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Research Article

Existence Results for ψ-Hilfer Fractional Integro-Differential Hybrid Boundary Value Problems for Differential Equations and Inclusions

Chanakarn Kiataramkul,[1] Sotiris K. Ntouyas,[2,3] and Jessada Tariboon[1]

[1]*Intelligent and Nonlinear Dynamic Innovations Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand*
[2]*Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece*
[3]*Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*

Correspondence should be addressed to Jessada Tariboon; jessada.t@sci.kmutnb.ac.th

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In this research work, we study a new class of ψ-Hilfer hybrid fractional integro-differential boundary value problems with nonlocal boundary conditions. Existence results are established for single and multivalued cases, by using suitable fixed-point theorems for the product of two single or multivalued operators. Examples illustrating the main results are also constructed.

1. Introduction

Some real-world problems in physics, mechanics, and other fields can be described better with the help of fractional differential equations. So, differential equations of fractional order has recently received a lot of attention and now constitutes a significant branch of nonlinear analysis. Numerous monographs have appeared devoted to fractional differential equations, for example, see [1–8]. Recently, differential equations and inclusions equipped with various boundary conditions have been widely investigated by many researchers, see [9–14] and the references cited therein.

Hybrid fractional differential equations have also been studied by several researchers. This class of equations involves the fractional derivative of an unknown function hybrid with the nonlinearity depending on it. Some recent results on hybrid differential equations can be found in a series of papers [15–18].

We will give a brief history on the subject of hybrid differential and fractional differential equations. In 2010, Dhage and Lakshmikantham [19] initiated the study of the first-order hybrid differential equation

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{x(t)}{f(t,x(t))}\right) = g(t,x(t)), t \in J := [0,T], \\ x(0) = x_0 \in \mathbb{R}, \end{cases} \tag{1}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g \in C(J \times \mathbb{R}, \mathbb{R})$. They established the existence, uniqueness results, and some fundamental differential inequalities.

In 2011, Zhao et al. [15] discussed the following hybrid fractional initial value problem

$$\begin{cases} D^q\left(\dfrac{x(t)}{f(t,x(t))}\right) = g(t,x(t)), \quad t \in J, \\ x(0) = x_0 \in \mathbb{R}, \end{cases} \tag{2}$$

where D^q is the Riemann-Liouville fractional derivative of order $0 < q < 1, f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, and $g \in C(J \times \mathbb{R}, \mathbb{R})$.

Sun et al. [16] studied the following hybrid fractional boundary value problem

$$\begin{cases} D^q\left(\dfrac{x(t)}{f(t, x(t))}\right) = g(t, x(t)), & t \in [0, 1], \\ x(0) = x(1) = 0, \end{cases} \tag{3}$$

where D^q is the Riemann-Liouville fractional derivative of order $1 < q < 2, f \in C([0, 1] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, and $g \in C([0, 1] \times \mathbb{R}, \mathbb{R})$.

In [20], the authors studied the existence of solutions for a nonlocal boundary value problem of hybrid fractional integro-differential equations given by

$$\begin{cases} D^\alpha\left[\dfrac{x(t) - \sum_{i=1}^n I^{\beta_i} h_i(t, x(t))}{g(t, x(t))}\right] = f(t, x(t)), & t \in [0, 1], \\ x(0) = m(x), x(1) = A, \end{cases} \tag{4}$$

where D^α is the Caputo fractional derivative of order α with $1 < \alpha \le 2$, I^{β_i} is the Riemann-Liouville fractional integral of order $\beta_i > 0$, $h_i \in C([0, 1] \times \mathbb{R}, \mathbb{R})$ for $i = 1, 2, \cdots, n$, $g \in C([0, 1] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$, a functional $m : C([0, 1], \mathbb{R}) \longrightarrow \mathbb{R}$, and $A \in \mathbb{R}$. The main result was obtained by means of a hybrid fixed-point theorem for three operators in a Banach algebra due to Dhage [21].

The existence of solutions for an initial value problem of hybrid fractional integro-differential equations, given by

$$\begin{cases} D^\alpha\left[\dfrac{D^\omega x(t) - \sum_{i=1}^n I^{\beta_i} h_i(t, x(t))}{g(t, x(t))}\right] = f(t, x(t)), & t \in [0, 1], \\ x(0) = 0, D^\omega x(0) = 0, \end{cases} \tag{5}$$

was studied in [22]. Here, D^θ is the Caputo fractional derivative of order $\theta \in \{\alpha, \omega\}$ with $0 < \alpha, \omega \le 1$; I^{β_i} is the Riemann-Liouville fractional integral of order $\beta_i > 0$, $h_i \in C([0, 1] \times \mathbb{R}, \mathbb{R})$ for $i = 1, 2, \cdots, n$, $g \in C([0, 1] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$. A generalization of Krasnoselskii fixed-point theorem due to Dhage [21] was used in the proof of the existence result.

The problem (5) was extended in [23] to boundary value problems of the form

$$\begin{cases} D^\alpha\left[\dfrac{D^\omega x(t) - \sum_{i=1}^n I^{\beta_i} h_i(t, x(t))}{g(t, x(t))}\right] = f(t, x(t)), & t \in [0, 1], \\ x(0) = 0, D^\omega x(0) = 0, x(1) = \delta x(\eta), & 0 < \delta < 1, 0 < \eta < 1, \end{cases} \tag{6}$$

where D^θ is the Caputo fractional derivative of order $\theta \in \{\alpha, \omega\}$ with $0 < \alpha \le 1$, $1 < \omega \le 2$; I^{β_i} is the Riemann-Liouville fractional integral of order $\beta_i > 0$, $h_i \in C([0, 1] \times \mathbb{R}, \mathbb{R})$ for $i = 1, 2, \cdots, n$, $g \in C([0, 1] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$. An existence result is proved via Dhage's [21] fixed-point theorem.

For recent results on hybrid boundary value problems of fractional differential equations and inclusions, we refer to [24–26] and references cited therein. In the literature, there do exist several definitions of fractional integrals and derivatives. One of them is the Hilfer fractional derivative, which composites both Riemann-Liouville and Caputo fractional derivatives [27]. Fractional differential equations involving Hilfer derivative have many applications, and we refer to [28] and the references cited therein. There are actual world occurrences with uncharacteristic dynamics such as atmospheric diffusion of pollution, signal transmissions through strong magnetic fields, the effect of the theory of the profitability of stocks in economic markets, the theoretical simulation of dielectric relaxation in glass forming materials, and network traffic. See [29, 30] and references cited therein.

In [31], an initial value problem was discussed for hybrid fractional differential equations involving ψ-Hilfer fractional derivative of the form

$$\begin{cases} {}^H\mathfrak{D}^{\alpha,\beta;\psi}\left(\dfrac{x(t)}{f(t, x(t))}\right) = g(t, x(t)), & t \in J, \\ I^{1-\gamma;\psi}\left(\dfrac{x(0)}{f(0, x(0))}\right) = x_0 \in \mathbb{R} \end{cases} \tag{7}$$

where ${}^H\mathfrak{D}^{\alpha,\beta;\psi}$ is the ψ-Hilfer fractional derivative with $0 < \alpha < 1, 0 \le \beta \le 1, \alpha \le \gamma = \alpha + \beta - \alpha\beta < 1$, $f \in C_{1-\gamma;\psi}(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, and $g \in C_{1-\gamma;\psi}(J \times \mathbb{R}, \mathbb{R})$. For some recent results on ψ-Hilfer fractional initial value problems, see [32–37] and references cited therein.

In the present work, we study a ψ-Hilfer hybrid fractional integro-differential nonlocal boundary value problem of the form

$$\begin{cases} {}^H\mathfrak{D}_{a^+}^{\alpha,\rho;\psi}\left[\left(\dfrac{x(t)}{g(t, x(t))}\right) - \displaystyle\sum_{i=1}^n \mathcal{I}_{a^+}^{\beta_i;\psi} h_i(t, x(t))\right] = f(t, x(t)), & t \in [a, b], \\ x(a) = 0, x(b) = m(x), \end{cases} \tag{8}$$

where ${}^H\mathfrak{D}_{a^+}^{\alpha,\rho;\psi}$ is the ψ-Hilfer fractional derivative operator of order α, with $0 < \alpha \le 2$, $0 \le \rho \le 1$; $\mathcal{I}_{a^+}^{\beta_i;\psi}$ is ψ-Riemann-Liouville fractional integral of order $\beta_i > 0$, for $i = 1, 2, \cdots, n$, $g \in C([a, b] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $f \in C([a, b] \times \mathbb{R}, \mathbb{R})$, $m : C([a, b], \mathbb{R}) \longrightarrow \mathbb{R}, h_i \in C([a, b] \times \mathbb{R}, \mathbb{R})$ with $h_i(a, 0) = 0$ for $i = 1, 2, \cdots, n$. An existence result is established via a fixed-point theorem for the product of two operators due to Dhage [21].

As a second problem, we investigate the existence of solutions for the following inclusion ψ-Hilfer fractional hybrid integro-differential equations with nonlocal boundary conditions of the form

$$\begin{cases} {}^{H}\mathfrak{D}^{\alpha,\beta;\psi}\left[\left(\dfrac{x(t)}{g(t,x(t))}\right)-\displaystyle\sum_{i=1}^{n}\mathcal{I}_{a^+}^{\beta_i;\psi}h_i(t,x(t))\right]\in F(t,x(t)), & t\in[a,b],\\ x(a)=0, x(b)=m(x), \end{cases} \tag{9}$$

where $F:[a,b]\times\mathbb{R}\longrightarrow\mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$, and the other quantities are the same as in boundary value problem (8). Here, the existence result is based on a multivalued fixed-point theorem for the product of two operators due to Dhage [38].

The rest of the paper is arranged as follows: in Section 2, we recall some notations, definitions, and lemmas from fractional calculus needed in our study. Also, we prove an auxiliary lemma helping us to transform the hybrid boundary value problem (8) into an equivalent integral equation. The main existence result for the ψ-Hilfer hybrid boundary value problem (8) is contained in Section 3. The obtained result is illustrated by a numerical example. Section 4 is devoted in the study of the inclusion case of the hybrid boundary value problem (8) by considering the multivalued hybrid boundary value problem (9). Some special cases are discussed in Section 5.

2. Preliminaries

This section is assigned to recall some notation in relation to fractional calculus. We denote by $\mathcal{AC}^n([a,b],\mathbb{R})$ the n-times absolutely continuous functions given by

$$\mathcal{AC}^n([a,b],\mathbb{R})=\left\{f:[a,b]\longrightarrow\mathbb{R};f^{(n-1)}\in\mathcal{AC}([a,b],\mathbb{R})\right\}. \tag{10}$$

Definition 1 (see [2]). Let (a,b), $(-\infty\le a<b\le\infty)$, be a finite or infinite interval of the half-axis $(0,\infty)$ and $\alpha>0$. Also, let ψ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $\psi'(x)$ on (a,b). The ψ-Riemann-Liouville fractional integral of a function f with respect to another function ψ on $[a,b]$ is defined by

$$\mathcal{I}_{a^+}^{\alpha;\psi}f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}f(s)ds,\quad t>a>0, \tag{11}$$

where $\Gamma(\cdot)$ is the Euler Gamma function.

Definition 2 (see [2]). Let $\psi'(t)\neq0$ and $\alpha>0$, $n\in\mathbb{N}$. The Riemann-Liouville derivatives of a function f with respect to another function ψ of order α is defined by

$$\begin{aligned}\mathfrak{D}_{a^+}^{\alpha;\psi}f(t)&=\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n\mathcal{I}_{a^+}^{n-\alpha;\psi}f(t)\\&=\frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n\int_a^t\psi'(s)(\psi(t)\\&\quad-\psi(s))^{n-\alpha-1}f(s)ds,\end{aligned} \tag{12}$$

where $n=[\alpha]+1$, $[\alpha]$ is represent the integer part of the real number α.

Definition 3 (see [32]). Let $n-1<\alpha<n$ with $n\in\mathbb{N}$, $[a,b]$ is the interval such that $-\infty\le a<b\le\infty$ and $f,\psi\in C^n([a,b],\mathbb{R})$ two functions such that ψ is increasing and $\psi'(t)\neq0$, for all $t\in[a,b]$. The ψ-Hilfer fractional derivative of a function f of order α and type $0\le\rho\le1$ is defined by

$$\begin{aligned}{}^{H}\mathfrak{D}_{a^+}^{\alpha,\rho;\psi}f(t)&=\mathcal{I}_{a^+}^{\rho(n-\alpha);\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n\mathcal{I}_{a^+}^{(1-\rho)(n-\alpha);\psi}f(t)\\&=\mathcal{I}_{a^+}^{\gamma-\alpha;\psi}\mathfrak{D}_{a^+}^{\gamma;\psi}f(t),\end{aligned} \tag{13}$$

where $n=[\alpha]+1$; $[\alpha]$ represents the integer part of the real number α with $\gamma=\alpha+\rho(n-\alpha)$.

Lemma 4 (see [2]). *Let $\alpha,\beta>0$, then we have the following semigroup property given by*

$$\mathcal{I}_{a^+}^{\alpha;\psi}\mathcal{I}_{a^+}^{\beta;\psi}f(t)=\mathcal{I}_{a^+}^{\alpha+\beta;\psi}f(t),\quad t>a. \tag{14}$$

Next, we present the ψ-fractional integral and derivatives of a power function.

Proposition 5 (see [2, 32]). *Let $\alpha\ge0$, $\upsilon>0$, and $t>a$, then, ψ-fractional integral and derivative of a power function are given by*

$$\begin{aligned}&\mathcal{I}_{a^+}^{\alpha;\psi}(\psi(s)-\psi(a))^{\upsilon-1}(t)=\frac{\Gamma(\upsilon)}{\Gamma(\upsilon+\alpha)}(\psi(t)-\psi(a))^{\upsilon+\alpha-1},\\&{}^{H}\mathfrak{D}_{a^+}^{\alpha,\rho;\psi}(\psi(s)-\psi(a))^{\upsilon-1}(t)\\&\quad=\frac{\Gamma(\upsilon)}{\Gamma(\upsilon-\alpha)}(\psi(t)-\psi(a))^{\upsilon-\alpha-1},n-1<\alpha<n,\upsilon>n.\end{aligned} \tag{15}$$

Lemma 6 (see [33]). *Let $m-1<\alpha<m$, $n-1<\beta<n$, $n,m\in\mathbb{N}$, $n\le m$, $0\le\rho\le1$, and $\alpha\ge\beta+\rho(n-\beta)$. If $f\in C^n([a,b],\mathbb{R})$, then*

$${}^{H}\mathfrak{D}_{a^+}^{\beta,\rho;\psi}\mathcal{I}_{a^+}^{\alpha;\psi}f(t)=\mathcal{I}_{a^+}^{\alpha-\beta;\psi}f(t). \tag{16}$$

Lemma 7 (see [32]). *If $f\in C^n([a,b],\mathbb{R})$, $n-1<\alpha<n$, $0\le\rho\le1$, and $\gamma=\alpha+\rho(n-\alpha)$, then*

$$\begin{aligned}\mathcal{I}_{a^+}^{\alpha;\psi}{}^{H}\mathfrak{D}_{a^+}^{\alpha,\rho;\psi}f(t)=f(t)-\sum_{k=1}^{n}\frac{(\psi(t)-\psi(a))^{\gamma-k}}{\Gamma(\gamma-k+1)}\\\cdot\nabla_{\psi}^{[n-k]}\mathcal{I}_{a^+}^{(1-\rho)(n-\alpha);\psi}f(a),\end{aligned} \tag{17}$$

for all $t\in[a,b]$, where $\nabla_{\psi}^{[n]}f(t)\coloneqq((1/\psi'(t))(d/dt))^nf(t)$.

Lemma 8. *Let $1<\alpha\le2$, $0\le\rho\le1$, $\gamma=\alpha+\rho(2-\alpha)$, and $z\in C([a,b],\mathbb{R})$. Then, x is a solution of the ψ-Hilfer hybrid fractional integro-differential nonlocal boundary value problem* *of the form*

$$\begin{cases} \mathfrak{D}_{a^+}^{\alpha,\rho;\psi}\left[\left(\dfrac{x(t)}{g(t,x(t))}\right) - \sum_{i=1}^{n} \mathcal{I}_{a^+}^{\beta_i;\psi} h_i(t,x(t)) = z(t), t \in [a,b]\right], \\ x(a) = 0, x(b) = m(x), \end{cases} \tag{18}$$

if and only if x satisfies the equation

$$\begin{aligned} x(t) = g(t,x(t)) \Bigg\{ & \sum_{i=1}^{n} \mathcal{I}_{a^+}^{\beta_i;\psi} h_i(t,x(t)) + \mathcal{I}_{a^+}^{\alpha;\psi} z(t) \\ & + \frac{(\psi(t)-\psi(a))^{\gamma-1}}{(\psi(b)-\psi(a))^{\gamma-1}} \Bigg(\frac{m(x)}{g(b,m(x))} \\ & - \sum_{i=1}^{n} \mathcal{I}_{a^+}^{\beta_i;\psi} h_i(b,m(x)) - \mathcal{I}_{a^+}^{\alpha;\psi} z(b) \Bigg) \Bigg\}. \end{aligned} \tag{19}$$

Proof. Let $x \in C([a,b],\mathbb{R})$ be a solution of the problem (18). Applying the operator $\mathcal{I}_{a^+}^{\alpha;\psi}$ to both sides of (18) and using Lemma 7, we obtain

$$\begin{aligned} \frac{x(t)}{g(t,x(t))} - \sum_{i=1}^{n} \mathcal{I}_{a^+}^{\beta_i;\psi} h_i(t,x(t)) & = \mathcal{I}_{a^+}^{\alpha;\psi} z(t) + \frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)} c_1 \\ & + \frac{(\psi(t)-\psi(a))^{\gamma-2}}{\Gamma(\gamma-1)} c_2, \end{aligned} \tag{20}$$

where c_1, $c_2 \in \mathbb{R}$. By using the first boundary condition, $x(a) = 0$, we get the constant $c_2 = 0$. From the second boundary condition, $x(b) = m(x)$, we find that

$$\begin{aligned} c_1 = \frac{\Gamma(\gamma)}{(\psi(b)-\psi(a))^{\gamma-1}} \Bigg(& \frac{m(x)}{g(b,m(x))} \\ & - \sum_{i=1}^{n} \mathcal{I}_{a^+}^{\beta_i;\psi} h_i(b,m(x)) - \mathcal{I}_{a^+}^{\alpha;\psi} z(b) \Bigg). \end{aligned} \tag{21}$$

Substituting the value of c_1 and c_2 in (20), we obtain (19).

Conversely, by a direct computation, it is easy to show that the solution x given by (19) satisfies the problem (18). The proof of Lemma 8 is completed.□ □

Let $\mathbb{X} = C(J,\mathbb{R})$ be the Banach space of continuous real-valued functions defined on $[a,b]$, equipped with the norm $\|x\| = \sup_{t\in[a,b]} |x(t)|$ and a multiplication $(xy)(t) = x(t)y(t)$, $\forall t \in [a,b]$. Then, clearly, $\mathbb{X}$ is a Banach algebra with above-defined supremum norm and multiplication in it.

Lemma 9 (see [21]). *Let S be a nonempty, closed convex, and bounded subset of the Banach algebra X and $A : X \longrightarrow X$, $B : S \longrightarrow X$ two operators such that*

(a) A is Lipschitzian with a Lipschitz constant k

(b) B is completely continuous

(c) $x = AxBy \Longrightarrow x \in S$ for all $y \in S$

(d) $Mk < 1$, where $M = \|B(S)\| = \sup\{|Bx|: x \in S\}$

Then, the operator equation $x = AxBx$ has a solution.

3. Existence Result for the Problem (8)

In view of Lemma 8, we define an operator $\mathcal{Q} : \mathbb{X} \longrightarrow \mathbb{X}$ by

$$\begin{aligned} \mathcal{Q}x(t) = g(t,x(t)) \Bigg\{ & \sum_{i=1}^{n} \mathcal{I}_{a^+}^{\beta_i;\psi} h_i(t,x(t)) + \mathcal{I}_{a^+}^{\alpha;\psi} f(t,x(t)) \\ & + \frac{(\psi(t)-\psi(a))^{\gamma-1}}{(\psi(b)-\psi(a))^{\gamma-1}} \Bigg(\frac{m(x)}{g(b,m(x))} \\ & - \sum_{i=1}^{n} \mathcal{I}_{a^+}^{\beta_i;\psi} h_i(b,m(x)) - \mathcal{I}_{a^+}^{\alpha;\psi} f(b,m(x)) \Bigg) \Bigg\}. \end{aligned} \tag{22}$$

Notice that the problem (8) has solutions if and only if the operator $\mathcal{Q}$ has fixed points.

Theorem 10. *Assume that:*

(A_1) The function $g : [a,b] \times \mathbb{R} \longrightarrow \mathbb{R} \setminus \{0\}$ is continuous and there exists a positive function ϕ, with bound $\|\phi\|$, such that

$$|g(t,x) - g(t,y)| \le \phi(t)|x-y|, \tag{23}$$

for $t \in [a,b]$ and $x, y \in \mathbb{R}$

(A_2) $|h_i(t,x)| \le \lambda_i(t), \lambda_i \in C([a,b],\mathbb{R}), \forall (t,x) \in [a,b] \times \mathbb{R}$, $i = 1, 2, \cdots, n, |f(t,x)| \le \mu(t), |g(t,x)| \le \nu(t)$, $\forall (t,x) \in [a,b] \times \mathbb{R}, \mu, \nu \in C([a,b],\mathbb{R})$

(A_3) There exists a positive constant $\mathcal{K} > 0$ such that

$$\frac{|m(x)|}{|g(b,m(x))|} \le \mathcal{K}, \quad \forall x \in C([a,b],\mathbb{R}), \tag{24}$$

(A_4) There exists a positive real number $r > 0$ such that

$$\|\nu\| \left[2\sum_{i=1}^{n} \frac{(\psi(b)-\psi(a))^{\beta_i}}{\Gamma(\beta_i+1)} \|\lambda_i\| + 2\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} \|\mu\| + \mathcal{K} \right] \le r. \tag{25}$$

Then, the ψ-Hilfer hybrid fractional integro-differential nonlocal boundary value problem (8) has at least one solution on $[a,b]$, provided that

$$\|\phi\| \left[2\sum_{i=1}^{n} \frac{(\psi(b)-\psi(a))^{\beta_i}}{\Gamma(\beta_i+1)} \|\lambda_i\| + 2\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} \|\mu\| + \mathcal{K} \right] < 1. \tag{26}$$

Proof. We consider a subset S of $\mathbb{X}$ defined by $S = \{x \in \mathbb{X} : \|x\| \le r\}$, where r satisfies the inequality (25). Observe that

S is a closed, convex, and bounded subset of the Banach space $\mathbb{X}$. We set $\sup_{t\in[a,b]}|\lambda_i(t)| = \|\lambda_i\|, i = 1, 2, \cdots, m$, $\sup_{t\in[a,b]}|\mu(t)| = \|\mu\|$, and $\sup_{t\in[a,b]}|\nu(t)| = \|\nu\|$.

Next, we define two more operators $\mathscr{A} : \mathbb{X} \longrightarrow \mathbb{X}$ and $\mathscr{B} : S \longrightarrow \mathbb{X}$ as follows:

$$\begin{aligned} \mathscr{A}x(t) &= g(t, x(t)), \quad t \in [a, b], \\ \mathscr{B}x(t) &= \sum_{i=1}^{n} \mathscr{I}_{a^+}^{\beta_i;\psi} h_i(t, x(t)) + \mathscr{I}_{a^+}^{\alpha;\psi} f(t, x(t)) \\ &\quad + \frac{(\psi(t)-\psi(a))^{\gamma-1}}{(\psi(b)-\psi(a))^{\gamma-1}} \left(\frac{m(x)}{g(b, m(x))} \right. \\ &\quad \left. - \sum_{i=1}^{n} \mathscr{I}_{a^+}^{\beta_i;\psi} h_i(b, m(x)) - \mathscr{I}_{a^+}^{\alpha;\psi} f(b, m(x)) \right). \end{aligned} \tag{27}$$

Clearly, $\mathscr{Q}x = \mathscr{A}x\mathscr{B}x$. In the next steps, we show that the operators $\mathscr{A}$ and $\mathscr{B}$ fulfil all the assumptions of Lemma 9. The proof is divided into three steps.□ □

Step 1. We show that the operator $\mathscr{A}$ is Lipschitzian with Lipschitz constant k, i.e., condition (a) of Lemma 9 is fulfilled. Let $x, y \in S$. Then we have

$$\begin{aligned} |Ax(t) - Ay(t)| &= |g(t, x(t)) - g(t, y(t))| \\ &\le \phi(t)|x(t) - y(t)| \le \|\phi\|\|x - y\|. \end{aligned} \tag{28}$$

Consequently,

$$\|\mathscr{A}x - \mathscr{A}y\| \le \|\phi\|\|x - y\|. \tag{29}$$

Hence, the operator $\mathscr{A}$ is Lipschitzian with Lipschitz constant $k = \|\phi\|$.

Step 2. We show that the condition (b) of Lemma 9 is satisfied, i.e., the operator $\mathscr{B}$ is completely continuous on S. First, we will prove its continuity. Let $\{x_j(t)\}$ be a sequence of functions in S converging to a function $x(t) \in S$. Then, by the Lebesgue dominant theorem, for each $t \in [a, b]$, we have

$$\begin{aligned} \lim_{j\longrightarrow\infty} \mathscr{B}x_j(t) &= \lim_{j\longrightarrow\infty} \left\{ \sum_{i=1}^{n} \mathscr{I}_{a^+}^{\beta_i;\psi} h_i\left(t, x_j(t)\right) + \mathscr{I}_{a^+}^{\alpha;\psi} f\left(t, x_j(t)\right) \right. \\ &\quad + \frac{(\psi(t)-\psi(a))^{\gamma-1}}{(\psi(b)-\psi(a))^{\gamma-1}} \left(\frac{m(x_j)}{g(b, m(x_j))} \right. \\ &\quad \left.\left. - \sum_{i=1}^{n} \mathscr{I}_{a^+}^{\beta_i;\psi} h_i\left(b, m(x_j)\right) - \mathscr{I}_{a^+}^{\alpha;\psi} f\left(b, m(x_j)\right) \right) \right\} \\ &= \sum_{i=1}^{n} \mathscr{I}_{a^+}^{\beta_i;\psi} \lim_{j\longrightarrow\infty} h_i\left(t, x_j(t)\right) + \mathscr{I}_{a^+}^{\alpha;\psi} \lim_{j\longrightarrow\infty} f\left(t, x_j(t)\right) \\ &\quad + \frac{(\psi(t)-\psi(a))^{\gamma-1}}{(\psi(b)-\psi(a))^{\gamma-1}} \left(\lim_{j\longrightarrow\infty} \frac{m(x_j)}{g(b, m(x_j))} \right. \\ &\quad - \sum_{i=1}^{n} \mathscr{I}_{a^+}^{\beta_i;\psi} \lim_{j\longrightarrow\infty} h_i\left(b, m(x_j)\right) \\ &\quad \left. - \mathscr{I}_{a^+}^{\alpha;\psi} \lim_{j\longrightarrow\infty} f\left(b, m(x_j)\right) \right) \\ &= \sum_{i=1}^{n} \mathscr{I}_{a^+}^{\beta_i;\psi} h_i(t, x(t)) + \mathscr{I}_{a^+}^{\alpha;\psi} f(t, x(t)) \\ &\quad + \frac{(\psi(t)-\psi(a))^{\gamma-1}}{(\psi(b)-\psi(a))^{\gamma-1}} \left(\frac{m(x)}{g(b, m(x))} \right. \\ &\quad \left. - \sum_{i=1}^{n} \mathscr{I}_{a^+}^{\beta_i;\psi} h_i(b, m(x)) - \mathscr{I}_{a^+}^{\alpha;\psi} f(b, m(x)) \right) \\ &= \mathscr{B}x(t). \end{aligned} \tag{30}$$

Therefore, $\mathscr{B}$ is a continuous operator on S. Next, we show that the operator $\mathscr{B}$ is uniformly bounded on S. For any $x \in S$, we have

$$\begin{aligned} |\mathscr{B}x(t)| &\le \sum_{i=1}^{n} \mathscr{I}_{a^+}^{\beta_i;\psi} |h_i(t, x(t))| + \mathscr{I}_{a^+}^{\alpha;\psi} |f(t, x(t))| \\ &\quad + \frac{(\psi(t)-\psi(a))^{\gamma-1}}{(\psi(b)-\psi(a))^{\gamma-1}} \left(\frac{|m(x)|}{|g(b, m(x))|} \right. \\ &\quad \left. + \sum_{i=1}^{n} \mathscr{I}_{a^+}^{\beta_i;\psi} |h_i(b, m(x))| + \mathscr{I}_{a^+}^{\alpha;\psi} |f(b, m(x))| \right) \\ &\le 2\sum_{i=1}^{n} \frac{(\psi(b)-\psi(a))^{\beta_i}}{\Gamma(\beta_i+1)} \|\lambda_i\| + 2\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} \|\mu\| \\ &\quad + \mathscr{K} := M. \end{aligned} \tag{31}$$

Hence, $\|\mathscr{B}x\| \le M, \forall t \in [a, b]$, which shows that the operator $\mathscr{B}$ is uniformly bounded on S. Now, we show that the operator $\mathscr{B}$ is equicontinuous. Let $t_1 < t_2$ and $x \in S$. Then, we have

$$\begin{aligned} &|\mathscr{B}x(t_2) - \mathscr{B}x(t_1)| \\ &\quad \le \sum_{i=1}^{n} \left| \frac{1}{\Gamma(\beta_i)} \int_a^{t_1} \psi'(s) \left[(\psi(t_2)-\psi(s))^{\beta_i-1} \right. \right. \\ &\quad \left. - (\psi(t_1)-\psi(s))^{\beta_i-1} \right] h_i(s, x(s)) ds \\ &\quad \left. + \frac{1}{\Gamma(\beta_i)} \int_{t_1}^{t_2} \psi'(s) (\psi(t_2)-\psi(s))^{\beta_i-1} h_i(s, x(s)) ds \right| \end{aligned}$$

$$
\begin{aligned}
&+\left|\frac{1}{\Gamma(\alpha)}\int_a^{t_1}\psi'(s)\left[(\psi(t_2)-\psi(s))^{\alpha-1}\right.\right.\\
&\left.-(\psi(t_1)-\psi(s))^{\alpha-1}\right]f(s,x(s))ds\\
&\left.+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}f(s,x(s))ds\right|\\
&+\frac{\left|(\psi(t_2)-\psi(a))^{\gamma_1-1}-(\psi(t_1)-\psi(a))^{\gamma_1-1}\right|}{(\psi(b)-\psi(a))^{\gamma-1}}\\
&\cdot\left(\frac{|m(x)|}{|g(b,m(x))|}+\sum_{i=1}^{n}\mathcal{I}_{a^+}^{\beta_i;\psi}|h_i(b,m(x))|+\mathcal{I}_{a^+}^{\alpha;\psi}|f(b,m(x))|\right)\\
\le&\sum_{i=1}^{n}\frac{\|\lambda_i\|}{\Gamma(\beta_i+1)}\left|(\psi(t_2)-\psi(a))^{\beta_i}-(\psi(t_1)-\psi(a))^{\beta_i}\right|\\
&+\frac{\|\mu\|}{\Gamma(\alpha+1)}|(\psi(t_2)-\psi(a))^{\alpha}-(\psi(t_1)-\psi(a))^{\alpha}|\\
&+\frac{\left|(\psi(t_2)-\psi(a))^{\gamma_1-1}-(\psi(t_1)-\psi(a))^{\gamma_1-1}\right|}{(\psi(b)-\psi(a))^{\gamma-1}}\\
&\cdot\left[\mathcal{K}+\sum_{i=1}^{n}\frac{(\psi(b)-\psi(a))^{\beta_i}}{\Gamma(\beta_i+1)}\|\lambda_i\|+\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\|\mu\|\right].
\end{aligned}
\tag{32}
$$

As $t_2-t_1\longrightarrow 0$, the right-hand side tends to zero, independently of x. Thus, $\mathcal{B}$ is equicontinuous. Therefore, it follows by Aezelá-Ascoli theorem that $\mathcal{B}$ is a completely continuous operator on S.

Step 3. We show that the third condition (c) of Lemma 9 is fulfilled. For any $y\in S$, we have

$$
\begin{aligned}
|x(t)|=&|\mathcal{A}x(t)\mathcal{B}y(t)|=|\mathcal{A}x(t)||\mathcal{B}y(t)|\\
\le&|g(t,x(t))|\left[\sum_{i=1}^{n}\mathcal{I}_{a^+}^{\beta_i;\psi}|h_i(t,y(t))|+\mathcal{I}_{a^+}^{\alpha;\psi}|f(t,y(t))|\right.\\
&+\frac{(\psi(t)-\psi(a))^{\gamma-1}}{(\psi(b)-\psi(a))^{\gamma-1}}\left(\frac{|m(y)|}{|g(b,m(y))|}\right.\\
&\left.\left.+\sum_{i=1}^{n}\mathcal{I}_{a^+}^{\beta_i;\psi}|h_i(b,m(y))|+\mathcal{I}_{a^+}^{\alpha;\psi}|f(b,m(y))|\right)\right]\\
\le&\|\nu\|\left[2\sum_{i=1}^{n}\frac{(\psi(b)-\psi(a))^{\beta_i}}{\Gamma(\beta_i+1)}\|\lambda_i\|\right.\\
&\left.+2\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\|\mu\|+\mathcal{K}\right]\le r,
\end{aligned}
\tag{33}
$$

which implies $\|x\|\le r$, and so, $x\in S$.

Moreover, by (26), it holds $kM<1$ which is fulfilled condition (d) of Lemma 9. Hence, all the conditions of Lemma 9 are satisfied, and consequently, the operator equation $x(t)=\mathcal{A}x(t)\mathcal{B}x(t)$ has at least one solution in S. Therefore, there exists a solution of the ψ-Hilfer hybrid fractional integro-differential nonlocal boundary value problem (8) in $[a,b]$. The proof is finished.

Now, we present an example of ψ-Hilfer hybrid fractional integro-differential boundary value problem to illustrate our main result.

Example 11. Consider the boundary value problem of the form

$$
\begin{cases}
{}^{H}\mathfrak{D}_{1/2}^{3/2,1/4;\left(1-e^{-2t}\right)}\left[\dfrac{x(t)}{g(t,x(t))}-\mathcal{I}_{1/2}^{3/4;\left(1-e^{-2t}\right)}h_1(t,x(t))-\mathcal{I}_{1/2}^{5/4;\left(1-e^{-2t}\right)}h_2(t,x(t))\right]=\dfrac{e^{-x2(t)}}{4t+6}\left(\dfrac{x^4(t)}{1+x^4(t)}\right)+\dfrac{\cos^8x(t)}{26t+3}+\dfrac{1}{16},t\in\left[\dfrac{1}{2},\dfrac{7}{2}\right],\\
x\left(\dfrac{1}{2}\right)=0,x\left(\dfrac{7}{2}\right)=2\sin\left(x\left(\dfrac{3}{2}\right)\right)+3\cos\left(x\left(\dfrac{7}{4}\right)\right)+4\tan^{-1}\left(x\left(\dfrac{5}{2}\right)\right),
\end{cases}
\tag{34}
$$

where

$$
\begin{aligned}
g(t,x)=&\frac{t|x|}{7(1+|x|)}+2t+5,h_1(t,x)\\
=&\frac{1-e^{-\left(x^4/\left(1+x^4\right)\right)}}{2t+1},h_2(t,x)=\frac{\sin^4x}{3+\cos^2\pi t}.
\end{aligned}
\tag{35}
$$

Here, $\alpha=3/2$, $\rho=1/4$, $\psi(t)=1-e^{-2t}$, $a=1/2$, $b=7/2$, $\beta_1=3/4$, $\beta_2=5/4$, and $m(x)=2\sin x+3\cos x+4\tan^{-1}x$. Then, we can find that

$$
|g(t,x)-g(t,y)|\le\frac{t}{7}|x-y|,
\tag{36}
$$

and we choose $\phi(t)=t/7$. In addition, $|h_1(t,x)|\le(1/(2t+1)):=\lambda_1(t)$, $|h_2(t,x)|\le(1/(3+\cos^2\pi t)):=\lambda_2(t)$, $|f(t,x)|\le(1/(4t+6))+(1/(26t+3))+1/16:=\mu(t)$, $|g(t,x)|\le t/7+2t+5:=\nu(t)$. Then, we have $\|\phi\|=1/2$, $\|\lambda_1\|=1/2$, $\|\lambda_2\|=1/3$, $\|\mu\|=1/4$, $\|\nu\|=25/2$ and

$$
\frac{|m(x)|}{|g(7/2,m(x))|}\le\frac{5+2\pi}{12}:=\mathcal{K},
\tag{37}
$$

which lead to

$$\|\phi\|\left[2\sum_{i=1}^{n}\frac{(\psi(b)-\psi(a))^{\beta_i}}{\Gamma(\beta_i+1)}\|\lambda_i\|+2\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\|\mu\|+\mathscr{K}\right] \approx 0.8524738760<1. \tag{38}$$

Therefore, by Theorem 10, the ψ-Hilfer hybrid fractional integro-differential nonlocal boundary value problem (34)-(35) has at least one solution $x(t)$ on $[1/2, 7/2]$, such that $\|x\|\le r$, where r is satisfies

$$\|\nu\|\left[2\sum_{i=1}^{n}\frac{(\psi(b)-\psi(a))^{\beta_i}}{\Gamma(\beta_i+1)}\|\lambda_i\|+2\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\|\mu\|+\mathscr{K}\right] \approx 21.31184690\le r. \tag{39}$$

4. Existence Result for the Problem (9)

First of all, we recall some basic concepts for multivalued maps [39–41]. For a normed space $(X,\|\cdot\|)$, let $\mathscr{P}_{cp,cv}(X)=\{Y\in\mathscr{P}(X)\colon Y \text{ is compact and convex}\}$. For each $x\in\mathbb{X}$, define the set of selections of F by

$$S_{F,x}:=\left\{v\in L^1([a,b],\mathbb{R})\colon v(t)\in F(t,x(t)),\quad \text{for } a.e. t\in[a,b]\right\}. \tag{40}$$

Lemma 12 (see [42]). *Let $F:[a,b]\times\mathbb{R}\longrightarrow\mathscr{P}_{cp,cv}(\mathbb{R})$ be an L^1 – Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([a,b],\mathbb{R})$ to $C([a,b],\mathbb{R})$. Then, the operator*

$$\begin{aligned}&\Theta\circ S_F: C([a,b],\mathbb{R})\longrightarrow\mathscr{P}_{cp,cv}(C([a,b],\mathbb{R})),\\ &x\longmapsto(\Theta\circ S_F)(x)=\Theta(S_{F,x}),\end{aligned} \tag{41}$$

is a closed graph operator in $C([a,b],\mathbb{R})\times C([a,b],\mathbb{R})$.

Remark 13. We recall that a multivalued map $F:[a,b]\times\mathbb{R}\longrightarrow\mathscr{P}(\mathbb{R})$ is said to be L^1 – Carathéodory if (i) $t\longmapsto F(t,x)$ is measurable for each $x\in\mathbb{R}$; (ii) $x\longmapsto F(t,x)$ is upper semicontinuous for almost all $t\in[a,b]$; (iii) for each $\alpha>0$, there exists $\varphi_\alpha\in L^1([a,b],\mathbb{R}^+)$ such that $\|F(t,x)\|=\sup\{|v|\colon v\in F(t,x)\}\le\varphi_\alpha(t)$ for all $x\in\mathbb{R}$ with $\|x\|\le\alpha$ and for *a.e.* $t\in[a,b]$.

The following multivalued fixed-point theorem for the product of two operators in a Banach algebra, due to Dhage [38, Theorem 4.13], plays a key role in proving the existence result for the nonlocal boundary value problem (9).

Lemma 14 (see [38]). *Let X be a Banach algebra and let $A:X\longrightarrow X$ be a single valued and $B:X\longrightarrow\mathscr{P}_{cp,cv}(X)$ be a multivalued operator satisfying the conditions:*

(a) A is single-valued Lipschitz with a Lipschitz constant k

(b) B is compact and upper semicontinuous

(c) $2Mk<1$, where $M=\|B(S)\|=\sup\{\|Bx\|\colon x\in S\}$

Then, either (i) the operator inclusion $x\in AxBx$ has a solution or (ii) the set $\mathscr{E}=\{u\in X\mid\lambda u\in AuBu,\lambda>1\}$ is unbounded.

Definition 15. A function $x\in AC([a,b],\mathbb{R})$ is a solution of the problem (9) if $x(a)=0, x(b)=m(x)$, and there exists function $v\in L^1([a,b],\mathbb{R})$ such that $v(t)\in F(t,x(t))$ *a.e.* on $[a,b]$ and

$$\begin{aligned}x(t)=g(t,x(t))\Bigg\{&\sum_{i=1}^{n}\mathscr{I}_{a^+}^{\beta_i;\psi}h_i(t,x(t))+\mathscr{I}_{a^+}^{\alpha;\psi}v(t)\\ &+\frac{(\psi(t)-\psi(a))^{\gamma-1}}{(\psi(b)-\psi(a))^{\gamma-1}}\left(\frac{m(x)}{g(b,m(x))}\right.\\ &\left.-\sum_{i=1}^{n}\mathscr{I}_{a^+}^{\beta_i;\psi}h_i(b,m(x))-\mathscr{I}_{a^+}^{\alpha;\psi}v(b)\right)\Bigg\}.\end{aligned} \tag{42}$$

Theorem 16. *Assume that (A_1) and (A_3) hold. In addition, we suppose that*

(B_1) $F:[a,b]\times\mathbb{R}\longrightarrow\mathscr{P}_{cp,cv}(\mathbb{R}^+)$ is L^1-Carathéodory multivalued map;

(B_2) The functions g and $h_i, i=1,2,\cdots,n$ satisfy condition (A_2);

(B_3) There exists a continuous function $q\in C([a,b],\mathbb{R}^+)$ such that

$$\begin{aligned}\|F(t,x)\|_{\mathscr{P}}&:=\sup\{|y|\colon y\in F(t,x)\}\\ &\le q(t),\quad \textit{for each } (t,x)\in[a,b]\times\mathbb{R}.\end{aligned} \tag{43}$$

Then, the nonlocal ψ-Hilfer hybrid inclusion boundary value problem (9) has at least one solution on $[a,b]$, provided that

$$\begin{aligned}\Omega:=\|q\|\Bigg\{&\frac{2\|q\|(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\\ &+2\sum_{i=1}^{n}\frac{(\psi(b)-\psi(a))^{\beta_i}}{\Gamma(\beta_i+1)}\|\lambda_i\|+\mathscr{K}\Bigg\}<\frac{1}{2}.\end{aligned} \tag{44}$$

Proof. To transform the boundary value problem (9) into a fixed-point problem, by using Lemma 8, we define a multivalued operator $Q_1:\mathbb{X}\longrightarrow\mathscr{P}(\mathbb{X})$ as

$$\mathcal{Q}_1 x = \left\{ \begin{array}{c} h \in \mathbb{X} : \\ h(t) = \left\{ g(t, x(t)) \left\{ \sum_{i=1}^{n} \mathcal{I}_{a^+}^{\beta_i;\psi} h_i(t, x(t)) + \mathcal{I}_{a^+}^{\alpha;\psi} v(t) \frac{(\psi(t)-\psi(a))^{\gamma-1}}{(\psi(b)-\psi(a))^{\gamma-1}} \left(\frac{m(x)}{g(b, m(x))} - \sum_{i=1}^{n} \mathcal{I}_{a^+}^{\beta_i;\psi} h_i(b, m(x)) - \mathcal{I}_{a^+}^{\alpha;\psi} v(b) \right) \right\}, v \in S_{F,x}, t \in [a, b]. \end{array} \right\}. \tag{45}$$

Next, we introduce the operator $\mathcal{A} : \mathbb{X} \longrightarrow \mathbb{X}$ as in (27) and the multivalued operator $\mathcal{B}_1 : \mathbb{X} \longrightarrow \mathcal{P}(\mathbb{X})$ by

$$\mathcal{B}_1 x = \left\{ \begin{array}{c} h \in \mathbb{X} : \\ h(t) = g(t, x(t)) \left\{ \sum_{i=1}^{n} \mathcal{I}_{a^+}^{\hat{a}_i;\psi} h_i(t, x(t)) + \mathcal{I}_{a^+}^{\alpha;\psi} v(t) \frac{(\psi(t)-\psi(a))^{\gamma-1}}{(\psi(b)-\psi(a))^{\gamma-1}} \left(\frac{m(x)}{g(b, m(x))} - \sum_{i=1}^{n} \mathcal{I}_{a^+}^{\beta_i;\psi} h_i(b, m(x)) - \mathcal{I}_{a^+}^{\alpha;\psi} v(b) \right), v \in S_{F,x}, t \in [a, b]. \end{array} \right\}. \tag{46}$$

We will show that the operators $\mathcal{A}$ and $\mathcal{B}_1$ satisfy the hypotheses of Lemma 14. The proof is given in a series of steps.

Step 1. B_1 is convex valued. Let $z_1, z_2 \in \mathcal{B}_1$. Then, there exist $v_1, v_2 \in S_{F,x}$ such that

$$\begin{aligned} z_j(t) = {} & \sum_{i=1}^{n} \mathcal{I}_{a^+}^{\beta_i;\psi} h_i(t, x(t)) + \mathcal{I}_{a^+}^{\alpha;\psi} v_j(t) + \frac{(\psi(t)-\psi(a))^{\gamma-1}}{(\psi(b)-\psi(a))^{\gamma-1}} \\ & \cdot \left(\frac{m(x)}{g(b, m(x))} - \sum_{i=1}^{n} \mathcal{I}_{a^+}^{\beta_i;\psi} h_i(b, m(x)) - \mathcal{I}_{a^+}^{\alpha;\psi} v_j(b) \right), \\ & j = 1, 2. \end{aligned} \tag{47}$$

For any $\theta \in [0, 1]$, we have

$$\begin{aligned} & \theta z_1(t) + (1-\theta) z_2(t) \\ & \quad = \sum_{i=1}^{n} \mathcal{I}_{a^+}^{\beta_i;\psi} h_i(t, x(t)) + \mathcal{I}_{a^+}^{\alpha;\psi} [\theta v_1(t) + (1-\theta) v_2(t)] \\ & \qquad + \frac{(\psi(t)-\psi(a))^{\gamma-1}}{(\psi(b)-\psi(a))^{\gamma-1}} \left(\frac{m(x)}{g(b, m(x))} \right. \\ & \qquad \left. - \sum_{i=1}^{n} \mathcal{I}_{a^+}^{\beta_i;\psi} h_i(b, m(x)) - \mathcal{I}_{a^+}^{\alpha;\psi} [\theta v_1(b) + (1-\theta) v_2(b)] \right). \end{aligned} \tag{48}$$

Since $F(t, x(t))$ is convex, $\theta v_1(t) + (1-\theta) v_2(t) \in F(t, x(t))$ for all $t \in [a, b]$, and so, $\theta z_1 + (1-\theta) z_2 \in S_{F,x}$. Thus, $\theta z_1 + (1-\theta) z_2 \in F(t, x(t))$, which means that $\mathcal{B}_1$ is convex valued on $\mathbb{X}$.

Step 2. $\mathcal{A}_1$ is single-valued Lipschitz operator on $\mathbb{X}$. It is proved in Step 1 of Theorem 10.

Step 3. The operator $\mathcal{B}_1$ is completely continuous and upper semicontinuous on $\mathbb{X}$. Let S be a bounded set of $\mathbb{X}$. Then, there exists a constant r such that $\|x\| \leq r$, for all $x \in S$. We prove first that the operator $\mathcal{B}_1$ is completely continuous. Let $x \in \mathcal{B}_1(S)$. Then, there exists $v \in S_{F,x}$ such that

$$\begin{aligned} h(t) = {} & \sum_{i=1}^{n} \mathcal{I}_{a^+}^{\beta_i;\psi} h_i(t, x(t)) + \mathcal{I}_{a^+}^{\alpha;\psi} v(t) + \frac{(\psi(t)-\psi(a))^{\gamma-1}}{(\psi(b)-\psi(a))^{\gamma-1}} \\ & \cdot \left(\frac{m(x)}{g(b, m(x))} - \sum_{i=1}^{n} \mathcal{I}_{a^+}^{\beta_i;\psi} h_i(b, m(x)) - \mathcal{I}_{a^+}^{\alpha;\psi} v(b) \right), \end{aligned} \tag{49}$$

for any $x \in S$. Then, we have

$$\begin{aligned} |h(t)| & \leq 2 \mathcal{I}_{a^+}^{\alpha;\psi} |v(b)| + \left| \frac{m(x)}{g(b, m(x))} \right| + 2 \sum_{i=1}^{n} \mathcal{I}_{a^+}^{\beta_i;\psi} |h_i(b, m(x))| \\ & \leq \frac{2\|q\| (\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} + 2 \sum_{i=1}^{n} \frac{(\psi(b)-\psi(a))^{\beta_i}}{\Gamma(\beta_i+1)} \|\lambda_i\| \\ & \quad + \mathcal{K} = M_1, \end{aligned} \tag{50}$$

for all $t \in [a, b]$, which implies that $\|x\| \leq M_1$. Therefore, the operator $\mathcal{B}_1$ is uniformly bounded on $\mathbb{X}$.

Next, we show that $\mathcal{B}_1(S)$ is an equicontinuous set in $\mathbb{X}$. Let $t_1, t_2 \in [a, b]$ with $t_1 < t_2$ and $x \in \mathcal{B}_1(S)$. Then, we have

$$\begin{aligned} & |\mathcal{B}_1 x(t_2) - \mathcal{B}_1 x(t_1)| \\ & \quad \leq \sum_{i=1}^{n} \left| \frac{1}{\Gamma(\beta_i)} \int_a^{t_1} \psi'(s) \left[(\psi(t_2)-\psi(s))^{\beta_i-1} \right. \right. \\ & \qquad \left. - (\psi(t_1)-\psi(s))^{\beta_i-1} \right] h_i(s, x(s)) ds \\ & \qquad \left. + \frac{1}{\Gamma(\beta_i)} \int_{t_1}^{t_2} \psi'(s) (\psi(t_2)-\psi(s))^{\beta_i-1} h_i(s, x(s)) ds \right| \end{aligned}$$

$$
\begin{aligned}
&+\left|\frac{1}{\Gamma(\alpha)}\int_a^{t_1}\psi'(s)\left[(\psi(t_2)-\psi(s))^{\alpha-1}\right.\right.\\
&\left.-(\psi(t_1)-\psi(s))^{\alpha-1}\right]v(s)ds\\
&\left.+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}v(s)ds\right|\\
&+\frac{\left|(\psi(t_2)-\psi(a))^{\gamma-1}-(\psi(t_1)-\psi(a))^{\gamma-1}\right|}{(\psi(b)-\psi(a))^{\gamma-1}}\\
&\cdot\left(\left|\frac{m(x)}{g(b,m(x))}\right|+\sum_{i=1}^{n}\mathcal{I}_{a^+}^{\beta_i;\psi}|h_i(b,m(x))+\mathcal{I}_{a^+}^{\alpha;\psi}|v(b)|\right)\\
\le&\sum_{i=1}^{n}\frac{\|\lambda_i\|}{\Gamma(\beta_i+1)}\left|(\psi(t_2)-\psi(a))^{\beta_i}-(\psi(t_1)-\psi(a))^{\beta_i}\right|\\
&+\frac{\|q\|}{\Gamma(\alpha+1)}|(\psi(t_2)-\psi(a))^{\alpha}-(\psi(t_1)-\psi(a))^{\alpha}|\\
&+\frac{\left|(\psi(t_2)-\psi(a))^{\gamma-1}-(\psi(t_1)-\psi(a))^{\gamma-1}\right|}{(\psi(b)-\psi(a))^{\gamma-1}}\\
&\cdot\left(\sum_{i=1}^{n}\frac{(\psi(b)-\psi(a))^{\beta_i}}{\Gamma(\beta_i+1)}\|\lambda_i\|+\|q\|\frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}+\mathcal{K}\right).
\end{aligned}
\tag{51}
$$

As $t_2-t_1\longrightarrow 0$, the right-hand side of the above inequality tends to zero independently of x, and thus, the operator $\mathcal{B}_1$ is equicontinuous. In consequence, the operator $\mathcal{B}_1$ is completely continuous by the Arzelá-Ascoli theorem.

Next, we show that $\mathcal{B}_1$ is an upper semicontinuous multivalued mapping on $\mathbb{X}$. It is known, by [[39], Proposition 1.2], that $\mathcal{B}_1$ will be upper semicontinuous if we establish that it has a closed graph, since already shown to be completely continuous. Thus, we will prove that $\mathcal{B}_1$ has a closed graph.

Let $\{x_n\}$ be a sequence in $\mathbb{X}$ such that $x_n\longrightarrow x^*$. Let $\{y_n\}$ be a sequence such that $y_n\in\mathcal{B}_1x_n$ and $y_n\longrightarrow y^*$. We shall show that $y^*\in\mathcal{B}_1x^*$. Since $y_n\in\mathcal{B}_1x_n$, there exists a $v_n\in S_{F,x_n}$ such that

$$
\begin{aligned}
x_n(t)=&\sum_{i=1}^{n}\mathcal{I}_{a^+}^{\beta_i;\psi}h_i(t,x(t))+\mathcal{I}_{a^+}^{\alpha;\psi}v_n(t)+\frac{(\psi(t)-\psi(a))^{\gamma-1}}{(\psi(b)-\psi(a))^{\gamma-1}}\\
&\cdot\left(\frac{m(x)}{g(b,m(x))}-\sum_{i=1}^{n}\mathcal{I}_{a^+}^{\beta_i;\psi}h_i(b,m(x))-\mathcal{I}_{a^+}^{\alpha;\psi}v_n(b)\right),\\
&t\in[a,b].
\end{aligned}
\tag{52}
$$

We must prove that there is a $v^*\in S_{F,x^*}$ such that

$$
\begin{aligned}
x^*(t)=&\sum_{i=1}^{n}\mathcal{I}_{a^+}^{\beta_i;\psi}h_i(t,x(t))+\mathcal{I}_{a^+}^{\alpha;\psi}v^*(t)+\frac{(\psi(t)-\psi(a))^{\gamma-1}}{(\psi(b)-\psi(a))^{\gamma-1}}\\
&\cdot\left(\frac{m(x)}{g(b,m(x))}-\sum_{i=1}^{n}\mathcal{I}_{a^+}^{\beta_i;\psi}h_i(b,m(x))-\mathcal{I}_{a^+}^{\alpha;\psi}v^*(b)\right),\\
&t\in[a,b].
\end{aligned}
\tag{53}
$$

Consider the continuous linear operator $L:L^1([a,b],\mathbb{R})\longrightarrow C([a,b],\mathbb{R})$ defined by

$$
\begin{aligned}
Lv(t)=&\sum_{i=1}^{n}\mathcal{I}_{a^+}^{\beta_i;\psi}h_i(t,x(t))+\mathcal{I}_{a^+}^{\alpha;\psi}v(t)+\frac{(\psi(t)-\psi(a))^{\gamma-1}}{(\psi(b)-\psi(a))^{\gamma-1}}\\
&\cdot\left(\frac{m(x)}{g(b,m(x))}-\sum_{i=1}^{n}\mathcal{I}_{a^+}^{\beta_i;\psi}h_i(b,m(x))-\mathcal{I}_{a^+}^{\alpha;\psi}v(b)\right),\\
&t\in[a,b].
\end{aligned}
\tag{54}
$$

Observe that

$$
\|x_n(t)-x^*(t)\|=\left\|2\mathcal{I}_{a^+}^{\alpha;\psi}[v_n(s)-v^*(s)]\right\|\longrightarrow 0,
\tag{55}
$$

as $n\longrightarrow\infty$. From Lemma 12, it follows that $L\circ S_{F,x}$ is a closed graph operator. Further, we have $y_n(t)\in L(S_{F,x})$. Since $y_n\longrightarrow y^*$, therefore, we have

$$
\begin{aligned}
x^*(t)=&\sum_{i=1}^{n}\mathcal{I}_{a^+}^{\beta_i;\psi}h_i(t,x(t))+\mathcal{I}_{a^+}^{\alpha;\psi}v^*(t)\\
&+\frac{(\psi(t)-\psi(a))^{\gamma-1}}{(\psi(b)-\psi(a))^{\gamma-1}}\left(\frac{m(x)}{g(b,m(x))}\right.\\
&\left.-\sum_{i=1}^{n}\mathcal{I}_{a^+}^{\beta_i;\psi}h_i(b,m(x))-\mathcal{I}_{a^+}^{\alpha;\psi}v^*(b)\right),\quad t\in[a,b].
\end{aligned}
\tag{56}
$$

As a result, we have that $\mathcal{B}_1$ is a compact and upper semicontinuous operator on X.

Step 4. We show that the condition (c) of Lemma 14 holds, that is, $kM<1/2$. This is obvious by (44).

Step 5. Finally, we show that the conclusion (ii) of Lemma 14 does not hold.

Let x be any solution of the boundary value problem (9) such that $\lambda x\in\mathcal{A}x\mathcal{B}_1x$ for some $\lambda>1$. Then, there is a $v\in S_{F,x}$ such that

$$
\begin{aligned}
x(t)=&\frac{1}{\lambda}g(t,x(t))\left\{\sum_{i=1}^{n}\mathcal{I}_{a^+}^{\beta_i;\psi}h_i(t,x(t))+\mathcal{I}_{a^+}^{\alpha;\psi}v(t)\right.\\
&+\frac{(\psi(t)-\psi(a))^{\gamma-1}}{(\psi(b)-\psi(a))^{\gamma-1}}\left(\frac{m(x)}{g(b,m(x))}\right.\\
&\left.\left.-\sum_{i=1}^{n}\mathcal{I}_{a^+}^{\beta_i;\psi}h_i(b,m(x))-\mathcal{I}_{a^+}^{\alpha;\psi}v(b)\right)\right\},\quad t\in[a,b].
\end{aligned}
\tag{57}
$$

Then, we have

$$
\begin{aligned}
|x(t)| \le{}& |g(t,x(t))| \Bigg\{ \sum_{i=1}^{n} \mathscr{I}_{a^+}^{\beta_i;\psi} |h_i(t,x(t))| \\
&+ \mathscr{I}_{a^+}^{\alpha;\psi} |v(t)| + \frac{|m(x)|}{|g(b,m(x))|} \\
&+ \sum_{i=1}^{n} \mathscr{I}_{a^+}^{\beta_i;\psi} |h_i(b,m(x))| + \mathscr{I}_{a^+}^{\alpha;\psi} |v(b)| \Bigg\} \\
\le{}& \|v\| \Bigg\{ 2\|q\| \frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} \\
&+ 2\sum_{i=1}^{n} \frac{(\psi(b)-\psi(a))^{\beta_i}}{\Gamma(\beta_i+1)} \|\lambda_i\| + \mathscr{K} \Bigg\}.
\end{aligned} \tag{58}
$$

Taking the supremum for $t \in [a,b]$ of the above inequality, we obtain a constant $M > 0$ such that

$$
\begin{aligned}
\|x\| \le \|v\| \Bigg\{ & 2\|q\| \frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} \\
&+ 2\sum_{i=1}^{n} \frac{(\psi(b)-\psi(a))^{\beta_i}}{\Gamma(\beta_i+1)} \|\lambda_i\| + \mathscr{K} \Bigg\} := M,
\end{aligned} \tag{59}
$$

which means that the set $\mathscr{E} = \{x \in \mathbb{X} : \lambda x \in \mathscr{A} x \mathscr{B}_1 x, \lambda > 1\}$ is bounded.

As a result, the conclusion (ii) of Lemma 14 does not hold. Hence, the conclusion (i) holds, and consequently, the boundary value problem (9) has at least one solution on $[a,b]$. This completes the proof.□ □

Example 17. Consider the boundary value problem of the form

$$
\begin{cases}
{}^{H}\mathfrak{D}_{1/4}^{7/4,3/5;\log_e(t^2+1)} \left[\dfrac{x(t)}{g(t,x(t))} - \mathscr{I}_{1/4}^{1/4;\log_e(t^2+1)} h_1(t,x(t)) - \mathscr{I}_{1/4}^{7/4;\log_e(t^2+1)} h_2(t,x(t)) - \mathscr{I}_{1/4}^{11/4;\log_e(t^2+1)} h_3(t,x(t)) \right] \in \left[0, = \dfrac{e^{-x^2(t)}}{4t+15} \left(\dfrac{x^{12}}{1+x^{12}} \right) + \dfrac{1}{(4t+3)^2} \right], & t \in \left[\dfrac{1}{4}, \dfrac{11}{4} \right], \\
x\left(\dfrac{1}{4}\right) = 0, x\left(\dfrac{11}{4}\right) = 5e^{-x^2\left(\frac{3}{4}\right)} + 2\dfrac{x^2(5/4)}{x^2(5/4)+1} + 7\sin^2 x\dfrac{7}{4} + 3\cos^6 x\dfrac{9}{4}.
\end{cases} \tag{60}
$$

where

$$
\begin{aligned}
g(t,x) &= \frac{8t}{4t+1} \left(\frac{x^2 + 2|x|}{1+|x|} \right) + 4t + 7, h_1(t,x) \\
&= \frac{|x|}{(8t+1)(|x|+1)}, h_2(t,x) \\
&= \frac{\sin^2 x}{(12t+2)}, h_3(t,x) = \frac{1-e^{-|x|}}{(16t+3)}.
\end{aligned} \tag{61}
$$

Here, $\alpha = 7/4$, $\rho = 3/5$, $\psi(t) = \log_e(t^2+1)$, $a = 1/4$, $b = 11/4$, $\beta_1 = 1/4$, $\beta_2 = 7/4$, $\beta_3 = 11/4$, $m(x) = 5e^{-x^2} + (2x^2/(x^2+1)) + 7\sin^2 x + 3\cos^6 x$. Now, we find that $|h_1(t,x)| \le (1/(8t+1)) := \lambda_1(t)$, $|h_2(t,x)| \le (1/(12t+2)) := \lambda_2(t)$, and $|h_3(t,x)| \le (1/(16t+3)) := \lambda_3(t)$ which yield $\|\lambda_1\| = 1/3$, $\|\lambda_2\| = 1/5$, $\|\lambda_3\| = 1/7$. In addition, we have

$$
\begin{aligned}
&\frac{|m(x)|}{|g(11/4, m(x))|} \le \frac{17}{18} := \mathscr{K}, \\
&\|F(t,x)\|_{\mathscr{P}} \le \frac{1}{4t+15} + \frac{1}{(4t+3)^2} := q(t),
\end{aligned} \tag{62}
$$

which implies $\|q\| = 1/8$. Hence,

$$
\Omega \approx 0.4726347379 < \frac{1}{2}. \tag{63}
$$

Therefore, by applying Theorem 16, the ψ-Hilfer hybrid fractional integro-differential nonlocal boundary value problem (60)-(61) has at least one solution on $[1/4, 11/4]$.

5. Special Cases

The problem (8) considered in the present work is general in the sense that it includes the following classes of new boundary value problems of ψ-Hilfer fractional differential equations.

(I) Let $g(t,x) = 1$ and $h_i(t,x) = 0, i = 1, 2, \cdots, n$ for all $t \in [a,b]$ and $x \in \mathbb{R}$. Then, the problem (8) reduces to the following ψ-Hilfer fractional boundary value problem

$$
\begin{cases}
{}^{H}\mathfrak{D}_{a^+}^{\alpha,\rho;\psi} x(t) = f(t,x(t)), & t \in [a,b], \\
x(a) = 0, x(b) = m(x).
\end{cases} \tag{64}
$$

In the case when $\mu(x) = \sum_{i=1}^{m} \lambda_i x(\theta_i)$, where $a < \theta_1 < \theta_2 < \cdots < \theta_n < b$, the corresponding ψ-Hilfer fractional boundary value problem was studied in [34] for $k = 0$.

(II) Let $h_i(t,x) = 0, i = 1, 2, \cdots, n$ for all $t \in [a,b]$ and $x \in \mathbb{R}$. Then, the problem (8) reduces to the following ψ-Hilfer fractional boundary value problem

$$\begin{cases} {}^{H}\mathfrak{D}_{a^+}^{\alpha,\rho;\psi}\left[\dfrac{x(t)}{g(t,x(t))}\right] = f(t,x(t)), t \in [a,b], \\ x(a) = 0, x(b) = m(x). \end{cases} \tag{65}$$

(III) Let $g(t,x) = 1$ for all $t \in [a,b]$ and $x \in \mathbb{R}$. Then, the problem (8) reduces to the following ψ-Hilfer fractional boundary value problem

$$\begin{cases} {}^{H}\mathfrak{D}_{a^+}^{\alpha,\rho;\psi}\left[x(t) - \sum_{i=1}^{n} \mathcal{I}_{a^+}^{\beta_i;\psi} h_i(t,x(t))\right] = f(t,x(t)), \quad t \in [a,b], \\ x(a) = 0, x(b) = m(x). \end{cases} \tag{66}$$

Therefore, the main result of this paper also includes the existence results for the solutions of the above-mentioned ψ-Hilfer boundary value problems of fractional differential equations as special cases.

6. Conclusions

In this paper, we studied a new class of ψ-Hilfer hybrid fractional integro-differential boundary value problems with nonlocal boundary conditions. After proving a basic lemma, helping us to transform the considered system into a fixed-point problem, an existence result is proved via a fixed-point theorem for the product of two operators due to Dhage [21]. The multivalued analogue is also studied, and an existence theorem was established with the help of a multivalued fixed-point theorem for the product of two operators due to Dhage [38]. Numerical examples illustrating the obtained results are also presented. Some special cases are also discussed. The obtained results are new and enrich the existing literature on hybrid fractional differential equations and inclusions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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