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Research Article

Characters of Explicit Solutions for a Semidiscrete Integrable Coupled Equation

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A semidiscrete integrable coupled system is obtained by embedding a free function into the discrete zero-curvature equation. Then, explicit solutions of the first two nontrivial equations in this system are derived directly by the Darboux transformation method. Finally, in order to compare the solutions before and after coupling intuitively, their structure figures are presented and analyzed.

1. Introduction

Integrable coupled equations have attracted more attention in soliton theory in recent years. For a given integrable system, we can construct a nontrivial system of differential equations which is still integrable and includes the original integrable system as a subsystem [1]. It is interesting to find integrable coupled systems for a given integrable equation. Hirota and Satsuma in 1981 introduced a coupled KdV system [2]. Fuchssteiner in 1982 proposed the important question: how should completely systems interact without losing complete integrability [3]? Then, the method for constructing integrable coupling systems by perturbation was first proposed by Ma and Fuchssteiner [4]. Later, the method has been developed. So far, it mainly includes perturbations, enlarging spectral problems [5, 6], creating new loop algebras [7], and multi-integrable couplings. With the development of integrable coupling theory, it is verified that integrable coupled equations are usually used for describing phenomena related to dark and antidark solitons. Such as the integrable coupled generalized nonlinear Schrödinger equations can exhibit N-bright-bright and N-dark-dark soliton solutions [8].

Obtaining explicit solutions for integrable equations is the main mission in nonlinear science research. Many methods have been developed to solve integrable equations, such as Darboux transformation [9–15], inverse scattering transformation [16], algebra-geometric approach [17], Lie symmetry method [18], and Hirota bilinear method [19]. Darboux transformation is a useful tool for solving integrable equations. It can obtain its nontrivial solutions in accordance with an arbitrary seed solution of the integrable equations. Solving integrable coupled equations by the Darboux transformation method is a meaningful investigation. Explicit solutions of an integrable coupled system of Merola-Ragnisco-Tu lattice equation (24) are investigated, and explicit solutions of a new discrete integrable soliton hierarchy with 4×4 Lax pair [20] are discussed by the Darboux transformation.

In this paper, our main consideration is the following semidiscrete integrable coupled equations:

$$\begin{cases} q_{n,t} = -\dfrac{q_n}{r_n q_{n+1}} + \dfrac{1}{r_n} + q_n, \\ r_{n,t} = \dfrac{r_n}{r_{n-1} q_n} - \dfrac{1}{q_n} - r_n, \\ u_{n,t} = \dfrac{u_{n+1} q_n}{q_{n+1}^2 r_{n-1}} - \dfrac{u_n}{r_n q_{n+1}} + \dfrac{1}{r_n} - \dfrac{s_n}{r_n^2} + u_n, \\ s_{n,t} = \dfrac{u_n (r_{n-1} - r_n)}{q_n^2 r_{n-1}} + \dfrac{s_n (r_{n-1} - r_n)}{q_n r_{n-1}^2} + \dfrac{r_n}{r_{n-1} q_n} - \dfrac{1}{q_n} - s_n, \end{cases} \tag{1}$$

where q_n, r_n, u_n, and s_n are the potentials. The spectral problems deriving the semidiscrete integrable coupled system are an extension of a spectral problem introduced by Sun et al. [21]. Another extension of the spectral problems in [21] was investigated by Xue et al. [22]. They also constructed infinitely many conversation laws and Darboux transformations for the first nonlinear integrable equations. In this paper, we will concentrate on investigating explicit solutions of Equation (1) by means of the Darboux transformation method based on its Lax pair.

The outline of this paper is as follows. In Section 2, we will derive Equation (1) by means of the discrete zero-curvature equations and construct the Darboux transformation for Equation (1) based on its spectral problems. In Section 3, we will obtain the explicit solutions of Equation (1) and discuss the properties of solutions by means of different figures. In Section 4, some conclusions will be given.

2. A New Integrable Coupled Equation and Its Darboux Transformation

2.1. Constructing for a New Integrable Coupled Equation. In order to obtain Equation (1), we consider the following discrete spectral problem:

$$E\varphi_n = U_n(u,\lambda)\varphi_n,\ U_n = \begin{pmatrix} 0 & \lambda q_n & 0 & \lambda u_n \\ -\lambda r_n & \lambda^2 q_n r_n + 1 & -\lambda s_n & \lambda^2(q_n s_n + u_n r_n) \\ 0 & 0 & 0 & \lambda q_n \\ 0 & 0 & -\lambda r_n & \lambda^2 q_n r_n + 1 \end{pmatrix}, \tag{2}$$

and its auxiliary problem

$$\varphi_{n,t_m} = \Gamma_n \varphi_n,\ \Gamma_n = \begin{pmatrix} \sum_{i\geq 0} a_n^{(i)}\lambda^{-2i} & \sum_{i\geq 0} b_n^{(i)}\lambda^{-2i+1} & \sum_{i\geq 0} e_n^{(i)}\lambda^{-2i} & \sum_{i\geq 0} f_n^{(i)}\lambda^{-2i+1} \\ \sum_{i\geq 0} c_n^{(i)}\lambda^{-2i+1} & -\sum_{i\geq 0} a_n^{(i)}\lambda^{-2i} & \sum_{i\geq 0} g_n^{(i)}\lambda^{-2i+1} & -\sum_{i\geq 0} e_n^{(i)}\lambda^{-2i} \\ 0 & 0 & \sum_{i\geq 0} a_n^{(i)}\lambda^{-2i} & \sum_{i\geq 0} b_n^{(i)}\lambda^{-2i+1} \\ 0 & 0 & \sum_{i\geq 0} c_n^{(i)}\lambda^{-2i+1} & -\sum_{i\geq 0} a_n^{(i)}\lambda^{-2i} \end{pmatrix}, \tag{3}$$

where λ is independent of t and E is a shift operator defined by $Ef(n) = Ef(n+1)$.

Then, we embed a free function $\delta_n^{(m)}$ into the following modification term $\Delta_n^{(m)}$:

$$\Delta_n^{(m)} = \begin{pmatrix} \delta_n^{(m)} & 0 & 0 & 0 \\ 0 & a_n^{(m)} & 0 & e_n^{(m)} \\ 0 & 0 & \delta_n^{(m)} & 0 \\ 0 & 0 & 0 & a_n^{(m)} \end{pmatrix}. \tag{4}$$

Let $V_n^{(m)} = \lambda^{2m}\Gamma_n^{(m)} + \Delta_n^{(m)}$. Solving the stationary discrete zero-curvature equations $(E\Gamma_n)U_n - U_n\Gamma_n = 0$ and the discrete zero-curvature equations $U_{n,t} - E(V_n^{(m)})U_n + U_n V_n^{(m)} = 0$, we can obtain the following integrable coupled hierarchy:

$$\begin{cases} q_{n,t} = a_{n+1}^m q_n + b_{n+1}^{(m)} + q_n \delta_{n+1}^{(m)}, \\ r_{n,t} = -a_n^{(m)} r_n + c_n^{(m)} - r_n \delta_n^{(m)}, \\ u_{n,t} = e_{n+1}^{(m)} q_n + a_{n+1}^{(m)} u_n + f_{n+1}^{(m)} + u_n \delta_{n+1}^{(m)}, \\ s_{n,t} = -e_n^{(m)} r_n - a_n^{(m)} s_n + g_n^{(m)} - s_n \delta_n^{(m)}. \end{cases} \tag{5}$$

Then, the semidiscrete integrable coupled Equation (1) can be obtained by setting $m = 1$ and $\delta_n^{(1)} = 1$. The detailed symbolic computation steps are similar to [22].

2.2. Darboux Transformation. In this section, we will investigate the Darboux transformation of Equation (1).

The spectral problems of Equation (1) are presented by

$$E\varphi_n = U_n(u,\lambda)\varphi_n,\ U_n = \begin{pmatrix} 0 & \lambda q_n & 0 & \lambda u_n \\ -\lambda r_n & \lambda^2 q_n r_n + 1 & -\lambda s_n & \lambda^2(q_n s_n + u_n r_n) \\ 0 & 0 & 0 & \lambda q_n \\ 0 & 0 & -\lambda r_n & \lambda^2 q_n r_n + 1 \end{pmatrix}, \tag{6}$$

$$\varphi_{n,t_1} = V_n^{(1)}\varphi_n, \quad V_n^{(1)} = \begin{pmatrix} V_{11} & \dfrac{\lambda}{r_{n-1}} & V_{13} & \lambda\dfrac{r_{n-1} - s_{n-1}}{r_{n-1}^2} \\ -\dfrac{\lambda}{q_n} & \dfrac{1}{2}\lambda^2 & \lambda\dfrac{u_n - q_n}{q_n^2} & \dfrac{1}{2}\lambda^2 \\ 0 & 0 & V_{11} & \dfrac{\lambda}{r_{n-1}} \\ 0 & 0 & -\dfrac{\lambda}{q_n} & \dfrac{1}{2}\lambda^2 \end{pmatrix}, \tag{7}$$

where

$$\begin{aligned} V_{11} &= -\frac{1}{2}\lambda^2 - \frac{1}{q_n r_{n-1}} + 1, \\ V_{13} &= -\frac{1}{2}\lambda^2 + \frac{u_n}{q_n^2 r_{n-1}} - \frac{1}{r_{n-1}q_n} + \frac{s_n}{q_n r_{n-1}^2}. \end{aligned} \tag{8}$$

Firstly, we choose a proper Darboux matrix:

$$T_n = \begin{pmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{pmatrix} = \begin{pmatrix} a_{2,n}\lambda^2 + a_{0,n} & b_{1,n}\lambda & e_{2,n}\lambda^2 + e_{0,n} & f_{1,n}\lambda \\ c_{1,n}\lambda & a_{0,n}\lambda^2 + 1 & g_{1,n}\lambda & e_{0,n}\lambda^2 + 1 \\ 0 & 0 & a_{2,n}\lambda^2 + a_{0,n} & b_{1,n}\lambda \\ 0 & 0 & c_{1,n}\lambda & a_{0,n}\lambda^2 + 1 \end{pmatrix}. \tag{9}$$

The Darboux transformation can transform U_n, V_n into $\tilde{U}_n, \tilde{V}_n$, i.e.,

$$\begin{aligned} E\tilde{\varphi}_n &= \tilde{U}_n\tilde{\varphi}_n, \\ \tilde{\varphi}_t &= \tilde{V}_n\tilde{\varphi}_n, \end{aligned} \tag{10}$$

where

$$\begin{aligned} \tilde{\varphi}_n &= T_n\varphi_n, \\ \tilde{U}_n &= T_{n+1}U_nT_n^{-1}, \\ \tilde{V}_n &= (T_{n,t} + T_nV_n)T_n^{-1}. \end{aligned} \tag{11}$$

And the potential functions $\tilde{U}_n$ and $\tilde{V}_n$ have the same form as U_n and V_n, respectively. Then, we need to define a solution matrix Φ_n; it can be represented as

$$\Phi_n = \begin{pmatrix} \varphi_1[n] & \psi_1[n] \\ \varphi_2[n] & \psi_2[n] \\ \varphi_3[n] & \psi_3[n] \\ \varphi_4[n] & \psi_4[n] \end{pmatrix}. \tag{12}$$

If we assume that the terms both $\varphi[n] = (\varphi_1[n], \varphi_2[n], \varphi_3[n], \varphi_4[n])^T$ and $\psi[n] = (\psi_1[n], \psi_2[n], \psi_3[n], \psi_4[n])^T$ are two linear independent solutions of the 4×4 Lax pair of Equation (1), we can obtain

$$\begin{aligned} \delta_{i,1}[n] &= \frac{\varphi_{2,n} - \kappa_i\psi_{2,n}}{\varphi_{1,n} - \kappa_i\psi_{1,n}}, \\ \delta_{i,2}[n] &= \frac{\varphi_{3,n} - \kappa_i\psi_{3,n}}{\varphi_{1,n} - \kappa_i\psi_{1,n}}, \\ \delta_{i,3}[n] &= \frac{\varphi_{4,n} - \kappa_i\psi_{4,n}}{\varphi_{1,n} - \kappa_i\psi_{1,n}}, \end{aligned} \tag{13}$$

by means of the transformation

$$\tilde{\Phi}_n = T_n\Phi_n. \tag{14}$$

So we have an algebraic system:

$$\begin{cases} \lambda_i^2 a_{2,n} + a_{0,n} + b_{1,n}\lambda_i\delta_{i,1}[n] + \left(e_{2,n}\lambda_i^2 + e_{0,n}\right)\delta_{i,2}[n] + f_{1,n}\lambda_i = 0, \\ c_{1,n}\lambda_i + \left(\lambda_i^2 + a_{0,n} + 1\right)\delta_{i,1}[n] + g_{1,n}\lambda_i\delta_{i,2}[n] + \left(\lambda_i^2 e_{0,n} + 1\right)\delta_{i,3}[n] = 0, \\ \left(\lambda_i^2 a_{2,n} + a_{0,n}\right)\delta_{i,1}[n] + b_{1,n}\lambda_i\delta_{i,3}[n] = 0, \\ c_{1,n}\lambda_i\delta_{i,2}[n] + \left(a_{0,n}\lambda_i^2 + 1\right)\delta_{i,3}[n] = 0. \end{cases} \tag{15}$$

Solving the linear algebraic system, we can obtain the expressions for $a_{0,n}, a_{2,n}, b_{1,n}, c_{1,n}, e_{0,n}, e_{2,n}, f_{1,n}, g_{1,n}$ in Appendix A. Furthermore, the following equations can be obtained by Equation (2):

$$\begin{aligned} \delta_{i,1}[n+1] &= \frac{-\lambda_i + \left(1 + \lambda_i^2\right)\delta_{i,1}[n] - \lambda_i\delta_{i,2}[n] + 2\lambda^2\delta_{i,3}[n]}{\lambda_i\delta_{i,1}[n] + \lambda_i\delta_{i,3}[n]}, \\ \delta_{i,2}[n+1] &= \frac{\delta_{i,2}[n]}{\delta_{i,1}[n] + \delta_{i,3}[n]}, \\ \delta_{i,3}[n+1] &= \frac{-\lambda_i\delta_{i,2}[n] + \left(1 + \lambda_i^2\right)\delta_{i,3}[n]}{\lambda_i\delta_{i,1}[n] + \lambda_i\delta_{i,3}[n]}. \end{aligned} \tag{16}$$

From Equation (9) to Equation (16), it can be seen that $\pm\lambda_1$ and $\pm\lambda_2$ are the roots of $\mathrm{Det}(T_n) = 0$. So, we have

$$\mathrm{Det}(T_n) = \alpha_n(t)(\lambda - \lambda_1)^2(\lambda + \lambda_1)^2(\lambda - \lambda_2)^2(\lambda + \lambda_2)^2, \tag{17}$$

where $\alpha_n(t)$ is a function with respect to t. From the above conclusion, we can prove the following proposition.

Proposition 1. *The form of the matrix* $\tilde{U}_n$ *is*

$$\tilde{U}_n = \begin{pmatrix} 0 & \lambda\tilde{q}_n & 0 & \lambda\tilde{u}_n \\ -\lambda\tilde{r}_n & \lambda^2\tilde{q}_n\tilde{r}_n + 1 & -\lambda\tilde{s}_n & \lambda^2(\tilde{q}_n\tilde{s}_n + \tilde{u}_n\tilde{r}_n) \\ 0 & 0 & 0 & \lambda\tilde{q}_n \\ 0 & 0 & -\lambda\tilde{r}_n & \lambda^2\tilde{q}_n\tilde{r}_n + 1 \end{pmatrix}, \tag{18}$$

and the transformation from the old potentials q_n, r_n, u_n, s_n *to new ones* $\tilde{q}_n, \tilde{r}_n, \tilde{u}_n, \tilde{s}_n$ *is given by*

$$\begin{cases} \tilde{q}_n = a_{0,n+1}q_n + b_{1,n+1}, \\ \tilde{r}_n = \dfrac{r_n + c_{1,n}}{a_{0,n}}, \\ \tilde{u}_n = a_{0,n+1}(u_n - q_n) + e_{0,n+1}q_n + f_{1,n+1} - b_{1,n+1}, \\ \tilde{s}_n = \dfrac{s_n}{a_{0,n}} + \dfrac{r_n}{a_{0,n}} + \dfrac{g_{1,n}}{a_{0,n}} - \dfrac{r_n e_{0,n}}{a_{0,n}^2} - \dfrac{c_{1,n}e_{0,n}}{a_{0,n}^2}. \end{cases} \tag{19}$$

Proof. Let $T_n^{-1} = T_n^*/\det(T_n)$ and

$$T_{n+1}U_nT_n^* = \begin{pmatrix} \Theta_{11,n}(\lambda) & \Theta_{12,n}(\lambda) & \Theta_{13,n}(\lambda) & \Theta_{14,n}(\lambda) \\ \Theta_{21,n}(\lambda) & \Theta_{22,n}(\lambda) & \Theta_{23,n}(\lambda) & \Theta_{24,n}(\lambda) \\ \Theta_{31,n}(\lambda) & \Theta_{32,n}(\lambda) & \Theta_{33,n}(\lambda) & \Theta_{34,n}(\lambda) \\ \Theta_{41,n}(\lambda) & \Theta_{42,n}(\lambda) & \Theta_{43,n}(\lambda) & \Theta_{44,n}(\lambda) \end{pmatrix}, \quad (20)$$

where all the expressions $\Theta_{ij}(\lambda), i, j = 1, 2, 3, 4$ are the functions with respect to n and t. It is easy to verify that the terms $\pm\lambda_1$ and $\pm\lambda_2$ are the roots of $\Theta_{ij}(\lambda), i, j = 1, 2, 3, 4$ except $\Theta_{31}(\lambda), \Theta_{32}(\lambda), \Theta_{41}(\lambda), \Theta_{42}(\lambda), \Theta_{11}(\lambda), \Theta_{13}(\lambda)$, and $\Theta_{33}(\lambda)$. We can also prove that $\Theta_{31}(\lambda), \Theta_{32}(\lambda), \Theta_{41}(\lambda), \Theta_{42}(\lambda), \Theta_{11}(\lambda), \Theta_{13}(\lambda), \Theta_{33}(\lambda) \equiv 0$, the terms $\Theta_{12,n}(\lambda)$, $\Theta_{14,n}(\lambda), \Theta_{21,n}(\lambda), \Theta_{23,n}(\lambda), \Theta_{34,n}(\lambda)$, and $\Theta_{43,n}(\lambda)$ are ninth-order polynomials with respect to λ, and the terms $\Theta_{22,n}(\lambda)$, $\Theta_{24,n}(\lambda)$, and $\Theta_{44,n}(\lambda)$ are tenth-order polynomials with respect to λ. The equation $T_{n+1}U_nT_n^* = \det(T_n)P_n$ can be written as $T_{n+1}U_n = P_nT_n$, with

$$P_n = \begin{pmatrix} 0 & P_{12}\lambda & 0 & P_{14}\lambda \\ P_{21}\lambda & P_{22}^{(2)}\lambda^2 + P_{22}^{(0)} & P_{23}\lambda & P_{24}\lambda^2 \\ 0 & 0 & 0 & P_{12}\lambda \\ 0 & 0 & P_{21}\lambda & P_{22}^{(2)}\lambda^2 + P_{22}^{(0)} \end{pmatrix}. \quad (21)$$

By comparing the coefficients in $T_{n+1}U_n = P_nT_n$, we find

$$\begin{aligned} P_{12} &= a_{0,n+1}q_n + b_{1,n+1} = \tilde{q}_n, \\ P_{21} &= \frac{-r_n - c_{1,n}}{a_{0,n}} = -\tilde{r}_n, P_{22}^{(0)} = 1, \\ P_{22}^{(2)} &= \frac{1}{a_{0,n}}(a_{0,n+1}q_n + b_{1,n+1})(r_n + c_{1,n}) = \tilde{q}_n\tilde{r}_n, \\ P_{14} &= a_{0,n+1}(u_n - q_n) + e_{0,n+1}q_n + f_{1,n+1} - b_{1,n+1} = \tilde{u}_n, \\ P_{23} &= -\frac{s_n}{a_{0,n}} - \frac{r_n}{a_{0,n}} - \frac{g_{1,n}}{a_{0,n}} + \frac{r_ne_{0,n}}{a_{0,n}^2} + \frac{c_{1,n}e_{0,n}}{a_{0,n}^2} = -\tilde{s}_n, \\ P_{24} &= (a_{0,n+1}q_n + b_{1,n+1})\left(\frac{s_n}{a_{0,n}} + \frac{r_n}{a_{0,n}} + \frac{g_{1,n}}{a_{0,n}} - \frac{r_ne_{0,n}}{a_{0,n}^2} - \frac{c_{1,n}e_{0,n}}{a_{0,n}^2}\right) \\ &\quad + \left(a_{0,n+1}(u_n - q_n) + a_{0,n+1}q_n + f_{1,n+1} - b_{1,n+1}\right)\left(\frac{r_n + c_{1,n}}{a_{0,n}}\right) \\ &= \tilde{q}_n\tilde{s}_n + \tilde{u}_n\tilde{r}_n. \end{aligned} \quad (22)$$

From the above equations, we can see $P_n = \tilde{U}_n$. The proof is completed. □

Proposition 2. *Under the transformations Equations (10) and (19), the term $\tilde{V}_n$ has the form*

$$\tilde{V}_n = \begin{pmatrix} -\frac{1}{2}\lambda^2 - \frac{1}{\tilde{q}_n\tilde{r}_{n-1}} + 1 & \frac{\lambda}{\tilde{r}_{n-1}} & -\frac{1}{2}\lambda^2 + \frac{\tilde{u}_n}{\tilde{q}_n^2\tilde{r}_{n-1}} - \frac{1}{\tilde{r}_{n-1}\tilde{q}_n} + \frac{\tilde{s}_n}{\tilde{q}_n\tilde{r}_{n-1}^2} & \lambda\frac{\tilde{r}_{n-1} - \tilde{s}_{n-1}}{\tilde{r}_{n-1}^2} \\ -\frac{\lambda}{\tilde{q}_n} & \frac{1}{2}\lambda^2 & \lambda\frac{\tilde{u}_n - \tilde{q}_n}{\tilde{q}_n^2} & \frac{1}{2}\lambda^2 \\ 0 & 0 & -\frac{1}{2}\lambda^2 - \frac{1}{\tilde{q}_n\tilde{r}_{n-1}} + 1 & \frac{\lambda}{\tilde{r}_{n-1}} \\ 0 & 0 & -\frac{\lambda}{\tilde{q}_n} & \frac{1}{2}\lambda^2 \end{pmatrix}. \quad (23)$$

Proof.

$$\left(T_{n,t} + T_nV_n^{(1)}\right)T_n^* = \begin{pmatrix} \Omega_{11,n}(\lambda) & \Omega_{12,n}(\lambda) & \Omega_{13,n}(\lambda) & \Omega_{14,n}(\lambda) \\ \Omega_{21,n}(\lambda) & \Omega_{22,n}(\lambda) & \Omega_{23,n}(\lambda) & \Omega_{24,n}(\lambda) \\ \Omega_{31,n}(\lambda) & \Omega_{32,n}(\lambda) & \Omega_{33,n}(\lambda) & \Omega_{34,n}(\lambda) \\ \Omega_{41,n}(\lambda) & \Omega_{42,n}(\lambda) & \Omega_{43,n}(\lambda) & \Omega_{44,n}(\lambda) \end{pmatrix}, \quad (24)$$

where all the expressions $\Omega_{ij}(i, j = 1, 2)$ are the functions with respect to n and t. It is obvious that the terms $\Omega_{11,n}(\lambda), \Omega_{13,n}(\lambda), \Omega_{22,n}(\lambda), \Omega_{24,n}(\lambda), \Omega_{33,n}(\lambda)$, and $\Omega_{34,n}(\lambda)$ are tenth-order polynomials with respect to λ, the terms $\Omega_{14,n}(\lambda), \Omega_{21,n}(\lambda), \Omega_{23,n}(\lambda), \Omega_{34,n}(\lambda)$, and $\Omega_{43,n}(\lambda)$ are ninth-order polynomials with respect to λ, the term $\Omega_{12,n}(\lambda)$ is eighth-order polynomials with respect to λ, and the terms $\Omega_{31,n}(\lambda), \Omega_{32,n}(\lambda), \Omega_{41,n}(\lambda)$, and $\Omega_{42,n}(\lambda)$ are all zero. In addition, the terms $\pm\lambda_1$ and $\pm\lambda_2$ are the roots of the $\Omega_{ij}(\lambda)$. So, we have

$$\left(T_{n,t}+T_nV_n^{(1)}\right)T_n^*=(\det T_n)Q_n, \tag{25}$$

with

$$Q_n=\begin{pmatrix} Q_{11}^{(2)}\lambda^2+Q_{11}^{(0)} & Q_{12}^{(1)} & Q_{13}^{(2)}\lambda^2+Q_{13}^{(0)} & Q_{14}^{(1)}\lambda \\ Q_{21}^{(1)}\lambda & Q_{22}^{(2)}\lambda^2 & Q_{13}^{(1)}\lambda & Q_{24}^{(2)}\lambda^2 \\ 0 & 0 & Q_{11}^{(2)}\lambda^2+Q_{11}^{(0)} & Q_{12}^{(1)}\lambda \\ 0 & 0 & Q_{21}^{(1)}\lambda & Q_{22}^{(2)}\lambda^2 \end{pmatrix}. \tag{26}$$

Equation (25) can be rewritten as

$$T_{n,t}+T_nV_n^{(1)}=Q_nT_n. \tag{27}$$

By comparing the coefficients in Equation (27), we obtain

$$\begin{aligned} Q_{11}^{(2)}&=-\frac{1}{2},\\ Q_{11}^{(0)}&=-\frac{1}{\tilde{q}_n\tilde{r}_{n-1}}+1,\\ Q_{12}&=\frac{1}{r_{n-1}},\\ Q_{13}^{(2)}&=-\frac{1}{2},\\ Q_{13}^{(0)}&=\frac{\tilde{u}_n}{\tilde{q}_n^2\tilde{r}_{n-1}}-\frac{1}{\tilde{q}_n\tilde{r}_{n-1}}+\frac{\tilde{s}_n}{\tilde{q}_n\tilde{r}_{n-1}^2},\\ Q_{14}&=\frac{\tilde{r}_{n-1}-\tilde{s}_{n-1}}{\tilde{r}_{n-1}^2},\\ Q_{21}&=-\frac{1}{\tilde{q}_n},\\ Q_{22}&=\frac{1}{2},\\ Q_{23}&=\frac{\tilde{u}_n-\tilde{q}_n}{\tilde{q}_n^2},\\ Q_{24}&=\frac{1}{2}. \end{aligned} \tag{28}$$

It is obvious that $Q_n=\tilde{V}_n$. The proof is completed. □

Equations (14) and (19) constitute the Darboux transformation for the semidiscrete integrable coupled systems based on Equations (6) and (7).

3. Explicit Solutions

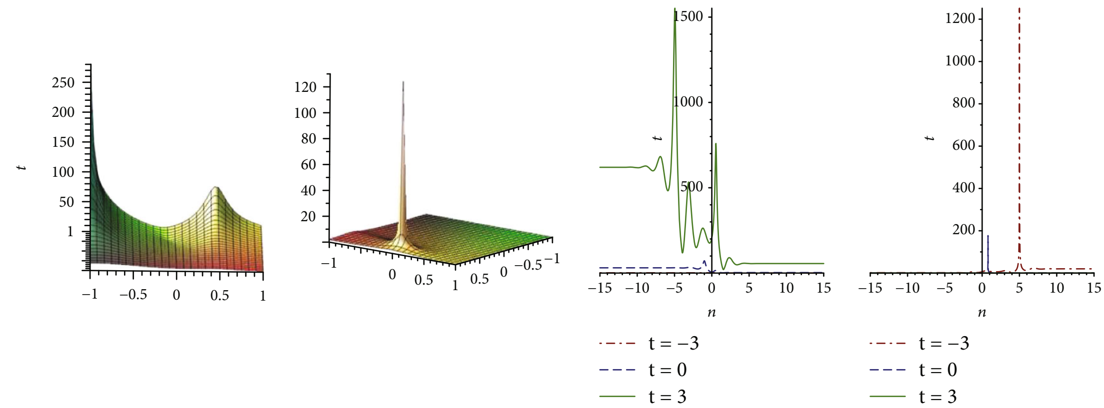
3.1. Explicit Solutions for Equation (1). In order to obtain the explicit solutions of the semidiscrete integrable coupled equations (1), we choose the trivial solution $q_n=u_n=e^t$, $r_n=s_n=e^{-t}$. Then, the spectral problems Equations (2) and (7) become

$$\begin{aligned} E\varphi_n&=\begin{pmatrix} 0 & \lambda e^t & 0 & \lambda e^t \\ -\lambda e^{-t} & \lambda^2 & -\lambda e^{-t} & 2\lambda^2 \\ 0 & 0 & 0 & \lambda e^t \\ 0 & 0 & -\lambda e^{-t} & \lambda^2 \end{pmatrix}\varphi_n,\\ \varphi_{n,t}&=\begin{pmatrix} -\frac{1}{2}\lambda^2 & \lambda e^t & -\frac{1}{2}\lambda^2+1 & 0 \\ -\lambda e^{-t} & \frac{1}{2}\lambda^2 & 0 & \frac{1}{2}\lambda^2 \\ 0 & 0 & -\frac{1}{2}\lambda^2 & \lambda e^t \\ 0 & 0 & -\lambda e^{-t} & \frac{1}{2}\lambda^2 \end{pmatrix}\varphi_n. \end{aligned} \tag{29}$$

By solving Equation (29), we achieve the two basic solutions of Equation (1), φ_n and ψ_n,

$$\varphi_n=\begin{pmatrix} \varphi_{1,n} \\ \varphi_{2,n} \\ \varphi_{3,n} \\ \varphi_{4,n} \end{pmatrix}=\begin{pmatrix} -\dfrac{\lambda^3\left(\lambda^2t-t+1\right)}{2\lambda^2-2}e^{-1/2\left(\lambda^2-2\right)t} \\ \dfrac{2-\lambda^4+(t-3)\lambda^2}{2\lambda^2-2}e^{-1/2\lambda^2t} \\ \lambda e^{-1/2\left(\lambda^2-2\right)t} \\ e^{1/2\lambda^2t} \end{pmatrix},$$

$$\psi_n=\begin{pmatrix} \psi_{1,n} \\ \psi_{2,n} \\ \psi_{3,n} \\ \psi_{4,n} \end{pmatrix}=\begin{pmatrix} \dfrac{\lambda^{2n-1}\left(\lambda^2+2\right)\left(\lambda^2t-t-1\right)}{2\lambda^2-2}e^{1/2\lambda^2t} \\ \dfrac{\lambda^{2n-2}\left[\left(\lambda^2+2\right)\left(\lambda^4t-\lambda^2t-1\right)+\left(\lambda^2-2\right)\left(\lambda^2-1\right)\right]}{2\lambda^2-2}e^{\left(\frac{1}{2}\lambda^2-1\right)t} \\ \lambda^{2n-1}e^{1/2\lambda^2t} \\ \lambda^{2n}e^{1/2\left(\lambda^2-2\right)t} \end{pmatrix}. \tag{30}$$



(a) q_n (b) r_n (c) q_n in Equation (24) (d) r_n in Equation (29)

Figure 1: Three-dimensional structure figures of explicit solutions from Equation (32) and the evolutions of these solutions with parameters $\lambda_1 = 0.3$, $\lambda_2 = 0.6I$, $\kappa_1 = \sqrt{3}I$, and $\kappa_2 = \sqrt{2}$.

Then, based on Equation (13), we can obtain

$$
\begin{aligned}
\delta_{i,1}[n] &= \frac{-\left((-2t-4)\lambda_i^{2n+2} + (t+1)\lambda_i^{2n+4} + t\lambda_i^{2n+6}\right)\kappa_i e^{t\left(\lambda_i^2-1\right)} + \left(-\lambda_i^6 + \lambda_i^4\right)t - 3\lambda_i^4 + 2\lambda_i^2}{2\left(\kappa_i\left((-1/2t+1/2)\lambda_i^{2n+2} - 1/2\lambda_i^{2n+4}t + \lambda_i^{2n}(t+1)\right)e^{t\lambda_i^2} - 1/2\lambda_i^3 e^t(\lambda_i+1)\left(t\lambda_i^2 - t + 1\right)\right)\lambda_i},\\
\delta_{i,2}[n] &= -\frac{\left(2\left(\lambda_i^2-1\right)\right)\left(\lambda_i^{2n} e^{\left(1/2t\lambda_i^2\right)}\kappa_i - \lambda_i^2\left(-1/2t\lambda_i^2 + t\right)\right)}{-\lambda_i^3(\lambda_i+1)\left(t\lambda_i^2 - t + 1\right)e^{-1/2t\lambda_i^2+t} + 2e^{\left(1/2t\lambda_i^2\right)}\left((-1/2t+1/2)\lambda_i^{2n+2} - 1/2\lambda_i^{2n+4}t + \lambda_i^{2n}(t+1)\right)\kappa_i},\\
\delta_{1,3}[n] &= \frac{\left(-2\kappa_i\lambda_i^2 + 2\kappa_i\right)\lambda_i^{2n+1}e^{t\left(\lambda_i^2-1\right)} + 2\lambda_i^3 - 2\lambda_i}{2\kappa_i\left((-1/2t+1/2)\lambda_i^{2n+2} - 1/2\lambda_i^{2n+4}t + \lambda_i^{2n}(t+1)\right)e^{t\lambda_i^2} - \lambda_i^3 e^t(\lambda_i+1)\left(t\lambda_i^2 - t + 1\right)}.
\end{aligned}
\tag{31}
$$

According to the transformations Equations (19), (16), and (31), we can obtain the explicit solutions of Equation (1) by symbolic computing software maple. It is presented by $\delta_{i,1}[n]$, $\delta_{i,2}[n]$, $\delta_{i,3}[n]$ in Appendix B.

3.2. Discussions. The integrable equations investigated in [22] are presented as follows:

$$
\begin{cases}
q_{n,t} = -\dfrac{q_n}{q_{n+1}r_n} + \dfrac{1}{r_n} + q_n,\\
r_{n,t} = \dfrac{r_n}{q_n r_{n-1}} - \dfrac{1}{q_n} - r_n.
\end{cases}
\tag{32}
$$

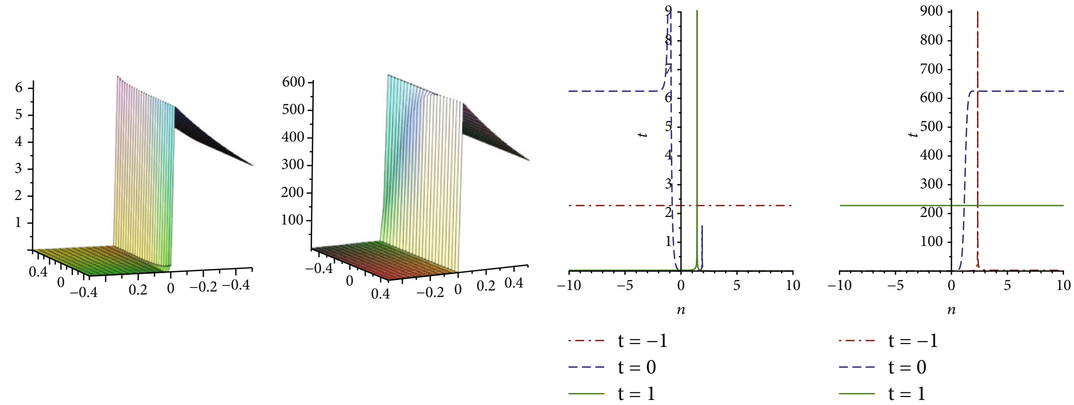
The explicit solutions of Equation (32) by Darboux transformation method are given in [22]

$$
\begin{cases}
\tilde{q}_n = \dfrac{-\rho_1\kappa_1\lambda_1^{2n}\lambda_2^2\left(\lambda_1^2-1\right)e^{2t} + \rho_2\kappa_2\lambda_2^{2n}\left(\rho_1\rho_3 e^t - \lambda_1^2 e^{2t} + \lambda_1^2\lambda_2^2 e^{2t}\right)}{\rho_1\kappa_1\lambda_1^{2n+2}\lambda_2^2\left(\lambda_1^2-1\right)e^t - \rho_2\kappa_2\lambda_1^2\lambda_2^{2n+2}\left(\rho_1\rho_3 - e^t + \lambda_2^2 e^t\right)},\\
\tilde{r}_n = \dfrac{\lambda_1^2 + \lambda_2^2 - \rho_1\kappa_1\lambda_1^{2n+2}\left(\lambda_2^2-1\right)e^{-t} + \rho_2\kappa_2\lambda_2^{2n+2}\left(\lambda_1^2-1\right)e^{-t}}{\lambda_1^2 e^t - \lambda_2^2 e^t + \rho_1\kappa_1\lambda_1^{2n}\left(\lambda_2^2-1\right) - \rho_2\kappa_2\lambda_2^{2n}\left(\lambda_1^2-1\right)},
\end{cases}
\tag{33}
$$

where $\rho_1 = e^{\lambda_1^2 t}$, $\rho_2 = e^{\lambda_2^2 t}$, and $\rho_3 = \kappa_1\lambda_1^{2n}(\lambda_1^2 - \lambda_2^2)$.

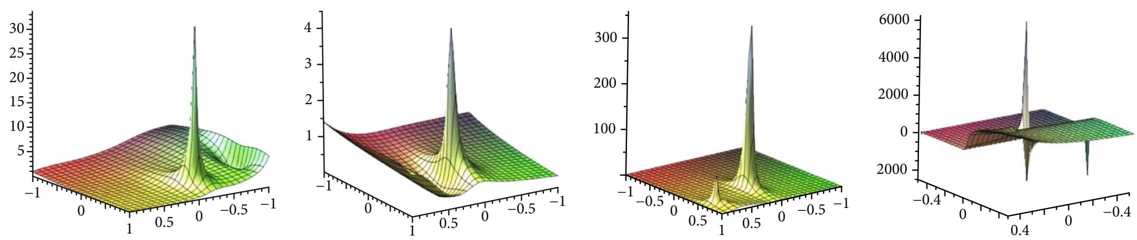
In order to compare the differences of the explicit solutions between integrable coupled Equation (1) and integrable Equation (32), we first investigate the three-dimensional structure and the evolution properties of explicit solutions Equation (33) with two sets of parameters, $\lambda_1 = 0.4$, $\lambda_2 = 25$, $\kappa_1 = 32$, and $\kappa_2 = 0.8$ and $\lambda_1 = 0.3$, $\lambda_2 = 0.6I$, $\kappa_1 = \sqrt{3}I$, and $\kappa_2 = \sqrt{2}$ in Figures 1 and 2. Then, we plot the three-dimensional structures, density figures, the evolution properties of $\tilde{q}_n$, $\tilde{r}_n$, and $\tilde{s}_n$ with parameters $\lambda_1 = 0.3$, $\lambda_2 = 0.6I$, and $\kappa_1 = \sqrt{3}I$, and $\tilde{u}_n$ with parameters $\lambda_1 = 0.4$, $\lambda_2 = 25$, $\kappa_1 = 32$, and $\kappa_2 = 0.8$ in Figures 3–5 (The expression for $\tilde{u}_n$ is so complicated that Maple cannot calculate its figure when the parameters contain a complex number. The specific expression of $\tilde{u}_n$ is in the appendix.).

From Figures 1 and 2, it can be observed that the solitary waves move from right to left whether the parameter contains complex numbers or not. From the evolutions of solutions in Figure 1, we see that these one-soliton solutions are not stable. From Figure 2, we observed that $\tilde{q}_n$ and $\tilde{r}_n$ have the kink-shaped structures when we take the appropriate parameters. From Figure 5, it is easy to see that the solitary waves of $\tilde{q}_n$, $\tilde{r}_n$, and $\tilde{s}_n$ also move from right to left, and they are also not stable. From Figures 1–5, we find that the



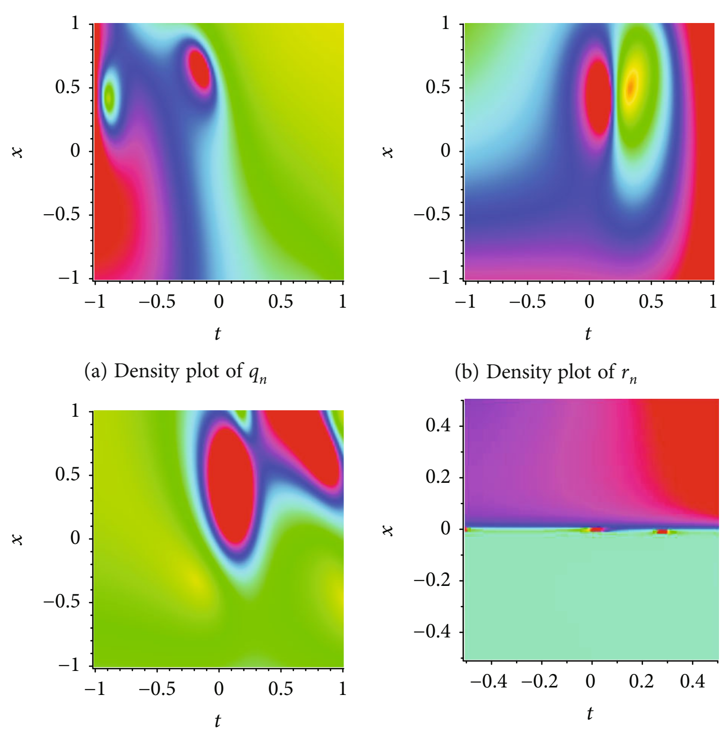
(a) q_n (b) r_n (c) q_n in Equation (29) (d) r_n in Equation (29)

Figure 2: Three-dimensional structure figures of explicit solutions from Equation (32) and the evolutions of these solutions with parameters $\lambda_1 = 0.4$, $\lambda_2 = 25$, $\kappa_1 = 32$, and $\kappa_2 = 0.8$.



(a) q_n (b) r_n (c) s_n (d) u_n

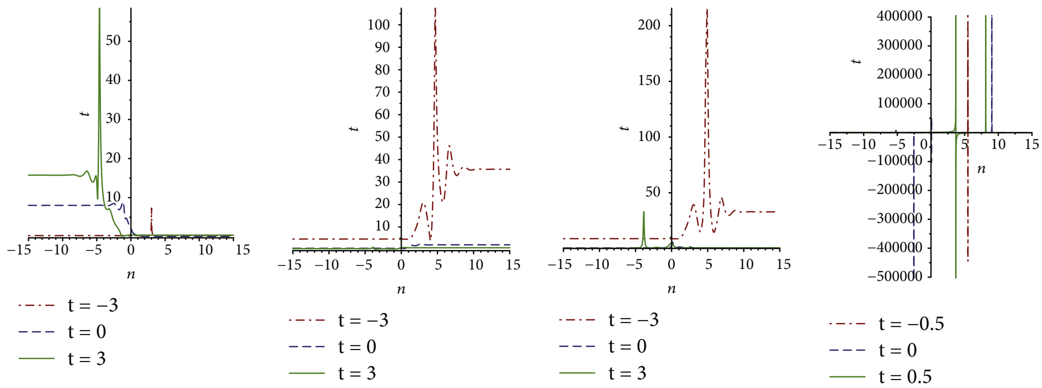
Figure 3: Three-dimensional structure figures of $\tilde{q}_n$, $\tilde{r}_n$, and $\tilde{s}_n$ with parameters $\lambda_1 = 0.3$, $\lambda_2 = 0.6I$, $\kappa_1 = \sqrt{3}I$, and $\kappa_2 = \sqrt{2}$ and density figures of $\tilde{u}_n$ with parameters $\lambda_1 = 0.4$, $\lambda_2 = 25$, $\kappa_1 = 32$, and $\kappa_2 = 0.8$.



(a) Density plot of q_n (b) Density plot of r_n

(c) Density plot of s_n (d) Density plot of u_n

Figure 4: Density figures of $\tilde{q}_n$, $\tilde{r}_n$, and $\tilde{s}_n$ with parameters $\lambda_1 = 0.3$, $\lambda_2 = 0.6I$, $\kappa_1 = \sqrt{3}I$, and $\kappa_2 = \sqrt{2}$ and density figures of $\tilde{u}_n$ with parameters $\lambda_1 = 0.4$, $\lambda_2 = 25$, $\kappa_1 = 32$, and $\kappa_2 = 0.8$.



(a) The evolution of q_n (b) The evolution of r_n (c) The evolution of s_n (d) The evolution of u_n

FIGURE 5: The evolution of solution $\tilde{q}_n$, $\tilde{r}_n$, and $\tilde{s}_n$ with parameters $\lambda_1 = 0.3$, $\lambda_2 = 0.6I$, $\kappa_1 = \sqrt{3}I$, and $\kappa_2 = \sqrt{2}$ and evolution of solution $\tilde{u}_n$ with parameters $\lambda_1 = 0.4$, $\lambda_2 = 25$, $\kappa_1 = 32$, and $\kappa_2 = 0.8$.

shapes, amplitudes, and wavelengths of solutions have a big difference before and after coupling, but the directions of solitary wave solution propagation have not changed.

4. Conclusions

In this paper, we have constructed a coupled 4×4 Lax pair by enlarging the 2×2 Lax pair and derived new semidiscrete integrable coupled equations which include the original integrable equations as subequations. Next, we have found a suitable Darboux matrix and obtained the explicit solutions of Equation (1) according to the seed solution $q_n = u_n = e^t$, $r_n = s_n = e^{-t}$ by means of the Darboux transformation method. Then, we plot three-dimensional structure figures, the evolution properties of solutions before and after coupling, and the density plot of solutions after coupling. Finally, we analyze the properties of solutions before and after coupling. All the results in this paper may be helpful in understanding some physical phenomena.

Appendix

A. Darboux Matrix in Equation (12)

$$a_{0,n} = \frac{-\delta_{1,2}[n]\delta_{2,3}[n]\lambda_1 + \delta_{1,3}[n]\delta_{2,2}[n]\lambda_2}{\lambda_1\lambda_2(\delta_{1,2}[n]\delta_{2,3}[n]\lambda_2 - \delta_{1,3}[n]\delta_{2,2}[n]\lambda_1)};$$

$$a_{2,n} = \frac{(\delta_{1,2}[n]\delta_{2,3}[n]\lambda_1 - \delta_{1,3}[n]\delta_{2,2}[n]\lambda_2)(-\delta_{1,1}[n]\delta_{2,3}[n]\lambda_2 + \delta_{1,3}[n]\delta_{2,1}[n]\lambda_1)}{\lambda_1^2\lambda_2^2(-\delta_{1,2}[n]\delta_{2,3}[n]\lambda_2 + \delta_{1,3}[n]\delta_{2,2}[n]\lambda_1)(\delta_{1,1}[n]\delta_{2,3}[n]\lambda_1 - \delta_{1,3}[n]\delta_{2,1}[n]\lambda_2)};$$

$$b_{1,n} = \frac{(\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2)(\delta_{1,2}[n]\delta_{2,3}[n]\lambda_1 - \delta_{1,3}[n]\delta_{2,2}[n]\lambda_2)\delta_{2,1}[n]\delta_{1,1}[n]}{\lambda_1^2\lambda_2^2(\delta_{1,2}[n]\delta_{2,3}[n]\lambda_2 - \delta_{1,3}[n]\delta_{2,2}[n]\lambda_1)(\delta_{1,1}[n]\delta_{2,3}[n]\lambda_1 - \delta_{1,3}[n]\delta_{2,1}[n]\lambda_2)};$$

$$c_{1,n} = -\frac{\delta_{1,3}[n]\delta_{2,3}[n]\left(\lambda_1^2 - \lambda_2^2\right)}{\lambda_1\lambda_2(\delta_{1,2}[n]\delta_{2,3}[n]\lambda_2 - \delta_{1,3}[n]\delta_{2,2}[n]\lambda_1)};$$

$$\begin{aligned}
e_{0,n} = {} & \frac{1}{\lambda_1\lambda_2(-\delta_{1,2}[n]\delta_{2,3}[n]\lambda_2 + \delta_{1,3}[n]\delta_{2,2}[n]\lambda_1)^2} \\
& \cdot \left(-\delta_{1,2}[n]^2\delta_{2,3}[n]^2\lambda_1\lambda_2 + \left(\left(\left(\left(\lambda_1^2 + \lambda_2^2\right)\delta_{2,3}[n] + \lambda_1^2\delta_{2,1}[n]\right.\right.\right.\right. \\
& \left. - \lambda_2^2\delta_{2,1}[n]\right)\delta_{2,2}[n] + \left(-\lambda_1^2 + \lambda_2^2\right)\delta_{2,3}[n]\big)\delta_{1,3}[n] \\
& - \delta_{1,1}[n]\delta_{2,2}[n]\delta_{2,3}[n](\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2)\big)\delta_{1,2}[n] \\
& + \delta_{1,3}[n]\delta_{2,2}[n]\left(-\lambda_1\lambda_2\delta_{2,2}[n]\delta_{1,3}[n] + \delta_{2,3}[n]\left(\lambda_1^2 - \lambda_2^2\right)\right)\big)\big);
\end{aligned}$$

$$\begin{aligned}
g_{1,n} = {} & -\frac{1}{\lambda_1\lambda_2(-\delta_{1,2}[n]\delta_{2,3}[n]\lambda_2 + \delta_{1,3}[n]\delta_{2,2}[n]\lambda_1)^2} \\
& \cdot \big((\lambda_1 - \lambda_2)\big(((\delta_{2,2}[n] - 1)\delta_{2,3}[n] + \delta_{2,1}[n]\delta_{2,2}[n])\lambda_1\delta_{1,3}[n]^2 \\
& - \lambda_2((\delta_{1,2}[n] - 1)\delta_{1,3}[n] + \delta_{1,1}[n]\delta_{1,2}[n])\delta_{2,3}[n]^2\big)(\lambda_1 + \lambda_2)\big);
\end{aligned}$$

$$\begin{aligned}
e_{2,n} = {} & \big(\big(-\delta_{1,2}[n]\delta_{1,3}[n]^2\delta_{2,1}[n] + \delta_{1,1}[n](\delta_{2,2}[n](\delta_{1,2}[n] + 1)\delta_{2,3}[n] \\
& + \delta_{1,2}[n]\delta_{2,1}[n](\delta_{1,1}[n] - \delta_{2,1}[n] + \delta_{2,2}[n]))\delta_{1,3}[n] \\
& - \delta_{1,1}[n]^2\delta_{1,2}[n]\delta_{2,2}[n]\delta_{2,3}[n])\delta_{2,2}[n]\delta_{2,3}[n]\lambda_1^4 \\
& - \big(-\delta_{1,3}[n]^3\delta_{2,1}[n]\delta_{2,2}[n]^2 + \delta_{2,2}[n]((\delta_{2,1}[n](\delta_{2,2}[n] - 1)\delta_{1,2}[n] \\
& + \delta_{2,2}[n](\delta_{1,1}[n]\delta_{2,2}[n] + \delta_{1,1}[n] + \delta_{2,1}[n]))\delta_{2,3}[n] \\
& + (\delta_{1,2}[n]\delta_{2,1}[n] + \delta_{1,1}[n](\delta_{1,1}[n] - \delta_{2,1}[n]))\delta_{2,2}[n]\delta_{2,1}[n])\delta_{1,3}[n]^2 \\
& + \delta_{1,2}[n](((\delta_{1,1}[n]\delta_{2,2}[n] - \delta_{1,1}[n] - \delta_{2,1}[n])\delta_{1,2}[n] \\
& + \delta_{1,1}[n]\delta_{2,2}[n])\delta_{2,3}[n] + \delta_{2,2}[n]\delta_{2,1}[n]\delta_{1,1}[n](\delta_{1,2}[n] \\
& - \delta_{2,2}[n]))\delta_{2,3}[n]\delta_{1,3}[n] + \delta_{1,2}[n]^2((\delta_{2,2}[n] + 1)\delta_{2,3}[n] \\
& - \delta_{1,1}[n]\delta_{2,2}[n] + \delta_{2,1}[n](\delta_{1,1}[n] - \delta_{2,1}[n]))\delta_{1,1}[n]\delta_{2,3}[n]^2\big)\lambda_2\lambda_1^3 \\
& + \lambda_2^2\big(\delta_{1,3}[n]^3\delta_{2,1}[n]\delta_{2,2}[n]^3 + \delta_{1,2}[n]((\delta_{2,1}[n](\delta_{2,2}[n] - 1)\delta_{1,2}[n] \\
& + \delta_{2,2}[n](\delta_{1,1}[n]\delta_{2,2}[n] + \delta_{2,1}[n]))\delta_{2,3}[n] + \delta_{1,2}[n]\delta_{2,1}[n]^2\delta_{2,2}[n])\delta_{1,3}[n]^2 \\
& - \delta_{2,2}[n]\delta_{2,3}[n]\big(\big(-\delta_{1,2}[n]^2\delta_{2,1}[n] - \delta_{1,1}[n](\delta_{2,2}[n] + 1)\delta_{1,2}[n] \\
& + \delta_{1,1}[n]\delta_{2,2}[n])\delta_{2,3}[n] + \delta_{1,2}[n]\delta_{2,1}[n]\delta_{1,1}[n](\delta_{1,2}[n] \\
& + \delta_{2,2}[n]))\delta_{1,3}[n] + \delta_{1,1}[n]\delta_{1,2}[n]\delta_{2,3}[n]^2\big(\delta_{1,1}[n]\delta_{2,2}[n]^2 \\
& + \delta_{1,2}[n]^2\delta_{2,3}[n]\big)\big)\lambda_1^2 + \lambda_2^3\big(-\delta_{2,1}[n]\delta_{2,2}[n]^2(\delta_{1,2}[n] + 1)\delta_{1,3}[n]^3 \\
& + \delta_{2,2}[n]((-\delta_{2,1}[n](\delta_{2,2}[n] + 1)\delta_{1,2}[n] + \delta_{2,2}[n](\delta_{1,1}[n] \\
& + \delta_{2,1}[n]))\delta_{2,3}[n] + (\delta_{1,2}[n]\delta_{2,1}[n] + \delta_{1,1}[n](\delta_{1,1}[n] \\
& - \delta_{2,1}[n]))\delta_{2,2}[n]\delta_{2,1}[n])\delta_{1,3}[n]^2 + \delta_{1,2}[n]\big(\big(-\delta_{1,2}[n]^2\delta_{2,1}[n] \\
& + (-\delta_{1,1}[n]\delta_{2,2}[n] - \delta_{1,1}[n] - \delta_{2,1}[n])\delta_{1,2}[n] \\
& + \delta_{1,1}[n]\delta_{2,2}[n])\delta_{2,3}[n] + \delta_{2,2}[n]\delta_{2,1}[n]\delta_{1,1}[n](\delta_{1,2}[n] \\
& - \delta_{2,2}[n]))\delta_{2,3}[n]\delta_{1,3}[n] + \delta_{1,2}[n]^2(\delta_{2,3}[n] - \delta_{1,1}[n]\delta_{2,2}[n] \\
& + \delta_{2,1}[n](\delta_{1,1}[n] - \delta_{2,1}[n]))\delta_{1,1}[n]\delta_{2,3}[n]^2\big)\lambda_1 \\
& - \lambda_2^4\delta_{1,2}[n]\delta_{1,3}[n](\delta_{1,2}[n]\delta_{2,1}[n]((-\delta_{2,2}[n] - 1)\delta_{2,3}[n] \\
& + \delta_{2,1}[n]\delta_{2,2}[n])\delta_{1,3}[n] + (\delta_{2,3}[n] + \delta_{2,1}[n](\delta_{1,1}[n] \\
& - \delta_{1,2}[n] - \delta_{2,1}[n]))\delta_{2,2}[n]\delta_{1,1}[n]\delta_{2,3}[n]))/\big(\lambda_1^2\lambda_2^2(\delta_{1,2}[n]\lambda_1 \\
& - \delta_{2,2}[n]\lambda_2)(-\delta_{1,2}[n]\delta_{2,3}[n]\lambda_2 + \delta_{1,3}[n]\delta_{2,2}[n]\lambda_1)^2 \\
& \cdot (\delta_{1,1}[n]\delta_{2,3}[n]\lambda_1 - \delta_{1,3}[n]\delta_{2,1}[n]\lambda_2)\big);
\end{aligned}$$

$$\begin{aligned}
f_{1,n} = {} & (\lambda_1 - \lambda_2)(\delta_{1,2}[n]\delta_{2,2}[n]\delta_{1,1}[n]((((-\delta_{2,1}[n] - \delta_{2,3}[n])\delta_{1,3}[n] \\
& + \delta_{1,1}[n]\delta_{2,3}[n])\delta_{2,2}[n] + \delta_{1,3}[n]\delta_{2,1}[n]^2)\delta_{1,2}[n] \\
& - \delta_{2,2}[n]\delta_{2,3}[n]\delta_{1,3}[n])\delta_{2,3}[n]\lambda_1^3 \\
& - \lambda_2\delta_{1,2}[n]\big(\delta_{1,1}[n]\delta_{1,2}[n]^2\delta_{2,3}[n]^2\big(\delta_{2,1}[n]^2
\end{aligned}$$

$$
\begin{aligned}
&-\delta_{2,2}[n]\delta_{2,3}[n]-\delta_{2,3}[n])+(((-\delta_{2,1}[n]-\delta_{2,3}[n])\delta_{1,3}[n]\\
&+\delta_{1,1}[n]\delta_{2,3}[n])\delta_{2,2}[n]+\delta_{1,3}[n]\delta_{2,3}[n])\delta_{1,3}[n]\delta_{2,2}[n]\delta_{2,1}[n]\delta_{1,2}[n]\\
&+(-\delta_{1,1}[n]\delta_{1,3}[n]\delta_{2,2}[n]\delta_{2,3}[n]+((-\delta_{1,1}[n]-2\delta_{2,1}[n])\delta_{2,3}[n]\\
&+\delta_{1,1}[n]\delta_{2,1}[n]^2)\delta_{1,3}[n]+\delta_{1,1}[n]^2\delta_{2,1}[n]\delta_{2,3}[n])\delta_{1,3}[n]\delta_{2,2}[n]^2)\lambda_1^2\\
&+\lambda_2^2\delta_{2,2}[n]\big(-\delta_{1,2}[n]^3\delta_{1,3}[n]\delta_{2,1}[n]\delta_{2,3}[n]^2+(-\delta_{1,1}[n]((-\delta_{2,1}[n]\\
&+\delta_{2,3}[n])\delta_{1,3}[n]+\delta_{1,1}[n]\delta_{2,3}[n])\delta_{2,2}[n]+((-2\delta_{1,1}[n]-\delta_{2,1}[n])\delta_{2,3}[n]\\
&+\delta_{1,1}[n]\delta_{2,1}[n]^2)\delta_{1,3}[n]+\delta_{1,1}[n]^2\delta_{2,1}[n]\delta_{2,3}[n])\delta_{2,3}[n]\delta_{1,2}[n]^2\\
&+\delta_{1,3}[n]\delta_{2,2}[n]\big(-\delta_{1,3}[n]^2\delta_{2,1}[n]\delta_{2,2}[n]+\delta_{1,1}[n]\delta_{2,3}[n]^2\big)\delta_{1,2}[n]\\
&+\delta_{1,3}[n]^2\delta_{2,1}[n]\delta_{2,2}[n]^2\big(\delta_{1,1}[n]^2-\delta_{1,3}[n]\big)\big)\lambda_1\\
&-\lambda_2^3\delta_{1,2}[n]\delta_{1,3}[n]((((\delta_{2,1}[n]-\delta_{2,3}[n])\delta_{1,3}[n]\\
&-\delta_{1,1}[n]\delta_{2,3}[n])\delta_{2,2}[n]-\delta_{1,3}[n]\delta_{2,3}[n])\delta_{1,2}[n]\\
&+\delta_{1,1}[n]^2\delta_{2,2}[n]\delta_{2,3}[n])\delta_{2,2}[n]\delta_{2,1}[n])(\lambda_1+\lambda_2)/\big(\lambda_1^2\lambda_2^2(\delta_{1,2}[n]\lambda_1\\
&-\delta_{2,2}[n]\lambda_2)(-\delta_{1,2}[n]\delta_{2,3}[n]\lambda_2\delta_{1,3}[n]\delta_{2,2}[n]\lambda_1)^2\\
&\cdot(\delta_{1,1}[n]\delta_{2,3}[n]\lambda_1-\delta_{1,3}[n]\delta_{2,1}[n]\lambda_2)). \qquad \text{(A.1)}
\end{aligned}
$$

B. The Solution to Equation (1)

$$
\begin{aligned}
\tilde{q}_n={}&\big(\delta_{1,1}[n+1](\delta_{2,3}[n+1]\lambda_2-\delta_{2,1}[n+1])\lambda_1^2-\lambda_2^2\delta_{1,3}[n+1]\delta_{2,1}[n+1]\lambda_1\\
&+\lambda_2^2\delta_{1,1}[n+1]\delta_{2,1}[n+1]\big)\big(\delta_{1,2}[n+1]\delta_{2,3}[n+1]\lambda_1\\
&-\delta_{1,3}[n+1]\delta_{2,2}[n+1]\lambda_2\big)/\big(\lambda_1^2\lambda_2^2(-\delta_{1,2}[n+1]\delta_{2,3}[n+1]\lambda_2\\
&+\delta_{1,3}[n+1]\delta_{2,2}[n+1]\lambda_1(\delta_{1,1}[n+1]\delta_{2,3}[n+1]\lambda_1\\
&-\delta_{1,3}[n+1]\delta_{2,1}[n+1]\lambda_2))\big);
\end{aligned}
$$

$$
\tilde{r}_n=\frac{(1+c_{1,n})\lambda_1\lambda_2(-\delta_{1,2}[n]\delta_{2,3}[n]\lambda_2+\delta_{1,3}[n]\delta_{2,2}[n]\lambda_1)}{\delta_{1,2}[n]\delta_{2,3}[n]\lambda_1-\delta_{1,3}[n]\delta_{2,2}[n]\lambda_2};
$$

$$
\begin{aligned}
\tilde{u}_n={}&(\delta_{1,1}[n+1]\delta_{1,3}[n+1]^2\delta_{2,2}[n+1]^2\delta_{2,3}[n+1]f_{1,n+1}\lambda_1^5\lambda_2^2\\
&+\big(-\delta_{2,2}[n+1]f_{1,n+1}\delta_{1,3}[n+1]\big(2\delta_{1,1}[n+1]\delta_{1,2}[n+1]\delta_{2,3}[n+1]^2\\
&+\delta_{1,3}[n+1]^2\delta_{2,1}[n+1]\delta_{2,2}[n+1]\big)\lambda_2^3-\delta_{2,3}[n+1]\delta_{1,1}[n+1]\\
&\cdot\big((((-\delta_{2,2}[n+1]+1)\delta_{1,2}[n+1]-\delta_{2,2}[n+1])\delta_{2,3}[n+1]\\
&-\delta_{2,2}[n+1]\delta_{1,2}[n+1]\delta_{2,1}[n+1]\big)\delta_{1,3}[n+1]\\
&+q_n\delta_{2,2}[n+1]\delta_{2,3}[n+1]\delta_{1,1}[n+1]\delta_{1,2}[n+1]\big)\lambda_2\\
&+\delta_{2,2}[n+1]\delta_{2,3}[n+1]\delta_{1,1}[n+1]\delta_{1,2}[n+1]\delta_{1,3}[n+1]\delta_{2,1}[n+1]\lambda_1^4\\
&+\lambda_2\big(2\delta_{1,2}[n+1]\delta_{2,3}[n+1]\big(\delta_{2,2}[n+1]\delta_{1,3}[n+1]^2\delta_{2,1}[n+1]\\
&+(1/2)\delta_{2,3}[n+1]^2\delta_{1,1}[n+1]\delta_{1,2}[n+1]\big)f_{1,n+1}\lambda_2^3\\
&+\big(((\delta_{2,1}[n+1](-\delta_{2,2}[n+1]+1)\delta_{1,2}[n+1]-\delta_{2,2}[n+1]^2\delta_{1,1}[n+1]\\
&-\delta_{2,2}[n+1]\delta_{2,1}[n+1])\delta_{2,3}[n+1]-\delta_{2,2}[n+1]\delta_{1,2}[n+1]\delta_{2,1}[n+1]^2)\delta_{1,3}[n+1]^2\\
&+\delta_{2,2}[n+1]\delta_{2,3}[n+1]\delta_{1,1}[n+1]\delta_{1,2}[n+1]\delta_{1,3}[n+1]\delta_{2,1}[n+1]\\
&-\delta_{2,3}[n+1]^3\delta_{1,1}[n+1]\delta_{1,2}[n+1]^2)\lambda_2\\
&-\delta_{1,1}[n+1]\delta_{2,1}[n+1]\big(\delta_{1,2}[n+1]^2\delta_{2,3}[n+1]^2\\
&+\delta_{1,3}[n+1]^2\delta_{2,2}[n+1]^2)\big)\lambda_1^3\\
&-\delta_{2,3}[n+1]^2f_{1,n+1}\delta_{1,2}[n+1]^2\delta_{1,3}[n+1]\delta_{2,1}[n+1]\lambda_2^2\\
&+\delta_{2,2}[n+1]^2\delta_{2,1}[n+1]\delta_{1,3}[n+1]^3-\big(\big(-\delta_{2,1}[n+1]\delta_{1,2}[n+1]^2\\
&-\delta_{1,1}[n+1](\delta_{2,2}[n+1]+1)\delta_{1,2}[n+1]\\
&+\delta_{2,2}[n+1]\delta_{1,1}[n+1]\big)\delta_{2,3}[n+1]\\
&+\delta_{2,2}[n+1]\delta_{1,1}[n+1]\delta_{1,2}[n+1]\delta_{2,1}[n+1])\delta_{2,3}[n+1]\delta_{1,3}[n+1]\\
&+\delta_{2,2}[n+1]\delta_{2,3}[n+1]^2\delta_{1,1}[n+1]^2\delta_{1,2}[n+1]\lambda_2^3\lambda_1^2\\
&-\delta_{2,1}[n+1]\lambda_2^3(((((\delta_{2,2}[n+1]+1)\delta_{1,2}[n+1]\\
&-\delta_{2,2}[n+1])\delta_{2,3}[n+1]-\delta_{2,2}[n+1]\delta_{1,2}[n+1]\delta_{2,1}[n+1])\delta_{1,3}[n+1]\\
&+\delta_{2,2}[n+1]\delta_{2,3}[n+1]\delta_{1,1}[n+1]\delta_{1,2}[n+1])\delta_{1,3}[n+1]\lambda_2\\
&-\delta_{1,1}[n+1]\big(\delta_{1,2}[n+1]^2\delta_{2,3}[n+1]^2+\delta_{1,3}[n+1]^2\delta_{2,2}[n+1]^2\big)\big)\lambda_1\\
&-\delta_{1,1}[n+1]\delta_{1,2}[n+1]\delta_{1,3}[n+1]\delta_{2,1}[n+1]\delta_{2,2}[n+1]\delta_{2,3}[n+1]\lambda_2^4)/\\
&\cdot\big(\lambda_1^2\lambda_2^2(-\delta_{1,2}[n+1]\delta_{2,3}[n+1]\lambda_2+\delta_{1,3}[n+1]\delta_{2,2}[n+1]\lambda_1)^2\\
&\cdot(\delta_{1,1}[n+1]\delta_{2,3}[n+1]\lambda_1-\delta_{1,3}[n+1]\delta_{2,1}[n+1]\lambda_2));
\end{aligned}
$$

$$
\begin{aligned}
\tilde{s}_n={}&(2(\lambda_1-\lambda_2)((((\delta_{1,2}[n]-\delta_{2,2}[n])\delta_{2,3}[n]-\delta_{1,2}[n]\delta_{2,1}[n]\delta_{2,2}[n])\delta_{1,3}[n]\\
&+\delta_{1,1}[n]\delta_{1,2}[n]\delta_{2,2}[n]\delta_{2,3}[n]\big)\delta_{2,2}[n]\lambda_2\\
&-(1/2)\delta_{1,2}[n]\delta_{2,3}[n]((\delta_{2,2}[n]-1)\delta_{2,3}[n]\\
&+\delta_{2,1}[n]\delta_{2,2}[n])\delta_{1,3}[n])\delta_{1,3}[n]\lambda_1^2-\lambda_2(\delta_{1,2}[n]\delta_{2,3}[n](((\delta_{1,2}[n]\\
&-\delta_{2,2}[n])\delta_{2,3}[n]-\delta_{1,2}[n]\delta_{2,1}[n]\delta_{2,2}[n]\big)\delta_{1,3}[n]\\
&+\delta_{1,1}[n]\delta_{1,2}[n]\delta_{2,2}[n]\delta_{2,3}[n])\lambda_2-(1/2((\delta_{2,2}[n]-1)\delta_{2,3}[n]\\
&+\delta_{2,1}[n]\delta_{2,2}[n]))\delta_{2,2}[n]\delta_{1,3}[n]^3-(1/2)\delta_{1,2}[n]\delta_{2,3}[n]^3(\delta_{1,2}[n]-1)\delta_{1,3}[n]\\
&-(1/2)\delta_{1,1}[n]\delta_{1,2}[n]^2\delta_{2,3}[n]^3)\lambda_1-(1/2)\lambda_2^2\delta_{2,3}[n]^2((\delta_{1,2}[n]-1)\delta_{1,3}[n]\\
&+\delta_{1,1}[n]\delta_{1,2}[n])\delta_{2,2}[n]\delta_{1,3}[n]\lambda_2+\lambda_1)/((-\delta_{1,2}[n]\delta_{2,3}[n]\lambda_2\\
&+\delta_{1,3}[n]\delta_{2,2}[n]\lambda_1)(\delta_{1,2}[n]\delta_{2,3}[n]\lambda_1-\delta_{1,3}[n]\delta_{2,2}[n]\lambda_2)^2). \qquad \text{(B.1)}
\end{aligned}
$$

Data Availability

The data in the manuscript are available from the corresponding authors upon reasonable request.

Conflicts of Interest

The authors declare that they have no conflict of interest.

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