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Research Article

G-Chain Mixing and G-Chain Transitivity in Metric G-Space

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Firstly, we introduce the concept of G-chain mixing, G-mixing, and G-chain transitivity in metric G-space. Secondly, we study their dynamical properties and obtain the following results. (1) If the map f has the G-shadowing property, then the map f is G-chain mixed if and only if the map f is G-mixed. (2) The map f is G-chain transitive if and only if for any positive integer $k \geq 2$, the map f^k is G-chain transitive. (3) If the map f is G-pointwise chain recurrent, then the map f is G-chain transitive. (4) If there exists a nonempty open set U satisfying $G(U) = U$, $\bar{U} \neq X$, and $f(\bar{U}) \subset U$, then we have that the map f is not G-chain transitive. These conclusions enrich the theory of G-chain mixing, G-mixing, and G-chain transitivity in metric G-space.

1. Introduction

Chain mixing, mixing, and chain transitivity are very important concepts in topological dynamical systems. Many scholars studied their dynamical properties and obtained some valuable results (see [1–9]). Fatehi [1] showed that chain mixing, chain transitivity, and chain recurrence properties are equivalent in iterated function systems; Ji [2] proved that if the sequence map $\{f_n\}$ is G-mixed, then the limit map f is also G-mixed under G-strong uniform convergence; Zeng et al. [3] gave that if the sequence map $\{f_n\}$ is weakly mixed, then the limit map f is also weakly mixed under strong uniform convergence; Liu and Yi [4] pointed out that if the map F is chain mixed, then F^k is chain mixed for any positive integer k in nonautonomous dynamical system; Diao [5] proved that the map f is chain mixed if and only if there exists positive integer r and s such that $f^r \times f^s$ is chain transitive.

In this paper, firstly, we introduce the concept of G-chain mixing, G-mixing, and G-chain transitivity in metric G-space. Secondly, we study their dynamical properties and obtain the following theorem.

Theorem 1. *Let (X, d) be a compact metric G-space, $f : X \longrightarrow X$ be a pseudo equivalent, and isometric map and the metric d be invariant to G. If the map f has the G-shadowing property, then the map f is G-chain mixed if and only if the map f is G-mixed.*

Theorem 2. *Let (X, d) be a compact metric G-space, $f : X \longrightarrow X$ be a pseudo equivalent map, and topological group G be compact. Then, the map f is G-chain transitive if and only if for any positive integer $k \geq 2$, the map f^k is G-chain transitive.*

Theorem 3. *Let (X, d) be a compact metric G-space, $f : X \longrightarrow X$ be a continuous map, and topological group G be compact. If the map f is a G-pointwise chain recurrent, then the map f is G-chain transitive.*

Theorem 4. *Let (X, d) be a compact metric G-space, $f : X \longrightarrow X$ be a continuous closed map, and topological group G be compact. If there exists a nonempty open set U satisfying $G(U) = U$, $\bar{U} \neq X$, and $f(\bar{U}) \subset U$, then we have that the map f is not G-chain transitive.*

The above conclusions enrich the theory of G-chain mixing, G-mixing, and G-chain transitivity in metric G-space.

Next, we prove Theorem 1 in Section 2 and Theorems 2–4 in Section 3.

2. G-Chain Mixing and G-Mixing in Metric G-Space

Definition 5 (see [10]). Let (X, d) be a metric G-space, G be a topological group, and $\varphi: G \times X \longrightarrow X$ be a continuous map. The (X, G, θ) or X is called a metric G-space if the following conditions are satisfied:

(1) $\varphi(e, x) = x$ where e is the identity of G and for all $x \in X$

(2) $\varphi(g_1, \varphi(g_2, x)) = \varphi(g_1 g_2, x)$ where for all $x \in X$ and all $g_1, g_2 \in G$

If X is compact, then X is also said to be compact metric G-space. For the convenience of writing, $\varphi(g, x)$ is usually abbreviated as gx.

Definition 6 (see [11]). Let (X, d) be a metric G-space and f be a continuous map from X to X. f is said to be a pseudo equivariant map if for all $x \in X$ and all $p \in G$, there exists $p' \in G$ such that

$$f(px) = p' f(x). \tag{1}$$

Definition 7 (see [12]). Let (X, d) be a metric G-space. The metric d is said to be invariant to the topological group G provided that $d(x, y) = d(gx, gy)$ for all $x, y \in X$ and $g \in G$.

Definition 8 (see [7]). Let (X, d) be a metric space and f be a continuous map from X to X. The map f is said to be an isometry map if we have $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$.

Definition 9 (see [11]). Let (X, d) be a metric G-space and f be a continuous map from X to X. The sequence $\{x_i\}_{i=0}^{\infty}$ is called to be a (G, δ)-pseudo orbit of f if for any $i \geq 0$, there exists $t_i \in G$ such that $d(t_i f(x_i), x_{i+1}) < \delta$.

Definition 10 (see [11]). Let (X, d) be a metric G-space and f be a continuous map from X to X. The sequence $\{x_i\}_{i=0}^{\infty}$ is said to be (G, ε)-shadowed by a point y in X if for any $i \geq 0$, there exists $t_i \in G$ such that $d(f^i(y), t_i x_i) < \varepsilon$.

Definition 11 (see [11]). Let (X, d) be a metric G-space and f be a continuous map from X to X. The map f has G-shadowing property if any $\varepsilon > 0$, there exists $\delta > 0$ such that for any (G, δ)-pseudo orbit $\{x_i\}_{i=0}^{\infty}$ of f, there exists a point y in X such that the sequence $\{x_i\}_{i=0}^{\infty}$ is (G, ε)-shadowed by the point y.

Definition 12. Let (X, d) be a metric G-space and f be a continuous map from X to X. The map f is said to be G-mixed if for any nonempty open set U and V in X, there exists a positive integer N such that for any positive integer n greater than or equal to N, there exists $g_n \in G$ such that

$$g_n f^n(U) \cap V \neq \varnothing. \tag{2}$$

Definition 13 (see [13]). Let (X, d) be a metric G-space and f be a continuous map from X to X. Write $x_0 = x$ and $y_0 = y$. The sequence $\{x_i\}_{i=0}^{n}$ is called to be a (G, δ)-chain with length n from x to y under the action of f if for any $0 \leq i < n$, there exists $g_i \in G$ such that

$$d(g_i f(x_i), x_{i+1}) < \delta. \tag{3}$$

Definition 14. Let (X, d) be a metric G-space and f be a continuous map from X to X. The map f is said to be G-chain mixed if for any $\varepsilon > 0$ and $x, y \in X$, there exists a positive integer N such that for any positive integer n greater than or equal to N, there exists a (G, δ)-chain $\{x_i\}_{i=0}^{n}$ with length n from x to y under the action of f.

Now we start to prove Theorem 15.

Theorem 15. *Let (X, d) be a compact metric G-space, $f: X \longrightarrow X$ be a pseudo equivalent and isometric map, and the metric d be invariant to G. If the map f has the G-shadowing property, then the map f is G-chain mixed if and only if the map f is G-mixed.*

Proof. Suppose that the map f is G-chain mixed. For any $x, y \in X$, let $B(x, r_1)$ and$B(x, r_2)$ be any spherical field in X. According to that the map f has the G-shadowing property, for any $0 < \varepsilon < \min\{r_1, r_2\}$, there exists $0 < \delta < \varepsilon$ such that for any (G, δ)-pseudo orbit $\{x_i\}_{i=0}^{\infty}$, there exists $z \in X$ and $g_i \in G$ such that for any $i \geq 0$, we have that

$$d\left(f^i(z), g_i x_i\right) < \varepsilon. \tag{4}$$

Since the map f is G-chain mixed, there exists a positive integer N_1 such that for any positive integer n greater than or equal to N_1, there exists a (G, δ)-chain with length n from x to y under the action of f. By (4), there exists $z \in X$ and $g_0, g_n \in G$ such that

$$\begin{aligned} &d(z, g_0 x) < \varepsilon, \\ &d(f^n(z), g_n y) < \varepsilon. \end{aligned} \tag{5}$$

Since the metric d is invariant to G, we can get that

$$\begin{aligned} &d\left(g_0^{-1} z, x\right) < \varepsilon, \\ &d\left(g_n^{-1} f^n(z), y\right) < \varepsilon. \end{aligned} \tag{6}$$

Hence, we have that

$$\begin{aligned} &g_n^{-1} f^n(z) \in B(y, r_2), \\ &d\left(g_0^{-1} z, x\right) < r_1. \end{aligned} \tag{7}$$

Combined with the definition of isometric map f, we can obtain that

$$d\left(f^n\left(g_0^{-1}z\right), f^n(x)\right) = d\left(g_0^{-1}z, x\right). \tag{8}$$

So we can get that

$$d\left(f^n\left(g_0^{-1}z\right), f^n(x)\right) < r_1. \tag{9}$$

Combined with the definition of the pseudo equivariant map f, there exists $(g_0^{-1})' \in G$ such that

$$d\left(\left(g_0^{-1}\right)' f^n(z), f^n(x)\right) < r_1. \tag{10}$$

Let $t_n \in G$ satisfying $t_n(g_0^{-1})' = g_n^{-1}$. According to that the metric d is invariant to G, we have that

$$d\left(g_n^{-1} f^n(z), t_n f^n(x)\right) < r_1. \tag{11}$$

So

$$g_n^{-1} f^n(z) \in t_n f^n(B(x, r_1)). \tag{12}$$

Thus,

$$t_n f^n(B(x, r_1)) \cap B(y, r_2) \neq \varnothing. \tag{13}$$

Hence, the map f is G-mixed.

Suppose that the map f is G-mixed. For any $\eta > 0$ and $x, y \in X$, there exists a positive integer N_2 such that for any positive integer n greater than or equal to N_2, there exists $t_n \in G$ such that

$$t_n f^n\left(B\left(x, \frac{\eta}{2}\right)\right) \cap B\left(y, \frac{\eta}{2}\right) \neq \varnothing. \tag{14}$$

Let

$$u \in t_n f^n\left(B\left(x, \frac{\eta}{2}\right)\right) \cap B\left(y, \frac{\eta}{2}\right). \tag{15}$$

Then, we have that

$$d(y, u) < \frac{\eta}{2}. \tag{16}$$

In addition, there exists $z' \in B(x, \eta/2)$ such that $u = t_n f^n(z')$. Since the metric d is invariant to G and the map f is an isometric map, we can get that

$$d\left(t_n f^n\left(z'\right), t_n f^n(x)\right) < \frac{\eta}{2}. \tag{17}$$

So

$$d(u, t_n f^n(x)) < \frac{\eta}{2}. \tag{18}$$

Hence, we have that

$$d(t_n f^n(x), y) < d(t_n f^n(x), u) + d(u, y) < \eta. \tag{19}$$

Thus, $\{x, f(x), f^2(x) \cdots f^{n-1}(x), y\}$ is a (G, η)-chain with length n from x to y under the action of f. So the map f is G-chain mixed. This completes the proof. □

3. G-Chain Transitive in Metric G-Space

Definition 16 (see [13]). Let (X, d) be a metric space and f be a continuous map from X to X. The $CE_G(x, f)$ is called to be a G-chain equivalent set of the point x if we write

$$CE_G(x, f) = \left\{ y \in X, x \overset{ch\,G\,f}{\longrightarrow} y, y \overset{ch \quad G \quad f}{\longrightarrow} x \right\}. \tag{20}$$

Definition 17. Let (X, d) be a metric G-space and f be a continuous map from X to X. The map f is said to be G-chain transitive if for any $\varepsilon > 0$ and $x, y \in X$, there exists a (G, ε)-chain $\{x_i\}_{i=0}^m$ from x to y under the action of f.

Definition 18 (see [14]). Let (X, d) be a metric G-space and f be a continuous map from X to X. A point x is called to be a G-chain recurrent point if for any $\varepsilon > 0$, there exists (G, ε)-chain $\{x_i\}_{i=0}^m$ from x to x under the action of f, denoted by $CR_G(f)$ the G-chain recurrent point set of the map f.

Definition 19. Let (X, d) be a metric G-space and f be a continuous map from X to X. if $CR_G(f) = X$, then f is said to be G-pointwise chain recurrent.

Definition 20 (see [12]). Let (X, d) be a metric G-space and $A \subset X$. Write

$$GA \equiv \{gx : g \in G, x \in X\}. \tag{21}$$

Now we give Lemma 21 in order to prove Theorems 2 and 3 in this section.

Lemma 21 (see [12]). *Let (X, d) be a compact metric G-space and G be compact topological group. Then, for any $\varepsilon > 0$, there exists $0 < \delta < \varepsilon$ such that $d(x, y) < \delta$ implies $d(gx, gy) < \varepsilon$ for any $g \in G$.*

Theorem 22. *Let (X, d) be a compact metric G-space, $f : X \longrightarrow X$ be a pseudo equivalent map, and topological group G be compact. Then, the map f is G-chain transitive if and only if for any positive integer $k \geq 2$, the map f^k is G-chain transitive.*

Proof. Suppose that the map f is G-chain transitive. By Lemma 21, for any $\varepsilon > 0$, there exists a positive integer m_1 greater than or equal to k such that $d(x, y) < \varepsilon/m_1$ implies

$$d(gx, gy) < \frac{\varepsilon}{k} \text{ for any } g \in G. \tag{22}$$

According to the uniform continuity of the map f, for $\varepsilon/m_1 > 0$, there exists a positive integer m_2 greater than or equal to m_1 such that $d(x, y) < \varepsilon/m_2$ implies

$$d\left(f^i(x), f^i(y)\right) < \frac{\varepsilon}{m_1} \text{ for any } 0 \le i < k. \tag{23}$$

Since the map f is G-chain transitive, for any $x, y \in X$, there exists $(G, \varepsilon/m_2)$-chain $\{x_i\}_{i=0}^n$ from x to y under the action of f. So we have that $x_0 = x$ and $x_n = y$. Now we have two cases. □

Case 1. If k is a factor of n, then there exists a positive integer m_3 such that $n = m_3 k$. According to that $\{x_i\}_{i=0}^n$ is $(G, \varepsilon/m_2)$-chain under the action of f, for any $0 \le i < n$, there exists $g_i \in G$ such that

$$d(g_i f(x_i), x_{i+1}) < \frac{\varepsilon}{m_2}. \tag{24}$$

Hence, for any $0 \le j \le m_3 - 1$, we have that

$$\begin{aligned} &d\left(g_{jk} f(x_{jk}), x_{jk+1}\right) < \frac{\varepsilon}{m_2},\\ &d\left(g_{jk+1} f(x_{jk+1}), x_{jk+2}\right) < \frac{\varepsilon}{m_2} \cdots\cdots d\left(g_{jk+k-2} f(x_{jk+k-2}), x_{jk+k-1}\right) < \frac{\varepsilon}{m_2},\\ &d\left(g_{jk+k-1} f(x_{jk+k-1}), x_{(j+1)k}\right) < \frac{\varepsilon}{m_2}. \end{aligned} \tag{25}$$

Combined with the definition of pseudo equivariant map f and (23), there exists $g'_{jk}, g'_{jk+1}, g'_{jk+2} \cdots g'_{jk+k-2}, g'_{jk+k-1} \in G(g'_{jk+k-1} = g_{jk+k-1})$ such that

$$\begin{aligned} &d\left(g'_{jk} f^k(x_{jk}), f^{k-1}(x_{jk+1})\right) < \frac{\varepsilon}{m_1},\\ &d\left(g'_{jk+1} f^{k-1}(x_{jk+1}), f^{k-2}(x_{jk+2})\right) < \frac{\varepsilon}{m_1} \cdots\cdots d\left(g'_{jk+k-2} f^2(x_{jk+k-2}), f(x_{jk+k-1})\right) < \frac{\varepsilon}{m_1},\\ &d\left(g'_{jk+k-1} f(x_{jk+k-1}), x_{(j+1)k}\right) < \frac{\varepsilon}{m_1}. \end{aligned} \tag{26}$$

By (22), we can obtain that

$$\begin{aligned} &d\left(g'_{jk+k-1} g'_{jk+k-2} g'_{jk+k-3} \cdots g'_{jk+2} g'_{jk+1} g'_{jk} f^k(x_{jk}), g'_{jk+k-1} g'_{jk+k-2} g'_{jk+k-3} \cdots g'_{jk+2} g'_{jk+1} f^{k-1}(x_{jk+1})\right) < \frac{\varepsilon}{k},\\ &d\left(g'_{jk+k-1} g'_{jk+k-2} g'_{jk+k-3} \cdots g'_{jk+2} g'_{jk+1} f^{k-1}(x_{jk+1}), g'_{jk+k-1} g'_{jk+k-2} g'_{jk+k-3} \cdots g'_{jk+2} f^{k-2}(x_{jk+2})\right)\\ &\quad < \frac{\varepsilon}{k} \cdots\cdots d\left(g'_{jk+k-1} g'_{jk+k-2} f^2(x_{jk+k-2}), g'_{jk+k-1} f(x_{jk+k-1})\right) < \frac{\varepsilon}{k}, d\left(g'_{jk+k-1} f(x_{jk+k-1}), x_{(j+1)k}\right) < \frac{\varepsilon}{k}. \end{aligned} \tag{27}$$

Hence, we can get that

$$\begin{aligned} &d\left(g'_{jk+k-1} g'_{jk+k-2} g'_{jk+k-3} \cdots g'_{jk+2} g'_{jk+1} g'_{jk} f^k(x_{jk}), x_{(j+1)k}\right)\\ &\quad < d\left(g'_{jk+k-1} g'_{jk+k-2} g'_{jk+k-3} \cdots g'_{jk+2} g'_{jk+1} g'_{jk} f^k(x_{jk}),\right.\\ &\qquad \left. \cdot\, g'_{jk+k-1} g'_{jk+k-2} g'_{jk+k-3} \cdots g'_{jk+2} g'_{jk+1} f^{k-1}(x_{jk+1})\right)\\ &\qquad + d\left(g'_{jk+k-1} g'_{jk+k-2} g'_{jk+k-3} \cdots g'_{jk+2} g'_{jk+1} f^{k-1}(x_{jk+1}),\right.\\ &\qquad \left. \cdot\, g'_{jk+k-1} g'_{jk+k-2} g'_{jk+k-3} \cdots g'_{jk+2} f^{k-2}(x_{jk+2})\right)\\ &\qquad + \cdots\cdots + d\left(g'_{jk+k-1} g'_{jk+k-2} f^2(x_{jk+k-2}), g'_{jk+k-1} f(x_{jk+k-1})\right)\\ &\qquad + d\left(g'_{jk+k-1} f(x_{jk+k-1}), x_{(j+1)k}\right) < \frac{\varepsilon}{k} + \frac{\varepsilon}{k} + \frac{\varepsilon}{k} + \cdots + \frac{\varepsilon}{k} = \varepsilon, \end{aligned} \tag{28}$$

so $\{x_0, x_k, x_{2k} \cdots x_{m_3 k}\}$ is a (G, ε)-chain from x to y under the action of f^k. Hence, the map f^k is G-chain transitive.

Case 2. If k is not a factor of n, then there exists a positive integer m_4 such that

$$n = k m_4 + r, 1 \le r \le k - 1. \tag{29}$$

By the above same way, we can get that $\{x_0, x_k, x_{2k} \cdots x_{m_4 k}\}$ is a (G, ε)-chain under the action of f^k. In addition, we have that

$$d\left(g_{m_4 k} f(x_{m_4 k}), x_{m_4 k+1}\right) < \frac{\varepsilon}{m_2},$$

$$d\left(g_{m_4k+1}f\left(x_{m_4k+1}\right), x_{m_4k+2}\right) < \frac{\varepsilon}{m_2} \cdots\cdots d\left(g_{m_4k+r-2}f\left(x_{m_4k+r-2}\right), x_{m_4k+r-1}\right) < \frac{\varepsilon}{m_2},$$

$$d\left(g_{m_4k+r-1}f\left(x_{m_4k+r-1}\right), x_{m_4k+r}\right) < \frac{\varepsilon}{m_2}. \quad (30)$$

According to the definition of pseudo equivariant map f and (23), there exists $t_{m_4k}, t_{m_4k+1}, t_{m_4k+2} \cdots t_{m_4k+r-2}, t_{m_4k+r-1}$ $(t_{m_4k+r-1} = g_{m_4k+r-1}) \in G$ such that

$$d\left(t_{m_4k}f^k\left(x_{m_4k}\right), f^{k-1}\left(x_{m_4k+1}\right)\right) < \frac{\varepsilon}{m_1}, \quad (31)$$

$d(t_{m_4k+1}f^{k-1}(x_{m_4k+1}), f^{k-2}(x_{m_4k+2})) < \varepsilon/m_1 \cdots \cdots d(t_{m_4k+r-2}f^2(x_{m_4k+r-2}), f(x_{m_4k+r-1})) < \varepsilon/m_1$,

$$d\left(t_{m_4k+r-1}f\left(x_{m_4k+r-1}\right), x_{m_4k+r}\right) < \frac{\varepsilon}{m_1}. \quad (32)$$

By (22), we can obtain that.

$$\begin{aligned} & d\left(t_{m_4k+r-1}t_{m_4k+r-2}t_{m_4k+r-3} \cdots t_{m_4k+2}t_{m_4k+1}t_{m_4k}f^k \right.\\ & \quad \left. \cdot \left(x_{m_4k}\right), t_{m_4k+r-1}t_{m_4k+r-2}t_{m_4k+r-3} \cdots t_{m_4k+2}t_{m_4k+1}f^{k-1}\left(x_{m_4k+1}\right)\right)\\ & < \frac{\varepsilon}{k} d\left(t_{m_4k+r-1}t_{m_4k+r-2}t_{m_4k+r-3} \cdots t_{m_4k+2}t_{m_4k+1}f^{k-1}\right.\\ & \quad \left. \left(x_{m_4k+1}\right), t_{m_4k+r-1}t_{m_4k+r-2}t_{m_4k+r-3} \cdots t_{m_4k+2}f^{k-2}\left(x_{m_4k+2}\right)\right)\\ & < \frac{\varepsilon}{k} \cdots\cdots d\left(t_{m_4k+r-1}t_{m_4k+r-2}f^2\left(x_{m_4k+r-2}\right), t_{m_4k+r-1}f\left(x_{m_4k+r-1}\right)\right)\\ & < \frac{\varepsilon}{k} d\left(t_{m_4k+r-1}f\left(x_{m_4k+r-1}\right), x_{m_4k+r}\right) < \frac{\varepsilon}{k}. \end{aligned} \quad (33)$$

Hence, we can get that

$$\begin{aligned} & d\left(t_{m_4k+r-1}t_{m_4k+r-2}t_{m_4k+r-3} \cdots t_{m_4k+2}t_{m_4k+1}t_{m_4k}f^k\left(x_{m_4k}\right), x_{m_4k+r}\right)\\ & < d\left(t_{m_4k+r-1}t_{m_4k+r-2}t_{m_4k+r-3} \cdots t_{m_4k+2}t_{m_4k+1}t_{m_4k}f^k\right.\\ & \quad \cdot \left(x_{m_4k}\right), t_{m_4k+r-1}t_{m_4k+r-2}t_{m_4k+r-3} \cdots t_{m_4k+2}t_{m_4k+1}f^{k-1}\\ & \quad \left. \cdot \left(x_{m_4k+1}\right)\right) + + d\left(t_{m_4k+r-1}t_{m_4k+r-2}t_{m_4k+r-3} \cdots t_{m_4k+2}t_{m_4k+1}f^{k-1}\right.\\ & \quad \cdot \left(x_{m_4k+1}\right), t_{m_4k+r-1}t_{m_4k+r-2}t_{m_4k+r-3} \cdots t_{m_4k+2}f^{k-2}\\ & \quad \cdot \left(x_{m_4k+2}\right)) + \cdots\cdots + d\left(t_{mk+r-1}t_{mk+r-2}f^2\right.\\ & \quad \left. \cdot \left(x_{m_4k+r-2}\right), t_{mk+r-1}f\left(x_{m_4k+r-1}\right)\right)\\ & \quad + d\left(t_{m_4k+r-1}f\left(x_{m_4k+r-1}\right), x_{m_4k+r}\right)\\ & < \frac{\varepsilon}{k} + \frac{\varepsilon}{k} + \frac{\varepsilon}{k} + \cdots + \frac{\varepsilon}{k} = \frac{r\varepsilon}{k} < \varepsilon, \end{aligned} \quad (34)$$

so $\{x_0, x_k, x_{2k} \cdots x_{m_4k}, x_{m_4k+r}\}$ is a (G, ε)-chain from x to y under the action of f^k. Hence, the map f^k is G-chain transitive.

Suppose that the map f^k is G-chain transitive. For any $\eta > 0$ and $x, y \in X$, there exists (G, η)-chain $\{x_i\}_{i=0}^{l}$ from x to y under the action of f^k. Let

$$y_{mk+i} = f^i(x_m), 0 \le m < l, 0 \le i \le k-1. \quad (35)$$

Then, $\{y\}_{j=0}^{lk}$ is a (G, η)-chain from x to y under the action of f. Hence, f is G-chain transitive. Thus, we end the proof.

Theorem 23. *Let (X, d) be a compact metric G-space, $f : X \longrightarrow X$ be a continuous map, and topological group G be compact. If the map f is a G-pointwise chain recurrent, then the map f is G-chain transitive.*

Proof. Suppose that the map f is G-pointwise chain recurrent. Let $x, z \in X$. According to [13], the $CE_G(x, f)$ is a closed set. Suppose $y \in CE_G(x, f)$. By Lemma 21, for any $\varepsilon > 0$, there exists $0 < \varepsilon_0 < \varepsilon$ such that $d(u, v) < \varepsilon_0/2$ implies

$$d(gu, gv) < \frac{\varepsilon}{2} \text{ for any } g \in G. \quad (36)$$

According to $y \in CE_G(x, f)$, there exists $(G, \varepsilon/2)$-chain under the action of f

$$\{x_0, x_1 \cdots x_{n-1}, x_n, y_1, y_2 \cdots y_{m-1}, y_m\}, \quad (37)$$

where $x_0 = y_m = x$ and $x_n = y$. Hence, there exists $g_{n-1}, g_n \in G$ such that

$$d(g_{n-1}f(x_{n-1}), y) < \frac{\varepsilon}{2}, \quad (38)$$

$$d(g_nf(y), y_1) < \frac{\varepsilon}{2}. \quad (39)$$

According to the continuity of the map f, there exists $0 < \delta < \varepsilon_0/2$ such that $d(z, y) < \delta$ implies

$$d(f(z), f(y)) < \frac{\varepsilon_0}{2}. \quad (40)$$

Let $y' \in B(y, \delta)$. By (36) and (40), we have that

$$d\left(g_nf(y), g_nf\left(y'\right)\right) < \frac{\varepsilon}{2}. \quad (41)$$

According to $y' \in B(y, \delta)$ and (38), we have that

$$d\left(g_{n-1}f(x_{n-1}), y'\right) < d(g_{n-1}f(x_{n-1}), y) + d\left(y, y'\right) < \varepsilon. \quad (42)$$

By (39) and (41), we can get that

$$d\left(g_nf\left(y'\right), y_1\right) < d\left(g_nf\left(y'\right), g_nf(y)\right) + d(g_nf(y), y_1) < \varepsilon. \quad (43)$$

Hence, $\{x_0, x_1 \cdots x_{n-1}, y', y_1, y_2 \cdots y_{m-1}, y_m\}$ is a (G, ε)-chain under the action of f. So we have that

$$y' \in CE_G(x, f). \tag{44}$$

Thus,

$$B(y, \delta) \subset CE_G(x, f). \tag{45}$$

Hence, the $CE_G(x, f)$ is an open set. Since f is G-pointwise chain recurrent, we can get that

$$x \in CR_G(f). \tag{46}$$

Hence, $CE_G(x, f)$ is not an empty set. Because the metric space is connected, so we have that

$$CE_G(x, f) = X. \tag{47}$$

Hence,

$$z \in CE_G(x, f). \tag{48}$$

Thus, there exists a (G, ε)-chain from x to y under the action of f. So the map f is G-chain transitive. Thus, we end the proof. □

Theorem 24. *Let (X, d) be a compact metric G-space, $f : X \longrightarrow X$ be a continuous closed map, and topological group G be compact. If there exists a nonempty open set U satisfying $G(U) = U$, $\bar{U} \neq X$, and $f(\bar{U}) \subset U$, then we have that the map f is not G-chain transitive.*

Proof. Let $y \in X - U$ and $\delta = d(f(\bar{U}), X - U)$. Since X is compact metric space, δ is greater than 0. By Lemma 21, for the above $\delta/2 > 0$, there exists $0 < \delta_0 < \delta/2$ such that $d(u, v) < \delta_0$ implies

$$d(gu, gv) < \frac{\delta}{2} \text{ for any } g \in G. \tag{49}$$

Let $z \in f(\bar{U})$. It remains to show that there is no (G, δ_0)-chain from z to y under the action of f. If there exists a (G, δ_0)-chain $\{z_i\}_{i=0}^{n}$ from z to y under the action of f, then for any $0 \leq i \leq n$, there exists $g_i \in G$ such that

$$d(g_i f(z_i), z_{i+1}) < \delta_0. \tag{50}$$

By (49), we have that

$$d\left(f(z_i), g_i^{-1} z_{i+1}\right) < \frac{\delta}{2}. \tag{51}$$

In particular, we have that

$$d\left(f(z), g_0^{-1} z_1\right) < \frac{\delta}{2}. \tag{52}$$

According to $z \in f(\bar{U})$ and $f(\bar{U}) \subset U$, we can obtain that

$$f(z) \in f\left(\bar{U}\right). \tag{53}$$

Combined with $\delta = d(f(\bar{U}), X - U)$, we can get that

$$g_0^{-1} z_1 \in U. \tag{54}$$

According to $G(U) = U$, we have that the point z_1 is in U. So

$$f(z_1) \in f\left(\bar{U}\right). \tag{55}$$

By the same way, we can get $z_i \in U$ for any $2 \leq i \leq n$. Hence, $y \in U$. This contradicts that the point y is in $X - U$. So there is no (G, δ_0)-chain from z to y under the action of f. Then, the map f is not G-chain transitive. Thus, we end the proof. □

4. Conclusion

Firstly, we introduce the concept of G-chain mixing, G-mixing, and G-chain transitivity in metric G-space. Secondly, we study their dynamical properties and obtain the following results. (1) If the map f has the G-shadowing property, then the map f is G-chain mixed if and only if the map f is G-mixed. (2) The map f is G-chain transitive if and only if for any positive integer $k \geq 2$, the map f^k is G-chain transitive. (3) If the map f is G-pointwise chain recurrent, then the map f is G-chain transitive. (4) If there exists a nonempty open set U satisfying $G(U) = U$, $\bar{U} \neq X$, and $f(\bar{U}) \subset U$, then we have that the map f is not G-chain transitive. These conclusions enrich the theory of G-chain mixing, G-mixing, and G-chain transitivity in metric G-space.

Most importantly, it provided the theoretical basis and scientific foundation for the application of G-chain mixing, G-mixing, and G-chain transitivity in computational mathematics and biological mathematics.

Data Availability

The data used to support the findings of this study are included within references [1–14] in the article.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

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