

## Research Article

# Lax Representation and Darboux Solutions of the Classical Painlevé Second Equation

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In this article, we present Darboux solutions of the classical Painlevé second equation. We reexpress the classical Painlevé second Lax pair in new setting introducing gauge transformations to yield its Darboux expression in additive form. The new linear system of that equation carries similar structure as other integrable systems possess in the AKNS scheme. Finally, we generalize the Darboux transformation of the classical Painlevé second equation to the  $N$ -th form in terms of Wronskian.

## 1. Introduction

The six classical Painlevé equations first were introduced by Paul Painlevé and his colleagues while classifying nonlinear second-order ordinary differential equations with respect to their solutions [1]. In the beginning, these were well entertained in mathematics because of their connection to nonlinear partial differential equation, where they appear as ordinary differential equation reduction of higher dimensional integrable that further pursued to establish the Painlevé test for these partial differential equations [2, 3]. For example, the Painlevé second (P-II) equation arises as ODE reduction of the Korteweg-de Vries (KdV) equation [4, 5]. Although various properties of these equations have been studied from the mathematical point of such as their integrability through the Painlevé test, their zero-curvature representation is a compatibility condition of associated set of linear systems [6–8] as well as in the framework of hamiltonian formalism [9]. These equation got considerable attention from the physical point of view because the number of problems in applied mathematics and in physics is made to be integrable in terms of their solutions, Painlevé transcendents. The Painlevé second equation is one of these six equation

$$v_{xx} = v^3 + xv - \alpha \quad (1)$$

applied as a model to study electric field in semiconductor [10], where  $v_{xx} = d^2v/dx^2$  and  $\alpha$  is parameter, whose rational solutions have been constructed in [11] by means of Yablonski-Vorobév polynomials and further, the determinantal generalization of its rational solutions is presented by [12]. Moreover, the classical Painlevé II equations can also be represented by the Noumi-Yamada systems [13], these systems are discovered by Noumi and Yamada while studying symmetry of Painlevé equations, and these systems also possess the affine Weyl group symmetry of type  $A_1^{(1)}$ . In the Noumi-Yamada setting, the symmetric form of the Painlevé II equation (1) is given by

$$\begin{cases} v_0' = v_0 v_2 + v_2 u_0 + \alpha_0 \\ v_1' = -v_1 v_2 - v_2 u_1 + \alpha_1 \\ v_2' = v_1 - v_0 \end{cases} \quad (2)$$

where  $v_i' = dv_i/dz$  and  $\alpha_0, \alpha_1$ , are constant parameters. This can be shown that on eliminating  $v_1$  and  $v_0$ , one can obtain Painlevé II equation (1) for  $v_2$ . In proposition 1 below, and we introduce the Lax representation of the Painlevé II symmetric form (2). We also present the Darboux solutions of the Painlevé II equation with the help of Darboux transformations incorporating its gauged Lax pair. The Darboux

transformation significantly has got considerable attention in theory of integrable systems to find their exact solutions. The successful implementation of these transformations can be found in [14–16], where the solitonic solutions have been generated to few integrable systems with their computational presentations.

**Proposition 1.** *The Painlevé II symmetric 2 can be represented by Lax operators  $L_z = [P, L]$ , where  $L$  and  $M$  are Lax operators.*

*Proof.* For this purpose, consider a linear system

$$L_z \psi = \lambda \psi, \quad (3)$$

and the evolution of  $\psi$  with respect to  $z$  is given by

$$\psi_z = P\psi. \quad (4)$$

The compatibility condition of this systems yields the Lax equation

$$L_z = [P, L]. \quad (5)$$

Here,  $\lambda$  is a spectral parameter, and  $\lambda_z = 0$ . Now, we consider the Lax pair  $L, P$  in the following form

$$L = \begin{pmatrix} L_1 & O & O \\ O & L_2 & O \\ O & O & L_3 \end{pmatrix}, \quad (6)$$

$$P = \begin{pmatrix} P_1 & O & O \\ O & P_2 & O \\ O & O & P_3 \end{pmatrix},$$

where

$$L_1 = \begin{pmatrix} 1 & 0 \\ -v_0 & -1 \end{pmatrix}, L_2 = \begin{pmatrix} 1 & 0 \\ -v_1 & -1 \end{pmatrix}, L_3 = \begin{pmatrix} -1 & 0 \\ -v_2 & 1 \end{pmatrix}, \quad (7)$$

and the elements of the matrix  $P$  are given by

$$P_1 = \begin{pmatrix} \rho_1 & 0 \\ 0 & -\rho_1 \end{pmatrix}, L_2 = \begin{pmatrix} -\rho_2 & 0 \\ 0 & \rho_2 \end{pmatrix}, L_3 = \begin{pmatrix} -1 & 0 \\ \frac{1}{2}\sigma & 1 \end{pmatrix}, \quad (8)$$

where

$$\rho_1 = -v_2 - \frac{1}{2}\alpha_0 v_0^{-1}, \rho_2 = -v_2 + \frac{1}{2}\alpha_1 v_1^{-1}, \sigma = u_0 - v_1 + 2v_2$$

$$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \quad (9)$$

When the above Lax matrices  $L$  and  $P$  are subjected to the Lax equation (5), we obtain the Painlevé II symmetric form (2), and one may expand determinant  $|I + \mu L|_{ii}$  in terms of the eigenvalues of  $L$  similar as done for the quasideterminants in [17].

The Painlevé second equation (1) arises as the compatibility condition of linear systems

$$\Psi_\lambda = A(x; \lambda)\Psi, \quad \Psi_x = B(x; \lambda)\Psi \quad (10)$$

with the Lax pair

$$\begin{cases} A = (4\lambda v + \alpha\lambda^{-1})\sigma_1 - 2v'\sigma_2 - (4i\lambda^2 + ix + 2iv^2)\sigma_3 \\ B = v\sigma_1 - i\lambda\sigma_3 \end{cases} \quad (11)$$

that yields the Painlevé second equation (1), where  $\sigma_1, \sigma_2$ , and  $\sigma_3$  are Pauli spin matrices.

*Remark 2.* Under the standard Darboux transformation [18, 19] on components of the column vector  $\Psi = (X/Y)$  as

$$\begin{cases} X \rightarrow X[1] = \lambda Y - \lambda_1 \left( \frac{Y_1}{X_1} \right) X \\ Y \rightarrow Y[1] = \lambda X - \lambda_1 \left( \frac{X_1}{Y_1} \right) Y \end{cases}, \quad (12)$$

one can construct the 1-fold Darboux transformation to the solution of equation (1) by using its associated linear system (10) in the following form  $v[1] = v(YX^{-1})^2$  that yields trivial Darboux solutions to equation (1) taking  $v = 0$  as a seed solution at  $\alpha = 0$ , which are meaningless. In the next section, we will present nontrivial Darboux solutions of that equation introducing gauge transformation that brings its Lax pair (11) to a new form as similar as we have for many integrable systems in the AKNS scheme. Further, by iteration, the  $N$ -fold Darboux transformation will be generalized to determinantal form in terms of Wronskian.

## 2. Gauge Transformation and Nontrivial Darboux Solutions

**Proposition 3.** *Let us consider a matrix  $G = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$  that transforms the old Lax pair (11)  $(A, B)$  to a new set as  $A \rightarrow \tilde{A} = GAG^{-1}, B \rightarrow \tilde{B} = GBG^{-1}$ , where the new Lax pair  $(\tilde{A}, \tilde{B})$*

reproduces Painlevé second equation (1) as subjected to the zero-curvature condition  $\tilde{A}_x - \tilde{B}_\lambda = [\tilde{B}, \tilde{A}]$ .

*Proof.* Under above gauge transformations, the new Lax will take the following form

$$\begin{cases} \tilde{A} = -2v'\sigma_1 - (4i\lambda^2 + ix + 2iv^2)\sigma_2 + (4\lambda v + \alpha\lambda^{-1})\sigma_3 \\ \tilde{B} = v\sigma_3 - i\lambda\sigma_2, \end{cases} \quad (13)$$

with the linear system

$$\Psi_\lambda = \tilde{A}(x; \lambda)\Psi, \Psi_x = \tilde{B}(x; \lambda)\Psi, \quad (14)$$

whose compatibility condition gives zero-curvature form in terms of Lax matrices  $(\tilde{A}, \tilde{B})$  as  $\tilde{A}_x - \tilde{B}_\lambda = [\tilde{B}, \tilde{A}]$ . Now, having values for  $\tilde{A}_x, \tilde{B}_\lambda$ , and for commutator  $[\tilde{B}, \tilde{A}]$  from gauged Lax pair (13) and substituting in the corresponding zero-curvature condition, some computations system reproduces the classical Painlevé second equation (1).

**Proposition 4.** *The linear system (14) under the Darboux transformation (12) connects the next new solution and says  $v[1]$  of (1) to its old one  $v$  as*

$$v[1] = -v + \frac{d}{dx} \ln \frac{X_1}{Y_1}, \quad (15)$$

where  $X_1$  and  $Y_1$  are particular solutions of the linear system (14) at  $\lambda = \lambda_1$ .

*Proof.* Let us start with standard Darboux transformations (12) on components of the column vector and consider a column matrix  $\Psi = XY$  with the linear system (14), and we have the following transformed equation

$$X'[1] = v[1]X[1] - \lambda Y[1], \quad (16)$$

and now from (12)

$$X'[1] = \lambda_o Y'_o - \lambda_1 \left( \frac{Y'_1}{X_1} \right) X_o + \lambda \left( \frac{Y_1}{X_1^2} \right) X'_1 X_o - \lambda_1 \left( \frac{Y_1}{X_1} \right) X'_o. \quad (17)$$

Now, after combining (16) and (17) and then comparing the coefficients of  $X_o$  and  $Y_o$ , at  $\lambda_o = \lambda_1 = \lambda X_o$ , we obtain

$$v \left( \frac{Y_1}{X_1} \right) - \frac{Y_1^2}{X_1^2} \lambda = -v[1] \left( \frac{Y_1}{X_1} \right) \lambda - \lambda. \quad (18)$$

The above expression yields after simplification and the 1-fold Darboux transformation as follows:

$$v[1] = -v + \left( \frac{Y_1}{X_1} - \frac{X_1}{Y_1} \right) \lambda_1, \quad (19)$$

$$v[1] = -v + (Y_1 X_1^{-1} - X_1 Y_1^{-1}) \lambda_1, \quad (20)$$

where in the above expression (20),  $v$  is the old solution of the classical Painlevé II equation that generates its new solution  $v[1]$  having  $X_1$  and  $Y_1$  as particular solutions of the system (14) at  $\lambda = \lambda_1$ . Now, this can be shown that after substituting the values of  $\lambda_1 X_1$  and  $\lambda_1 X_1$  in (20) from the linear system (14), the 1-fold Darboux transformation can be expressed as

$$v[1] = -v + \frac{d}{dx} \ln \frac{X_1}{Y_1}. \quad (21)$$

Above result shows that the classical Painlevé second Darboux transformation in the additive structure where as if we pursue with the old Lax pair, the Darboux expression will be in product form which mostly happened for noncommutative case not recommended in the classical framework.

### 3. N-Fold Darboux Solutions in Terms of Wronskians

Here, we generalize the Darboux transformation in terms of Wronskian. The iterated Darboux solutions can be describe in a more compact way as  $N$ th wronskian of the components of the eigenfunctions.

*3.1. 1-Fold Darboux Solution.* With the following settings,

$$\begin{aligned} Y_i^{(1)} &= \lambda_i Y_i, \\ X_i^{(1)} &= \lambda_i X_i. \end{aligned} \quad (22)$$

The 1-fold Darboux transformation  $X[1]$ ,  $Y[1]$ , and  $v[1]$  can be written as below

$$X[1] = \frac{W_1(X_1, Y_1, X_o, Y_o)[2]}{W_1(X_1, Y_1)[1]}, \quad (23)$$

$$Y[1] = \frac{W_2(X_1, Y_1, X_o, Y_o)[2]}{W_2(X_1, Y_1)[1]},$$

$$v[1] = -v + \partial_x \ln \left( \frac{W_1(X_1, Y_1)[1]}{W_2(X_1, Y_1)[1]} \right), \quad (24)$$

where  $W_1$  and  $W_2$  are Wronskians defined as

$$\begin{aligned} W_1[2] &= \begin{vmatrix} X_1 & X_o \\ Y_1^{(1)} & Y_o^{(1)} \end{vmatrix}, \\ W_2[2] &= \begin{vmatrix} Y_1 & Y_o \\ X_1^{(1)} & X_o^{(1)} \end{vmatrix}, \\ W_1[1] &= X_1, \\ W_2[1] &= Y_1. \end{aligned} \quad (25)$$

3.2. *Wronskians for 2-Fold Darboux Transformation.* The 2-fold Darboux transformation for the  $X$  component can be written as

$$X[2] = \lambda_o Y_o[1] - \lambda_1 \left( \frac{Y_1[1]}{X_1[1]} \right) X_o[1], \quad (26)$$

or

$$X[2] = Y_o^{(1)}[1] - \left( \frac{Y_1^{(1)}[1]}{X_1^{(1)}[1]} \right) X_o[1]. \quad (27)$$

And now, in terms of Wronskian, it can be expressed as

$$X[2] = \frac{W_1(X_k, Y_k, X_o, Y_o)[3]}{W_1(X_k, Y_k)[2]}. \quad (28)$$

Similarly, for the  $Y$  component, we have

$$Y[2] = \lambda_o X_o[1] - \lambda_1 \left( \frac{X_1[1]}{Y_1[1]} \right) Y_o[1], \quad (29)$$

which equivalently can be expressed as

$$Y[2] = X_o^{(1)}[1] - \left( \frac{X_1^{(1)}[1]}{Y_1[1]} \right) Y_o[1], \quad (30)$$

or

$$Y[2] = \frac{W_2(X_k, Y_k, X_o, Y_o)[3]}{W_2(X_k, Y_k)[2]}, \quad (31)$$

where  $k = 1, 2$ . Now, the second iteration on (21) yields the next new solution to equation (1)

$$\nu[2] = -\nu + \partial_x \ln \left( \frac{W_1(X_1, Y_1)[2]}{W_2(X_1, Y_1)[2]} \right), \quad (32)$$

where the Wronskians are defined as follows:

$$\begin{aligned} W_1[3] &= \begin{vmatrix} Y_2 & Y_1 & Y_o \\ X_2^{(1)} & X_1^{(1)} & X_o^{(1)} \\ Y_2^{(2)} & Y_1^{(2)} & Y_o^{(2)} \end{vmatrix}, \\ W_2[3] &= \begin{vmatrix} X_2 & X_1 & X_o \\ Y_2^{(1)} & Y_1^{(1)} & Y_o^{(1)} \\ X_2^{(2)} & X_1^{(2)} & X_o^{(2)} \end{vmatrix}, \\ W_1[2] &= \begin{vmatrix} X_2 & X_1 \\ Y_2^{(1)} & Y_1^{(1)} \end{vmatrix}, \\ W_2[2] &= \begin{vmatrix} Y_2 & Y_1 \\ X_2^{(1)} & X_1^{(1)} \end{vmatrix}. \end{aligned} \quad (33)$$

3.3. *N-Fold Darboux Transformation.* For the  $N$ -time iterated Darboux transformation, we have the following results for  $X$  and  $Y$  components, respectively

$$\begin{aligned} X[N] &= \frac{W_1(X_k, Y_k, X_o, Y_o)[N+1]}{W_1(X_k, Y_k)[N]}, \\ Y[N] &= \frac{W_2(X_k, Y_k, X_o, Y_o)[N+1]}{W_2(X_k, Y_k)[N]}, \end{aligned} \quad (34)$$

where  $k = 1, 2, \dots, N$  and  $X_k$  and  $Y_k$  are particular solutions of the Lax pair at  $\lambda = \lambda_k$ .

For odd values of  $N$ , determinants  $W_1$  and  $W_2$  are

$$\begin{aligned} W_1[N+1] &= \begin{vmatrix} X_N & X_{N-1} & \cdots & X_1 & X_o \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_N^{(N-1)} & Y_{N-1}^{(N-1)} & \cdots & Y_1^{(N-1)} & Y_o^{(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ X_N^{(N)} & X_{N-1}^{(N)} & \cdots & X_1^{(N)} & X_o^{(N)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_N & Y_{(N-1)} & \cdots & Y_1 & Y_o \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ X_N^{(N-1)} & X_{(N-1)}^{(N-1)} & \cdots & X_1^{(N-1)} & X_o^{(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_N^{(N)} & Y_{(N-1)}^{(N)} & \cdots & Y_1^{(N)} & Y_o^{(N)} \end{vmatrix}, \\ W_2[N+1] &= \begin{vmatrix} X_N & X_{N-1} & \cdots & X_1 & X_o \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_N^{(N-1)} & Y_{N-1}^{(N-1)} & \cdots & Y_1^{(N-1)} & Y_o^{(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ X_N^{(N)} & X_{N-1}^{(N)} & \cdots & X_1^{(N)} & X_o^{(N)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_N & Y_{(N-1)} & \cdots & Y_1 & Y_o \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ X_N^{(N-1)} & X_{(N-1)}^{(N-1)} & \cdots & X_1^{(N-1)} & X_o^{(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_N^{(N)} & Y_{(N-1)}^{(N)} & \cdots & Y_1^{(N)} & Y_o^{(N)} \end{vmatrix}. \end{aligned} \quad (35)$$

For even values of  $N$ , we have the following determinants:

$$\begin{aligned} W_1[N+1] &= \begin{vmatrix} Y_N & Y_{N-1} & \cdots & Y_1 & Y_o \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ X_N^{(N-1)} & X_{N-1}^{(N-1)} & \cdots & X_1^{(N-1)} & X_o^{(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_N^{(N)} & Y_{N-1}^{(N)} & \cdots & Y_1^{(N)} & Y_o^{(N)} \end{vmatrix}, \\ W_2[N+1] &= \begin{vmatrix} X_N & X_{N-1} & \cdots & X_1 & X_o \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Y_N^{(N-1)} & Y_{N-1}^{(N-1)} & \cdots & Y_1^{(N-1)} & Y_o^{(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ X_N^{(N)} & X_{N-1}^{(N)} & \cdots & X_1^{(N)} & X_o^{(N)} \end{vmatrix}. \end{aligned} \quad (36)$$

Finally, the all possible Darboux solutions of (1) can generalized to the  $N$ -th form in terms of Wronskian as below

$$\nu[N] = -\nu + \partial_x \ln \left( \frac{W_1(X_k, Y_k)[N]}{W_2(X_k, Y_k)[N]} \right), \quad (37)$$

where  $v$  is the initial or “seed” solution that can be taken as trivial at  $\alpha = 0$  in (1), and with this simplest choice, one can construct all possible nontrivial solutions to that equation.

#### 4. Conclusion

Here, we have presented other equivalent Lax representation classical Painlevé second equation through the gauge-like transformations. This has been shown that the Lax pair in the new setting remained helpful to construct the Darboux transformation in additive structure as other integrable systems possess. That transformations may yield all possible nontrivial Darboux solutions of the classical Painlevé second equation taking seed solution as zero in that transformation. Finally,  $N$ -fold Darboux transformations have been presented in terms of Wronskians. Still, work is in progress to construct the exact solution through its Darboux transformation involving the associated Riccati forms of classical Painlevé second.

#### Data Availability

This data is applicable specially in the study of Painlevé equations and Integrable systems. Here, the Darboux approach is adopted to obtain the results of the Painlevé II equation which are discussed by Flashka and Newel in different formalisms.

#### Conflicts of Interest

The authors declare that they have no conflicts of interest.

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#### References

- [1] P. Painlevé, “Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme,” *Acta Mathematica*, vol. 25, pp. 1–85, 1902.
- [2] J. Weiss, M. Tabor, and G. Carnevale, “The Painlevé property for partial differential equations,” *Journal of Mathematical Physics*, vol. 24, no. 3, pp. 522–526, 1983.
- [3] J. Weiss, *Solitons in Physics, Mathematics and Nonlinear Optics*, P. J. Olver and D. H. Sattinger, Eds., Springer, Berlin, 1990.
- [4] N. Joshi, “The second Painlevé hierarchy and the stationary KdV hierarchy,” *Publications of the Research Institute for Mathematical Sciences*, vol. 40, no. 3, pp. 1039–1061, 2004.
- [5] N. Joshi and M. Mazzocco, “Existence and uniqueness of tri-tronquée solutions of the second Painlevé hierarchy,” *Nonlinearity*, vol. 16, no. 2, pp. 427–439, 2003.
- [6] A. N. W. Hone, “Painlevé Tests, singularity structure and integrability,” in *Integrability. Lecture Notes in Physics*, vol. 767, A. V. Mikhailov, Ed., pp. 245–277, Springer, Berlin, Heidelberg, 2009.
- [7] S. P. Balandin and V. V. Sokolov, “On the Painlevé test for non-Abelian equations,” *Physics Letters A*, vol. 246, no. 3–4, pp. 267–272, 1998.
- [8] A. S. Fokas, A. R. Its, A. A. Kapaev, and V. Y. Novokshenov, *Painlevé transcendents: The Riemann Hilbert approach (Mathematical Surveys and Monographs) vol 128*, American Mathematical Society, Providence, RI, USA, 2006.
- [9] K. Okamoto, “The Hamiltonians associated to the Painlevé equations,” in *The Painlevé Property, One Century Later, CRM Series in Mathematical Physics*, R. Conte, Ed., pp. 735–787, Springer, Berlin, 1999.
- [10] N. A. Kudryashov, “The second Painlevé equation as a model for the electric field in a semiconductor,” *Physics Letters A*, vol. 233, no. 4–6, pp. 397–400, 1997.
- [11] P. A. Clarkson, “Painlevé equations—nonlinear special functions,” *Journal of Computational and Applied Mathematics*, vol. 153, no. 1–2, pp. 127–140, 2003.
- [12] K. Kajiwara and Y. Ohta, “Determinant structure of the rational solutions for the Painlevé II equation,” *Journal of Mathematical Physics*, vol. 37, pp. 4393–4704, 1996.
- [13] M. Noumi and Y. Yamada, “Higher order Painlevé equations of type A1,” *Funkcialaj Ekvacioj*, vol. 41, pp. 483–503, 1998.
- [14] W.-Q. Peng, S.-F. Tian, X.-B. Wang, and T.-T. Zhang, “Characteristics of rogue waves on a periodic background for the Hirota equation,” *Wave Motion*, vol. 93, article 102454, 2020.
- [15] L.-L. Feng, S.-F. Tian, and T.-T. Zhang, “Solitary wave, breather wave and rogue wave solutions of an inhomogeneous fifth-order nonlinear Schrödinger equation from Heisenberg ferromagnetism,” *Rocky Mountain Journal of Mathematics*, vol. 49, no. 1, pp. 29–45, 2019.
- [16] W.-Q. Peng, S.-F. Tian, and T.-T. Zhang, “Breather waves, high-order rogue waves and their dynamics in the coupled nonlinear Schrödinger equations with alternate signs of nonlinearities,” *EPL*, vol. 127, no. 5, article 50005, 2019.
- [17] I. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. S. Retakh, and J. Y. Thibon, “Noncommutative symmetric functions,” <http://arxiv.org/abs/hep-th/9407124v1>.
- [18] V. B. Matveev and M. A. Salle, *Darboux Transformations and Soliton*, Springer, Berlin, 1991.
- [19] M. Hassan, “Darboux transformation of the generalized coupled dispersionless integrable system,” *Journal of Physics A: Mathematical and Theoretical*, vol. 42, no. 6, article 065203, 2009.