Hindawi
Advances in Mathematical Physics
Volume 2021, Article ID 9978633, 14 pages
https://doi.org/10.1155/2021/9978633

*Research Article*

# Variationally Improved Bézier Surfaces with Shifted Knots

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Received 20 March 2021; Accepted 5 August 2021; Published 25 August 2021

Academic Editor: Manuel De Le n



The Plateau-Bézier problem with shifted knots is to find the surface of minimal area amongst all the Bézier surfaces with shifted knots spanned by the admitted boundary. Instead of variational minimization of usual area functional, the quasi-minimal Bézier surface with shifted knots is obtained as the solution of variational minimization of Dirichlet functional that turns up as the sum of two integrals and the vanishing condition gives us the system of linear algebraic constraints on the control points. The coefficients of these control points bear symmetry for the pair of summation indices as well as for the pair of free indices. These linear constraints are then solved for unknown interior control points in terms of given boundary control points to get quasi-minimal Bézier surface with shifted knots. The functional gradient of the surface gives possible candidate functions as the minimizers of the aforementioned Dirichlet functional; when solved for unknown interior control points, it results in a surface of minimal area called quasi-minimal Bézier surface. In particular, it is implemented on a biquadratic Bézier surface by expressing the unknown control point $P_{11}$ as the linear combination of the known control points in this case. This can be implemented to Bézier surfaces with shifted knots of higher degree, as well if desired.

## 1. Introduction

We observe that the nature behaves in a way that certain quantity is either a maximum or a minimum of some quantity, in various phenomena occurring in our universe. For example, the shortest distance between two points is a straight line in the case of no constraint; otherwise, it is said to be a geodesic in the case of some given constraint that may be in differential or in integral form. It is well known that a light ray takes its shortest path while moving with constant speed in the shortest possible time. This is in conformity with the well-known principle called the Fermat's principle which states that a light ray will follow the path that requires the least possible time. Fermat's principle is applicable only to behaviour of light. Other phenomena in which principle of minima are applicable are, e.g., a rubber balloon when blown up takes a spherical shape, and similar is true about a soap bubble. In fact, amongst all the surfaces containing the same volume, a sphere has the least area. Another example is that of Brachistochrone problem, which is related to finding the shape of the curve along which an object slides down from rest to another point in the least possible time. This phenomenon is related to the broader discipline of mathematics, the optimization theory, which encompasses the minimization of some principle given in the form of energy functional, linear programming, network analysis, and so on. The optimization problem, in its simplest form, seeks for a function that maximizes or minimizes a given constraint usually given in the form of an integral. The function itself is called the objective function that maximizes or minimizes the given functional. Such phenomena are studied in the variational calculus. The goal of variational calculus is to study the conditions for an objective function in a given domain $\Omega$ in which we can find a solution of an optimization problem and to study the characteristics of the extreme values that correspond to the solution. One of the active areas of research is to seek for variational improvement in an objective function that means to find quasi-minimal functions that are closer to the minimal objective functions [1, 2]. These minimization principles are helpful in constructing mathematical models describing some symmetry structure of a given physical system in the fields of solid and fluid mechanics, electromagnetism, and many more. The basic idea behind a variational technique is to minimize an objective function subject to given boundary constraints. Its target is to find the value (s) of a parameter (s) by the vanishing condition of gradient of a certain functional based on some constraints. This vanishing condition then can be used to find solution of a system of constraints obtained in this way. The scheme of the constraints may be in the form of an integral, a differential, or an algebraic condition.

In surface modeling, a relevant problem is the Plateau problem [3], which was presented by a Belgian physicist named Joseph Plateau. Plateau demonstrated the minimal surfaces related to soap films. In 1849, he showed that a wire frame dipped in soapy water behaves as the boundary of the spanned surface. In differential geometry, its use is to construct surfaces of minimal area, a surface called a minimal surface. A minimal surface is a surface whose mean curvature vanishes at its every point for all of its possible parameterizations. The minimal surface theory combines the objects being studied, their origin, and their relation to the physical world. Minimal surfaces find their applications in different branches of mathematics like computer science, operations research, economics, and its affiliated disciplines. The availability of computer graphics has increased the visual understanding of these surfaces. The importance of the Plateau problem was soon realized by Douglas who minimized a quantity which is now known as Dirichlet integral [4]. The theory of minimal surfaces [5, 6] arose from an attempt made by Lagrange (1762) to find a surface $z = z(u, v)$ of minimal area spanned by a given curve by minimizing the area functional, which resulted in an equation now known as the Euler-Lagrange equation. He obtained the solution only in the form of plane. In 1776, Meusnier showed that the only ruled surfaces that are minimal and satisfy the Euler-Lagrange equation are the catenoid and the helicoid. However, he was able to show that the corresponding Euler-Lagrange equation for these surfaces reduces to twice of the mean curvature of the surface. He inferred that the minimal surfaces are the surfaces for which the mean curvature is zero everywhere on the surface. The vanishing condition of the mean curvature of a surface given in the form $z = z(u, v)$ reduces to the Euler-Lagrange equation $(1 + z_v^2)z_{uu} - 2z_u z_v z_{uv} + (1 + z_u^2)z_{vv} = 0$. The Euler-Lagrange equation is a partial differential equation of second order, and its general solution $z = z(u, v)$ does exist only for some particular cases. A variety of minimal surfaces can be seen in literature starting from the minimal surfaces spanned by simple closed contours to a boundary comprising a finite number of curves, polygonal, or smooth regular curves. This includes the work of Gaspard Monge and Legendre (1795) who found the representation formulas for their solution surfaces, the Heinrich Scherk surfaces (1830), Douglas's work [4], the American mathematician, who found a form of solution by minimizing a quantity now known as Douglas integral, the Hungarian Radó [7] found a form of solution by minimizing an energy integral. The significant work in this field of minimal surfaces can be seen by other mathematicians like McShane [8], Tonelli [9], Shiffman [10], Courant [11, 12], Tompkins [13], Morrey [14, 15], Osserman [16], Gulliver [17], and Karcher [18]. For a comprehensive survey of the minimal surfaces, the work of Osserman. [16] and Nitsche [6] is important. Other work, whose importance though was realized later, was due to Bernstein [19] whose innovative approach to partial differential equations led to his famous Bernstein's theorem which helped to study a wider class of surfaces for higher dimensions, now known as Bézier surfaces.

The applications of minimal surfaces are found in the emerging disciplines of the computer-aided geometric design, computer graphics, computer-aided design, and geometric modeling. Séquin's studies [20–22] are based on a variety of applications of these minimal surfaces in constructing, for example, for motor vehicles, the interactive CAD tools can be used for engineering purpose. The minimal surfaces can be built through a variational technique by finding solution of vanishing condition of gradient of a constraint usually in the form of an integral having the property of minimizing something, for example, minimizing an energy integral. These variational techniques are advantageous for a class of surfaces, better known as Bézier surfaces admitting useful properties appropriate for applications in CAGD. The Bézier surfaces are based on the Bernstein polynomials [23] $B_j^m(u)$ and $B_k^n(v)$ of degree $m$ and $n$, where

$$\begin{aligned} B_j^m(u) &= \binom{m}{j} u^j (1-u)^{m-j}, \\ B_k^n(v) &= \binom{n}{k} v^k (1-v)^{n-k}, \end{aligned} \tag{1}$$

$\binom{m}{j}$ and $\binom{n}{k}$ being the binomial coefficients. These polynomials are named after his creator, Sergei Natanovich Bernstein, and the so called the Bernstein polynomial is a linear combination of exponents of a variable as mentioned above, in a form called the Bernstein form. For a given control net of points $P_{jk}$, a Bézier surface is defined as

$$x(u, v) = \sum_{j=0}^{m} \sum_{k=0}^{n} B_j^m(u) B_k^n(v) P_{jk}. \tag{2}$$

Bézier surfaces, first described by the French engineer Pierre Bézier (in 1962), used these surfaces in automobile industry. These surfaces justify their importance in CAGD; one of the instances is the designing of freedom surfaces which are constructed using the patches of triangular and rectangular Bézier surfaces. These freedom surfaces are used as the surface fitting tools in CAGD, CAD, and CAID. In CAGD, the Plateau problem is one of the fundamental surface design problems. For prescribed boundary, we can have arbitrary number of surfaces stretched over the given boundary; one of these surfaces is of minimal area. The search for such a surface of minimal area is called the so-called Plateau problem. An interconnected problem is the Plateau-Bézier problem which is related to search for a Bézier surface of minimal area amongst all the possible Bézier surfaces spanned by the same boundary. Cosín and Monterde [24] dealt with minimal surfaces as the solution in Bézier surface form; further, they analysed the properties of these surfaces associated with control point mesh. In particular, in bicubicle case, for an affine transformation, they show the Bézier form of solution for the minimal surfaces as the pieces of the Enneper surface. For given prescribed border as the control points for a Bézier surface, we can find the gradient of the constraint integral (area integral or some energy integral) and the vanishing condition of the gradient of the functional gives system of linear algebraic equations expressing the unknown interior control points in terms of known boundary control points. Monterde [25] worked out the Plateau-Bézier problem by replacing the usual area integral by Dirichlet energy functional and found the Bézier solution as the extremal of Dirichlet energy functional. Monterde and Ugail [26] extremize general quadratic functional to obtain the Euler-Lagrange equation for Bézier surfaces as its solution, utilizing the boundary information from a general fourth order PDE. The Plateau-Bézier problem for triangular Bézier patches was studied by Arnal et al. [27]. Hao et al. [28, 29] dealt with the quasi-Plateau-Bézier problem for a broader variety of boundaries consisting of polynomial boundary curves obtaining Bézier surfaces as the solution of the extremal of Dirichlet functional, the harmonic and biharmonic functionals, and Multiresolution Analysis (MRA) employing B-splines. The properties related to the minimal surfaces spanned by a boundary with quintic form of parametric polynomials can be seen in Xu and Wang's study [30]. Chen et al. [31] discuss the Plateau-Bézier problem in context to minimization of extended Dirichlet functional (by estimating suitable value of (Equation (4) in Section 2.2) and extended bending energy functional by estimating suitable value of (Equation (19) in Section 3). The authors find out the value of $\lambda$ (in the case of extended Dirichlet functional) and $\alpha$ (in the case of the extended energy functional) rather than minimizing the functional itself. The idea may be extended to more generalized surfaces called toric Bézier surfaces. Ahmad and Naeem [32] obtained the toric Bézier surfaces as the extremal of quasi-harmonic energy functional. The algorithms proposed by Ahmad and Masud [33–35] for generating quasi-minimal surfaces spanned by a finite number of boundary curves as the variational minimization of curvature can be used for further surface topography; for an application, see Zhu et al.'s study [36]. The geometric shapes can be used for informative visualization in engineering by the adjustment of control points or the weights for another approach for controlling the shape [37–39].

As mentioned above, one of the widely used restrictions is to find the Bézier surface as the extremal of an energy functional depending on the desired characteristics of a surface, which is done by investigating the vanishing condition of gradient of such a functional. In return, the vanishing condition of the gradient of the deliberately chosen functional reduces to an algebraic system of linear constraints. We consider a class of surfaces called the Bézier surfaces with shifted knots which generalize the classic Bézier surfaces and give the intertwined quantities of differential geometry for these surfaces. The modification of bases of Bernstein polynomials have been used to study these surfaces. Bernstein operator with different modifications is an active area of research in approximation theory useful in CAGD as well. In 2010, Gadjiev and Gorbanalizadeh [40] introduced Bernstein-Stancu-type polynomials with shifted knots.

Khan et al. [41] study Bézier curves and surfaces based on modified Bernstein basis functions with shifted knots for $t \in [\alpha/n + \beta, n + \alpha/n + \beta]$ with the parameters $\alpha$ and $\beta$ enabling the authors to shift Bernstein basis functions over subintervals of [0,1]. Khan and Lobiyal [42] deal with the extension of rational Lupaş Bernstein functions, Lupaş Bézier curves, and surfaces involving $(p, q)$-integers for $p > 0$ and $q > 0$. Motivated by the work of Khan et al. [41] and Mursaleen et al. [43], Nisar et al. [44] find Lupaş $q$-Bernstein basis functions (blending functions with shifted knots) for $(t \in [a/[\mu]_q + b, [\mu]_q + a/[\mu]_q + b])$ to construct Bernstein Bézier curves and surfaces with shifted knots and study various properties of these functions. As mentioned above, one of these modifications is to construct Bézier surfaces with shifted knots as the extremal of a suitable energy functional.

The energy functional deliberately chosen is the Dirichlet functional instead of the usual vexing area functional which involves square root in its integrand and is inconvenient to manage. This functional can be used to generate a surface of least area to reflect the properties of a minimal surface, the so called quasi-minimal Bézier surface. As for any candidate functional, we can find the quasi-minimal Bézier surface with shifted knots by solving the vanishing condition of the gradient of Dirichlet functional for Bézier surface with shifted knots for unknown control points. The vanishing condition of the gradient of the Dirichlet condition reduces to an algebraic system of equations for these unknown control points. Bézier surface with shifted knots appears quite frequent in the mathematical models of surface formation in CAGD and other disciplines of mathematics. The functional gradient of a surface $x(u, v)$ gives us the possible candidate functions as the minimizers of the proposed functional; the analogue of this technique in calculus of several variables finds the extremal points of a function of several variables by equating its gradient to zero. The vanishing condition of a functional gradient generates a system of algebraic constraints that can be solved for the interior control points to be determined as the linear combination of accepted boundary control points. We shall term the surface as the quasi-minimal Bézier surface with shifted knots. These surfaces have applications in the field of engineering, in analysis of objects in physics, and mechanism for cellular materials. The related Plateau-Bézier surface with shifted knot problem comprises of identifying the shifted knots Bézier surfaces of minimal area amongst all the shifted knots Bézier surfaces with accepted boundary.

## 2. Preliminaries

It is well-recognized that the Bernstein polynomial $B_n(f\,;u)$ converges uniformly to every continuous function $f(u)$ in the interval [0,1]. In 1968, Stancu [45] introduced the following generalization of the Bernstein polynomials.

$$P_{n,\eta,\lambda}(f\,;u) = \sum_{k=0}^{n} f\left(\frac{k+\eta}{n+\lambda}\right)\binom{n}{k} u^k (1-u)^{n-k}, \tag{3}$$

and the polynomials $P_{n,\eta,\lambda}(f\,;u)$ converge uniformly to the continuous function $f(u)$ in [0,1], where $0 \le u \le 1$ and $0 \le \eta \le \lambda$. In 2010, Gadjiev and Gorbanalizadeh [40] introduced the Stancu-type polynomials for one and two variables, given by

$$\begin{aligned} S_{n,\eta,\lambda}(f\,;u) = {} & \left(\frac{n+\lambda_2}{n}\right)^2 \sum_{k=0}^{n}\binom{n}{k}\left(u-\frac{\eta^2}{n+\lambda_2}\right)^k \\ & \cdot \left(\frac{n+\eta_2}{n+\lambda_2}-u\right)^{n-k} f\left(\frac{k+\eta_1}{n+\lambda_1}\right), \end{aligned} \tag{4}$$

where $\eta_2/n+\lambda_2 \le u \le n+\eta_2/n+\lambda_2$ and $\eta_k, \lambda_k (k=1,2)$ are positive real numbers provided that $0 \le \eta_1 \le \eta_2 \le \lambda_1 \le \lambda_2$. It is to be noted that these Stancu-type polynomials reduce to Bernstein-Stancu-type polynomials for $\eta_2 = \lambda_2 = 0$. However, for $\eta_1 = \lambda_1 = \eta_2 = \lambda_2 = 0$, these polynomials become classical Bernstein polynomials. Recently, Khalid et al. [41] introduced Bézier surfaces based on one of the modified Bernstein basis functions, the blending functions with shifted knots in the following form:

$$\mathfrak{B}(u,v) = \sum_{j,k=0}^{m,n} G^j_{m,\eta,\lambda}(u) G^k_{n,\eta,\lambda}(v) P_{jk}, \tag{5}$$

in which the Bernstein functions $G^j_{m,\eta,\lambda}(u)$ and $G^k_{n,\eta,\lambda}(v)$ with shifted knots are of degree $m$ and $n$, respectively, and

$$G^j_{m,\eta,\lambda}(u) = \binom{m}{j}\left(\frac{m+\lambda}{m}\right)^m \left(u-\frac{\eta}{m+\lambda}\right)^j \left(\frac{m+\eta}{m+\lambda}-u\right)^{m-j}. \tag{6}$$

$P_{jk} = \{x^a_{jk}\}^n_{j,k=1}$ (for $a = 1, 2, 3$), control points of Bézier surface with shifted knots. Khalid et al. [41] established the following properties of Bernstein functions with shifted knots, in comparison with the properties satisfied by classical Bernstein polynomials, and these are as follows:

(1) The Bernstein functions are nonnegative, and the same is true when they are of shifted knots for all $u \in [\eta/n+\lambda, n+\eta/n+\lambda]$. This can be readily seen from Equation (6) that each Bernstein polynomial with shifted knots $G^j_{m,\eta,\lambda}(u) \ge 0$ for $j = 0, 1, 2, \cdots, n$ and $t \in [\eta/m+\lambda, m+\eta/m+\lambda]$

(2) The collection of all the Bernstein functions with shifted knots $G^j_{m,\eta,\lambda}(u)$ for $j = 0, 1, 2, \cdots, m$ and $u \in [\eta/m+\lambda, m+\eta/m+\lambda]$ set up a partition of unity, that is, when the Bernstein functions are summed up for shifted knots for $j = 0, 1, 2, \cdots, m$ and $u \in [\eta/m+\lambda, m+\eta/m+\lambda]$, they equate to 1. In notation, it appears as

$$\sum_{j=0}^{m} G^j_{m,\eta,\lambda}(u) = 1, u \in \left[\frac{\eta}{m+\lambda}, \frac{m+\eta}{m+\lambda}\right]. \tag{7}$$

Likewise, the sum of product of Bernstein polynomials forms a partition of unity, which can be shown by writing

$$\sum_{i,j=0}^{m,n} G^i_{m,\eta,\lambda}(u) G^j_{n,\eta,\lambda}(v) = 1, \tag{8}$$

and Bézier surface with shifted knots $\mathfrak{B}(u,v)$ (Equation (5)) represents an affine combination in terms of its own control points. Thus, the Bézier surface with shifted knots $\mathfrak{B}(u,v)$ lies within the convex hull of its own points and represents a convex combination of its own control points $P_{jk}$.

(3) The Bernstein polynomials with shifted knots $G^j_{m,\eta,\lambda}(u)$ obey the reducibility property that for $\eta = \lambda = 0$, Equation (6) reduces to the classical Bernstein polynomial over [0,1], and this means that

$$G^i_{m,0,0}(u) = \binom{m}{j}(u)^j (1-u)^{m-j} = B^j_m(u). \tag{9}$$

As noted above, the Bernstein polynomials with shifted knots reduce to the classical Bernstein polynomials of the same degree for $\eta = \lambda = 0$; thus, the Bézier surface with shifted knots (Equation (5)) reduces to the classical Bézier surface with shifted knots for $\eta = \lambda = 0$, which can be established easily from Equations (5) and (6), that

$$\mathfrak{B}(u,v) = \sum_{j,k=0}^{m,n} G^i_{m,0,0}(v) P_{jk} = \sum_{j,k=0}^{m,n} B^j_m(u) B^k_n(v) P_{jk} = \mathrm{x}(u,v). \tag{10}$$

(4) The Bernstein polynomials with shifted knots satisfy the following end-point interpolation property that

$$G^k_{n,\eta,\lambda}\left(\frac{\eta}{n+\lambda}\right)=\begin{cases}1, & k=0,\\ 0, & k\neq 0,\end{cases}$$
$$G^k_{n,\eta,\lambda}\left(\frac{n+\eta}{n+\lambda}\right)=\begin{cases}1, & k=n,\\ 0, & k\neq n.\end{cases} \quad (11)$$

(5) An $n$ degree Bernstein function with shifted knots can be expressed as the linear combination of two $n+1$ degree Bernstein functions with shifted knots or two $n-1$ degree Bernstein functions, with shifted knots, which is analogous to writing a classical Bernstein polynomial as the two Bernstein functions linearly combined together, of higher degree or lower degree. This helps to write the Bézier curves or surfaces in more useful form by determining a new set for its control points for the same curve or a surface as the convex combination for its old set of points. With the help of Equation (6), we note that the following identities are satisfied

$$\left(u-\frac{\eta}{m+\lambda}\right)G^i_{m,\eta,\lambda}(u)=\left(\frac{i+1}{m+1}\right)\left(\frac{m}{m+\lambda}\right)G^{i+1}_{m+1,\eta,\lambda}(u),$$
$$\left(\frac{m+\eta}{m+\lambda}-u\right)G^i_{m,\eta,\lambda}(u)=\left(\frac{m+1-i}{m+1}\right)\left(\frac{m}{m+\lambda}\right)G^i_{m+1,\eta,\lambda}(u). \quad (12)$$

(6) A Bernstein polynomial with shifted knots of degree $m$ can be written as a linear combination of the two Bernstein polynomials with shifted knots of degree $m+1$, and the relation is given by

$$G^i_{m,\eta,\lambda}(u)=\left(\frac{m+1-i}{m+1}\right)G^i_{m+1,\eta,\lambda}(u)+\left(\frac{i+1}{m+1}\right)G^{i+1}_{m+1,\eta,\lambda}(u), \quad (13)$$

where $\eta/m+\lambda\le u\le m+\eta/m+\lambda$ and $\eta,\lambda$ are positive real numbers fulfilling $0\le\eta\le\lambda$.

(7) A Bernstein polynomial with shifted knots of degree $m$ can be expressed as a linear combination of the two Bernstein polynomials with shifted knots of degree $m-1$, and the relation can be written as

$$G^i_{m,\eta,\lambda}(u)=\frac{m+\lambda}{m}\left(u-\frac{\eta}{m+\lambda}\right)G^{i-1}_{m-1,\eta,\lambda}(u)+\frac{m+\lambda}{m}\left(\frac{m+\eta}{m+\lambda}-u\right)G^i_{m-1,\eta,\lambda}(u), \quad (14)$$

where $\eta/m+\lambda\le u\le m+\eta/m+\lambda$ and $\eta,\lambda$ are positive real numbers with $0\le\eta\le\lambda$ (see Khalid et al. [41], for further related properties and discussion).

Motivated by the work [25] and the geometry explored in the work [41] of Bézier surfaces with shifted knots, in the following sections, we apply a variational technique to the Bézier surfaces depending on the Bernstein blending functions with shifted knots. We call the problem of finding the quasi-minimal surface in this way as the Plateau-Bézier problem with shifted knots. This can be done by replacing the usual area functional by the Dirichlet functional, and the solution of the vanishing condition for the gradient of the Dirichlet functional appears as the algebraic system of linear constraints on the unknown control points.

## 3. The Integral of the Bernstein-Like Functions with Shifted Knots

Bernstein function with shifted knots are given above in Equation (6), where $\eta/m+\lambda\le u\le m+\eta/n+\lambda$ and $\eta,\lambda$ for $0\le\eta\le\lambda$ are positive real numbers. Taking the derivative of both sides of the Bernstein functions with shifted knots (Equation (6)) and denote this derivative by $\left(G^j_{m,\eta,\lambda}(u)\right)_u$, it follows that

$$\left(G^j_{m,\eta,\lambda}(u)\right)_u=j\binom{m}{j}\left(\frac{m+\lambda}{m}\right)^m\left(u-\frac{\eta}{m+\lambda}\right)^{j-1}\cdot\left(\frac{m+\eta}{m+\lambda}-u\right)^{m-j}-(m-j)\binom{m}{j}\left(\frac{m+\lambda}{m}\right)^m\times\left(u-\frac{\eta}{m+\lambda}\right)^j\left(\frac{m+\eta}{m+\lambda}-u\right)^{m-j-1}, \quad (15)$$

where the binomial coefficients can be written in the form

$$j\binom{m}{j}=m\binom{m-1}{j-1},$$
$$(m-j)\binom{m}{j}=m\binom{m-1}{j}. \quad (16)$$

Equation (15) together with (16) reduces to the following expression:

$$\left(G^j_{m,\eta,\lambda}(u)\right)_u=(m+\lambda)\left[\left(\binom{(m-1)}{(j+1)}\left(\frac{m+\lambda}{m}\right)^{m-1}\cdot\left(u-\frac{\eta}{m+\lambda}\right)^{j-1}\left(\frac{m+\eta}{m+\lambda}-u\right)^{(m-1)-(j+1)}\right)-\left(\binom{m-1}{j}\left(\frac{m+\lambda}{m}\right)^{m-1}\left(u-\frac{\eta}{m+\lambda}\right)^j\cdot\left(\frac{m+\eta}{m+\lambda}-u\right)^{m-1-j}\right)\right]. \quad (17)$$

Equation (17) can be written in terms of Bernstein-like functions $H^{j-1}_{m-1,\eta,\lambda}(u)$ and $H^{j}_{m-1,\eta,\lambda}(u)$,

$$\left(G^{j}_{m,\eta,\lambda}(u)\right)_u = (m+\lambda)\left(H^{j-1}_{m-1,\eta,\lambda}(u) - H^{j}_{m-1,\eta,\lambda}(u)\right), \quad (18)$$

and the derivative of Bernstein function $G^{j}_{n,\eta,\lambda}(v)$ with shifted knots $w.r.t.v$ can be written in the similar notation as follows:

$$\left(G^{k}_{n,\eta,\lambda}(v)\right)_v = (n+\lambda)\left(H^{k-1}_{n-1,\eta,\lambda}(v) - H^{k}_{n-1,\eta,\lambda}(v)\right), \quad (19)$$

where the Bernstein-like function $H^{j}_{m-1,\eta,\lambda}(u)$ is

$$H^{j}_{m-1,\eta,\lambda}(u) = \binom{m-1}{j}\left(\frac{m+\lambda}{m}\right)^{m-1}\left(u-\frac{\eta}{m+\lambda}\right)^{j}\left(\frac{m+\eta}{m+\lambda}-u\right)^{m-1-j}. \quad (20)$$

In Section 4 below, we shall need the integral of product of these Bernstein-like polynomials, and we denote the product of these Bernstein-like polynomials by $H^{j,k}_{m-1,m-1,\eta,\lambda}(u)$ where

$$H^{j,k}_{m-1,m-1,\eta,\lambda}(u) = H^{j}_{m-1,\eta,\lambda}(u)H^{k}_{m-1,\eta,\lambda}(u). \quad (21)$$

By virtue of Equation (20), the product $H^{j,k}_{m-1,m-1,\eta,\lambda}(u)$ of Bernstein-like polynomials (Equation (21)) appears in the following form:

$$H^{j,k}_{m-1,m-1,\eta,\lambda}(u) = A^{m-1}_{(j)(k)}\left(\frac{\lambda+m}{m}\right)^{2m-2}\left(u-\frac{\eta}{\lambda+m}\right)^{j+k}\cdot\left(\frac{\eta+m}{\lambda+m}-u\right)^{(2m-2)-(j+k)}, \quad (22)$$

where for some convenience, we introduce

$$A^{m-1}_{(j)(k)} = \binom{m-1}{j}\binom{m-1}{k}. \quad (23)$$

These polynomials (Equation (22)) pop up in Section 4 for different values of $j$ and $k$ while finding the gradient of Dirichlet functional for Bézier surfaces with shifted knots, and they stand for $H^{i-1,k-1}_{m-1,m-1,\eta,\lambda}(u)$, $H^{i-1,k}_{m-1,m-1,\eta,\lambda}(u)$, $H^{i,k-1}_{m-1,m-1,\eta,\lambda}(u)$ and $H^{i,k}_{m-1,m-1,\eta,\lambda}(u)$.

For $a, b \in \mathbf{R}$ and $0 \le j \le n$, $j, n \in Z$, note that

$$\int_a^b (u-a)^j(b-u)^{n-j}du = \frac{j!(b-a)^{n+1}(n-j)!}{(n+1)!}, \quad (24)$$

so that for $a = \eta/n+\lambda$, $b = n+\eta/n+\lambda$, the integral of Bernstein polynomials with shifted knots (given by Equation (6)) can be written as

$$\begin{aligned}\int_{\eta/\lambda+n}^{n+\eta/\lambda+n} G^{j}_{n,\eta,\lambda}(u)du &= \binom{n}{j}\left(\frac{\lambda+n}{n}\right)^{n}\int_{\eta/\lambda+n}^{n+\eta/\lambda+n}\left(u-\frac{\eta}{\lambda+n}\right)^{j}\cdot\left(\frac{\eta+n}{\lambda+n}-u\right)^{n-j}du = \binom{n}{j}\left(\frac{\lambda+n}{n}\right)^{n}\\ &\cdot\frac{j!(n-j)!}{(n+1)!}\left(\frac{n}{\lambda+n}\right)^{n+1}\\ &= \frac{n}{(n+1)(\lambda+n)}.\end{aligned} \quad (25)$$

Note that the above integral of Bernstein polynomials with shifted knots is independent of $\eta$ and $j$ but depends only on $n$ and $\lambda$. For example, for $n=3$, Equation (25) implies that $\int_{\eta/\lambda+3}^{\eta+3/\lambda+3} G^{0}_{3,\eta,\lambda}(u)du = 3/4(\lambda+3)$ and hence for $j=0,1,2,3$, we have $\int_{\eta/\lambda+3}^{\eta+3/\lambda+3} G^{0}_{3,\eta,\lambda}(u)du = \int_{\eta/\lambda+3(\eta/\lambda+3)}^{\eta+3/\lambda+3} G^{1}_{3,\eta,\lambda}(u)du = \int_{\eta/\lambda+3}^{\eta+3/\lambda+3} G^{2}_{3,\eta,\lambda}(u)du = \int_{\eta/\lambda+3}^{\eta+3/\lambda+3} G^{3}_{3,\eta,\lambda}(u)du = 3/4(\lambda+3)$. Let us denote the product of two Bernstein polynomials of the same degree $n$ (Equation (6)) by $G^{j,k}_{n,n,\eta,\lambda}(u) = G^{j}_{n,\eta,\lambda}(u)G^{k}_{n,\eta,\lambda}(u)$ for $j, k = 0, 1, 2 \cdots n$, and, hence using Equation (6), $G^{j,k}_{n,n,\eta,\lambda}(u)$ reduces to

$$G^{j,k}_{n,n,\eta,\lambda}(u) = A^{n}_{(j)(k)}\left(\frac{\lambda+n}{n}\right)^{2n}\left(u-\frac{\eta}{\lambda+n}\right)^{j+k}\left(\frac{\eta+n}{\lambda+n}-u\right)^{2n-(j+k)}. \quad (26)$$

The integral of above product (26) of two Bernstein polynomials $G^{j}_{n,\eta,\lambda}(u)$ and $G^{k}_{n,\eta,\lambda}(u)$ of the same degree $n$ can be computed by using Equation (25), which is given by

$$\int_{\eta/\lambda+n}^{\eta+n/\lambda+n} G^{j,k}_{n,n,\eta,\lambda}(u)du = n((\lambda+n)(2n+1)!)^{-1}A^{n}_{(j)(k)}B^{r}_{t}, \quad (27)$$

where

$$B^{r}_{t} = (j+k)!(2n-j-k)!, r = j+k, t = 2n-(j+k). \quad (28)$$

The integral of the product of the Bernstein-like polynomials (Equation (21))

$$\begin{aligned}\int_{\eta/\lambda+m}^{\eta+m/\lambda+m} H^{j,k}_{m-1,m-1,\eta,\lambda}(u)du &= \left(\frac{\lambda+m}{m}\right)^{2m-2}A^{m-1}_{(j)(k)}\\ &\cdot\int_{\eta/\lambda+m}^{\eta+m/\lambda+m}\left(u-\frac{\eta}{\lambda+m}\right)^{j+k}\\ &\cdot\left(\frac{\eta+m}{\lambda+m}-u\right)^{(2m-2)-(j+k)}du,\end{aligned} \quad (29)$$

can be computed in view of Equation (24), and it is given by

$$\int_{\eta/\lambda+m}^{\eta+m/\lambda+m} H^{j,k}_{m-1,m-1,\eta,\lambda}(u)du = m((\lambda+m)(2m-1)!)^{-1}A^{m-1}_{(j)(k)} \cdot (j+k)!((2m-2)-(j+k))!. \tag{30}$$

Similarly, we can find the integrals of the products (Equation (22)) of other Bernstein-like functions for different values of $j$ and $k$. In the following section, we find now the vanishing condition for gradient of the Dirichlet functional for Bézier surfaces with shifted knots, utilizing the results established in this section.

## 4. Vanishing Condition for Gradient of Dirichlet Functional for Bézier Surfaces with Shifted Knots

Let $x : U \subset R^2 \longrightarrow R^3$ be a regular parameterized surface, and the surface area of $x(u, v)$ is given by the so-called area integral

$$A(\mathscr{P}) = \int_R \|x_u \wedge x_v\| dudv = \int_R \left(EG - F^2\right)^{1/2} dudv, \tag{31}$$

where $E = E(u, v)$, $F = F(u, v)$, and $G = G(u, v)$ represent the coefficients of the first fundamental form $ds^2 = Edu^2 + 2Fd udv + Gdv^2$, given by

$$E(u, v) = \|x_u\|^2 = \langle x_u, x_u\rangle, F(u, v) = \langle x_u x_v\rangle, G(u, v) = \|x_v\|^2 = \langle x_v, x_v\rangle, \tag{32}$$

and $R = [0, 1] \times [0, 1]$; then, a surface $x$ is minimal; for a differentiable function $g : \bar{D} \longrightarrow R$ where $\bar{D} = DU\partial D$ over a finite domain $D \subset R$, if and only if $A'(0) = 0$ for the domain $D$ and the normal variation of $x(\bar{D})$ (do Carmo [46] pages 197-201), which is possible only when $H(u, v)$, the mean curvature of the surface $x(u, v)$ vanishes for every parametrization. Thus, a surface of minimal area can be obtained as an extremal of the area functional (31). The area functional (31) involves square root in its integrand; and thus, the integrand is highly nonlinear in its surface parameters. The boundary of Bézier surface with shifted knots (Equation (5)) can be completely determined by its control points. Then, an equivalent statement of the Plateau-Bézier problem with shifted knots can be adopted. This comprises finding the interior control points such that the resulting Bézier surface with shifted knots is minimal surface amongst all the Bézier surfaces with shifted knots. The high nonlinearity of the area functional (31) makes it less plausible for the purpose; instead, we use the Dirichlet functional as has been done by Monterde [25] for his work on Bézier surfaces of minimal area.

The Dirichlet energy functional for a surface $x(u, v)$ is given by the following expression:

$$D(\mathscr{P}) = \frac{1}{2}\int_R (E(u, v) + G(u, v))dudv. \tag{33}$$

For the coefficients of the first fundamental form as defined in Equation (32), the Dirichlet functional can be written in the following form:

$$D(\mathscr{P}) = \frac{1}{2}\int_R \left(\|x_u\|^2 + \|x_v\|^2\right)dudv = \frac{1}{2}\int_R (\langle x_u, x_u\rangle + \langle x_v . x_v\rangle)dudv. \tag{34}$$

Due to the nonlinearity of the area functional, this Dirichlet functional is comparable with the area functional and it is well known that area minimizing property is preserved for the Bézier surfaces [25]. We intend to find the gradient of the above Dirichlet integral $w.r.t.$ given control points and equate it to zero to find the extremal condition on the inner control points. These constraints express the interior control points (to be determined) as the linear combination of boundary accepted control points of the Bézier surface with shifted knots. For the Bézier surface (5) with shifted knots, above Dirichlet integral (34) takes the following form:

$$D(\mathfrak{B}) = \frac{1}{2}\int_R (\langle \mathfrak{B}_u, \mathfrak{B}_u\rangle + \langle \mathfrak{B}_v, \mathfrak{B}_v\rangle)dudv. \tag{35}$$

We shall find the gradient of above Dirichlet integral (35) $w.r.t.$ given control points and equate it to zero for the constraints expressing the interior control points linearly as the combination of control points of the prescribed boundary of the Bézier surface with shifted knots (Equation (5)). Let us take the gradient of the Dirichlet integral (35) and write it as the sum of two integrals denoted by $M_{ij}$ and $N_{ij}$ as

$$\frac{\partial D(\mathfrak{B})}{\partial x^a_{ij}} = M_{ij} + N_{ij}, \tag{36}$$

where $M_{ij}$ and $N_{ij}$ are the constituent integrals of above Equation (36) given by

$$\begin{aligned} M_{ij} &= \int_R \left\langle \frac{\partial \mathfrak{B}_u}{\partial x^a_{ij}}, \mathfrak{B}_u \right\rangle dudv, \\ N_{ij} &= \int_R \left\langle \frac{\partial \mathfrak{B}_v}{\partial x^a_{ij}}, \mathfrak{B}_v \right\rangle dudv. \end{aligned} \tag{37}$$

Let us find these integrals $M_{ij}$ and $N_{ij}$ in terms of control points. We first target the integral $M_{ij}$ and note that the expression $\partial \mathfrak{B}_u / \partial x^a_{ij}$ in the first integral $M_{ij}$ in Equation

(37) can be written as

$$\frac{\partial}{\partial x_{ij}^a}(\mathfrak{B}_u(u,v)) = \frac{\partial}{\partial u}\left(\frac{\partial}{\partial x_{ij}^a}\left(\sum_{i,j=0}^{m,n} G_{m,\eta,\lambda}^i(u)G_{n,\eta,\lambda}^j(v)P_{kl}\right)\right), \tag{38}$$

which is reduced to

$$\begin{aligned}\frac{\partial}{\partial x_{ij}^a}(\mathfrak{B}_u(u,v)) &= \frac{\partial}{\partial u}\left(G_{m,\eta,\lambda}^i(u)G_{n,\eta,\lambda}^j(v)e^a\right)\\ &= \left(G_{m,\eta,\lambda}^i(u)\right)_u G_{n,\eta,\lambda}^j(v)e^a.\end{aligned} \tag{39}$$

Plugging the value of $(G_{m,\eta,\lambda}^i(u))_u$ from Equation (18) in Equation (39) gives us

$$\frac{\partial}{\partial x_{ij}^a}(\mathfrak{B}_u(u,v)) = (m+\lambda)\left(H_{m-1,\eta,\lambda}^{i-1}(u) - H_{m-1,\eta,\lambda}^i(u)\right)G_{n,\eta,\lambda}^j(v)e^a. \tag{40}$$

Insert the expression for $\partial\mathfrak{B}_u/\partial x_{ij}^a$ (Equation (40)) in the first integral $M_{ij}$ given by Equation (37) to get

$$M_{ij} = (m+\lambda)\int_R \left(H_{m-1,\eta,\lambda}^{i-1}(u) - H_{m-1,\eta,\lambda}^i(u)\right)G_{n,\eta,\lambda}^j(v)\langle e^a, \mathfrak{B}_u\rangle du dv, \tag{41}$$

where $H_{m-1,\eta,\lambda}^i(u)$ is Bernstein-like function given by Equation (20). Proceeding in the same way as above, note that

$$\begin{aligned}\frac{\partial}{\partial x_{ij}^a}(\mathfrak{B}_v(u,v)) &= \frac{\partial}{\partial v}\left(\frac{\partial}{\partial x_{ij}^a}\left(\sum_{i=j=0}^{m,n} G_{m,\eta,\lambda}^i(u)G_{n,\eta,\lambda}^j(v)P_{ij}\right)\right)\\ &= G_{m,\eta,\lambda}^i(u)\left(G_{n,\eta,\lambda}^j(v)\right)_v e^a.\end{aligned} \tag{42}$$

Substituting Equation (19) in above Equation (42), we find that

$$\begin{aligned}\frac{\partial}{\partial x_{ij}^a}(\mathfrak{B}_v(u,v)) &= G_{m,\eta,\lambda}^i(u)\left(G_{n,\eta,\lambda}^j(v)\right)_v e^a\\ &= (n+\lambda)G_{m,\eta,\lambda}^i(u)\\ &\quad\cdot\left(H_{n-1,\eta,\lambda}^{j-1}(v) - H_{n-1,\eta,\lambda}^j(v)\right)e^a.\end{aligned} \tag{43}$$

Substitute the expression for $\partial\mathfrak{B}_v/\partial x_{ij}^a$ from Equation (43) in Equation (37), so that the 2$^{\text{nd}}$ integral $N_{ij}$ may be rewritten as

$$\begin{aligned}N_{ij} &= \int_R \left\langle \frac{\partial\mathfrak{B}_v}{\partial x_{ij}^a}, \mathfrak{B}_v\right\rangle du dv = (n+\lambda)\int_R G_{m,\eta,\lambda}^i(u)\\ &\quad\cdot\left(H_{n-1,\eta,\lambda}^{j-1}(v) - H_{n-1,\eta,\lambda}^j(v)\right)\langle e^a, \mathfrak{B}_v\rangle du dv.\end{aligned} \tag{44}$$

For the integrals (41) and (44), we rewrite the expression for $B_u$ and $B_v$ in terms of Bernstein functions with shifted knots (using Equations (18) and (19), respectively).

$$\begin{aligned}\mathfrak{B}_u(u,v) &= \sum_{k,l=0}^{m,n}\left(G_{m,\eta,\lambda}^k(u)\right)_u G_{n,\eta,\lambda}^l(v)P_{kl} = (m+\lambda)\sum_{k,l=0}^{m,n}\\ &\quad\cdot\left(H_{m-1,\eta,\lambda}^{k-1}(u) - H_{m-1,\eta,\lambda}^k(u)\right)G_{n,\eta,\lambda}^l(v)P_{kl},\end{aligned} \tag{45}$$

$$\begin{aligned}\mathfrak{B}_v(u,v) &= \sum_{k,l=0}^{m,n} G_{m,\eta,\lambda}^k(u)\left(G_{n,\eta,\lambda}^l(v)\right)_v P_{kl} = (n+\lambda)\sum_{k,l=0}^{m,n} G_{m,\eta,\lambda}^k(u)\\ &\quad\cdot\left(H_{n-1,\eta,\lambda}^{l-1}(v) - H_{n-1,\eta,\lambda}^l(v)\right)P_{kl}.\end{aligned} \tag{46}$$

For the above value of $\mathfrak{B}_u(u,v)$ (Equation (45)), the integral $M_{ij}$ (Equation (41)) may be written as

$$\begin{aligned}M_{ij} &= (m+\lambda)^2\int_R\left[H_{m-1,\eta,\lambda}^{i-1}(u) - H_{m-1,\eta,\lambda}^i(u)\right]G_{n,\eta,\lambda}^j(v)\\ &\quad\cdot\left\langle e^a, \sum_{k,l=0}^{m,n}\left(H_{m-1,\eta,\lambda}^{k-1}(u) - H_{m-1,\eta,\lambda}^k(u)\right)G_{n,\eta,\lambda}^l(v)P_{kl}\right\rangle du dv,\end{aligned} \tag{47}$$

and plugging the value of $\mathfrak{B}_u(u,v)$ (Equation (46)) in the integral $N_{ij}$ (44) gives us

$$\begin{aligned}N_{ij} &= (n+\lambda)^2\int_R G_{m,\eta,\lambda}^i(u)\left(H_{n-1,\eta,\lambda}^{j-1}(v) - H_{n-1,\eta,\lambda}^j(v)\right)\\ &\quad\cdot\left\langle e^a, \sum_{k,l=0}^{m,n} G_{m,\eta,\lambda}^k(u)\left(H_{n-1,\eta,\lambda}^{l-1}(v) - H_{n-1,\eta,\lambda}^l(v)\right)P_{kl}\right\rangle du dv,\end{aligned} \tag{48}$$

in which the inner product $\langle e^a, P_{kl}\rangle$ of basis vectors $e^a$ and the control points $P_{kl}$ reduces to $x_{kl}^a$ for $a = 1, 2, 3$ and $P_{kl} = (x_{kl}^1, x_{kl}^2, x_{kl}^3)$. The constituent integrals (47) and (48) can be further simplified in the following form

$$\begin{aligned}M_{ij} &= (m+\lambda)^2\sum_{k,l=0}^{m,n}\int_R\left(H_{m-1,\eta,\lambda}^{i-1}(u) - H_{m-1,\eta,\lambda}^i(u)\right)\\ &\quad\cdot\left(H_{m-1,\eta,\lambda}^{k-1}(u) - H_{m-1,\eta,\lambda}^k(u)\right)G_{n,\eta,\lambda}^l(v)G_{n,\eta,\lambda}^j(v)\langle e^a, P_{kl}\rangle du dv,\end{aligned} \tag{49}$$

$$\begin{aligned}N_{ij} &= (n+\lambda)^2\sum_{k,l=0}^{m,n}\int_R\left(H_{n-1,\eta,\lambda}^{j-1}(v) - H_{n-1,\eta,\lambda}^j(v)\right)\\ &\quad\cdot\left(H_{n-1,\eta,\lambda}^{l-1}(v) - H_{n-1,\eta,\lambda}^l(v)\right)G_{m,\eta,\lambda}^i(u)G_{m,\eta,\lambda}^k(u)\langle e^a, P_{kl}\rangle du dv.\end{aligned} \tag{50}$$

Let us denote the integrand of the first integral $M_{ij}$ ((49)) by $Q_{ij}^{kl}(u,v)$, where

$$Q_{ij}^{kl}(u,v) = \left(H_{m-1,\eta,\lambda}^{i-1}(u) - H_{m-1,\eta,\lambda}^{i}(u)\right) \cdot \left(H_{m-1,\eta,\lambda}^{k-1}(u) - H_{m-1,\eta,\lambda}^{k}(u)\right) G_{n,\eta,\lambda}^{l}(v) G_{n,\eta,\lambda}^{j}(v), \tag{51}$$

where indices $i$ and $j$ are the free indices and the $k$ and $l$ are the dummy indices used in Equations (49) and (50). With this notation, the integral (49) takes the form

$$M_{ij} = (m+\lambda)^2 \sum_{k,l=0}^{m,n} R_{ij}^{kl} \langle e^a, P_{kl} \rangle, \tag{52}$$

where

$$R_{ij}^{kl} = \int_R Q_{ij}^{kl}(u,v)\,du\,dv. \tag{53}$$

Expand the integrand $Q_{ij}^{kl}(u,v)$ given by Equation (51) as the product of Bernstein polynomials with shifted knots

$$Q_{ij}^{kl}(u,v) = \left(H_{m-1,\eta,\lambda}^{i-1}(u) H_{m-1,\eta,\lambda}^{k-1}(u) - H_{m-1,\eta,\lambda}^{i-1}(u) H_{m-1,\eta,\lambda}^{k}(u) - H_{m-1,\eta,\lambda}^{i}(u) H_{m-1,\eta,\lambda}^{k-1}(u) + H_{m-1,\eta,\lambda}^{i}(u) H_{m-1,\eta,\lambda}^{k}(u)\right) G_{n,\eta,\lambda}^{j}(v) G_{n,\eta,\lambda}^{l}(v). \tag{54}$$

Let us denote the products of Bernstein-like polynomials with shifted knots by

$$H_{m-1,m-1,\eta,\lambda}^{i-1,k-1}(u) = H_{m-1,\eta,\lambda}^{i-1}(u) H_{m-1,\eta,\lambda}^{k-1}(u), \tag{55}$$

and that of Bernstein polynomials with shifted knots by

$$G_{n,n,\eta,\lambda}^{j,l}(v) = G_{n,\eta,\lambda}^{j}(v) G_{n,\eta,\lambda}^{l}(v). \tag{56}$$

In view of the notation introduced in (55) and (56), the integral Equation (53), for the bivariate function $Q_{ij}^{kl}(u,v)$ given by (54), takes the following convenient form

$$R_{ij}^{kl} = C_{i-1j}^{k-1l} - C_{i-1j}^{k,l} - C_{ij}^{k-1l} + C_{ij}^{kl}, \tag{57}$$

which is the sum of four integrals denoted by $C_{i-1j}^{k-1l}$, $C_{i-1j}^{k,l}$, $C_{ij}^{k-1l}$, $C_{ij}^{kl}$, and these are

$$\begin{aligned} C_{i-1j}^{k-1l} &= \iint_R H_{m-1,m-1,\eta,\lambda}^{i-1,k-1}(u) G_{n,n,\eta,\lambda}^{j,l}(v)\,dv\,du, \; C_{i-1j}^{k,l} = \iint_R H_{m-1,m-1,\eta,\lambda}^{i-1,k}(u) G_{n,n,\eta,\lambda}^{j,l}(v)\,dv\,du, \\ C_{ij}^{k-1l} &= \iint_R H_{m-1,m-1,\eta,\lambda}^{i,k-1}(u) G_{n,n,\eta,\lambda}^{j,l}(v)\,du\,dv, \; C_{ij}^{kl} = \iint_R H_{m-1,m-1,\eta,\lambda}^{i,k}(u) G_{n,n,\eta,\lambda}^{j,l}(v)\,du\,dv. \end{aligned} \tag{58}$$

The integrals of the product of Bernstein polynomials with shifted knots given by Equation (27) and that for the product of Bernstein-like polynomials given by Equation (30) allow us to find the constituent integrals of (58), and these are

$$C_{i-1j}^{k-1l} = D_j^l A_{(k-1)(i-1)}^{m-1} B_q^{p-2}, \; C_{i-1j}^{k,l} = D_j^l A_{(k)(i-1)}^{m-1} B_{q-1}^{p-1}, \; C_{ij}^{k-1l} = D_j^l A_{(k-1)(i)}^{m-1} B_{q-1}^{p-1}, \; C_{ij}^{kl} = D_j^l A_{(k)(i)}^{m-1} B_{q-2}^{p}, \tag{59}$$

where $B_q^p = (i+k)!(2m-(i+k))!$, $p = i+k$, $q = 2m-(i+k)$, $Btr = (j+l)!(2n-(j+l))!$, $r = j+l$, $t = 2n-(j+l)$ and

$$D_j^l = mn\gamma A_{(j)(l)}^{n} B_t^r, \quad A_{(k)(i)}^{m-1} = \begin{pmatrix} m-1 \\ k \end{pmatrix} \begin{pmatrix} m-1 \\ i \end{pmatrix}, \tag{60}$$

where $\gamma^{-1} = ((\lambda+m)(\lambda+n)(2n+1)!(2m-1)!)$. Plugging the values of the integrals $C_{i-1j}^{k-1l}$, $C_{i-1j}^{k,l}$, $C_{ij}^{k-1l}$, $C_{ij}^{kl}$ given by Equation (59) along with Equation (60) in Equation (57), we get

$$R_{ij}^{kl} = D_j^l \left(A_{(k-1)(i-1)}^{m-1} B_q^{p-2} - A_{(k)(i-1)}^{m-1} B_{q-1}^{p-1} - A_{(k-1)(i)}^{m-1} B_{q-1}^{p-1} + A_{(k)(i)}^{m-1} B_{q-2}^{p}\right). \tag{61}$$

Substituting value of $R_{ij}^{kl}$ from above Equation (61) in Equation (52) gives us

$$M_{ij} = (m+\lambda)^2 \sum_{k,l=0}^{m,n} D_j^l \left(A_{(k-1)(i-1)}^{m-1} B_q^{p-2} - A_{(k)(i-1)}^{m-1} B_{q-1}^{p-1} - A_{(k-1)(i)}^{m-1} B_{q-1}^{p-1} + A_{(k)(i)}^{m-1} B_{q-2}^{p}\right) \langle e^a, P_{kl} \rangle. \tag{62}$$

In the similar way, we can find the second integral $N_{ij}$ (Equation (50)) by identifying its integrand by

$$T_{ij}^{kl}(u,v) = \left(H_{n-1,\eta,\lambda}^{j-1}(v) - H_{n-1,\eta,\lambda}^{j}(v)\right) \cdot \left(H_{n-1,\eta,\lambda}^{l-1}(v) - H_{n-1,\eta,\lambda}^{l}(v)\right) G_{m,\eta,\lambda}^{i}(u) G_{m,\eta,\lambda}^{k}(u), \tag{63}$$

along with

$$S_{ij}^{kl} = \int_R T_{ij}^{kl}(u,v)\,du\,dv. \tag{64}$$

Equation (50) can be written as

$$Nij = (n+\lambda)^2 \sum_{k,l=0}^{m,n} S_{ij}^{kl} \langle e^a, P_{kl} \rangle. \tag{65}$$

Expand the integrand $T_{ij}^{kl}(u,v)$ given by Equation (63) as the product of Bernstein polynomials with shifted knots

$$T_{ij}^{kl}(u,v)=\begin{pmatrix} H_{n-1,\eta,\lambda}^{j-1}(v)H_{n-1,\eta,\lambda}^{l-1}(v)-H_{n-1,\eta,\lambda}^{j-1}(v)H_{n-1,\eta,\lambda}^{l}(v)- \\ H_{n-1,\eta,\lambda}^{j}(v)H_{n-1,\eta,\lambda}^{l-1}(v)+H_{n-1,\eta,\lambda}^{j}(v)H_{n-1,\eta,\lambda}^{l}(v) \end{pmatrix} G_{m,\eta,\lambda}^{i}(u)G_{m,\eta,\lambda}^{k}(u). \tag{66}$$

Let us denote the products of Bernstein polynomials with shifted knots by

$$H_{n-1,\eta,\lambda}^{j-1,l-1}(v)=H_{n-1,\eta,\lambda}^{j-1}(v)H_{n-1,\eta,\lambda}^{l-1}(v), \tag{67}$$

We can write the above bivariate function $T_{ij}^{kl}(u,v)$ in Equation (63) along with Equation (67) in the following form:

$$T_{ij}^{kl}(u,v)=\left(H_{n-1,\eta,\lambda}^{j-1,l-1}(v)-H_{n-1,\eta,\lambda}^{j-1,l}(v)-H_{n-1,\eta,\lambda}^{j,l-1}(v)+H_{n-1,\eta,\lambda}^{j,l}(v)\right)G_{m,\eta,\lambda}^{i,k}(u). \tag{68}$$

Substituting Equation (68) in Equation (64), we find

$$\begin{aligned} S_{ij}^{kl}=\int_R \Big(&H_{n-1,n-1,\eta,\lambda}^{j-1,l-1}(v)-H_{n-1,n-1,\eta,\lambda}^{j-1,l}(v)-H_{n-1,n-1,\eta,\lambda}^{j,l-1}(v) \\ &+H_{n-1,n-1,\eta,\lambda}^{j,l}(v)\Big)G_{m,\eta,\lambda}^{i,k}(u)dudv, \end{aligned} \tag{69}$$

which appears as the sum of four integrals

$$S_{ij}^{kl}=E_{ij-1}^{kl-1}-E_{ij-1}^{kl}-E_{ij}^{kl-1}+E_{ij}^{kl}, \tag{70}$$

where

$$\begin{aligned} E_{ij-1}^{kl-1}&=\int_R G_{m,m,\eta,\lambda}^{i,k}(u)H_{n-1,n-1,\eta,\lambda}^{j-1,l-1}(v)dudv, E_{ij-1}^{kl} \\ &=\int_R G_{m,m,\eta,\lambda}^{i,k}(u)H_{n-1,n-1,\eta,\lambda}^{j-1,l}(v)dudv, \end{aligned} \tag{71}$$

$$\begin{aligned} E_{ij}^{kl-1}&=\int_R G_{m,m,\eta,\lambda}^{i,k}(u)H_{n-1,n-1,\eta,\lambda}^{j,l-1}(v)dudv, \\ E_{ij}^{kl}&=\int_R G_{m,m,\eta,\lambda}^{i,k}(u)H_{n-1,n-1,\eta,\lambda}^{j,l}(v)dudv. \end{aligned} \tag{72}$$

Equations (27) and (30) allow us to find the integrals of Equation (71) which when substituted in Equation (70) and gives us $S_{ij}^{kl}$ and replacing back $S_{ij}^{kl}$ in Equation (65), and we obtain the second integral $N_{ij}$ with the similar process as

$$\begin{aligned} N_{ij}=(n+\lambda)^2\sum_{k,l=0}^{m,n} D_i^k\Big(&A_{(l-1)(j-1)}^{m-1}B_s^{r-2}+A_{(l)(j)}^{m-1}B_{s-2}^{r} \\ &-A_{(l)(j-1)}^{m-1}B_{s-1}^{r-1}-A_{(l-1)(j)}^{m-1}B_{s-1}^{r-1}\Big)P_{kl}, \end{aligned} \tag{73}$$

where $B_s^r=(j+1)!(2n-(j+l))!, r=j+l, s=2m-(j+1)$. The vanishing condition for the gradient of the Dirichlet functional is thus given by equating the sum of Equations (62) and (73) to zero that means

$$M_{ij}+N_{ij}=0, \tag{74}$$

which gives us a system of algebraic constraints in terms of control points, and this system of algebraic constraints can be solved for unknown interior control points in terms of given boundary control points. For the given boundary points and the interior points obtained by solving the linear constraints as mentioned above, it gives us the so-called quasi-minimal Bézier surface with shifted knots. The above analysis gives us the following result:

**Theorem 1.** *A Bézier surface of degree $(m,n)$ with shifted knots characterised by its $(m+1)\times(n+1)$ number of control points $P_{kl}$(for $k=0,1,2,\cdots,m$, $l=0,1,2,\cdots,n$) is the extremal of the Dirichlet functional for*

$$\sum_{k,l=0}^{m,n} C_{kl}^{ij}P_{kl}=0, \Leftrightarrow C_{kl}^{ij}=(m+\lambda)^2\sigma_{kl}^{ij}+(n+\lambda)^2\sigma_{lk}^{ji}, \tag{75}$$

*where*

$$\sigma_{kl}^{ij}=D_j^l\left(A_{(k-1)(i-1)}^{m-1}B_q^{p-2}-A_{(k)(i-1)}^{m-1}B_{q-1}^{p-1}-A_{(k-1)(i)}^{m-1}B_{q-1}^{p-1}+A_{(k)(i)}^{m-1}B_{q-2}^{p}\right). \tag{76}$$

*$\sigma_{kl}^{ij}$ is symmetric with respect to its summation pair of indices $k,l$ and the pair of free indices $i,j$. That is, $\sigma_{kl}^{ji}$ can be obtained by interchanging $i,j$ and $k,l$ in $\sigma_{kl}^{ij}\cdot D_j^l$ and $A_{(k)(i)}^{m-1}$ are the constants indicated by Equation (60).*

**Corollary 2.** *A biquadratic Bézier surface with shifted knots is the extremal of the Dirichlet functional if it satisfies the extremal condition given by*

$$\sum_{j,k=0}^{2} C_{jk}P_{jk}=0, \tag{77}$$

*where the coefficients $C_{jk}$ of the control points are given below.*

$$
\begin{aligned}
C_{00} &= -\frac{(\lambda+2)^2}{(3\lambda+6)(5\lambda+10)} - \frac{1}{15},\\
C_{01} &= \frac{2}{15} - \frac{4(\lambda+2)}{15}\left(\frac{1}{3\lambda+6}\right),\\
C_{02} &= -\frac{(\lambda+2)^2}{(3\lambda+6)(5\lambda+10)} - \frac{1}{15},\\
C_{10} &= \frac{2(\lambda+2)^2}{(3(\lambda+2))(5\lambda+10)} - \frac{4}{45},\\
C_{11} &= \frac{8(\lambda+2)}{15}\left(\frac{1}{3\lambda+2}\right) + \frac{8}{45},\\
C_{12} &= \frac{4(\lambda+2)}{3(5\lambda+10)} - \frac{4}{45},\\
C_{20} &= -\frac{(\lambda+2)}{3(5\lambda+10)} - \frac{1}{15},\\
C_{21} &= \frac{2}{15} - \frac{4(\lambda+2)}{15}\left(\frac{1}{3\lambda+6}\right),\\
C_{22} &= -\frac{(\lambda+2)^2}{(3(\lambda+2))(5\lambda+10)} - \frac{1}{15}.
\end{aligned}
\tag{78}
$$

*Equation (77) can be solved for the coefficient $P_{11}$, which is the only unknown coefficient in the case of $2\times 2$ grid of points given by $P_{00}, P_{01}, P_{02}, P_{10}, P_{11}, P_{12}, P_{20}, P_{21}, P_{22}$. This simply means that we can write*

$$
\begin{aligned}
P_{11} = -\Bigg(&\frac{C_{00}}{C_{11}}P_{00} + \frac{C_{01}}{C_{11}}P_{01} + \frac{C_{02}}{C_{11}}P_{02} + \frac{C_{10}}{C_{11}}P_{10}\\
&+ \frac{C_{12}}{C_{11}}P_{12} + \frac{C_{20}}{C_{11}}P_{20} + \frac{C_{21}}{C_{11}}P_{21} + \frac{C_{22}}{C_{11}}P_{22}\Bigg).
\end{aligned}
\tag{79}
$$

*The adjustment provided by this point along with already known points generates a quasi-minimal Bézier surface with shifted knots.*

## 5. Extremal of a Biquadratic Bézier Surface with Shifted Knots´

A biquadratic Bézier surface $\mathfrak{B}(u,v)$ with shifted knots (Equation (5))

$$
\mathfrak{B}(u,v) = \sum_{j,k=0}^{2,2} G^{j}_{2,\eta,\lambda}(u)G^{j}_{2,\eta,\lambda}(v)P_{jk}, \tag{80}
$$

for the Bernstein functions defined by Equation (6) for $m=n=2$ can be written in the form

$$
\begin{aligned}
\mathfrak{B}(u,v) = {}& \frac{1}{16}(\lambda+2)^4\left(\frac{\eta+2}{\lambda+2}-u\right)^2\left(\frac{\eta+2}{\lambda+2}-v\right)^2 P_{00} + \frac{1}{8}(\lambda+2)^4\\
&\cdot\left(\frac{\eta+2}{\lambda+2}-u\right)^2\left(v-\frac{\eta}{\lambda+2}\right)\left(\frac{\eta+2}{\lambda+2}-v\right)P_{01} + \frac{1}{16}(\lambda+2)^4\\
&\cdot\left(\frac{\eta+2}{\lambda+2}-u\right)^2\left(v-\frac{\eta}{\lambda+2}\right)^2 P_{02} + \frac{1}{8}(\lambda+2)^4\left(u-\frac{\eta}{\lambda+2}\right)\\
&\cdot\left(\frac{\eta+2}{\lambda+2}-u\right)\left(\frac{\eta+2}{\lambda+2}-v\right)^2 P_{10} + \frac{1}{4}(\lambda+2)^4\left(u-\frac{\eta}{\lambda+2}\right)\\
&\cdot\left(\frac{\eta+2}{\lambda+2}-u\right)\left(v-\frac{\eta}{\lambda+2}\right)\left(\frac{\eta+2}{\lambda+2}-v\right)P_{11} + \frac{1}{8}(\lambda+2)^4\\
&\cdot\left(u-\frac{\eta}{\lambda+2}\right)\times\left(\frac{\eta+2}{\lambda+2}-u\right)\left(v-\frac{\eta}{\lambda+2}\right)P_{12} + \frac{1}{16}(\lambda+2)^4\\
&\cdot\left(u-\frac{\eta}{\lambda+2}\right)^2\left(\frac{\eta+2}{\lambda+2}-v\right)^2 P_{20} + \frac{1}{8}(\lambda+2)^4\times\left(u-\frac{\eta}{\lambda+2}\right)^2\\
&\cdot\left(v-\frac{\eta}{\lambda+2}\right)\left(\frac{\eta+2}{\lambda+2}-v\right)P_{21} + \frac{1}{16}(\lambda+2)^4\left(u-\frac{\eta}{\lambda+2}\right)^2\\
&\cdot\left(v-\frac{\eta}{\lambda+2}\right)^2 P_{22}.
\end{aligned}
\tag{81}
$$

A little simplification, in particular, for $\eta = 0.2$ and $\lambda = 0.2$, a biquadratic Bézier surface $\mathfrak{B}(u,v) = (x(u,v), y(u,v), z(u,v))$ with shifted knots for control points $P_{jk} = (j, k, (-1)^{j+k})$ where $j, k = 0, 1, 2$, reduces to

$$
x(u,v) = -0.2 + 2.2u, \tag{82}
$$

$$
y(u,v) = -0.2 + 2.2v, \tag{83}
$$

$$
\begin{aligned}
z(u,v) = {}& 2.1 - 7.6u + 7u^2 - 7.6v + 27.9uv\\
&- 25.6u^2v + 7v^2 - 25.6uv^2 + 23.4u^2v^2.
\end{aligned}
\tag{84}
$$

The fundamental magnitudes of this surface can be computed which are as follows:

$$
\begin{aligned}
E = {}& 62.65 - 211.97u + 194.301u^2 - 423.93v + 1554.41uv\\
&- 1424.88u^2v + 1165.81v^2 - 4274.63uv^2 + 3918.41u^2v^2\\
&- 1424.88v^3 + 5224.55uv^3 - 4789.17u^2v^3 + 653.068v^4\\
&- 2394.58uv^4 + 2195.03u^2v^4,
\end{aligned}
$$

$$
\begin{aligned}
F = {}& 57.8087 - 317.948u + 582.90u^2 - 356.22u^3 - 317.95v\\
&+ 1748.71uv - 3205.97u^2v + 1959.20u^3v + 582.90v^2\\
&- 3205.97uv^2 + 5877.61u^2v^2 - 3591.88u^3v^2 - 356.22v^3\\
&+ 1959.20uv^3 - 3591.88u^2v^3 + 2195.03u^3v^3,
\end{aligned}
$$

$$
\begin{aligned}
G = {}& 62.6487 - 423.93u + 1165.81u^2 - 1424.88u^3\\
&+ 653.068u^4 - 211.97v + 1554.41uv - 4274.63u^2v\\
&+ 5224.55u^3v - 2394.58u^4v + 194.30v^2 - 1424.88uv^2\\
&+ 3918.41u^2v^2 - 4789.17u^3v^2 + 2195.03u^4v^2,
\end{aligned}
$$

$$
e = 67.47 - 247.37v + 226.76v^2,
$$

$$
f = 453.52uv - 247.37u - 247.37v + 134.93,
$$

$$g = 226.76u^2 - 247.37u + 67.47. \tag{85}$$

A unit normal to the biquadratic Bézier surface with shifted knots can be computed using $N = (\mathfrak{B}_u \times \mathfrak{B}_v)/|\mathfrak{B}_u \times \mathfrak{B}_v|$. The coefficients of the fundamental forms along with well-defined unit normal to the two dimensional parametric surface (84) gives us the differential geometry related quantities like mean and Gaussian curvature of the surface to further analyse the quasi-minimal surfaces that come out as the extremal of the Dirichlet functional. Though, for a minimal surface the mean curvature should be zero for all values of the surface parameters $u$ and $v$, however, the mean curvature in this case appears as the function of surface parameters $u$ and $v$ giving us a quasi-minimal surface, as the expression for the minimal surface involves approximate coefficients. For a quasi-minimal biquadratic Bézier surface with shifted knots for given control points and for $\eta = 0.2$ and $\lambda = 0.2$, the optimal point can be computed from Equation (79) and it turns out that the optimal point, for biquadratic surface is $P_{11} = (1, 1, 2)$. The numerator of the mean curvature of this quasi-minimal biquadratic Bézier surface (coefficients rounded off to the nearest decimal place) is given by

$$\begin{aligned} Hnum = {} & -2 \times 10^6 u^4 v^4 + 4 \times 10^6 u^4 v^3 - 3 \times 10^6 u^4 v^2 \\ & + 1 \times 10^6 u^4 v - 158496 u^4 + 4 \times 10^6 u^3 v^4 \\ & - 9 \times 10^6 u^3 v^3 + 7 \times 10^6 u^3 v^2 - 3 \times 10^6 u^3 v \\ & + 345810.u^3 - 3 \times 10^6 u^2 v^4 + 7 \times 10^6 u^2 v^3 \\ & - 6 \times 10^6 u^2 v^2 + 2 \times 10^6 u^2 v - 278076 u^2 \\ & + 1 \times 106 uv^4 - 3 \times 106 uv^3 + 2 \times 106 uv^2 \\ & - 762397 uv + 97584.7u - 158496 v^4 \\ & + 345810 v^3 - 278076 v^2 + 97584.7v - 12162.8, \end{aligned} \tag{86}$$

the numerator of the Gaussian curvature of the quasi-minimal biquadratic Bézier surface is

$$\begin{aligned} K_{num} = {} & 23174.1 + 81852.9u - 75031.8u^2 + 81852.9v \\ & - 286847uv + 262943u^2 v - 75031.8v^2 \\ & + 262943uv^2 - 241031u^2 v^2, \end{aligned} \tag{87}$$

and the Dirichlet integrand

$$\begin{aligned} D(\mathfrak{B}) = {} & 71.8179 - 407.989u + 981.181u^2 - 1113.18u^3 \\ & + 510.21u^4 - 407.989v + 2091.44uv \\ & - 4143.52u^2 v + 4081.68u^3 v - 1870.77u^4 v \\ & + 981.181v^2 - 4143.52uv^2 + 5839.07u^2 v^2 \\ & - 3741.54u^3 v^2 + 1714.87u^4 v^2 - 1113.18v^3 \\ & + 4081.68uv^3 - 3741.54u^2 v^3 + 510.21v^4 \\ & - 1870.77uv4 + 1714.87u^2 v^4. \end{aligned} \tag{88}$$

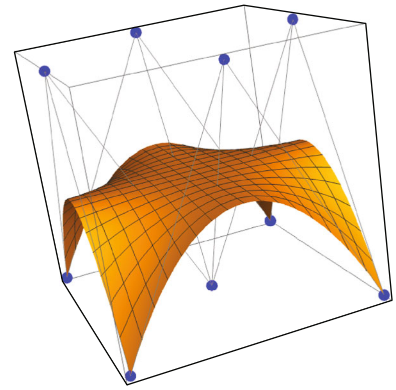

Figure 1: A biquadratic Bézier surface with shifted knots for $\eta = 0.2$ and $\lambda = 0.2$.

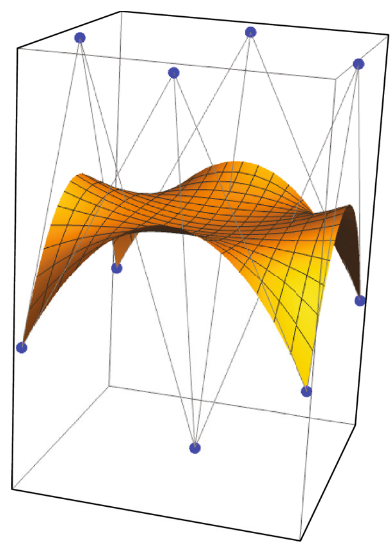

Figure 2: The quasi-minimal biquadratic Bézier surface with shifted knots for $\eta = 0.2$ and $\lambda = 0.2$ for given boundary control points and optimal point obtained from Equation (79).

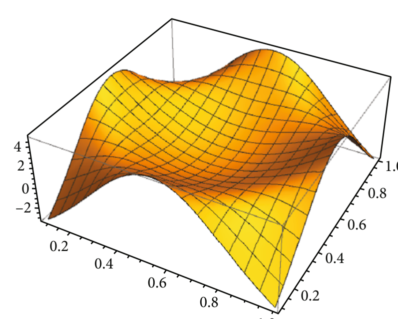


Figure 3: The mean curvature function of the biquadratic Bézier surface with shifted knots for $\eta = 0.2$ and $\lambda = 0.2$.

For given control points, the biquadratic Bézier surface with shifted knots (Equation (84)) for $\eta = 0.2$ and $\lambda = 0.2$ is shown in Figure 1, whereas the quasi-minimal biquadratic Bézier surface with shifted knots for the same values of $\eta = 0.2$ and $\lambda = 0.2$ along with the optimal point obtained from Equation (79) is shown in Figure 2. The numerator part of

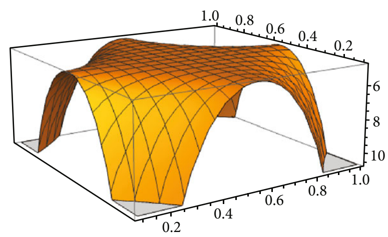


Figure 4: The Dirichlet integrand function for the biquadratic Bézier surface with shifted knots for $\eta = 0.2$ and $\lambda = 0.2$.

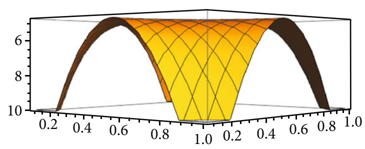


Figure 5: The area integrand function for the biquadratic Bézier surface with shifted knots for $\eta = 0.2$ and $\lambda = 0.2$.

the related mean curvature function of this quasi-minimal biquadratic Bézier surface with shifted knots is shown in Figure 3 and the Dirichlet integrand function in Figure 4, and the area functional integrand is shown in Figure 5. Thus, the variational minimization of the Dirichlet functional to obtain a quasi-minimal Bézier surface with shifted knots is quite useful for further geometric analysis of the surface obtained, in particular, for geometry-related quantities like Gaussian and mean curvature.

## 6. Conclusion

A surface is said to be minimal if its mean curvature vanishes everywhere on the surface, which is the outcome of variational minimization of area functional. Instead of variational minimization of the area function, we find the quasi-minimal Bézier surfaces with shifted knots as the solution of variational minimization of the Dirichlet functional, which is expressed as the sum of two functionals in which the respective coefficients of the control points come up with symmetry in the pair of summation and free indices that is helpful to solve the constraint equations for the interior control points as the linear combination of prescribed boundary control points. The functional gradient of the prospective functional for the surface gives us the possible quasi-minimal candidate functions as the minimizers of the functional. For such algebraic constraints, we find the gradient of Dirichlet function for the Bézier surface with shifted knots and the vanishing condition results in the aforesaid constraints on the interior control points as the linear combination of prescribed boundary control points. For illustration, the technique is implemented on a biquadratic Bézier surface to obtain a quasi-minimal surface. Similar type of process can be performed for a higher degree Bézier surface to attain the respective quasi-minimal Bézier surface with shifted knots.

## Data Availability

Data used to support the findings of this study are included within the article.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## References

[1] D. Barrera, M. A. Fortes, P. Gonzlez, and M. Pasadas, “Minimal energy–surfaces on uniform Powell–Sabin–type meshes for noisy data,” *Journal of Computational and Applied Mathematics*, vol. 2, pp. 592–602, 2008.

[2] X. Jiao, D. Wang, and H. Zha, “Simple and effective variational optimization of surface and volume triangulations,” *Engineering with Computers*, vol. 27, no. 1, pp. 81–94, 2011.

[3] C. Coppin and D. Greenspan, “A contribution to the particle modeling of soap films,” *Applied Mathematics and Computation*, vol. 26, no. 4, pp. 315–331, 1988.

[4] J. Douglas, “Solution of the problem of Plateau,” *Transactions of the American Mathematical Society*, vol. 33, no. 1, pp. 263–321, 1931.

[5] R. Osserman, *A Survey of Minimal Surfaces*, Dover Publications Inc., 1986.

[6] J. C. C. Nitsche, *Lectures on Minimal Surfaces*, Cambridge University Press, 1989.

[7] T. Radó, “On Plateau’s problem,” *Annals of Mathematics*, vol. 31, no. 3, pp. 457–469, 1930.

[8] E. J. Shane, “Parameterization of saddle surfaces, with applications to the problem of Plateau,” *Transactions of the American Mathematical Society*, vol. 35, pp. 716–733, 1933.

[9] L. Tonelli, “Sul problema di Plateau, I & II,” *Rend. R. Accad. dei Lincei*, vol. 24, 1936.

[10] M. Shiffman, “The Plateau problem for non-relative minima,” *Annals of Mathematics*, vol. 40, no. 4, pp. 834–854, 1939.

[11] R. Courant, “Plateau’s problem and Dirichlet’s principle,” *Annals of Mathematics*, vol. 38, no. 3, pp. 679–725, 1937.

[12] R. Courant, *Dirichlet’s Principle, Conformal Mapping and Minimal Surfaces*, Springer, 1977.

[13] M. Tompkins, “Minimal surfaces of non-minimum type by a new mode of aproximation,” *Annals of Mathematics*, vol. 42, pp. 443–472, 1941.

[14] C. B. Morrey, “The problem of Plateau on a Riemannian manifold,” *Annals of Mathematics*, vol. 49, no. 4, pp. 807–851, 1948.

[15] C. B. Morrey, “The higher-dimensional Plateau problem on a Riemannian manifold,” *Proceedings of the National Academy of Sciences of the United States of America*, vol. 54, no. 4, pp. 1029–1035, 1965.

[16] R. Osserman, “A proof of the regularity everywhere of the classical solution to Plateau’s problem,” *Annals of Mathematics*, vol. 91, no. 3, pp. 550–569, 1970.

[17] R. Gulliver, “Regularity of minimizing surfaces of prescribed mean curvature,” *Annals of Mathematics*, vol. 97, no. 2, pp. 275–305, 1973.

[18] H. Karcher, “The triply periodic minimal surfaces of Alan Schoen and their constant mean curvature companions,” *Manuscripta mathematica*, vol. 64, no. 3, pp. 291–357, 1989.

[19] F. J. Almgren, "Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem," *Annals of Mathematics*, vol. 84, no. 2, pp. 277–292, 1966.

[20] C. H. Séquin, "CAD tools for aesthetic engineering," *Computer-Aided Design*, vol. 37, no. 7, pp. 737–750, 2005.

[21] C. H. Séquin, "Interactive procedural computer-aided design," in *CAD/Graphics Conference, HKUST*, pp. 181–186, Hong Kong, 2005.

[22] P. Joshi and C. Squin, "Energy minimizers for curvature-based surface functionals," *Computer-Aided Design and Applications*, vol. 4, no. 5, pp. 607–617, 2007.

[23] S. Bernstein, "Démonstration du Theoreme du Weierstrass fondeé sur le calcul des probabilitié," *Commun. Soc. Math. Kharkow*, vol. 13, pp. 1-2, 1912.

[24] C. Cosín and J. Monterde, *Bézier Surfaces of Minimal Area*, Springer, Berlin Heidelberg, 2002.

[25] J. Monterde, "Bézier surfaces of minimal area: the Dirichlet approach," *Computer Aided Geometric Design*, vol. 21, no. 2, pp. 117–136, 2004.

[26] J. Monterde and H. Ugail, "A general 4th-order PDE method to generate Bézier surfaces from the boundary," *Computer Aided Geometric Design*, vol. 23, no. 2, pp. 208–225, 2006.

[27] A. Arnal, A. Lluch, and J. Monterde, *Triangular Bézier Surfaces of Minimal Area*, Springer, Berlin Heidelberg, 2003.

[28] Y.-X. Hao, R.-H. Wang, and C.-J. Li, "Minimal quasi-Bézier surface," *Applied Mathematical Modelling*, vol. 36, no. 12, pp. 5751–5757, 2012.

[29] Y.-X. Hao, C.-J. Li, and R.-H. Wang, "An approximation method based on MRA for the quasi-Plateau problem," *BIT Numerical Mathematics*, vol. 53, no. 2, pp. 411–442, 2013.

[30] G. Xu and G. Wang, "Quintic parametric polynomial minimal surfaces and their properties," *Differential Geometry and its Applications*, vol. 28, no. 6, pp. 697–704, 2010.

[31] X.-D. Chen, G. Xu, and Y. Wanga, "Approximation methods for the Plateau-Bézier problem," in *2009 11th IEEE International Conference on Computer-Aided Design and Computer Graphics*, pp. 588–591, Huangshan, China, 2009.

[32] D. Ahmad and S. Naeem, "Quasi-harmonic constraints for toric Bézier surfaces," *Sigma Journal of Engineering and Natural Sciences*, vol. 36, pp. 325–340, 2018.

[33] D. Ahmad and B. Masud, "A Coons patch spanning a finite number of curves tested for variationally minimizing its area," *Abstract and Applied Analysis*, vol. 2013, 15 pages, 2013.

[34] D. Ahmad and B. Masud, "Variational minimization on string-rearrangement surfaces, illustrated by an analysis of the bilinear interpolation," *Applied Mathematics and Computation*, vol. 233, pp. 72–84, 2014.

[35] D. Ahmad and B. Masud, "Near-stability of a quasi-minimal surface indicated through a tested curvature algorithm," *Computers & Mathematics with Applications*, vol. 69, no. 10, pp. 1242–1262, 2015.

[36] C. Zhu, G. Peng, D. Liu, X. Hu, and Y. Wu, "Evaluation of surface topography of SiCp/Al composite in grinding," *The International Journal of Advanced Manufacturing Technology*, vol. 102, no. 9-12, pp. 2807–2821, 2019.

[37] J.-B. Liu, J. Zhao, H. He, and Z. Shao, "Valency-based topological descriptors and structural property of the generalized Sierpiński networks," *Journal of Statistical Physics*, vol. 177, no. 6, pp. 1131–1147, 2019.

[38] J.-B. Liu, J. Zhao, J. Min, and J. Cao, "The Hosoya index of graphs formed by a fractal graph," *Fractals*, vol. 27, no. 8, article 1950135, 2019.

[39] J.-B. Liu, J. Zhao, and Z.-Q. Cai, "On the generalized adjacency, Laplacian and signless Laplacian spectra of the weighted edge corona networks," *Physica A: Statistical Mechanics and its Applications*, vol. 540, p. 123073, 2020.

[40] A. D. Gadjiev and A. M. Ghorbanalizadeh, "Approximation properties of a new type Bernstein-Stancu polynomials of one and two variables," *Applied Mathematics and Computation*, vol. 216, no. 3, pp. 890–901, 2010.

[41] K. Khan, D. Lobiyal, and A. Kilicman, "Bézier curves and surfaces based on modified Bernstein polynomials," *Azerbaijan Journal of Mathematics*, vol. 9, pp. 2218–6816, 2019.

[42] K. Khan and D. K. Lobiyal, "Bèzier curves based on Lupaş(p,q)-analogue of Bernstein functions in CAGD," *Journal of Computational and Applied Mathematics*, vol. 317, pp. 458–477, 2017.

[43] M. Mursaleen, K. J. Ansari, and A. Khan, "Approximation properties and error estimation of q-Bernstein shifted operators," *Numerical Algorithms*, vol. 84, no. 1, pp. 207–227, 2020.

[44] K. S. Nisar, V. Sharma, and A. Khan, "Lupaş blending functions with shifted knots and q-Bézier curves," *Journal of Inequalities and Applications*, vol. 184, no. 1, 2020.

[45] D. D. Stancu, "Approximation of functions by a new class of linear polynomial operators," *Revue Roumaine des Mathematiques Pures et Appliquees*, vol. 13, pp. 1173–1194, 1968.

[46] M. P. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice Hall, 1976.