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Research Article

Evolutoids of the Mixed-Type Curves

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The evolutoid of a regular curve in the Lorentz-Minkowski plane $\mathbb{R}_1^2$ is the envelope of the lines between tangents and normals of the curve. It is regarded as the generalized caustic (evolute) of the curve. The evolutoid of a mixed-type curve has not been considered since the definition of the evolutoid at lightlike point can not be given naturally. In this paper, we devote ourselves to consider the evolutoids of the regular mixed-type curves in $\mathbb{R}_1^2$. As the angle of lightlike vector and nonlightlike vector can not be defined, we introduce the evolutoids of the nonlightlike regular curves in $\mathbb{R}_1^2$ and give the conception of the σ-transform first. On this basis, we define the evolutoids of the regular mixed-type curves by using a lightcone frame. Then, we study when does the evolutoid of a mixed-type curve have singular points and discuss the relationship of the type of the points of the mixed-type curve and the type of the points of its evolutoid.

1. Introduction

The caustic, also called evolute, is an important study object in physics and nonlinear sciences. Compared with the neighbouring spaces, the field strength on caustics increases sharply; thus, many interesting phenomena emerged. The caustics have wide applications in many other research fields, such as optics, mechanics, and electromagnetism; thus, they have drawn the attention of many scientists. In [1], in order to confirm the location of caustic regression points, the researchers gave a new geometric variational criterion. Besides, computing parametric equations of the caustic were solved, in whatever Cartesian coordinate system. The light focuses are not on caustic as usually what is said in catastrophe optics but on regions that are defined as second order evolutes was proofed in [2]. When a particle is compelled to slide on tautochrone curves, it is isochronal to the gained oscillation is. If two guides are incorporated for the pendulum string to make its length short forcing the particle to follow the tautochrone, these guides were exactly connected with the evolute of the pendulum trajectory. In [3], a special isochronous pendulum that is asymmetric, in which a part of the trajectory accords with the nonisochronous simple pendulum and the rest of the part corresponds to a peculiar trajectory, was considered. It is in the singular points of the evolute consistent with the limitations of the edge segments and in the points where some walls meet that the limitations of domain walls appears (see [4]). Because of the importance of the caustics, more and deeper studies of them are necessary.

From the geometric point of view, the evolute of a curve in the Euclidean plane is usually defined by the locus of centres of osculating circles of the base curve or the envelope of the normal lines of the base curve. As it is an important object of studying in classical differential geometry, it is widely considered by many scholars (see [5, 6]). Since the singular curves are more familiar conditions in the natural world, there are more and more studies about the evolutes of the curves with singular points (see [5, 7, 8]). Also, the concept of evolutes is generalized to hyperbolic, pseudosphere, lightcone, Lorentz space, and some other spaces, to show the application about the singularity theory (see [9–12]).

As we all know, in a plane, it is a curve tangent to all of the family of lines the envelope of them. The evolute of a curve can be regarded as the envelope of the normal lines, and the envelope of the tangent lines is the curve itself. But how about the condition between the envelope of tangents and the envelope of normals? In fact, it is precisely the evolutoid of the curve. As a generalization of the evolutes, the studies of the evolutoids have great significance. There have been some studies about the evolutoids of the curves. The evolutoids of the regular curves in $\mathbb{R}^2$ were considered, and the family of parallels associated with the evolutoids was studied by Giblin and Warder [13]. In [14], Izumiya and Takeuchi considered the condition when the curves have singular points. They introduced the evolutoids of frontals in $\mathbb{R}^2$ and gave the relationship of the evolutoids and pedaloids of frontals. In [15], Aydın Şekerci and Izumiya studied the evolutoids of nonlightlike curves including not only regular condition but also the condition with singular points in $\mathbb{R}^2_1$.

In addition, because the Lorentz space is in connection with general relativity theory strongly, studying the Lorentz space has great significance, of course also including its subspaces. Plentiful relevant studies have appeared (see [9–11, 16–22]). As a subspace of the Lorentz space, the Lorentz-Minkowski plane, always denoted by $\mathbb{R}^2_1$, is interested by scholars. They take pains to study the curves in it. For the nonlightlike curves in $\mathbb{R}^2_1$, we always select their arc-length parameter and adopt Frenet-Serret frame to study them (see [23]). However, the mixed-type curves, which are consisted by three types of points, are more familiar conditions. But with regard to the mixed-type curves, since the appearance of lightlike points, the Frenet-Serret frame does not work and we can not study them through the old method. Until 2018, Izumiya et al. gave the lightcone frame and established the fundamental theory of the mixed-type curves in the Lorentz-Minkowski plane in [10]. As an application of the theory, they studied the evolutes of the regular mixed-type curves. The mixed-type curves with singular points were studied by us in [12]. The (n, m)-cusp mixed-type curves in $\mathbb{R}^2_1$ were considered, as well as the evolutes of the (n, m)-cusp mixed-type curves were given. In [24], we investigated the pedal curves of mixed-type curves in $\mathbb{R}^2_1$ and consider their properties. But as for the evolutoids of the mixed-type curves in $\mathbb{R}^2_1$, which is an interesting and worthy subject, nobody has been studying them.

In this paper, we focus on solving this question. We consider the evolutoids of the regular mixed-type curves in $\mathbb{R}^2_1$ and study their propositions. In Section 2, we review some essential knowledge about $\mathbb{R}^2_1$. In addition, to achieve the goal that defining the evolutoids of mixed-type curves, the evolutoids of the nonlightlike curves are introduced by us in this section first. In Section 3, we introduce a suitable frame for the mixed-type curves which is called lightcone frame in $\mathbb{R}^2_1$. Meanwhile, we define what the evolutoids of the mixed-type curves is. Then, we consider when the evolutoids of the curves in $\mathbb{R}^2_1$ have singular points. Also, we give the relationship of the types of the points on the evolutoid of a mixed-type curve in $\mathbb{R}^2_1$ and the types of the points on this base curve. Finally, for the purpose of showing the characteristics of the evolutoids of the mixed-type curves, we give two examples in Section 4.

If not specifically mentioned, all maps and manifolds in this paper are infinitely differentiable.

2. Preliminaries

2.1. Basic Conceptions in $\mathbb{R}^2_1$. Some essential knowledge about the Lorentz-Minkowski plane is given here to provide convenience for the following research.

Let $\mathbb{R}^2 = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$ be a 2-dimensional vector space. If $\mathbb{R}^2$ equipped with the metric which is induced by the *pseudo-scalar product*

$$\langle \mathbf{x}, \mathbf{y} \rangle = -x_1 y_1 + x_2 y_2, \tag{1}$$

where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$. Then, we call $\mathbb{R}^2_1 = (\mathbb{R}^2, \langle,\rangle)$ the Lorentz-Minkowski plane.

There are three types of vectors in $\mathbb{R}^2_1$. For a nonzero vector $\mathbf{x} \in \mathbb{R}^2_1$, when the pseudoscalar product of $\mathbf{x}$ and itself is positive, negative, and vanishing, it is called *spacelike*, *timelike*, or *lightlike*, respectively. A *nonlightlike* vector refers to the vector is spacelike or timelike.

For a vector $\mathbf{x} \in \mathbb{R}^2_1$, if there exists a vector $\mathbf{y} \in \mathbb{R}^2_1$, which satisfies $\langle \mathbf{x}, \mathbf{y} \rangle = 0$, we say $\mathbf{y}$ is pseudo-orthogonal to $\mathbf{x}$.

We define the *norm* of $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2_1$ by

$$\|\mathbf{x}\| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}, \tag{2}$$

and the pseudo-orthogonal complement of $\mathbf{x}$ is given by $\mathbf{x}^\perp = (x_2, x_1)$. By definition, $\mathbf{x}$ and $\mathbf{x}^\perp$ are pseudo-orthogonal to each other, and

$$\|\mathbf{x}\| = \|\mathbf{x}^\perp\|. \tag{3}$$

For a nonzero vector $\mathbf{x} \in \mathbb{R}^2_1$, it is obvious that $\mathbf{x}^\perp$ is spacelike (resp., timelike) if and only if $\mathbf{x}$ is timelike (resp., spacelike) and $\mathbf{x}^\perp = \pm\mathbf{x}$ if and only if $\mathbf{x}$ is lightlike.

Let $\boldsymbol{\rho} : I(\subseteq \mathbb{R}) \longrightarrow \mathbb{R}^2_1$ be a regular curve. Denote $\dot{\boldsymbol{\rho}}(t) = (d\boldsymbol{\rho}/dt)(t)$; then, we say $\boldsymbol{\rho}$ is a *spacelike* (resp., *timelike*, *lightlike*) curve if the pseudoscalar of $\dot{\boldsymbol{\rho}}(t)$ and itself is positive (resp., negative, vanishing) for any $t \in I$. Furthermore, the type of a point $\boldsymbol{\rho}(t)$ (or t) is determined by the type of $\dot{\boldsymbol{\rho}}(t)$. For more details, see [10].

Moreover, we say a curve is *nonlightlike* which means it is a spacelike or timelike curve and a point is *nonlightlike* if it is a spacelike or timelike point. If $\boldsymbol{\rho}(t)$ contains three types of points simultaneously, then, it is exactly a mixed-type curve, which is the main research object in this paper.

The lightcone in $\mathbb{R}^2_1$ with centre $\mathbf{p} \in \mathbb{R}^2_1$ is defined as

$$LC^*(\mathbf{p}, 0) = \{\mathbf{y} \in \mathbb{R}^2_1 \mid \langle \mathbf{y} - \mathbf{p}, \mathbf{y} - \mathbf{p} \rangle = 0\}. \tag{4}$$

The lightcone centred at the origin in $\mathbb{R}^2_1$ is denoted by LC^*, and we denote the part of LC^* in first quadrant, second quadrant, third quadrant, and forth quadrant by LC^*_{++}, LC^*_{-+}, LC^*_{+-}, and LC^*_{--}, respectively. It is obvious that LC^* divides

$\mathbb{R}_1^2$ into four parts. We write them as follows:

$$\begin{aligned} Sl_1^{2+} &= \{\mathbf{y} \in \mathbb{R}_1^2 \mid \langle \mathbf{y}, \mathbf{y} \rangle > 0, y_2 > 0\}, \\ Sl_1^{2-} &= \{\mathbf{y} \in \mathbb{R}_1^2 \mid \langle \mathbf{y}, \mathbf{y} \rangle > 0, y_2 < 0\}, \\ Tl_1^{2+} &= \{\mathbf{y} \in \mathbb{R}_1^2 \mid \langle \mathbf{y}, \mathbf{y} \rangle < 0, y_1 > 0\}, \\ Tl_1^{2-} &= \{\mathbf{y} \in \mathbb{R}_1^2 \mid \langle \mathbf{y}, \mathbf{y} \rangle < 0, y_1 < 0\}, \end{aligned} \tag{5}$$

where $\mathbf{y} = (y_1, y_2) \in \mathbb{R}_1^2$.

Now, we introduce Lorentz motion in $\mathbb{R}_1^2$.

Definition 1. Let $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2 : I \longrightarrow \mathbb{R}_1^2$ be two regular curves. $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ are called congruent through a Lorentz motion if there exist a matrix Q and a constant $\mathbf{a} \in \mathbb{R}_1^2$ such that $\boldsymbol{\rho}_1(t) = Q\boldsymbol{\rho}_2(t) + \mathbf{a}$ for all $t \in I$, where Q is given by

$$Q = \begin{pmatrix} \cosh\sigma & -\sinh\sigma \\ -\sinh\sigma & \cosh\sigma \end{pmatrix} \text{or } Q = \begin{pmatrix} -\cosh\sigma & \sinh\sigma \\ \sinh\sigma & -\cosh\sigma \end{pmatrix}, \tag{6}$$

for some $\sigma \in \mathbb{R}$.

The type vector in $\mathbb{R}_1^2$ is unaltered through a Lorentz motion. More specifically, if a vector $\mathbf{x} = (x_1, x_2)$ is in Sl_1^{2+} (resp. Sl_1^{2-}, Tl_1^{2+}, Tl_1^{2-}, LC^*_{++}, LC^*_{-+}, LC^*_{+-}, LC^*_{--}), when

$$Q(\mathbf{x}) = \begin{pmatrix} \cosh\sigma x_1 - \sinh\sigma x_2 \\ -\sinh\sigma x_1 + \cosh\sigma x_2 \end{pmatrix}, \tag{7}$$

it is still in Sl_1^{2+} (resp. Sl_1^{2-}, Tl_1^{2+}, Tl_1^{2-}, LC^*_{++}, LC^*_{-+}, LC^*_{+-}, LC^*_{--}); when

$$Q(\mathbf{x}) = \begin{pmatrix} -\cosh\sigma x_1 + \sinh\sigma x_2 \\ \sinh\sigma x_1 - \cosh\sigma x_2 \end{pmatrix}, \tag{8}$$

it is in Sl_1^{2-} (resp. Sl_1^{2+}, Tl_1^{2-}, Tl_1^{2+}, LC^*_{-+}, LC^*_{++}, LC^*_{--}, LC^*_{-+}).

Let $\boldsymbol{\rho} : I \longrightarrow \mathbb{R}_1^2$ be a regular nonlightlike curve. We take s as the arc-length parameter of $\boldsymbol{\rho}$. For arbitrary $s \in I$, $\|\boldsymbol{\rho}'(s)\| = 1$, the unit tangent vector of $\boldsymbol{\rho}(s)$ is given as $\mathbf{t}(s) = \boldsymbol{\rho}'(s)$ and the unit normal vector of $\boldsymbol{\rho}(s)$ is given as $\mathbf{n}(s) = (-1)^{\omega+1}\boldsymbol{\rho}'(s)^{\perp}$; then, the orientation of $\{\mathbf{t}(s), \mathbf{n}(s)\}$ is anticlockwise, where

$$\omega = \begin{cases} 1, & \text{if } s \text{ is timelike}, \\ 2, & \text{if } s \text{ is spacelike}. \end{cases} \tag{9}$$

So the Frenet-Serret formula is given by

$$\begin{pmatrix} \mathbf{t}'(s) \\ \mathbf{n}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) \\ \kappa(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \end{pmatrix}, \tag{10}$$

where $\kappa(s) = \langle \boldsymbol{\rho}''(s), \boldsymbol{\rho}'(s)^{\perp} \rangle$ is the curvature of $\boldsymbol{\rho}(s)$.

If taking not the arc-length parameter s but the general parameter t of $\boldsymbol{\rho}$, then, the unit tangent vector of $\boldsymbol{\rho}(t)$ can be given as $\mathbf{t}(t) = \dot{\boldsymbol{\rho}}(t)/\|\dot{\boldsymbol{\rho}}(t)\|$ and the unit normal vector of $\boldsymbol{\rho}(t)$ can be given as $\mathbf{n}(t) = (-1)^{\omega+1}(\dot{\boldsymbol{\rho}}(t)^{\perp}/\|\dot{\boldsymbol{\rho}}(t)\|)$, and the orientation of $\{\mathbf{t}(t), \mathbf{n}(t)\}$ is anticlockwise.

The Frenet-Serret formula is written as

$$\begin{pmatrix} \dot{\mathbf{t}}(t) \\ \dot{\mathbf{n}}(t) \end{pmatrix} = \|\dot{\boldsymbol{\rho}}(t)\| \begin{pmatrix} 0 & \kappa(t) \\ \kappa(t) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t}(t) \\ \mathbf{n}(t) \end{pmatrix}, \tag{11}$$

and the curvature of $\boldsymbol{\rho}(t)$ is given by $\kappa(t) = \langle \ddot{\boldsymbol{\rho}}(t), \dot{\boldsymbol{\rho}}(t)^{\perp} \rangle / \|\dot{\boldsymbol{\rho}}(t)\|^3$.

A point t on a curve $\boldsymbol{\rho}$ is an inflection point if and only if $\langle \ddot{\boldsymbol{\rho}}(t), \dot{\boldsymbol{\rho}}(t)^{\perp} \rangle = 0$. And for $\boldsymbol{\rho}(t)$, a regular nonlightlike curve without inflection points in $\mathbb{R}_1^2$, its evolute is defined to be

$$e(t) = \boldsymbol{\rho}(t) - \frac{1}{\kappa(t)}\mathbf{n}(t). \tag{12}$$

2.2. Evolutoids of the Nonlightlike Curves in $\mathbb{R}_1^2$. If $\boldsymbol{\rho}$ is a regular nonlightlike curve without inflection points in $\mathbb{R}_1^2$, then, the family of the normal lines to $\boldsymbol{\rho}$ has an envelope, and so does the family of the tangent lines to $\boldsymbol{\rho}$. As we all know, the envelope of a family of lines in $\mathbb{R}_1^2$ is a curve tangent to all of them. The envelope of the normal lines is the evolute of the curve, and the envelope of the tangent lines is $\boldsymbol{\rho}$ itself. But what happens between the envelope of tangents and the envelope of normals. Here, we shall consider this question. The condition in the Euclidean plane was studied by Giblin and Warder [13]. In [15], Aydın Şekerci and Izumiya have introduced the evolutoids of nonlightlike curves in $\mathbb{R}_1^2$. To better carry out the following research, we think about them by another expression approach.

In order to solve above question, we define the σ-transform first.

Let $\boldsymbol{\rho} : I \longrightarrow \mathbb{R}_1^2$ be a regular nonlightlike curve. Through Lorentz motion, the σ-transform of its unit tangent vector $\mathbf{t}(t)$ is defined by

$$\mathbf{t}(t)\cosh\sigma + \mathbf{n}(t)\sinh\sigma, \tag{13}$$

where $\sigma \in \mathbb{R}$. We discover that $\mathbf{t}(t)\cosh\sigma + \mathbf{n}(t)\sinh\sigma$ rotates in the part which $\mathbf{t}(t)$ is in and the norm of $\mathbf{t}(t)\cosh\sigma + \mathbf{n}(t)\sinh\sigma$ is the same as the norm of $\mathbf{t}(t)$. The line obtained by $\mathbf{t}(t)\cosh\sigma + \mathbf{n}(t)\sinh\sigma$ at t is written as l, and the envelope of lines l is denoted by τ_σ. It is obvious that when $\sigma = 0$, τ_0 is $\boldsymbol{\rho}$ itself. When $\sigma \neq 0$, τ_σ is so-called the evolutoid of $\boldsymbol{\rho}$.

Similarly, we can consider the σ-transform of $\mathbf{n}(t)$. It is defined by $\mathbf{t}(t)\sinh\sigma + \mathbf{n}(t)\cosh\sigma$. The line obtained by $\mathbf{t}(t)\sinh\sigma + \mathbf{n}(t)\cosh\sigma$ is written as l as well, and the envelope of lines l is also denoted by τ_σ. When $\sigma = 0$, τ_0 is the evolute of $\boldsymbol{\rho}$, and when $\sigma \neq 0$, τ_σ is also called the evolutoid of $\boldsymbol{\rho}$.

Remark 2. As the lightcone LC^* divides $\mathbb{R}_1^2$ into four parts and $\mathbf{t}(t)$, $\mathbf{n}(t)$, $-\mathbf{t}(t)$, and $-\mathbf{n}(t)$ are in different parts, we shall consider the σ-transform of $\mathbf{t}(t)$, $\mathbf{n}(t)$, $-\mathbf{t}(t)$, and $-\mathbf{n}(t)$, respectively. In fact, the line l obtained by the σ-transform of $\mathbf{t}(t)$ and the σ-transform of $-\mathbf{t}(t)$ is the same, either the σ-transform of $\mathbf{n}(t)$ or the σ-transform of $-\mathbf{n}(t)$. Therefore, we only need to consider the σ-transform of $\mathbf{t}(t)$ and $\mathbf{n}(t)$.

Let $\boldsymbol{\rho} : I \longrightarrow \mathbb{R}_1^2$ be a regular nonlightlike curve without inflection points, and we consider the evolutoids of $\boldsymbol{\rho}$. We suppose that l is obtained by the σ-transform of $\mathbf{t}(t)$.

As the direction of l is $\mathbf{t}(t)\cosh\sigma + \mathbf{n}(t)\sinh\sigma$, the vector pseudo-orthogonal to l can be expressed as $\mathbf{t}(t)\sinh\sigma + \mathbf{n}(t)\cosh\sigma$, and for any $\mathbf{u} = (u_1, u_2) \in \mathbb{R}_1^2$, a vector equation of line L is $F(\mathbf{u}, t) = 0$ such that

$$F(\mathbf{u}, t) = \langle \mathbf{u} - \boldsymbol{\rho}(t), \mathbf{t}(t)\sinh\sigma + \mathbf{n}(t)\cosh\sigma \rangle. \tag{14}$$

If we fix σ, then $F(\mathbf{u}, t) = 0$ describes a family of lines: for arbitrary $t \in I$, there is a line and when t varies the line moves in $\mathbb{R}_1^2$. For a fixed σ, the envelope τ_σ of the family of the lines given by equation (14) is the set of points $\mathbf{u} = (u_1, u_2)$ in $\mathbb{R}_1^2$; then, there exists $t \in I$ such that the following equation holds

$$F(\mathbf{u}, t) = F|_t(\mathbf{u}, t) = 0. \tag{15}$$

For any fixed σ,

$$\begin{aligned} F|_t(\mathbf{u}, t) &= \langle -\dot{\boldsymbol{\rho}}(t), \mathbf{t}(t)\sinh\sigma + \mathbf{n}(t)\cosh\sigma \rangle \\ &\quad + \langle \mathbf{u} - \boldsymbol{\rho}(t), \|\dot{\boldsymbol{\rho}}(t)\|\kappa(t)\mathbf{n}(t)\sinh\sigma + \|\dot{\boldsymbol{\rho}}(t)\|\kappa(t)\mathbf{t}(t)\cosh\sigma \rangle \\ &= \langle -\|\dot{\boldsymbol{\rho}}(t)\|\mathbf{t}(t), \mathbf{t}(t)\sinh\sigma + \mathbf{n}(t)\cosh\sigma \rangle \\ &\quad + \|\dot{\boldsymbol{\rho}}(t)\| \langle \mathbf{u} - \boldsymbol{\rho}(t), \kappa(t)\mathbf{n}(t)\sinh\sigma + \kappa(t)\mathbf{t}(t)\cosh\sigma \rangle \\ &= -\|\dot{\boldsymbol{\rho}}(t)\|(-1)^\omega \sinh\sigma \\ &\quad + \|\dot{\boldsymbol{\rho}}(t)\| \langle \mathbf{u} - \boldsymbol{\rho}(t), \kappa(t)\mathbf{n}(t)\sinh\sigma + \kappa(t)\mathbf{t}(t)\cosh\sigma \rangle \\ &= 0. \end{aligned} \tag{16}$$

Then,

$$\langle \mathbf{u} - \boldsymbol{\rho}(t), \kappa(t)\mathbf{n}(t)\sinh\sigma + \kappa(t)\mathbf{t}(t)\cosh\sigma \rangle = (-1)^\omega \sinh\sigma. \tag{17}$$

Let $\mathbf{u} - \boldsymbol{\rho}(t) = \xi_1\mathbf{t}(t) + \xi_2\mathbf{n}(t)$, where $\xi_1, \xi_2 \in \mathbb{R}$, and it can denote any vectors in $\mathbb{R}_1^2$. Applying $\mathbf{u} - \boldsymbol{\rho}(t) = \xi_1\mathbf{t}(t) + \xi_2\mathbf{n}(t)$ in $F = F|_t = 0$, we can obtain two equations about ξ_1 and ξ_2.

$$\begin{aligned} \xi_1 \sinh\sigma - \xi_2 \cosh\sigma &= 0, \\ \xi_1\kappa(t)\cosh\sigma - \xi_2\kappa(t)\sinh\sigma &= \sinh\sigma. \end{aligned} \tag{18}$$

By direct calculation, we can get the evolutoid $Ev(\sigma)_\rho(t)$ of the regular nonlightlike curve without inflection points $\boldsymbol{\rho}(t)$ in $\mathbb{R}_1^2$ through the σ-transform of $\mathbf{t}(t)$ can be expressed by

$$Ev(\sigma)_\rho(t) = \mathbf{u} = \boldsymbol{\rho}(t) + \frac{\cosh\sigma\sinh\sigma}{\kappa(t)}\mathbf{t}(t) + \frac{\sinh^2\sigma}{\kappa(t)}\mathbf{n}(t). \tag{19}$$

Since $\boldsymbol{\rho}(t)$ is a regular curve without inflection points, $\kappa(t) \neq 0$. If $\sigma = 0$, then, the lines $F = 0$ are tangent to $\boldsymbol{\rho}$ and the envelope is consistent with the original curve $\boldsymbol{\rho}$, namely, $Ev(\sigma)_\rho(t) = \boldsymbol{\rho}(t)$.

If l is obtained by the σ-transform of $\mathbf{n}(t)$, similar to the above method, we can get the evolutoid $Ev(\sigma)_\rho(t)$ of $\boldsymbol{\rho}(t)$ through the σ-transform of $\mathbf{n}(t)$ expressed as follows:

$$Ev(\sigma)_\rho(t) = \boldsymbol{\rho}(t) - \frac{\cosh\sigma\sinh\sigma}{\kappa(t)}\mathbf{t}(t) - \frac{\cosh^2\sigma}{\kappa(t)}\mathbf{n}(t). \tag{20}$$

Base on above all, for a regular nonlightlike curve $\boldsymbol{\rho}(t)$ in $\mathbb{R}_1^2$, we can obtain the definition of its evolutoid.

Definition 3. Let $\boldsymbol{\rho} : I \longrightarrow \mathbb{R}_1^2$ be a regular non-lightlike curve without inflection points in $\mathbb{R}_1^2$, then the evolutoid of $\boldsymbol{\rho}(t)$ is as follows:

$$Ev(\sigma)_\rho(t) = \begin{cases} \boldsymbol{\rho}(t) + \dfrac{\cosh\sigma\sinh\sigma}{\kappa(t)}\mathbf{t}(t) + \dfrac{\sinh^2\sigma}{\kappa(t)}\mathbf{n}(t), & \text{if the } \sigma\text{-transform is on } \mathbf{t}(t), \\ \boldsymbol{\rho}(t) - \dfrac{\cosh\sigma\sinh\sigma}{\kappa(t)}\mathbf{t}(t) - \dfrac{\cosh^2\sigma}{\kappa(t)}\mathbf{n}(t), & \text{if the } \sigma\text{-transform is on } \mathbf{n}(t). \end{cases} \tag{21}$$

Remark 4. In fact, when the σ-transform is on $\mathbf{t}(t)$ (or $\mathbf{n}(t)$), the evolutoid of $\boldsymbol{\rho}(t)$ is consistent with $\phi_{\mathbf{T}}$ -evolutoid (or $\phi_{\mathbf{N}}$ -evolutoid) in [15]. But for the further research of the evolutoids of the mixed-type curves, we adopted the definition mode of σ-transform by our own.

3. Evolutoids of the Mixed-Type Curves in $\mathbb{R}_1^2$

For a regular curve in $\mathbb{R}_1^2$, when it contains spacelike regions and timelike regions simultaneously, the lightlike point appears between them. This kind of curve is the mixed-type curve that we said. There are many interesting phenomena because of the appearance of lightlike points. In this section, we would like to consider the evolutoids of the regular mixed-type curves in $\mathbb{R}_1^2$, which is the focus of our work. But it is badly observed that as the curvature cannot be defined very properly at lightlike point, the Frenet-Serret frame is not applicative for the mixed-type curves. Actually, following frame will be introduced by us which is pretty suitable for a mixed-type curve. The frame is given by Izumiya et al. [10].

$(1, 1)$ and $(1,-1)$ are two lightlike vectors. Apparently, they are independent to each other. Denote $(1, 1)$ and $(1,-1)$ by $\mathbb{L}^+$ and $\mathbb{L}^-$, respectively; then, the vector pair $\{\mathbb{L}^+, \mathbb{L}^-\}$ is called a *lightcone frame* of $\boldsymbol{\rho}(t)$ in $\mathbb{R}_1^2$.

Let $\boldsymbol{\rho} : I \longrightarrow \mathbb{R}_1^2$ be a regular mixed-type curve. There exists a corresponding smooth map (α, β): $I \longrightarrow \mathbb{R}^2 \setminus \{0\}$, which satisfies

$$\dot{\boldsymbol{\rho}}(t) = \alpha(t)\mathbb{L}^+ + \beta(t)\mathbb{L}^-, \quad \forall t \in I. \tag{22}$$

If equation (22) is established, (α, β) is called *the lightlike tangential data* of $\boldsymbol{\rho}(t)$. The pseudo-orthogonal complement of $\dot{\boldsymbol{\rho}}(t)$ can be expressed by

$$\dot{\boldsymbol{\rho}}(t)^\perp = \alpha(t)\mathbb{L}^+ - \beta(t)\mathbb{L}^-. \tag{23}$$

Since

$$\langle \dot{\boldsymbol{\rho}}(t), \dot{\boldsymbol{\rho}}(t) \rangle = -4\alpha(t)\beta(t), \tag{24}$$

the type of $\boldsymbol{\rho}(t_0)$ can be determined by $\alpha(t_0)\beta(t_0)$.

Let $\boldsymbol{\rho} : I \longrightarrow \mathbb{R}_1^2$ be a regular mixed-type curve with the lightlike tangential date (α, β), then, $\boldsymbol{\rho}(t_0)$ is an inflection point of $\boldsymbol{\rho}$ if and only if

$$\dot{\alpha}(t_0)\beta(t_0) - \alpha(t_0)\dot{\beta}(t_0) = 0. \tag{25}$$

As $\dot{\boldsymbol{\rho}}(t) = \alpha(t)\mathbb{L}^+ + \beta(t)\mathbb{L}^-$, we use Lorentz motion on $\dot{\boldsymbol{\rho}}(t)$, and we can get

$$M(\sigma)\alpha(t)\mathbb{L}^+ + N(\sigma)\beta(t)\mathbb{L}^- \tag{26}$$

or

$$-M(\sigma)\alpha(t)\mathbb{L}^+ - N(\sigma)\beta(t)\mathbb{L}^-, \tag{27}$$

where $M(\sigma) = \cosh \sigma - \sinh \sigma$ and $N(\sigma) = \cosh \sigma + \sinh \sigma$. Formula (26) is called the σ-transform of $\alpha(t)\mathbb{L}^+ + \beta(t)\mathbb{L}^-$, and formula (27) is called the σ-transform of $-(\alpha(t)\mathbb{L}^+ + \beta(t)\mathbb{L}^-)$.

Similarly, we can obtain that the σ-transform of $\alpha(t)\mathbb{L}^+ - \beta(t)\mathbb{L}^-$ and $-(\alpha(t)\mathbb{L}^+ - \beta(t)\mathbb{L}^-)$ is $M(\sigma)\alpha(t)\mathbb{L}^+ - N(\sigma)\beta(t)\mathbb{L}^-$ and $-M(\sigma)\alpha(t)\mathbb{L}^+ + N(\sigma)\beta(t)\mathbb{L}^-$, respectively.

In fact, the line l obtained by $M(\sigma)\alpha(t)\mathbb{L}^+ + N(\sigma)\beta(t)\mathbb{L}^-$ and $-M(\sigma)\alpha(t)\mathbb{L}^+ - N(\sigma)\beta(t)\mathbb{L}^-$ is the same, either $M(\sigma)\alpha(t)\mathbb{L}^+ - N(\sigma)\beta(t)\mathbb{L}^-$ or $-M(\sigma)\alpha(t)\mathbb{L}^+ + N(\sigma)\beta(t)\mathbb{L}^-$. Therefore, we only need to consider the σ-transform of $\alpha(t)\mathbb{L}^+ + \beta(t)\mathbb{L}^-$ and $\alpha(t)\mathbb{L}^+ - \beta(t)\mathbb{L}^-$. In addition, the type of $M(\sigma)\alpha(t)\mathbb{L}^+ + N(\sigma)\beta(t)\mathbb{L}^-$ is the same as $\alpha(t)\mathbb{L}^+ + \beta(t)\mathbb{L}^-$, so does the type of $M(\sigma)\alpha(t)\mathbb{L}^+ - N(\sigma)\beta(t)\mathbb{L}^-$ and $\alpha(t)\mathbb{L}^+ - \beta(t)\mathbb{L}^-$; thus, we can avoid the angle of lightlike vector and nonlightlike vector through σ-transform.

The definition of the evolutoid of a regular mixed-type curve with the lightcone frame $\{\mathbb{L}^+, \mathbb{L}^-\}$ and the lightlike tangential data (α, β) can be given as follows:

Theorem 5. *Let $\boldsymbol{\rho} : I \longrightarrow \mathbb{R}_1^2$ be a regular mixed-type curve without inflection points in $\mathbb{R}_1^2$, then the evolutoid of $\boldsymbol{\rho}(t)$ is as follows:*

$$Ev(\sigma)_{\boldsymbol{\rho}}(t) = \begin{cases} \boldsymbol{\rho}(t) - \dfrac{2\alpha(t)\beta(t)\sinh\sigma}{\dot{\alpha}(t)\beta(t) - \alpha(t)\dot{\beta}(t)}(M(\sigma)\alpha(t)\mathbb{L}^+ + N(\sigma)\beta(t)\mathbb{L}^-), & \text{if the } \sigma\text{-transform is on } \alpha(t)\mathbb{L}^+ + \beta(t)\mathbb{L}^-, \\ \boldsymbol{\rho}(t) - \dfrac{2\alpha(t)\beta(t)\cosh\sigma}{\dot{\alpha}(t)\beta(t) - \alpha(t)\dot{\beta}(t)}(M(\sigma)\alpha(t)\mathbb{L}^+ - N(\sigma)\beta(t)\mathbb{L}^-), & \text{if the } \sigma\text{-transform is on } \alpha(t)\mathbb{L}^+ - \beta(t)\mathbb{L}^-. \end{cases} \tag{28}$$

Proof. First, supposing that the σ-transform is on $\alpha(t)\mathbb{L}^+ + \beta(t)\mathbb{L}^-$, then, the direction of l is

$$M(\sigma)\alpha(t)\mathbb{L}^+ + N(\sigma)\beta(t)\mathbb{L}^-. \tag{29}$$

The vector pseudo-orthogonal to l is

$$M(\sigma)\alpha(t)\mathbb{L}^+ - N(\sigma)\beta(t)\mathbb{L}^-. \tag{30}$$

For any $\mathbf{u} = (u_1, u_2) \in \mathbb{R}_1^2$, we have

$$F(\mathbf{u}, t) = F|_t(\mathbf{u}, t) = 0, \tag{31}$$

where

$$F(\mathbf{u}, t) = \langle \mathbf{u} - \boldsymbol{\rho}(t), M(\sigma)\alpha(t)\mathbb{L}^+ - N(\sigma)\beta(t)\mathbb{L}^- \rangle \tag{32}$$

and σ is fixed.

By direct calculation,

$$F|_t(\mathbf{u},t) = -4\alpha(t)\beta(t)\sinh\sigma + \left\langle \mathbf{u}-\boldsymbol{\rho}(t), M(\sigma)\dot{\alpha}(t)\mathbb{L}^+ - N(\sigma)\dot{\beta}(t)\mathbb{L}^- \right\rangle. \tag{33}$$

Let

$$\mathbf{u}-\boldsymbol{\rho}(t) = \zeta_1(t)\mathbb{L}^+ + \zeta_2(t)\mathbb{L}^-, \tag{34}$$

where $(\zeta_1,\zeta_2)\colon I \longrightarrow \mathbb{R}^2 \setminus \{0\}$ is a smooth map.

We have

$$\begin{aligned} &2\zeta_1(t)N(\sigma)\beta(t) - 2\zeta_2(t)M(\sigma)\alpha(t) = 0,\\ &2\zeta_1(t)N(\sigma)\dot{\beta}(t) - 2\zeta_2(t)M(\sigma)\dot{\alpha}(t) = 4\alpha(t)\beta(t)\sinh\sigma. \end{aligned} \tag{35}$$

By direct calculation, we get

$$\begin{aligned} \zeta_1(t) &= \frac{2\alpha^2(t)\beta(t)\sinh\sigma}{N(\sigma)\left(\alpha(t)\dot{\beta}(t)-\dot{\alpha}(t)\beta(t)\right)},\\ \zeta_2(t) &= \frac{2\alpha(t)\beta^2(t)\sinh\sigma}{M(\sigma)\left(\alpha(t)\dot{\beta}(t)-\dot{\alpha}(t)\beta(t)\right)}. \end{aligned} \tag{36}$$

Thus,

$$\mathbf{u} = \boldsymbol{\rho}(t) - \frac{2\alpha(t)\beta(t)\sinh\sigma}{\dot{\alpha}(t)\beta(t)-\alpha(t)\dot{\beta}(t)}\left(M(\sigma)\alpha(t)\mathbb{L}^+ + N(\sigma)\beta(t)\mathbb{L}^-\right) = Ev(\sigma)_\rho(t). \tag{37}$$

If the σ-transform is on $\alpha(t)\mathbb{L}^+ - \beta(t)\mathbb{L}^-$, similar to the above process, we can get the evolutoid $Ev(\sigma)_\rho(t)$ of the regular mixed-type curve $\boldsymbol{\rho}(t)$ as follows:

$$Ev(\sigma)_\rho(t) = \boldsymbol{\rho}(t) - \frac{2\alpha(t)\beta(t)\cosh\sigma}{\dot{\alpha}(t)\beta(t)-\alpha(t)\dot{\beta}(t)}\left(M(\sigma)\alpha(t)\mathbb{L}^+ - N(\sigma)\beta(t)\mathbb{L}^-\right). \tag{38}$$

□

Remark 6. If the σ-transform is on $\alpha(t)\mathbb{L}^+ + \beta(t)\mathbb{L}^-$, when $\sigma_0 = 0$, $Ev(\sigma_0)_\rho(t)$ is consistent with $\boldsymbol{\rho}(t)$. If the σ-transform is on $\alpha(t)\mathbb{L}^+ - \beta(t)\mathbb{L}^-$, when $\sigma_0 = 0$, $Ev(\sigma_0)_\rho(t) = \boldsymbol{\rho}(t) - ((2\alpha(t)\beta(t))/(\dot{\alpha}(t)\beta(t)-\alpha(t)\dot{\beta}(t)))(\alpha(t)\mathbb{L}^+ - \beta(t)\mathbb{L}^-)$, it is exactly the evolute of $\boldsymbol{\rho}(t)$.

As for a regular mixed-type curve $\boldsymbol{\rho}(t)$ in $\mathbb{R}^2_1$, is its evolutoid always regular? If not, when does the evolutoid of $\boldsymbol{\rho}(t)$ have singular points? We give the following theorem to answer this question.

Theorem 7. *Let $\boldsymbol{\rho}: I \longrightarrow \mathbb{R}^2_1$ be a regular mixed-type curve without inflection points in $\mathbb{R}^2_1$, $Ev(\sigma)_\rho : I \longrightarrow \mathbb{R}^2_1$ is the evolutoid of $\boldsymbol{\rho}$.*

(1) Take the arc-length parameter s, and suppose that $\boldsymbol{\rho}(s_0)$ is a nonlightlike point; then, s_0 is a singular point of $Ev(\sigma)_\rho(s)$ if and only if

(i) $\kappa^2(s_0)\cosh\sigma - \kappa'(s_0)\sinh\sigma = 0$, when the σ-transform is on $\mathbf{t}(s)$;

(ii) $-\kappa^2(s_0)\sinh\sigma + \kappa'(s_0)\cosh\sigma = 0$, when the σ-transform is on $\mathbf{n}(s)$.

(2) Suppose that $\boldsymbol{\rho}(t_0)$ is a lighlike point,

(i) When the σ-transform is on $\alpha(t)\mathbb{L}^+ + \beta(t)\mathbb{L}^-$,

(a) if $\alpha(t_0) = 0$, $\beta(t_0) \neq 0$, t_0 is a singular point of $Ev(\sigma)_\rho(t)$ if and only if $\sigma = \ln\sqrt{2}$,

(b) if $\alpha(t_0) \neq 0$, $\beta(t_0) = 0$, t_0 is a singular point of $Ev(\sigma)_\rho(t)$ if and only if $\sigma = -\ln\sqrt{2}$;

(ii) When the σ-transform is on $\alpha(t)\mathbb{L}^+ - \beta(t)\mathbb{L}^-$, $Ev(\sigma)_\rho(t_0)$ is always a regular point.

Proof. First, suppose that $\boldsymbol{\rho}(t_0)$ is a non-lightlike point and the σ-transform is on $\mathbf{t}(t)$. Using the Frenet-Serret formula we can get

$$\dot{E}v(\sigma)_\rho(t_0) = \left(\|\dot{\boldsymbol{\rho}}(t_0)\|\cosh\sigma - \sinh\sigma\frac{\kappa'(t_0)}{\kappa^2(t_0)}\right)\left(\cosh\mathbf{t}(t_0) + \sinh\mathbf{n}(t_0)\right). \tag{39}$$

$\dot{E}v(\sigma)_\rho(t_0) = 0$ if and only if $\kappa^2(t_0)\|\dot{\boldsymbol{\rho}}(t_0)\|\cosh\sigma - \kappa'(t_0)\sinh\sigma = 0$. If we take the arc-length parameter s, then, we have

$$\kappa^2(s_0)\cosh\sigma - \kappa'(s_0)\sinh\sigma = 0. \tag{40}$$

Similarly, if the σ-transform is on $\mathbf{n}(t)$, we have $\dot{E}v(\sigma)_\rho(s_0) = 0$ if and only if $-\kappa^2(s_0)\sinh\sigma + \kappa'(s_0)\cosh\sigma = 0$.

Next, suppose that $\boldsymbol{\rho}(t_0)$ is a lightlike point and the σ-transform is on $\alpha(t)\mathbb{L}^+ + \beta(t)\mathbb{L}^-$. By direct calculation, we can get

$$\begin{aligned} \dot{E}v(\sigma)_\rho(t_0) = {} & A(t_0)\big(-\dot{\alpha}^2(t_0)\beta^2(t_0)(4\sinh\sigma M(\sigma)-1)\\ & + \alpha^2(t_0)\dot{\beta}^2(t_0)(2\sinh\sigma M(\sigma)+1)\\ & + 2\alpha(t_0)\dot{\alpha}(t_0)\beta(t_0)\dot{\beta}(t_0)(\sinh\sigma M(\sigma)-1)\\ & + 2\alpha(t_0)\beta(t_0)\left(\ddot{\alpha}(t_0)\beta(t_0)-\alpha(t_0)\ddot{\beta}(t_0)\right)\sinh\sigma M(\sigma)\big)\mathbb{L}^+\\ & + B(t_0)\big(-\dot{\alpha}^2(t_0)\beta^2(t_0)(2\sinh\sigma N(\sigma)-1)\\ & + \alpha^2(t_0)\dot{\beta}^2(t_0)(4\sinh\sigma N(\sigma)+1)\\ & - 2\alpha(t_0)\dot{\alpha}(t_0)\beta(t_0)\dot{\beta}(t_0)(\sinh\sigma N(\sigma)+1)\\ & + 2\alpha(t_0)\beta(t_0)\left(\ddot{\alpha}(t_0)\beta(t_0)-\alpha(t_0)\ddot{\beta}(t_0)\right)\sinh\sigma N(\sigma)\big)\mathbb{L}^-, \end{aligned} \tag{41}$$

where

$$A(t_0) = \frac{\alpha(t_0)}{\left(\dot{\alpha}(t_0)\beta(t_0) - \alpha(t_0)\dot{\beta}(t_0)\right)^2},$$
$$B(t_0) = \frac{\beta(t_0)}{\left(\dot{\alpha}(t_0)\beta(t_0) - \alpha(t_0)\dot{\beta}(t_0)\right)^2}. \tag{42}$$

When $\alpha(t_0) = 0$, $\beta(t_0) \neq 0$, $\dot{E}v(\sigma)_\rho(t_0) = 0$ if and only if $2 \sinh \sigma N(\sigma) - 1 = 0$, that is, $\sigma = \ln \sqrt{2}$. When $\alpha(t_0) \neq 0$, $\beta(t_0) = 0$, $\dot{E}v(\sigma)_\rho(t_0) = 0$ if and only if $2 \sinh \sigma M(\sigma) + 1 = 0$, that is, $\sigma = -\ln \sqrt{2}$.

Similarly, if the σ-transform is on $\alpha(t)\mathbb{L}^+ - \beta(t)\mathbb{L}^-$, we can get

$$\begin{aligned}\dot{E}v(\sigma)_\rho(t_0) = {} & A(t_0)\big(-\dot{\alpha}^2(t_0)\beta^2(t_0)(4\cosh \sigma M(\sigma) - 1) \\ & + \alpha^2(t_0)\dot{\beta}^2(t_0)(2\cosh\sigma M(\sigma) + 1) \\ & + 2\alpha(t_0)\dot{\alpha}(t_0)\beta(t_0)\dot{\beta}(t_0)(\cosh\sigma M(\sigma) - 1) \\ & + 2\alpha(t_0)\beta(t_0)\left(\ddot{\alpha}(t_0)\beta(t_0) - \alpha(t_0)\ddot{\beta}(t_0)\right)\cosh\sigma M(\sigma)\big)\mathbb{L}^+ \\ & + B(t_0)\big(\dot{\alpha}^2(t_0)\beta^2(t_0)(2\cosh\sigma N(\sigma) + 1) \\ & - \alpha^2(t_0)\dot{\beta}^2(t_0)(4\cosh\sigma N(\sigma) - 1) \\ & + 2\alpha(t_0)\dot{\alpha}(t_0)\beta(t_0)\dot{\beta}(t_0)(\cosh\sigma M(\sigma) - 1) \\ & + 2\alpha(t_0)\beta(t_0)\left(\ddot{\alpha}(t_0)\beta(t_0) - \alpha(t_0)\ddot{\beta}(t_0)\right)\cosh\sigma M(\sigma)\big)\mathbb{L}^-.\end{aligned} \tag{43}$$

When $\alpha(t_0) = 0$, $\beta(t_0) \neq 0$, $\dot{E}v(\sigma)_\rho(t_0) = 0$ if and only if $2\cosh\sigma N(\sigma) + 1 = 0$, that is, $e^{2\sigma} = -2$. We can not work out such σ, so there not exist t_0 such that $Ev(\sigma)_\rho(t_0)$ is singular point. When $\alpha(t_0) \neq 0$ and $\beta(t_0) = 0$, there not exist t_0 such that $Ev(\sigma)_\rho(t_0)$ is a singular point. Thus, $Ev(\sigma)_\rho(t_0)$ is always a regular point □

In [10], we know that for a point $\boldsymbol{\rho}(t_0)$, if it is a lightlike point, then the corresponding point on the evolute $Ev(\boldsymbol{\rho})(t_0)$ is always a regular point. Here, we can conclude that for a lightlike point $\boldsymbol{\rho}(t_0)$, its evolutoid $Ev(\sigma)_\rho(t_0)$ may be singular, which is a different phenomenon from the evolute.

We would like to consider the relationship of the types of the points on the evolutoid $Ev(\sigma)_\rho(t_0)$ of a mixed-type curve $\boldsymbol{\rho}(t)$ and the types of the points $\boldsymbol{\rho}(t_0)$. By Theorem 7, the following proposition can be obtained.

Proposition 8. *Let $\boldsymbol{\rho} : I \longrightarrow \mathbb{R}^2_1$ be a regular mixed-type curve without inflection points in $\mathbb{R}^2_1$, $Ev(\sigma)_\rho : I \longrightarrow \mathbb{R}^2_1$ is the evolutoid of $\boldsymbol{\rho}(t)$.*

(1) Take the arc-length parameter s, and suppose that $\boldsymbol{\rho}(s_0)$ is a non-lightlike point; then,

(i) when the σ-transform is on $\mathbf{t}(s)$ and $\kappa^2(s_0)\cosh\sigma - \kappa'(s_0)\sinh\sigma \neq 0$, then, $Ev(\sigma)_\rho(s_0)$ is spacelike (resp., timelike) if and only if $\boldsymbol{\rho}(s_0)$ is spacelike (resp., timelike);

(ii) when the σ-transform is on $\mathbf{n}(s)$ and $-\kappa^2(s_0)\sinh\sigma + \kappa'(s_0)\cosh\sigma \neq 0$, then, $Ev(\sigma)_\rho(s_0)$ is timelike (resp., spacelike) if and only if $\boldsymbol{\rho}(s_0)$ is spacelike (resp., timelike).

(2) Suppose that $\boldsymbol{\rho}(t_0)$ is a lighlike point; then,

(i) when the σ-transform is on $\alpha(t)\mathbb{L}^+ + \beta(t)\mathbb{L}^-$,

(a) if $\alpha(t_0) = 0$, $\beta(t_0) \neq 0$, and $\sigma \neq \ln\sqrt{2}$, then $Ev(\sigma)_\rho(t_0)$ is lightlike;

(b) if $\alpha(t_0) \neq 0$, $\beta(t_0) = 0$, and $\sigma \neq -\ln\sqrt{2}$, then $Ev(\sigma)_\rho(t_0)$ is lightlike;

(ii) when the σ-transform is on $\alpha(t)\mathbb{L}^+ - \beta(t)\mathbb{L}^-$, $Ev(\sigma)_\rho(t_0)$ is always lightlike.

4. Examples

We would like to present the characteristics of the evolutoids of the regular mixed-type curves without inflections by the following two examples.

Example 9. Let $\boldsymbol{\rho} : (-1, 1) \longrightarrow \mathbb{R}^2_1$, $\boldsymbol{\rho}(t) = (t, t^2)$ be a regular mixed-type curve without inflections. See the blue curve in Figure 1.

We can get that the evolute of $\boldsymbol{\rho}(t)$ is

$$Ev(\boldsymbol{\rho})(t) = \left(4t^2, 3t^3 - \frac{1}{2}\right). \tag{44}$$

See the orange dashed curve in Figure 1.

If the σ-transform is on $\alpha(t)\mathbb{L}^+ - \beta(t)\mathbb{L}^-$ and $\sigma = \ln\sqrt{2}$, the evolutoid of $\boldsymbol{\rho}(t)$ is

$$Ev(\sigma)_\rho(t) = \frac{1}{2}\left(9t^3 - \frac{3}{2}t^2 - \frac{1}{4}t + \frac{3}{8}, -3t^3 + \frac{13}{2}t^2 + \frac{3}{4}t - \frac{9}{8}\right). \tag{45}$$

See the red dashed curve in Figure 1.

If the σ-transform is on $\alpha(t)\mathbb{L}^+ + \beta(t)\mathbb{L}^-$ and $\sigma = -\ln\sqrt{2}$, the evolutoid of $\boldsymbol{\rho}(t)$ is

$$Ev(\sigma)_\rho(t) = \frac{1}{2}\left(-t^3 - \frac{3}{2}t^2 + \frac{9}{4}t + \frac{3}{8}, -3t^3 + \frac{3}{2}t^2 + \frac{3}{4}t + \frac{1}{8}\right). \tag{46}$$

When $t_0 = 1/2$, $\boldsymbol{\rho}(t_0)$ is a lightlike point, and $Ev(\sigma)_\rho(t_0)$ is a singular point. See the green curve in Figure 1.

Example 10. Let $\boldsymbol{\rho} : [0, 2\pi) \longrightarrow \mathbb{R}^2_1$, $\boldsymbol{\rho}(t) = (\cos t, \sin t)$ be a regular mixed-type curve without inflections. See the blue curve in Figure 2.

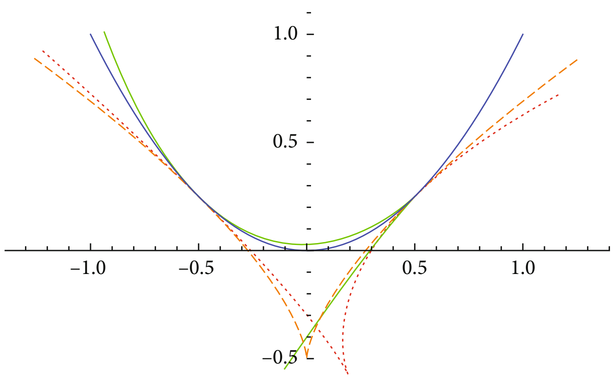

FIGURE 1: The mixed-type curve (blue) and the evolutoid (evolute) of it.

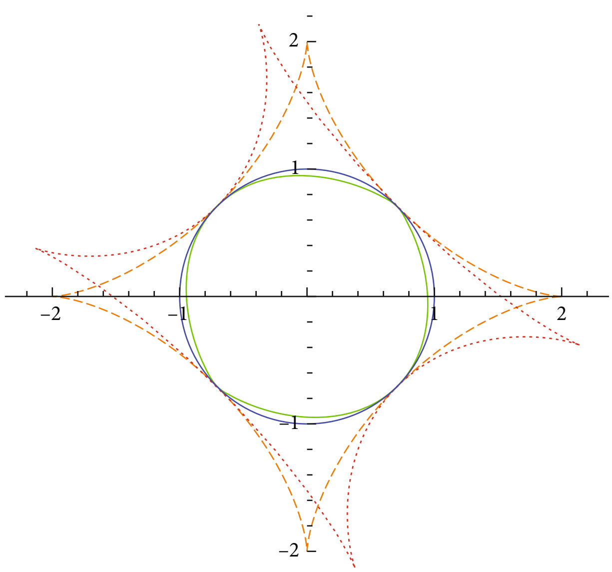

FIGURE 2: The mixed-type curve (blue) and the evolutoid (evolute) of it.

We can get that the evolute of $\boldsymbol{\rho}(t)$ is

$$Ev(\boldsymbol{\rho})(t) = 2\left(\cos^3 t, \sin^3 t\right). \tag{47}$$

See the orange dashed curve in Figure 2.

If the σ-transform is on $\alpha(t)\mathbb{L}^+ - \beta(t)\mathbb{L}^-$ and $\sigma = \ln\sqrt{2}$, the evolutoid of $\boldsymbol{\rho}(t)$ is

$$Ev(\sigma)_\rho(t) = \left(\cos t + \frac{3}{8}\left(1 - 2\sin^2 t\right)(3\cos t + \sin t), \sin t - \frac{3}{8}\left(1 - 2\sin^2 t\right)(\cos t + 3\sin t)\right). \tag{48}$$

See the red dashed curve in Figure 2.

If the σ-transform is on $\alpha(t)\mathbb{L}^+ + \beta(t)\mathbb{L}^-$ and $\sigma = \ln\sqrt{2}$, the evolutoid of $\boldsymbol{\rho}(t)$ is

$$Ev(\sigma)_\rho(t) = \left(\cos t - \frac{1}{8}\left(1 - 2\sin^2 t\right)(\cos t + 3\sin t), \sin t + \frac{1}{8}\left(1 - 2\sin^2 t\right)(3\cos t + \sin t)\right). \tag{49}$$

When $t_0 = \pi/4$ or $5\pi/4$, $\boldsymbol{\rho}(t_0)$ is a lightlike point, and $Ev(\sigma)_\rho(t_0)$ is a singular point. See the green curve in Figure 2.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interests in this work.

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