implies that

$$\begin{aligned} \max_{t\in[0,T]}\mathcal{Y}_\rho(t) &\le \mathcal{D}(\Omega, d, T, N)\frac{h^\alpha}{\alpha}\left(\|\varphi\|_{H^{2+m}(\Omega)} + \|\theta\|_{H^{2+m}(\Omega)}\right) \\ &\quad + 2\bar{\mathcal{D}}_1\frac{h^\alpha}{\alpha} + \bar{C}\left(\frac{T}{\rho}\right)^{\alpha/2}\max_{s\in[0,T]}\mathcal{Y}_\rho(s). \end{aligned} \tag{67}$$

Since $\bar{C}(T/\rho)^{\alpha/2} \longrightarrow 0$ when $\rho \longrightarrow +\infty$, we can choose $\rho^* > 0$ such that

$$\bar{C}\left(\frac{T}{\rho}\right)^{\alpha/2} < 1/2. \tag{68}$$

Hence, we follow from (67) and (64) that

$$\begin{aligned} \max_{t\in[0,T]}\mathcal{Y}_{\rho^*}(t) &\le 2\mathcal{D}(\Omega, d, T, N)\frac{h^\alpha}{\alpha} \\ &\quad \cdot \left(\|\varphi\|_{H^{2+m}(\Omega)} + \|\theta\|_{H^{2+m}(\Omega)}\right) + 4\bar{\mathcal{D}}_1\frac{h^\alpha}{\alpha}, \end{aligned} \tag{69}$$

which allows us to conclude that

$$\begin{aligned} &\|u(.,t+h) - u(.,t)\|_{H^m(\Omega)} + \|v(.,t+h) - v(.,t)\|_{H^m(\Omega)} \\ &\quad \le 2e^{\rho^* t}\mathcal{D}(\Omega, d, T, N)\frac{h^\alpha}{\alpha}\left(\|\varphi\|_{H^{2+m}(\Omega)} + \|\theta\|_{H^{2+m}(\Omega)}\right) \\ &\qquad + e^{\rho^* t}4\bar{\mathcal{D}}_1\frac{h^\alpha}{\alpha} red \cdot . \end{aligned} \tag{70}$$

This inequality says that

$$(u, v) \in C^\alpha([0, T]; H^m(\Omega)) \times C^\alpha([0, T]; H^m(\Omega)). \tag{71}$$

□

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

Both authors contributed equally and significantly in writing this paper. Four authors read and approved the final manuscript.

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