Research Article

Lie Symmetry Analysis of $C_1(m, a, b)$ Partial Differential Equations

Hengtai Wang,¹ Aminu Ma'aruf Nass²,² and Zhiwei Zou¹

¹School of Mathematics and Physics, University of South China, Hengyang, 421001 Hunan, China

²Department of Actuarial Science, Federal University, Dutse, P.M.B 7156 Jigawa State, Nigeria

Correspondence should be addressed to Aminu Ma'aruf Nass; aminu.nass@fud.edu.ng

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In this article, we discussed the Lie symmetry analysis of $C_1(m, a, b)$ fractional and integer order differential equations. The symmetry algebra of both differential equations is obtained and utilized to find the similarity reductions, invariant solutions, and conservation laws. In both cases, the symmetry algebra is of low dimensions.

1. Introduction

In the last century, fractional partial differential equations (FPDEs) have played important roles in the fields of science and engineering, for instance, physics, chemistry, biology, and control theory. Recently, those class of differential equations has also attracted much more interest of mathematicians and physicists [1–6].

Finding the best methods of obtaining the exact solutions of differential equations remains one of the unanswered questions in the field. Many approaches have been developed by mathematicians to study the solutions of PFDEs, such as Adomian decomposition method, the fractional subequation method, numerical method, the first integral method, and Lie symmetry method [7–14]. In this article, we consider one of the powerful techniques of solving and analyzing differential equations, i.e., the Lie symmetry method. The Lie symmetry method is widely used to transform partial differential equations (PDEs) into ordinary differential equations (ODEs), and the ODE is later solved numerically or analytically using similarity invariant [7, 9, 10, 12, 14–22]. Lie symmetry is also utilized in obtaining the conservation laws (CLs) [23]. The method developed by Noether theorem [24] and Ibragimov's [25] is one of the best and simplest methods of evaluating CLs of differential equations.

Consider general forms of fractional differential equations:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = F[u], \quad (1)$$

where $u = u(x, t)$ denotes the unknown function, and $F[u] = F(x, u, u_x, u_{xx}, \dots)$ is a known function. The fractional order α is a real number and $\partial^\alpha u / \partial t^\alpha$ denotes Riemann-Liouville (R-L) derivative defined in [1, 3, 26] as

$$D_t^\alpha u = \begin{cases} \frac{\partial^n u}{\partial t^n}, & \alpha = n \in \mathbb{N}, \\ \frac{1}{\Gamma(n - \alpha)} \frac{\partial^n}{\partial t^n} \int_0^t \frac{u(\tau, x)}{(t - \tau)^{\alpha + 1 - n}} d\tau, & n - 1 < \alpha < n, n \in \mathbb{N}, \end{cases} \quad (2)$$

where $\Gamma(x)$ denotes the standard gamma function defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt. \quad (3)$$

The $C_N(m, a, b)$ partial differential equations

$$u_t + (u^m)_x + \frac{1}{b} \left[u^a \nabla^2 u^b \right]_x = 0, m > 1, 1 < n = a + b, \quad (4)$$

where N denotes the spatial dimension, have been introduced in [27]. In particular, if $N = 1$, it becomes

$$u_t + (u^m)_x + \frac{1}{b} \left[u^a (u^b)_{xx} \right]_x = 0, \quad (5)$$

which is equivalent to

$$u_t + mu^{m-1}u_x + (b-1)(a+b-2)u^{a+b-3}u_x^3 + (a+3b-3)u^{a+b-2}u_x u_{xx} + u^{a+b-1}u_{xxx} = 0. \quad (6)$$

Therefore, the fractional form of $C_1(m, a, b)$ differential equation is define as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = -mu^{m-1}u_x - (b-1)(a+b-2)u^{a+b-3}u_x^3 - (a+3b-3)u^{a+b-2}u_x u_{xx} - u^{a+b-1}u_{xxx}, \quad (7)$$

where $\partial^\alpha u / \partial t^\alpha$ is given by (2).

2. Lie Symmetries of Eq. (6)

In this section, we first consider the Lie symmetry analysis of Eq.(6). To obtained the Lie symmetry analysis, we first consider a one-parameter Lie group of transformations

$$\begin{aligned} \bar{x} &= x + \varepsilon \xi(x, t, u) + O(\varepsilon^2), \\ \bar{t} &= t + \varepsilon \tau(x, t, u) + O(\varepsilon^2), \\ \bar{u} &= u + \varepsilon \phi(x, t, u) + O(\varepsilon^2), \end{aligned} \quad (8)$$

with a small parameter $\varepsilon \ll 1$. The vector field associated with the one-parameter group of transformation is

$$V = \xi(x, t, u) \frac{\partial}{\partial x} + \tau(x, t, u) \frac{\partial}{\partial t} + \phi(x, t, u) \frac{\partial}{\partial u}. \quad (9)$$

Thus, expanding the infinitesimals generator to include the transformation of the derivatives, we obtained the following third prolongation $pr^{(3)}V$

$$pr^{(3)}V = V + \phi^x \frac{\partial}{\partial u_x} + \phi^t \frac{\partial}{\partial u_t} + \phi_{xx} \frac{\partial}{\partial u_{xx}} + \phi^{xxx} \frac{\partial}{\partial u_{xxx}}. \quad (10)$$

In (10), ϕ^x , ϕ^t , ϕ^{xx} , and ϕ^{xxx} are all undetermined functions, which are given by the following formulae

$$\phi^x = D_x(\phi - \xi u_x - \tau u_t) + \xi u_{xx} + \tau u_{xt}, \quad (11)$$

$$\phi^t = D_t(\phi - \xi u_x - \tau u_t) + \xi u_{xt} + \tau u_{tt}, \quad (12)$$

$$\phi^{xx} = D_x^2(\phi - \xi u_x - \tau u_t) + \xi u_{xxx} + \tau u_{xxt}, \quad (13)$$

$$\phi^{xxx} = D_x^3(\phi - \xi u_x - \tau u_t) + \xi u_{xxx} + \tau u_{xxx}, \quad (14)$$

where D_x and D_t represent the total derivatives with respect to x and t , respectively. e refers the reader(s) to [28] for details of how to evaluate the prolongation formulae.

If the vector field (9) forms a symmetry of Eq.(6), the infinitesimal generator V must satisfy the following invariance criterion for Eq.(6), given as

$$pr^{(3)}V(\Delta) \Big|_{\Delta=0} = 0, \quad (15)$$

where

$$\Delta = u_t + mu^{m-1}u_x + (b-1)(a+b-2)u^{a+b-3}u_x^3 + (a+3b-3)u^{a+b-2}u_x u_{xx} + u^{a+b-1}u_{xxx}. \quad (16)$$

Substituting Eqs. (11)–(14) into Eq. (15) and equating the coefficients of various powers of partial derivatives of u to zero, an overdetermined system of equations known as determining equations is obtained. Solving the determining equations the following infinitesimals for $C_1(m, a, b)$ has been derived

$$\begin{aligned} \xi(x, t, u) &= pc_1 x + c_2, \\ \tau(x, t, u) &= qc_1 t + c_3, \\ \phi(x, t, u) &= c_1 u, \end{aligned} \quad (17)$$

where c_1 , c_2 , and c_3 are arbitrary constants, and throughout this paper, we denote

$$p = \frac{a+b-m}{2}, q = \frac{a+b-3m+2}{2}. \quad (18)$$

Therefore, we have the following conclusion.

Theorem 1. For the arbitrary parameters m, a, b , if $m > 1$, $a+b > 1$, the vector field admitted by the differential equation (6) is

$$V_1 = \partial_x, V_2 = \partial_t, V_3 = px\partial_x + qt\partial_t + u\partial_u. \quad (19)$$

The sketch of the proof has been stated above. In details, we should divide it into the following four cases. Obviously, the vector fields of each case are the special cases of (19)

(i) If $a+b=2$, $a+3b \neq 3$, the vector field is

$$V_1 = \partial_x, V_2 = \partial_t, V_3 = \left(1 - \frac{m}{2}\right)x\partial_x + \left(2 - \frac{3m}{2}\right)t\partial_t + u\partial_u \quad (20)$$

(ii) If $a+b=2$, $a+3b=3$, the vector field is the same as (i)

(iii) If $a + b \neq 2$, $a + 3b = 3$, the vector field is

$$V_1 = \partial_x, V_2 = \partial_t, V_3 = px\partial_x + qt\partial_t + u\partial_u \quad (21)$$

(iv) If $a + b \neq 2$, $a + 3b \neq 3$, $b \neq 1$, the vector field is the same as (iii).

The vector fields V_1 , V_2 , and V_3 form a Lie algebra under the following Lie bracket

$$[f\partial_i, g\partial_j] = f\partial_i(g)\partial_j - g\partial_j(f)\partial_i, \quad (22)$$

where i and j stand for x, t, u .

That is to say

$$[V_1, V_2] = 0, \quad [V_1, V_3] = pV_3, \quad [V_2, V_3] = qV_3. \quad (23)$$

By solving the following ordinary differential equations with the initial conditions:

$$\begin{aligned} \frac{dx^*}{d\varepsilon} &= \xi(x^*, t^*, u^*), x^*|_{\varepsilon=0} = x, \\ \frac{dt^*}{d\varepsilon} &= \tau(x^*, t^*, u^*), t^*|_{\varepsilon=0} = t, \\ \frac{du^*}{d\varepsilon} &= \phi(x^*, t^*, u^*), u^*|_{\varepsilon=0} = u. \end{aligned} \quad (24)$$

We therefore obtain the group transformation which is generated by infinitesimal generators V_1, V_2, V_3 , respectively

$$G_1 : (x, t, u) \mapsto (x + \varepsilon, t, u), \quad (25)$$

$$G_2 : (x, t, u) \mapsto (x, t + \varepsilon, u), \quad (26)$$

$$G_3 : (x, t, u) \mapsto (e^{p\varepsilon}x, e^{q\varepsilon}t, e^\varepsilon u). \quad (27)$$

Remark 2. In (25)–(27), an arbitrary element in G_i ($i = 1, 2, 3$) can transfer one solution of Eq. (6) to another one, so do the products of the elements from G_1, G_2 , and G_3 .

Remark 3. The Lie group $G_1 \times G_2$ is a normal Lie subgroup of $G_1 G_2 G_3$. The Lie algebra generated by V_1 and V_2 is an ideal of L .

Theorem 4. If $u = f(x, t)$ is a solution of Eq. (1.3), then $u^{(1)}$, $u^{(2)}$, and $u^{(3)}$ as follows are solutions of Eq. (6) as well.

$$\begin{aligned} u^{(1)} &= f(x - \varepsilon, t), \\ u^{(2)} &= f(x, t - \varepsilon), \\ u^{(3)} &= e^\varepsilon f(e^{-p\varepsilon}x, e^{-q\varepsilon}t). \end{aligned} \quad (28)$$

3. Similarity Reductions to Eq. (6)

In the preceding section, we obtained the group symmetry analysis of Eq. (6). In this section, the characteristic equa-

tions of vector fields are obtained by making use of (19) and utilized to perform the symmetry reduction.

For $V = V_1$, the characteristic equation can be presented as follows:

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0}. \quad (29)$$

Solving the characteristic equation (29), we have $u = f(\xi)$, where $\xi = t$ which yields a trivial solution.

For $V = V_2$, the characteristic equation can be expressed as follows:

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{0}, \quad (30)$$

from which we have $u = f(\xi)$, where $\xi = x$ which transformed (6) to an ODE

$$\begin{aligned} m^{f^{m-1}f'} + (b-1)(a+b-2)f^{a+b-3}f'^3 \\ + (a+3b-3)f^{a+b-2}f'f'' + f^{a+b-1}f''' = 0. \end{aligned} \quad (31)$$

For $V = V_3$, the characteristic equation can be written as follows:

$$\frac{dx}{px} = \frac{dt}{qt} = \frac{du}{u}. \quad (32)$$

Solving the characteristic equation, we have the following invariants $\xi = xt^{-p/q}$, $u = t^{1/q}f(\xi)$, which transformed (6) to an ODE

$$\begin{aligned} \frac{1}{q}t^{\frac{1}{q}-1}f - \frac{p}{q}\left(xt^{-\frac{p}{q}}\right)t^{\frac{1}{q}-1} - f' + mt^{\frac{p}{q}-1}f^{m-1}f' \\ + (b-1)(a+b-2)t^{\frac{a+b-2m}{q}}f^{a+b-3}f'^3 \\ + (a+3b-3)t^{\frac{a+b-3p}{q}}f'f'' + t^{\frac{a+b-3p}{q}}f'f'' \\ + t^{\frac{a+b-3p}{q}}f^{a+b-1}f''' = 0. \end{aligned} \quad (33)$$

4. Nonlinear Self-Adjointness of Eq. (6)

In this section, we shall show that Eq. (6) is nonlinearly self-adjoint. Let it start by presenting the definitions of nonlinear self-adjointness of differential equations according to Ibragimov's [13, 25].

Definition 5. Given a differential function F and the new dependent variable $v = v(x)$ known as the adjoint variable or local variable [13, 25], the formal Lagrangian for the differential equation $F = 0$ is the differential function given by

$$F^* := vF. \quad (34)$$

Definition 6. [25]. The differential equation (6) is said to be nonlinearly selfadjoint if there exists a substitution

$$v = \phi(x, t, u) \neq 0, \quad (35)$$

such that

$$F^* \Big|_{v=\phi(x,t,u)} = \lambda F, \quad (36)$$

for some undermine function λ .

Theorem 7. The differential equation (6) is self-adjoint.

Proof. The formal Lagrangian is

$$\begin{aligned} \mathcal{L} = v \Big(& u_t + mu^{m-1}u_x + (b-1)(a+b-2)u^{a+b-3}u_x^3 \\ & + (a+3b-3)u^{a+b-2}u_xu_{xx} + u^{a+b-1}u_{xxx} \Big). \end{aligned} \quad (37)$$

Substituting into (36) = 0, we have the adjoint equation to Eq. (6)

$$\begin{aligned} -mu^{m-1}v_{x-a}(a+b-2)u^{a+b-3}v_xu_x^2 - 2au^{a+b-2}v_xu_{xx} \\ - v_t - 2au^{a+b-2}v_{xx}u_x - u^{a+b-1}v_{xxx} = 0. \end{aligned} \quad (38)$$

Let $v = \phi(x, t, u)$ and the left hand of (38) be $\lambda \cdot \Delta$, we shall get $\lambda = -\phi_u$, and

$$\begin{cases} \varphi = c_1u + c_2, b = a + 1, \\ \varphi = c_3u^{b-a-1}, b \neq a + 1. \end{cases} \quad (39)$$

Then, we prove that Eq. (6) is self-adjoint. $\square$

Generally speaking, we are now to calculate the conserved vectors. However, by using the method of Ibragimov [25], so we here omit the process.

5. Lie Symmetry and Reductions of Eq. (7)

In this section, we deal with all of the point symmetries of Eq. (7). We now assume that $n-1 < \alpha < n$ in this section. Thus, Eq. (1.4) is an FPDE with the infinitesimal generator given by (2.2). If the vector field (9) generates a symmetry of Eq. (7), then V must satisfy the following Lie symmetry condition:

$$pr^{(3)}V(\Delta_1) \Big|_{\Delta_1=0} = 0, \quad (40)$$

where

$$\begin{aligned} \Delta_1 = \frac{\partial^\alpha u}{\partial t^\alpha} + mu^{m-1}u_x + (b-1)(a+b-2)u^{a+b-3}u_x^3 \\ + (a+3b-3)u^{a+b-2}u_xu_{xx} + u^{a+b-1}u_{xxx}. \end{aligned} \quad (41)$$

Also, the invariant condition yields [29]

$$\tau(x, t, u) \Big|_{t=0} = 0, \quad (42)$$

and the α th extended infinitesimal related to Riemann-Liouville fractional time derivative with (42) is given by [30, 31].

$$\begin{aligned} \phi_\alpha^0 = \frac{\partial^\alpha \phi}{\partial t^\alpha} + (\phi u - \alpha D_t(\tau)) \frac{\partial^\alpha u}{\partial t^\alpha} - u \frac{\partial^\alpha \phi_u}{\partial t^\alpha} + \mu \\ - \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n}(u_x) \\ + \sum_{n=1}^{\infty} \left[\binom{\alpha}{n} \frac{\partial^\alpha \phi_u}{\partial t^\alpha} - \binom{\alpha}{n+1} D_t^{n+1}(\tau) \right] D_t^{\alpha-n}(u), \end{aligned} \quad (43)$$

where

$$\begin{aligned} \mu = \sum_{n=2}^{\infty} \sum_{m=2}^n \sum_{k=2}^m \sum_{r=0}^{k-1} \binom{\alpha}{n} \binom{n}{m} \binom{k}{r} \frac{1}{k!} \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} \\ \cdot [-u]^{n-\alpha} \frac{\partial}{\partial t^m} \times [u^{k-r}] \frac{\partial^{n-m+k} \phi}{\partial t^{n-m} \partial u^k}. \end{aligned} \quad (44)$$

The expression of μ is complicated; however, it should converges to zero when the infinitesimal φ is linear in u , because of the existence of the derivatives $\partial^{n-m+k}/\partial t^{n-m} \partial u^k$, $k \geq 2$ in the above expression (44).

Thus, the Lie group classification method for the FPDE leads to the following result.

Theorem 8. The infinitesimal symmetry group of the equation (7) is spanned by the two vector fields

$$V_1 = \partial_x \quad V_2 = px \partial_x + \frac{1}{\alpha} qt \partial_t + u \partial_u. \quad (45)$$

Proof. Considering the invariance criterion (40), we have

$$\begin{aligned} \phi_\alpha^0 + m(m-1)u^{m-2}\phi u_x + mu^{m-1}\phi^x + (b-1)(a+b-2) \\ \cdot (a+b-3)u^{a+b-4}\phi u_x^3 + 3(b-1)(a+b-2)u^{a+b-3}\phi^x u_x^2 \\ + (a+3b-3)(a+b-2)u^{a+b-3}\phi u_x u_{xx} + (a+3b-3)\phi_{u_{xx}}^x \\ + (a+3b-3)\phi^{xx} u_x + (a+b-1)\phi u^{a+b-2} u_{xxx} + u^{a+b-1} \phi^{xxx} = 0 \end{aligned} \quad (46)$$

solving the determining equation (46), and we obtained the following infinitesimals

$$\begin{aligned} \xi(x, t, u) &= pc_1x + c_2, \\ \tau(x, t, u) &= \frac{1}{\alpha} qc_1t, \\ \phi(x, t, u) &= c_1u, \end{aligned} \quad (47)$$

which follows that the fractional differential equation (7) admitted two dimensional symmetries and is spanned by (45). $\square$

Next, we utilized the admitted Lie symmetry and perform similarity reductions, present the reduced nonlinear fractional ordinary differential equations (FODEs), and classify the corresponding group-invariant solutions of the fractional $C_1(m, a, b)$.

Case 9. For $V_1 = \partial/\partial x$, the characteristic equation is

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0}, \quad (48)$$

and the following similarity variables are obtained by solving the characteristic equation $u = f(\zeta)$, $\zeta = t$, which yields the following reduced ODE

$$\partial_t^\alpha f(t) = 0. \quad (49)$$

The fractional ODE has a polynomial general group-invariant solution

$$u = a_1 t^{\alpha-1}, \quad (50)$$

where a_1 is an arbitrary constant of integration [6, 15, 16].

Case 10. For $V_2 = px\partial_x + 1/2qt\partial_t + u\partial_u$, the characteristic equation is

$$\frac{dx}{px} = \frac{\alpha dt}{qt} = \frac{du}{u}, \quad (51)$$

and by solving the above equation, we get the group-invariant solution

$$u = t^{\frac{\alpha}{q}} g(\zeta), \quad \zeta = xt^{-\frac{px}{q}}, \quad (52)$$

where g is an arbitrary function of ζ . Using these invariants, Eq. (7) transforms to a special nonlinear ODE of fractional order. Thus, we have the following theorem corresponding to this case.

Theorem 11. *The transformation (52) reduces (7) to the following nonlinear ordinary differential equation of fractional order with the Erd'elyi-Kober fractional differential operator $P_{\beta}^{\tau, \alpha}$ of order [5]*

$$\left(P_{\beta}^{\tau, \alpha, n-\alpha} g\right) := \prod_{j=0}^{n-1} \left(\tau + j - \frac{1}{\beta} \zeta \frac{d}{d\zeta}\right) \left(K_{\beta}^{\tau, \alpha} g\right)(\zeta), \quad n = \begin{cases} |\alpha| + 1, & \alpha \notin \mathbb{N} \\ \alpha, & \alpha \in \mathbb{N} \end{cases}, \quad (53)$$

where

$$\left(K_{\beta}^{\tau, \alpha} g\right)(\zeta) := \begin{cases} \frac{1}{\Gamma(\alpha)} \int_1^{\infty} (u-1)^{\alpha-1} u^{-u(\tau+\alpha)} g\left(\zeta u^{\frac{1}{\beta}}\right) du, & \alpha > 0, \\ g(\zeta), & \alpha = 0. \end{cases} \quad (54)$$

is the Erd'elyi-Kober fractional integral operator.

Proof. Let $n-1 < \alpha < n$, $n = 1, 2, 3, \dots$, according to the Riemann-Liouville fractional derivative, and one can obtain

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n} \left[\frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} s^{\frac{\alpha}{q}} g\left(xs^{-\frac{px}{q}}\right) ds \right]. \quad (55)$$

Let $v = t/s$, and then $ds = -(t/v^2)dv$; so, (55) can be written as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n} \left[\frac{1}{\Gamma(n-\alpha)} \int_1^{\infty} (v-1)^{n-\alpha-1} v^{-(n-\alpha+\frac{\alpha}{q}+1)} g\left(\zeta v^{\frac{px}{q}}\right) dv \right]. \quad (56)$$

In view of the Erd'elyi-Kober fractional integral operator (54), one can get

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n} \left[t^{n-\alpha+\frac{\alpha}{q}} \left(K_{\frac{q}{px}}^{1+\frac{\alpha}{q}, n-\alpha} g\right)(\zeta) \right]. \quad (57)$$

Therefore, the right hand side of (57) becomes

$$\begin{aligned} \frac{\partial^n}{\partial t^n} \left[t^{n-\alpha+\frac{\alpha}{q}} \left(K_{\frac{q}{px}}^{1+\frac{\alpha}{q}, n-\alpha} g\right)(\zeta) \right] &= \frac{\partial^{n-1}}{\partial t^{n-1}} \left[\frac{\partial}{\partial t} \left(t^{n-\alpha+\frac{\alpha}{q}} \left(K_{\frac{q}{px}}^{1+\frac{\alpha}{q}, n-\alpha} g\right)(\zeta) \right) \right] \\ &= \frac{\partial^{n-1}}{\partial t^{n-1}} \left[t^{n-\alpha+\frac{\alpha}{q}-1} \left(n-\alpha + \frac{\alpha}{q} - 1 - \frac{px}{q} \zeta \frac{d}{d\zeta} \right) \left(K_{\frac{q}{px}}^{1+\frac{\alpha}{q}, n-\alpha} g\right)(\zeta) \right]. \end{aligned} \quad (58)$$

Repeating the same procedure $n-1$ times, one can obtain

$$\begin{aligned} \frac{\partial^n}{\partial t^n} \left[t^{n-\alpha+\frac{\alpha}{q}} \left(K_{\frac{q}{px}}^{1+\frac{\alpha}{q}, n-\alpha} g\right)(\zeta) \right] &= t^{-\alpha+\frac{\alpha}{q}} \prod_{j=0}^{n-1} \\ &\cdot \left(1-\alpha + \frac{\alpha}{q} + j - \frac{px}{q} \zeta \frac{d}{d\zeta} \right) \left(K_{\frac{q}{px}}^{1+\frac{\alpha}{q}, n-\alpha} g\right)(\zeta). \end{aligned} \quad (59)$$

Using the definition of the Erd'elyi-Kober fractional differential operator (53), we get

$$\frac{\partial^n}{\partial t^n} \left[t^{n-\alpha+\frac{\alpha}{q}} \left(K_{\frac{q}{px}}^{1+\frac{\alpha}{q}, n-\alpha} g\right)(\zeta) \right] = t^{-\alpha+\frac{\alpha}{q}} \left(P_{\frac{q}{px}}^{1-\alpha+\frac{\alpha}{q}, \alpha} g\right)(\zeta). \quad (60)$$

Substituting (60) into (57), we obtain an expression for the time fractional derivative

$$\frac{\partial^\alpha u}{\partial t^\alpha} = t^{-\alpha+\frac{\alpha}{q}} \left(P_{\frac{q}{px}}^{1-\alpha+\frac{\alpha}{q}, \alpha} g\right)(\zeta). \quad (61)$$

Thus, the fractional $C_1(m, a, b)$ equation (7) can be reduced into a fractional order.

ODE:

$$\begin{aligned} \left(P_{\frac{a}{p\alpha}}^{1-\alpha+\frac{a}{q}\alpha} g \right) (\zeta) = & -mg^{m-1}g_\zeta - (b-1)(a+b-2)g^{a+b-3}g_\zeta^3 \\ & - (a+3b-3)g^{a+b-2}g_\zeta^3 \\ & - (a+3b-3)g^{a+b-2}g_\zeta u_{\zeta\zeta} - g^{a+b-1}g_{\zeta\zeta\zeta}. \end{aligned} \quad (62)$$

□

6. Conservation Law of Eq. (7)

In this section, we will construct the conservation laws of the fractional $C_1(m, a, b)$ differential equation (7).

Let $\mathcal{L}$ be the formal Lagrangian of Eq. (7) written as $\mathcal{L} = v\Delta_1$, where $v = v(x, t)$ is the new introduced dependent variable; so, the adjoint equation of (1.4) is written as $\delta\mathcal{L}/\delta u = 0$, where $\delta/\delta u$ is the Euler-Lagrange operator with respect to u and defined by

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + (D_t^\alpha)^* \frac{\partial}{\partial D_t^\alpha u} + \sum_{k=1}^{\infty} (-1)^k D_{i_1} D_{i_2} \cdots D_{i_k} \frac{\partial}{\partial u_{i_1 i_2 \cdots i_k}}, \quad (63)$$

and $(D_t^\alpha)^*$ is the adjoint operator of D_t^α . For Riemann-Liouville fractional differential operators, we have

$$(D_t^\alpha)^* = (-1)^n I_r^{n-\alpha} (D_t^n) = \sum_r^C D_r^\alpha, \quad (64)$$

where

$$I_r^{n-\alpha} (D_t^n) = \frac{1}{\Gamma(n-\alpha)} \int_t^r (\tau-t)^{n-\alpha-1} f(x, \tau) d\tau, \quad n = [\alpha] + 1 \quad (65)$$

is the right-sided fractional integral operators of order α .

In this case, the formal Lagrangian of Eq. (7) is written as

$$\begin{aligned} \mathcal{L} = v(x, t) \left[\frac{\partial^\alpha u}{\partial t^\alpha} + mu^{m-1}u_x + (b-1)(a+b-2)u^{a+b-3}u_x^3 \right. \\ \left. + (a+3b-3)u^{a+b-2}u_x u_{xx} + u^{a+b-1}u_{xxx} \right]. \end{aligned} \quad (66)$$

Every Lie point symmetry (45) admitted by the Eq. (7)

leads to a conservation law $D_t(C^i) = 0$ where the components C^i constructed by the following formula [5, 32]:

$$\begin{aligned} C^t &= \sum_{k=0}^{n-1} (-1)^k D_t^{\alpha-1-k}(W) D^k \frac{\partial \mathcal{L}}{\partial D_t^\alpha u} - J \left(W, D_t^n \frac{\partial \mathcal{L}}{\partial D_t^\alpha u} \right), \\ C^x &= W \left[\frac{\partial \mathcal{L}}{\partial u_x} - D_x \left(\frac{\partial \mathcal{L}}{\partial u_{xx}} \right) + D_x^2 \left(\frac{\partial \mathcal{L}}{\partial u_{xxx}} \right) \right] \\ &\quad + D_x(W) \left[\frac{\partial \mathcal{L}}{\partial u_{xx}} - D_x \frac{\partial \mathcal{L}}{\partial u_{xxx}} \right] + D_x^2(W) \frac{\partial \mathcal{L}}{\partial u_{xxx}}, \end{aligned} \quad (67)$$

where $W = \phi - \xi u_x - \tau u_t$ and $\mathcal{L}$ are the formal Lagrangian which are written in the symmetric form, and the mixed derivatives J are given by the integral:

$$J(f, g) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \int_t^r f(x, \xi) g(x, \mu) (\mu - \xi)^{n-\alpha-1} d\mu d\xi, \quad (68)$$

for $0 < \alpha < 1$. For other cases, the calculations of conserved vector components are similar. In this case,

$$C^t = D_t^{\alpha-1}(W) \frac{\partial \mathcal{L}}{\partial D_t^\alpha u} - J \left(W, D_t \frac{\partial \mathcal{L}}{\partial D_t^\alpha u} \right). \quad (69)$$

The conserved vector components C^t, C^x are as follows:

$$\begin{aligned} C^t &= D_t^{\alpha-1}(W_i) v - J(W_{i,vt}), \\ C^x &= W_i \left[mvu^{m-1} + 3(b-1)(a+b-2)u^{a+b-3}vu_x^2 \right. \\ &\quad + (a+3b-3)u^{a+b-2}vu_{xx} - (a+3b-3)(a+b-2)u^{a+b-3}vu_x^2 \\ &\quad - (a+3b-3)u^{a+b-2}v_x u_x - (a+3b-3)u^{a+b-2}vu_{xx} \\ &\quad + (a+b-1)u^{a+b-2}vu_x + u^{a+b-1}v_x + (a+b-1) \\ &\quad \cdot (a+b-2)u^{a+b-3}vu_x^2 + 2(a+b-1)u^{a+b-2}v_x u_x \\ &\quad + (a+b-1)ua^{a+b-2}vu_{xx} + u^{a+b-1}v_{xx} \left. + D_x(W_i) \right. \\ &\quad \cdot \left[(a+3b-3)u^{a+b-2}vu_x - (a+b-1)u^{a+b-2}vu_x - u^{a+b-1}v_x \right] \\ &\quad \left. + D_x^2(W_i) \left(u^{a+b-1}v \right) \right]. \end{aligned} \quad (70)$$

where $W_1 = -u_x, W_2 = u - pxu_x - (1/\alpha)qtu_t$.

Data Availability

All the data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that this article has no conflicts of interest.

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