Research Article

A New Bivariate Extended Generalized Inverted Kumaraswamy Weibull Distribution

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This article presents a new bivariate extended generalized inverted Kumaraswamy Weibull (BIEGIKw-Weibull) distribution with nine parameters. Statistical properties of the new distribution are discussed. Forms of copulas, moments, conditional moments, bivariate reliability function, and bivariate hazard rate function are derived. Maximum likelihood estimators are formulated. Simulation is conducted for three different sets of parameters to verify the theoretical results and to discuss the new distribution properties. The performance of the maximum likelihood method is investigated via Monte Carlo simulation depending on the bias and the standard error. Simulated lifetime data is used as an application of the new model.

1. Introduction

The inverse Weibull (IW) distribution is widely used because of its applicability in various fields, like medicine, statistics, engineering, physics, and fluid mechanics [1–11]. To enhance such distributions, researchers introduced new generators by supplementing shape parameters to the base line distribution. The inverted Kumaraswamy (IK) with two shape parameters has been derived by Abd AL-Fattah et al. [12]. To accommodate both monotonic and nonmonotonic failure rates, the IK distribution has been generalized to involve three shape parameters (GIKum) by Iqbal et al. [13]. A new version with five parameters (GIKw-W) has been introduced by Jamal et al. [14]. Although the univariate continuous models suit many types of data sets, they cannot be used to model dependent sets of data; therefore, a lot of efforts have been done to develop bivariate distributions. Muhammed [15] proposed a bivariate generalized Kumaras-

wamy distribution. A bivariate inverse Weibull distribution has been developed by Mondal and Kundu [16]. Darwish and Shahbaz [17] formulated a bivariate transmuted Burr distribution; see also [18–25]. Most of the developed bivariate distributions have different shapes for the joint pdf and have singular part. In some cases, their joint probability distribution function can be expressed in compact forms. The maximum likelihood estimators cannot be expressed in explicit forms in most of the cases. Ganji et al. [26] generalized the method introduced by Alzaatreh et al. [27] to generate bivariate distributions with marginals having $T-X$ families. Let $g_{X,Y}(x,y)$ be the pdf of the bivariate random variable (X,Y) , with $x \in [a_1, b_1]$, $y \in [a_2, b_2]$, $-\infty < a_1 < b_1 < \infty$, $-\infty < a_2 < b_2 < \infty$. Consider $F_1(G_U(u))$ and $F_2(G_V(v))$ be functions of the cdfs of a random variables U and V , respectively, such that

$$(1) \quad F_1(G_U(u)) \in [a_1, b_1] \text{ and } F_2(G_V(v)) \in [a_2, b_2]$$

- (2) $F_1(G_U(u))$ and $F_2(G_V(v))$ are differentiable and monotonically nondecreasing functions
- (3) $F_1(G_U(u)) \rightarrow a_1$ as $u \rightarrow -\infty$, $F_1(G_U(u)) \rightarrow b_1$ as $u \rightarrow \infty$, $F_2(G_V(v)) \rightarrow a_2$ as $v \rightarrow -\infty$, and $F_2(G_V(v)) \rightarrow b_2$ as $v \rightarrow \infty$

The cdf of the random variable (U, V) is given by

$$F_{U,V}(F_1(G_X(u)), F_2(G_Y(z))) = \int_{c_1}^{F_1(G_X(u))} \int_{c_2}^{F_2(G_Y(z))} g_{X,Y}(x, y) dx dy. \quad (1)$$

In this paper, we introduce a new bivariate extended generalized inverted Kumaraswamy Weibull (BIEGIKw-Weibull) distribution; its joint pdf is absolutely continuous, takes only one form with no singular parts, and offers different shapes for different values of parameters, and its hazard function shows different shapes. Almost all statistical quantities of the new distribution can be obtained in closed forms including the maximum likelihood estimators. The new model is developed using the new six parameter distribution that is more flexible with so favorable properties [28]. Theoretical properties of the proposed distribution including marginal distributions, copulas, moments, conditional moments, bivariate reliability function, and bivariate hazard function are computed. Theoretical properties are investigated via simulation. Monte Carlo simulation is used to discuss the goodness of fit and the availability of the maximum likelihood method. A real data application is presented that proves the applicability of the new distribution. The paper is organized as follows. The new distribution is formulated in Section 2. In Section 3, closed forms of moments are derived. Reliability and hazard function are computed in Section 4. Estimation is performed in Section 5. Simulation for different three sets of parameters is performed in Section 6. A real data application is discussed in Section 7. Conclusion is given in Section 8.

2. Model Description

A one-dimensional random variable Z is said to have a GIKw-Weibull distribution if its cumulative distribution function (cdf) is given by

$$F(z) = \left(1 - \left\{1 + \left([1 - e^{-\delta z^\varphi}]^{-\lambda} - 1\right)^{-\gamma}\right\}^{-\alpha}\right)^\beta, \quad (2)$$

where $z > 0$, $\alpha, \beta, \gamma, \delta, \lambda, \varphi > 0$ are shape parameters [29]. Using (2) as a baseline distribution and $g_{X,Y}(x, y) = 1 + \delta_1(1-x) + \delta_2(1-y) + 2\delta_3(1-x-y)$ in (1), where $x, y \in [0, 1]$, $\delta_1, \delta_2, \delta_3 \in [-1, 1]$, $-1 \leq \delta_1 + \delta_3 \leq 1$, and $-1 \leq \delta_2 + \delta_3 \leq 1$, we formulate the following definition.

Definition 1. A bivariate random variable (U, V) is said to be a BIEGIKw-Weibull random variable if its cumulative probability function (cdf) and probability density function (pdf) are given by

$$F_{U,V}(u, v) = \left(1 - \left\{1 + \left([1 - e^{-\delta u^\varphi}]^{-\lambda} - 1\right)^{-\gamma}\right\}^{-\alpha}\right)^\beta \times \left(1 - \left\{1 + \left([1 - e^{-\delta v^\varphi}]^{-\lambda} - 1\right)^{-\gamma}\right\}^{-\alpha}\right)^\beta \cdot \left\{1 + (\delta_1 + \delta_3) \times \left(1 - \left\{1 + \left([1 - e^{-\delta u^\varphi}]^{-\lambda} - 1\right)^{-\gamma}\right\}^{-\alpha}\right)^\beta\right\} + (\delta_2 + \delta_3) \times \left(1 - \left\{1 + \left([1 - e^{-\delta v^\varphi}]^{-\lambda} - 1\right)^{-\gamma}\right\}^{-\alpha}\right)^\beta \right\}, \quad (3)$$

$$f_{U,V}(u, v) = (\alpha\beta\gamma\lambda\delta\varphi)^2 (uv)^{\varphi-1} e^{-\delta(u^\varphi + v^\varphi)} ([1 - e^{-\delta u^\varphi}][1 - e^{-\delta v^\varphi}])^{-(\lambda+1)} \times \{([1 - e^{-\delta u^\varphi}]^{-\lambda} - 1)([1 - e^{-\delta v^\varphi}]^{-\lambda} - 1)\}^{-(\gamma+1)} \times \{(1 + ([1 - e^{-\delta u^\varphi}]^{-\lambda} - 1)^{-\gamma})(1 + ([1 - e^{-\delta v^\varphi}]^{-\lambda} - 1)^{-\gamma})\}^{-(\alpha+1)} \times \{(1 - \{1 + ([1 - e^{-\delta u^\varphi}]^{-\lambda} - 1)^{-\gamma}\}^{-\alpha}) \times (1 - \{1 + ([1 - e^{-\delta v^\varphi}]^{-\lambda} - 1)^{-\gamma}\}^{-\alpha})\}^{\beta-1} \times \{1 + (\delta_1 + \delta_3)(1 - 2(1 - \{1 + ([1 - e^{-\delta u^\varphi}]^{-\lambda} - 1)^{-\gamma}\}^{-\alpha})^\beta) + (\delta_2 + \delta_3)(1 - 2(1 - \{1 + ([1 - e^{-\delta v^\varphi}]^{-\lambda} - 1)^{-\gamma}\}^{-\alpha})^\beta)\}, \quad (4)$$

where $u, v > 0$, $\alpha, \beta, \gamma, \delta > 0$, $\delta_1, \delta_2, \delta_3 \in [-1, 1]$, $-1 \leq \delta_1 + \delta_3 \leq 1$, and $-1 \leq \delta_2 + \delta_3 \leq 1$.

3. Marginals and Moments

Lemma 2. Let (U, V) be a BIEGIKw-Weibull random variable with cdf and pdf given in (3) and (4). Then, the marginals are

$$F_U(u) = \left(1 - \left\{1 + \left([1 - e^{-\delta u^\varphi}]^{-\lambda} - 1\right)^{-\gamma}\right\}^{-\alpha}\right)^\beta \times \left\{1 + (\delta_1 + \delta_3) \left(1 - \left\{1 + \left([1 - e^{-\delta u^\varphi}]^{-\lambda} - 1\right)^{-\gamma}\right\}^{-\alpha}\right)^\beta\right\}, \quad (5)$$

$$F_V(v) = \left(1 - \left\{1 + \left([1 - e^{-\delta v^\varphi}]^{-\lambda} - 1\right)^{-\gamma}\right\}^{-\alpha}\right)^\beta \times \left\{1 + (\delta_2 + \delta_3) \left(1 - \left\{1 + \left([1 - e^{-\delta v^\varphi}]^{-\lambda} - 1\right)^{-\gamma}\right\}^{-\alpha}\right)^\beta\right\}, \quad (6)$$

$$f_U(u) = \alpha\beta\gamma\lambda\delta\varphi u^{\varphi-1} e^{-\delta u^\varphi} [1 - e^{-\delta u^\varphi}]^{-(\lambda+1)} ([1 - e^{-\delta u^\varphi}]^{-\lambda} - 1)^{-(\gamma+1)} \times \left\{1 + \left([1 - e^{-\delta u^\varphi}]^{-\lambda} - 1\right)^{-\gamma}\right\}^{-(\alpha+1)} \times \left(1 - \left\{1 + \left([1 - e^{-\delta u^\varphi}]^{-\lambda} - 1\right)^{-\gamma}\right\}^{-\alpha}\right)^{\beta-1} \times \left\{1 + \delta_1 + \delta_3 - 2(\delta_1 + \delta_3) \left(1 - \left\{1 + \left([1 - e^{-\delta u^\varphi}]^{-\lambda} - 1\right)^{-\gamma}\right\}^{-\alpha}\right)^\beta\right\}, \quad (7)$$

$$\begin{aligned}
f_V(v) &= \alpha\beta\gamma\lambda\delta\varphi v^{\varphi-1} e^{-\delta v^\varphi} \left[1 - e^{-\delta v^\varphi}\right]^{-(\lambda+1)} \left(\left[1 - e^{-\delta v^\varphi}\right]^{-\lambda} - 1\right)^{-(\gamma+1)} \\
&\quad \times \left\{1 + \left(\left[1 - e^{-\delta v^\varphi}\right]^{-\lambda} - 1\right)^{-\gamma}\right\}^{-(\alpha+1)} \\
&\quad \times \left(1 - \left\{1 + \left(\left[1 - e^{-\delta v^\varphi}\right]^{-\lambda} - 1\right)^{-\gamma}\right\}^{-\alpha}\right)^{\beta-1} \\
&\quad \times \left\{1 + \delta_2 + \delta_3 - 2(\delta_2 + \delta_3) \left(1 - \left\{1 + \left(\left[1 - e^{-\delta v^\varphi}\right]^{-\lambda} - 1\right)^{-\gamma}\right\}^{-\alpha}\right)^\beta\right\}.
\end{aligned} \tag{8}$$

For $\delta_1 = \delta_2 = -\delta_3$, we get the baseline distribution EGIKw-Weibull.

Copula function is commonly used to investigate the dependence between two random variables.

Definition 3 (see [30]). Let (U, V) be a BIGIKw-Weibull random variable with cdf $F_{U,V}(u, v)$ and marginals $F_U(u)$ and $F_V(v)$, and then, its copula function can be defined as $F_{U,V}(u, v) \triangleq C(v, \phi)$, where $v = F_U(u)$, $\phi = F_V(v)$, and $v, \phi \in (0, 1)$ and the copula density function is defined as $c(v, \phi) = \partial^2 C(v, \phi) / \partial v \partial \phi$.

Using chain rule, we obtain

$$\begin{aligned}
f_{U,V}(u, v) &= \frac{\partial^2 F(u, v)}{\partial u \partial v} = \frac{\partial^2 C(F_U(u), F_V(v))}{\partial u \partial v} \\
&= \frac{\partial^2 C(v, \phi)}{\partial v \partial \phi} \frac{\partial F_U(u)}{\partial u} \frac{\partial F_V(v)}{\partial v} \triangleq c(v, \phi) f_U(u) f_V(v).
\end{aligned} \tag{9}$$

Lemma 4. Let (U, V) be a BIEGIKw-Weibull random variable with pdf and marginals given in (4), (7), and (8). Then, the copula density function is given by

$$\begin{aligned}
c(v, \phi) &= \frac{1}{H(u, \delta_1)H(v, \delta_2)} \\
&\quad \cdot \left\{1 + (\delta_1 + \delta_3) \times \left(1 - 2 \left(1 - \left\{1 + \left(\left[1 - e^{-\delta u^\varphi}\right]^{-\lambda} - 1\right)^{-\gamma}\right\}^{-\alpha}\right)^\beta\right)\right. \\
&\quad \left. + (\delta_2 + \delta_3) \left(1 - 2 \left(1 - \left\{1 + \left(\left[1 - e^{-\delta v^\varphi}\right]^{-\lambda} - 1\right)^{-\gamma}\right\}^{-\alpha}\right)^\beta\right)\right\}.
\end{aligned} \tag{10}$$

Lemma 5. Let (U, V) be a BIEGIKw-Weibull random variable with pdf and marginals given in (4), (7), and (8). Then, the conditional density functions are

$$\begin{aligned}
f_{U|V}(u/v) &= \frac{\alpha\beta\gamma\lambda\delta\varphi u^{\varphi-1} e^{-\delta u^\varphi} \left[1 - e^{-\delta u^\varphi}\right]^{-(\lambda+1)}}{H(v, \delta_2)} \\
&\quad \times \left(1 - \left[1 - e^{-\delta u^\varphi}\right]^\lambda\right)^{-(\gamma+1)} \left\{1 + \left(\left[1 - e^{-\delta u^\varphi}\right]^{-\lambda} - 1\right)^{-\gamma}\right\}^{-(\alpha+1)} \\
&\quad \times \left(1 - \left\{1 + \left(\left[1 - e^{-\delta u^\varphi}\right]^{-\lambda} - 1\right)^{-\gamma}\right\}^{-\alpha}\right)^{\beta-1} \\
&\quad \times \left\{1 + (\delta_1 + \delta_3) \left(1 - 2 \left(1 - \left\{1 + \left(\left[1 - e^{-\delta u^\varphi}\right]^{-\lambda} - 1\right)^{-\gamma}\right\}^{-\alpha}\right)^\beta\right)\right\}
\end{aligned}$$

$$+ (\delta_2 + \delta_3) \left(1 - 2 \left(1 - \left\{1 + \left(\left[1 - e^{-\delta v^\varphi}\right]^{-\lambda} - 1\right)^{-\gamma}\right\}^{-\alpha}\right)^\beta\right)\right\}, \tag{11}$$

$$\begin{aligned}
f_{V|U}(v/u) &= \frac{\alpha\beta\gamma\lambda\delta\varphi v^{\varphi-1} e^{-\delta v^\varphi} \left[1 - e^{-\delta v^\varphi}\right]^{-(\lambda+1)}}{H(u, \delta_1)} \left(1 - \left[1 - e^{-\delta v^\varphi}\right]^\lambda\right)^{-\gamma-1} \\
&\quad \times \left\{1 + \left(\left[1 - e^{-\delta v^\varphi}\right]^{-\lambda} - 1\right)^{-\gamma}\right\}^{-\alpha-1} \\
&\quad \times \left(1 - \left\{1 + \left(\left[1 - e^{-\delta v^\varphi}\right]^{-\lambda} - 1\right)^{-\gamma}\right\}^{-\alpha}\right)^{\beta-1} \\
&\quad \cdot \left\{1 + (\delta_1 + \delta_3) \left(1 - 2 \left(1 - \left\{1 + \left(\left[1 - e^{-\delta u^\varphi}\right]^{-\lambda} - 1\right)^{-\gamma}\right\}^{-\alpha}\right)^\beta\right)\right. \\
&\quad \left. + (\delta_2 + \delta_3) \left(1 - 2 \left(1 - \left\{1 + \left(\left[1 - e^{-\delta v^\varphi}\right]^{-\lambda} - 1\right)^{-\gamma}\right\}^{-\alpha}\right)^\beta\right)\right\},
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
H(x, \delta) &= 1 + \delta + \delta_3 3.7 - 2(\delta + \delta_3) \\
&\quad \cdot \left(1 - 2 \left(1 - \left\{1 + \left(\left[1 - e^{-\delta x^\varphi}\right]^{-\lambda} - 1\right)^{-\gamma}\right\}^{-\alpha}\right)^\beta\right).
\end{aligned} \tag{13}$$

Lemma 6. Let (U, V) be a BIEGIKw-Weibull random variable whose conditionals are given in (11) and (13). Then, the conditional moments are given by

$$\begin{aligned}
m_{V|U}^r(v) &= E\left(\frac{V^r}{V} = v\right) = \frac{1}{H(v, \delta_2)} \left\{ \frac{1}{\delta^{r/\varphi}} (1 + \delta_1 + \delta_2 + 2\delta_3) \right. \\
&\quad \left. - 2(\delta_2 + \delta_3) \left(1 - 2 \left(1 - \left\{1 + \left(\left[1 - e^{-\delta v^\varphi}\right]^{-\lambda} - 1\right)^{-\gamma}\right\}^{-\alpha}\right)^\beta\right) \right. \\
&\quad \times \sum_{i,j,k=0}^{\infty} \sum_{l=0}^{\lambda(\gamma j+k)-1} (-1)^l \xi_{i,j,k} \binom{\lambda(\gamma j+k)-1}{l} \frac{\Gamma((r/\varphi)+1)}{(l+1)^{(r/\varphi)+1}} \\
&\quad \left. - \frac{2(\delta_1 + \delta_3)}{\delta^{r/\varphi}} \sum_{i,j,k=0}^{\infty} \sum_{l=0}^{2\lambda(\gamma j+k)-1} (-1)^l \xi_{i,j,k} \binom{\lambda(\gamma j+k)-1}{l} \times \frac{\Gamma((r/\varphi)+1)}{(l+1)^{(r/\varphi)+1}} \right\}, \\
m_{V|U}^r(u) &= E\left(\frac{V^r}{U} = u\right) = \frac{1}{H(u, \delta_1)} \\
&\quad \cdot \left\{ \frac{1}{\delta^{r/\varphi}} (1 + \delta_1 + \delta_2 + 2\delta_3 - 2(\delta_2 + \delta_3)) \right. \\
&\quad \cdot \left(1 - 2 \left(1 - \left\{1 + \left(\left[1 - e^{-\delta u^\varphi}\right]^{-\lambda} - 1\right)^{-\gamma}\right\}^{-\alpha}\right)^\beta\right) \\
&\quad \times \sum_{i,j,k=0}^{\infty} \sum_{l=0}^{\lambda(\gamma j+k)-1} (-1)^l \xi_{i,j,k}(\beta) \binom{\lambda(\gamma j+k)-1}{l} \frac{\Gamma((r/\varphi)+1)}{(l+1)^{(r/\varphi)+1}} \\
&\quad \left. - \frac{2(\delta_1 + \delta_3)}{\delta^{r/\varphi}} \sum_{i,j,k=0}^{\infty} \sum_{l=0}^{\lambda(\gamma j+k)-1} (-1)^l \xi_{i,j,k}(2\beta) \right. \\
&\quad \cdot \left. \binom{\lambda(\gamma j+k)-1}{l} \times \frac{\Gamma((r/\varphi)+1)}{(l+1)^{(r/\varphi)+1}} \right\},
\end{aligned} \tag{14}$$

where

$$\xi_{i,j,k}(\rho) = (-1)^{i+j} \lambda(\gamma j+k) \binom{\rho}{i} \binom{\alpha i + j - 1}{j} \binom{\gamma j + k - 1}{k}. \tag{15}$$

γ is a positive integer and H is given by (13); for more details, see [28].

Lemma 7. Let (U, V) be a BIEGIKw-Weibull random variable whose density function is given in (4). Then, the joint moments are given by

$$m_{r,s} = E(U^r V^s) = \frac{(1 + \delta_1 + \delta_2 + 2\delta_3)}{\delta^{r/\varphi} \delta^{s/\varphi}} \times \left\{ \sum_{i,j,k=0}^{\infty} \sum_{l=0}^{\lambda(\gamma j+k)-1} (-1)^l \xi_{i,j,k}(\beta) \binom{\lambda(\gamma j+k)-1}{l} \frac{\Gamma((r/\varphi)+1)}{(l+1)^{(r/\varphi)+1}} \right. \\ \times \sum_{o,p,q=0}^{\infty} \sum_{t=0}^{\lambda(\gamma p+q)-1} (-1)^t \xi_{i,j,k}(\beta) \binom{\lambda(\gamma p+q)-1}{t} \frac{\Gamma((s/\varphi)+1)}{(t+1)^{(s/\varphi)+1}} \\ - \frac{2}{\delta^{r/\varphi} \delta^{s/\varphi}} \left[\sum_{i,j,k=0}^{\infty} \sum_{l=0}^{\lambda(\gamma j+k)-1} (-1)^l \xi_{i,j,k}(2\beta) \binom{2\lambda(\gamma j+k)-1}{l} \frac{\Gamma((r/\varphi)+1)}{(l+1)^{(r/\varphi)+1}} \right. \\ \times \sum_{o,p,q=0}^{\infty} \sum_{t=0}^{\lambda(\gamma p+q)-1} (-1)^t \xi_{o,p,q}(\beta) \binom{\lambda(\gamma p+q)-1}{t} \frac{\Gamma((s/\varphi)+1)}{(t+1)^{(s/\varphi)+1}} \\ - \frac{2(\delta_2 + \delta_3)}{\delta^{r/\varphi} \delta^{s/\varphi}} \left[\sum_{i,j,k=0}^{\infty} \sum_{l=0}^{\lambda(\gamma j+k)-1} (-1)^l \xi_{i,j,k}(\beta) \binom{\lambda(\gamma j+k)-1}{l} \frac{\Gamma((r/\varphi)+1)}{(l+1)^{(r/\varphi)+1}} \right. \\ \times \sum_{o,p,q=0}^{\infty} \sum_{t=0}^{\lambda(\gamma p+q)-1} (-1)^t \xi_{o,p,q}(2\beta) \binom{2\lambda(\gamma p+q)-1}{t} \frac{\Gamma((s/\varphi)+1)}{(t+1)^{(s/\varphi)+1}} \left. \right\}, \quad (16)$$

where γ is a positive integer and $\xi_{i,j,k}(\rho)$ is given by Equation (15).

4. Reliability and Hazard Functions

Bivariate hazard function can be used to characterize bivariate distributions. It describes the failure characteristics of the individual variables and their joint failure behavior. Here, we compute the bivariate reliability function and the hazard function defined by Navarro [30].

Lemma 8. Let (U, V) be a BIEGIKw-Weibull random variable whose cumulative and marginals are given in (3), (5), and (6), respectively. Then, its bivariate reliability function is given by

$$R(u, v) = R_1(u, v) + R_2(u, v), \quad (17)$$

where

$$R_1(u, v) = 1 - \left(1 - \left\{ 1 + \left([1 - e^{-\delta u^\varphi}]^{-\lambda} - 1 \right)^{-\gamma} \right\}^{-\alpha} \right)^\beta \\ \times \left(1 + (\delta_1 + \delta_3) \left(1 - \left\{ 1 + \left([1 - e^{-\delta u^\varphi}]^{-\lambda} - 1 \right)^{-\gamma} \right\}^{-\alpha} \right)^\beta \right) \\ - \left(1 - \left\{ 1 + \left([1 - e^{-\delta v^\varphi}]^{-\lambda} - 1 \right)^{-\gamma} \right\}^{-\alpha} \right)^\beta \\ \times \left(1 + (\delta_2 + \delta_3) \left(1 - \left\{ 1 + \left([1 - e^{-\delta v^\varphi}]^{-\lambda} - 1 \right)^{-\gamma} \right\}^{-\alpha} \right)^\beta \right),$$

$$R_2(u, v) = (\alpha\beta\gamma\lambda\delta\varphi)^2 (uv)^{\varphi-1} e^{-\delta(u^\varphi+v^\varphi)} \left([1 - e^{-\delta u^\varphi}] [1 - e^{-\delta v^\varphi}] \right)^{-(\lambda+1)} \\ \times \left(\left([1 - e^{-\delta u^\varphi}]^{-\lambda} - 1 \right) \left([1 - e^{-\delta v^\varphi}]^{-\lambda} - 1 \right) \right)^{-(\gamma+1)} \\ \times \left\{ \left(1 + \left([1 - e^{-\delta u^\varphi}]^{-\lambda} - 1 \right)^{-\gamma} \right) \left(1 + \left([1 - e^{-\delta v^\varphi}]^{-\lambda} - 1 \right)^{-\gamma} \right) \right\}^{-(\alpha+1)} \\ \times \left\{ \left(1 - \left\{ 1 + \left([1 - e^{-\delta u^\varphi}]^{-\lambda} - 1 \right)^{-\gamma} \right\}^{-\alpha} \right) \right. \\ \times \left(1 - \left\{ 1 + \left([1 - e^{-\delta v^\varphi}]^{-\lambda} - 1 \right)^{-\gamma} \right\}^{-\alpha} \right) \left. \right\}^{\beta-1} \\ \times \left\{ 1 + (\delta_1 + \delta_3) \left(1 - 2 \left(1 - \left\{ 1 + \left([1 - e^{-\delta u^\varphi}]^{-\lambda} - 1 \right)^{-\gamma} \right\}^{-\alpha} \right)^\beta \right) \right. \\ \left. + (\delta_2 + \delta_3) \left(1 - 2 \left(1 - \left\{ 1 + \left([1 - e^{-\delta v^\varphi}]^{-\lambda} - 1 \right)^{-\gamma} \right\}^{-\alpha} \right)^\beta \right) \right\}. \quad (18)$$

Lemma 9. Let (U, V) be a BIEGIKw-Weibull random variable whose density and reliability functions are given in (4) and (17). Then, its bivariate hazard rate function is given by

$$h(u, v) = \left\{ (\alpha\beta\gamma\lambda\delta\varphi)^2 (uv)^{\varphi-1} e^{-\delta(u^\varphi+v^\varphi)} \left([1 - e^{-\delta u^\varphi}] [1 - e^{-\delta v^\varphi}] \right)^{-(\lambda+1)} \right. \\ \times \left(\left([1 - e^{-\delta u^\varphi}]^{-\lambda} - 1 \right) \left([1 - e^{-\delta v^\varphi}]^{-\lambda} - 1 \right) \right)^{-(\gamma+1)} \\ \times \left\{ \left(1 + \left([1 - e^{-\delta u^\varphi}]^{-\lambda} - 1 \right)^{-\gamma} \right) \left(1 + \left([1 - e^{-\delta v^\varphi}]^{-\lambda} - 1 \right)^{-\gamma} \right) \right\}^{-(\alpha+1)} \\ \times \left\{ \left(1 - \left\{ 1 + \left([1 - e^{-\delta u^\varphi}]^{-\lambda} - 1 \right)^{-\gamma} \right\}^{-\alpha} \right) \right. \\ \times \left(1 - \left\{ 1 + \left([1 - e^{-\delta v^\varphi}]^{-\lambda} - 1 \right)^{-\gamma} \right\}^{-\alpha} \right) \left. \right\}^{\beta-1} \\ \times \left\{ 1 + (\delta_1 + \delta_3) \left(1 - 2 \left(1 - \left\{ 1 + \left([1 - e^{-\delta u^\varphi}]^{-\lambda} - 1 \right)^{-\gamma} \right\}^{-\alpha} \right)^\beta \right) \right. \\ \left. + (\delta_2 + \delta_3) \left(1 - 2 \left(1 - \left\{ 1 + \left([1 - e^{-\delta v^\varphi}]^{-\lambda} - 1 \right)^{-\gamma} \right\}^{-\alpha} \right)^\beta \right) \right\} \\ \cdot \left\{ 1 - \left(1 - \left\{ 1 + \left([1 - e^{-\delta u^\varphi}]^{-\lambda} - 1 \right)^{-\gamma} \right\}^{-\alpha} \right)^\beta \right. \\ \times \left(1 + (\delta_1 + \delta_3) \left(1 - \left\{ 1 + \left([1 - e^{-\delta u^\varphi}]^{-\lambda} - 1 \right)^{-\gamma} \right\}^{-\alpha} \right)^\beta \right) \\ - \left(1 - \left\{ 1 + \left([1 - e^{-\delta v^\varphi}]^{-\lambda} - 1 \right)^{-\gamma} \right\}^{-\alpha} \right)^\beta \\ \times \left(1 + (\delta_2 + \delta_3) \left(1 - \left\{ 1 + \left([1 - e^{-\delta v^\varphi}]^{-\lambda} - 1 \right)^{-\gamma} \right\}^{-\alpha} \right)^\beta \right) \\ \left. + (\alpha\beta\gamma\lambda\delta\varphi)^2 (uv)^{\varphi-1} e^{-\delta(u^\varphi+v^\varphi)} \left([1 - e^{-\delta u^\varphi}] [1 - e^{-\delta v^\varphi}] \right)^{-(\lambda+1)} \right. \\ \times \left(\left([1 - e^{-\delta u^\varphi}]^{-\lambda} - 1 \right) \left([1 - e^{-\delta v^\varphi}]^{-\lambda} - 1 \right) \right)^{-(\gamma+1)} \\ \times \left\{ \left(1 + \left([1 - e^{-\delta u^\varphi}]^{-\lambda} - 1 \right)^{-\gamma} \right) \left(1 + \left([1 - e^{-\delta v^\varphi}]^{-\lambda} - 1 \right)^{-\gamma} \right) \right\}^{-(\alpha+1)} \\ \times \left\{ \left(1 - \left\{ 1 + \left([1 - e^{-\delta u^\varphi}]^{-\lambda} - 1 \right)^{-\gamma} \right\}^{-\alpha} \right) \right. \\ \times \left(1 - \left\{ 1 + \left([1 - e^{-\delta v^\varphi}]^{-\lambda} - 1 \right)^{-\gamma} \right\}^{-\alpha} \right) \left. \right\}^{\beta-1} \\ \times \left\{ 1 + (\delta_1 + \delta_3) \left(1 - 2 \left(1 - \left\{ 1 + \left([1 - e^{-\delta u^\varphi}]^{-\lambda} - 1 \right)^{-\gamma} \right\}^{-\alpha} \right)^\beta \right) \right. \\ \left. + (\delta_2 + \delta_3) \left(1 - 2 \left(1 - \left\{ 1 + \left([1 - e^{-\delta v^\varphi}]^{-\lambda} - 1 \right)^{-\gamma} \right\}^{-\alpha} \right)^\beta \right) \right\} \right\}^{-1}. \quad (19)$$

FIGURE 1: Continued.

FIGURE 1: For the set of parameters $\alpha = 1, \beta = 1.8, \gamma = 2.0, \lambda = 1.5, \delta = 1.6, \delta_1 = -0.6, \delta_2 = -0.5, \delta_3 = 0.7$, and $\phi = 2.5$, the density function (4) is given in (a), the cumulative function (3) is given in (b), the hazard function (17) is given in (c), and the marginals (5), (7), (6), and (8) are given in (d–g), respectively.

5. Estimation

The maximum likelihood method is used to perform point estimators of the unknown parameters $\Theta = (\alpha, \beta, \gamma, \lambda, \phi, \delta, \delta_1, \delta_2, \delta_3)'$. Let $(u_1, v_1), (u_2, v_2), \dots, (u_n, v_n)$ be a random sample from the BIEGIKw-Weibull random variable. The maximum log-likelihood function is given by $L(\Theta) = L_1(\Theta) + L_2(\Theta)$, where

$$\begin{aligned}
 L_1(\Theta) &= 2n \ln(\alpha\beta\gamma\lambda\delta\phi) + (\phi - 1) \sum_{i=1}^n \ln u_i - \delta \sum_{i=1}^n u_i^\phi - (\lambda + 1) \\
 &\quad \times \sum_{i=1}^n \ln \left[1 - e^{-\delta u_i^\phi} \right] - (\gamma + 1) \sum_{i=1}^n \ln \left(\left[1 - e^{-\delta u_i^\phi} \right]^{-\lambda} - 1 \right) \\
 &\quad - (\alpha + 1) \times \sum_{i=1}^n \ln \left\{ 1 + \left(\left[1 - e^{-\delta u_i^\phi} \right]^{-\lambda} - 1 \right)^{-\gamma} \right\} \\
 &\quad + (\beta - 1) \times \sum_{i=1}^n \ln \left(1 - \left\{ 1 + \left(\left[1 - e^{-\delta u_i^\phi} \right]^{-\lambda} - 1 \right)^{-\gamma} \right\}^{-\alpha} \right) +, \\
 L_2(\Theta) &= (\phi - 1) \sum_{i=1}^n \ln v_i - \delta \sum_{i=1}^n v_i^\phi - (\lambda + 1) \sum_{i=1}^n \ln \left[1 - e^{-\delta v_i^\phi} \right] \\
 &\quad - (\gamma + 1) \times \sum_{i=1}^n \ln \left(\left[1 - e^{-\delta v_i^\phi} \right]^{-\lambda} - 1 \right) \\
 &\quad - (\alpha + 1) \sum_{i=1}^n \ln \left\{ 1 + \left(\left[1 - e^{-\delta v_i^\phi} \right]^{-\lambda} - 1 \right)^{-\gamma} \right\} \\
 &\quad + (\beta - 1) \sum_{i=1}^n \ln \left(1 - \left\{ 1 + \left(\left[1 - e^{-\delta v_i^\phi} \right]^{-\lambda} - 1 \right)^{-\gamma} \right\}^{-\alpha} \right) \\
 &\quad + \sum_{i=1}^n \ln \left\{ 1 + (\delta_1 + \delta_3) \left(1 - 2 \left(1 - \left\{ 1 + \left(\left[1 - e^{-\delta u_i^\phi} \right]^{-\lambda} - 1 \right)^{-\gamma} \right\}^{-\alpha} \right)^\beta \right) \right\} \\
 &\quad + (\delta_2 + \delta_3) \left(1 - 2 \left(1 - \left\{ 1 + \left(\left[1 - e^{-\delta v_i^\phi} \right]^{-\lambda} - 1 \right)^{-\gamma} \right\}^{-\alpha} \right)^\beta \right) \right\}. \tag{20}
 \end{aligned}$$

Consider the score vector $\Psi = (\Psi_\alpha, \Psi_\beta, \Psi_\gamma, \Psi_\lambda, \Psi_\phi, \Psi_\delta, \Psi_{\delta_1}, \Psi_{\delta_2}, \Psi_{\delta_3})'$, where $\Psi_\alpha = \Psi_{\alpha 1} + \Psi_{\alpha 2}, \Psi_\beta = \Psi_{\beta 1} + \Psi_{\beta 2}, \Psi_\gamma = \Psi_{\gamma 1} + \Psi_{\gamma 2}, \Psi_\lambda = \Psi_{\lambda 1} + \Psi_{\lambda 2}, \Psi_\phi = \Psi_{\phi 1} + \Psi_{\phi 2} + \Psi_{\phi 3}, \Psi_\delta = \Psi_{\delta 1} + \Psi_{\delta 2}$; then,

The system of equations can be solved via symbolic mathematical packages.

6. Simulation

In this section, simulation is performed for three different sets of parameters to verify our theoretical results and to discuss the new bivariate distribution properties. For the first set of parameters, $\alpha = 1, \beta = 1.8, \gamma = 2.0, \lambda = 1.5, \delta = 1.6, \delta_1 = -0.6, \delta_2 = -0.5, \delta_3 = 0.7$, and $\phi = 2.5$. The symmetry, unimodality, and high beakness of the density function (4) can be observed in Figure 1(a). For (u, v) approximately greater than $(2, 2)$, the cumulative function (3) approaches 1 (Figure 1(b)). The bivariate hazard function increases with the increase of (u, v) as displayed in Figure 1(c). Symmetry of marginals and bell shape densities with beakness can be noted in Figures 1(d)–1(g). For the second set of parameters, $\alpha = 1.5, \beta = 1.0, \gamma = 0.7, \lambda = 2.0, \delta = 0.6, \delta_1 = 0.7, \delta_2 = 0.8, \delta_3 = -0.5$, and $\phi = 1.0$. The density function (4) is unimodal and has high beak (Figure 2(a)). For (u, v) approximately greater than $(10, 10)$, the cumulative function (3) approaches 1 (Figure 2(b)). The bivariate hazard function decreases with the increase of (u, v) (Figure 2(c)). Unimodality and right tail property can be observed for marginals (Figures 2(d)–2(g)). For the third set of parameters, $\alpha = 0.5, \beta = 0.7, \gamma = 0.9, \lambda = 0.6, \delta = 0.2, \delta_1 = 0.5, \delta_2 = 0.4, \delta_3 = 0.1$, and $\phi = 0.3$. The density function (4) deformed and becomes an open surface (Figure 3(a)). The cumulative function (3) approaches 1 as (u, v) approaches infinity (Figure 3(b)). The bivariate hazard function decreases with the increase of

FIGURE 2: For the set of parameters $\alpha = 1.5$, $\beta = 1.0$, $\gamma = 0.7$, $\lambda = 2.0$, $\delta = 0.6$, $\delta_1 = 0.7$, $\delta_2 = 0.8$, $\delta_3 = -0.5$, and $\phi = 1.0$, the density function (4) is given in (a), the cumulative function (3) is given in (b), the hazard function (17) is given in (c), and the marginals (5), (7), (6), and (8) are given in (d–g), respectively.

FIGURE 3: Continued.

FIGURE 3: For the set of parameters $\alpha = 0.5$, $\beta = 0.7$, $\gamma = 0.9$, $\lambda = 0.6$, $\delta = 0.2$, $\delta_1 = 0.5$, $\delta_2 = 0.4$, $\delta_3 = 0.1$, and $\phi = 0.3$, the density function (4) is given in (a), the cumulative function (3) is given in (b), the hazard function (17) is given in (c), and the marginals (5), (7), (6), and (8) are given in (d–g), respectively.

TABLE 1: Average maximum likelihood estimates for the first set of observations.

Sample size	Parameter	Average estimate	Bias	SE
$n = 20$	α	0.3420	-0.158	0.0279
	β	2.510	-0.49	0.134
	γ	0.535	0.035	0.102
	λ	0.573	0.073	0.118
	δ	1.216	-0.284	0.143
	φ	2.902	-0.098	0.103
$n = 50$	α	0.4829	-0.017	0.0914
	β	4.508	1.508	0.667
	γ	0.5422	0.042	0.0986
	λ	0.4006	-0.099	0.0773
	δ	0.858	-0.642	0.153
	φ	2.801	-0.199	0.177
$n = 100$	α	0.2661	-0.244	0.0203
	β	3.206	0.206	0.243
	γ	0.9153	0.4153	0.0441
	λ	0.4187	-0.0813	0.0748
	δ	0.3047	-0.1953	0.0325
	φ	3.1596	0.1596	0.0850

TABLE 2: Average maximum likelihood estimates for the second set of observations.

Sample size	Parameter	Average estimate	Bias	SE
$n = 20$	α	1.463	-0.237	0.185
	β	1.98	0.18	1.40
	γ	0.96	-0.04	1.24
	λ	1.148	0.052	0.366
	δ	1.991	0.391	0.561
	φ	0.883	0.183	0.240
$n = 50$	α	1.405	-0.259	0.104
	β	2.424	0.624	0.956
	γ	1.39	0.39	1.21
	λ	1.497	0.297	0.579
	δ	1.941	0.341	0.378
	φ	0.714	-0.014	0.310
$n = 100$	α	1.448	-0.251	0.0804
	β	1.363	-0.437	0.332
	γ	1.08	0.08	1.76
	λ	1.403	0.203	0.309
	δ	1.323	-0.123	0.422
	φ	0.63	-0.07	0.276

(u, v) (Figure 3(c)). Marginals are given in Figures 3(d)–3(g). Monte Carlo simulation is used to generate samples for different sizes to discuss the performance of the new distribution. The first set of observations is generated for the parameters

$\alpha = 0.5$, $\beta = 3$, $\gamma = 0.5$, $\lambda = 0.6$, $\delta = 1.5$, $\delta_1 = -0.7$, $\delta_2 = -0.6$, $\delta_3 = 0.9$, and $\phi = 3$; the second set is generated for the parameters $\alpha = 1.7$, $\beta = 1.8$, $\gamma = 1.0$, $\lambda = 1.2$, $\delta = 1.6$, $\delta_1 = -0.6$, $\delta_2 = -0.5$, $\delta_3 = 0.8$, and $\phi = 0.7$. Considering δ_1 , δ_2 , and δ_3 as

TABLE 3: Simulated computer series system of two components.

Sys	Processor lifetime	Memory lifetime	Sys	Processor lifetime	Memory lifetime	Sys	Processor lifetime	Memory lifetime	Sys	Processor lifetime	Memory lifetime	Sys	Processor lifetime	Memory lifetime
1	1.9292	3.9291	11	1.9386	4.0043	21	1.1739	3.3857	31	0.1181	0.0884	41	0.6270	1.7289
2	3.6621	0.0026	12	2.1000	2.0513	22	1.3482	1.9705	32	5.0533	2.3238	42	0.7947	0.7947
3	3.9608	0.8323	13	0.9867	0.9867	23	3.0935	3.0935	33	1.6465	2.0197	43	0.5079	5.3535
4	2.3504	3.3364	14	0.1837	0.1837	24	2.1396	2.1548	34	0.9096	0.6214	44	2.5913	2.5913
5	1.0833	3.3059	15	1.3989	4.1268	25	1.3288	0.9689	35	1.7494	2.3643	45	2.5372	2.4923
6	2.8414	1.8438	16	2.3757	2.7953	26	0.1115	0.1115	36	0.1058	0.1058	46	1.1917	0.0801
7	0.3309	0.3309	17	3.5202	1.4095	27	0.8503	2.8578	37	0.4593	0.4593	47	1.5254	4.4088
8	2.9884	1.5961	18	2.3364	0.1624	28	0.1955	0.1955	38	0.9938	1.7689	48	1.0986	1.0986
9	0.5784	1.8795	19	0.8584	1.9556	29	0.4614	0.8584	39	5.7561	0.3212	49	1.0051	1.0051
10	0.5520	0.5520	20	4.3435	1.0001	30	3.3887	1.9796	40	6.2950	1.0495	50	1.3640	1.3640

FIGURE 4: Histogram of the data in Table 3.

TABLE 4: The average maximum likelihood estimates of the BIEGIKw-Weibull parameters for the data in Table 3.

Parameter	Estimate	AIC	BIC
α	0.82096924		
β	0.38096869		
γ	1.03547483		
λ	1.25168661		
δ	0.23330064	325.4836	342.6918
ϕ	1.65658910		
δ_1	-0.36575551		
δ_2	0.86741494		
δ_3	-0.05319925		

FIGURE 5: Continued.

FIGURE 5: (a) The joint density function, (b) the cumulative function, (c) the hazard function, and (d–g) the marginals for the estimated parameters in Table 4.

chosen parameters, the maximum likelihood method is used to estimate the rest parameters; results are given in Tables 1 and 2. Depending on the bias and standard error, the maximum likelihood method shows well performance.

7. Real Data Application

In this section, the data discussed by [31] is used to investigate the applicability of the new distribution. Let $(u_1, v_1), (u_2, v_2), \dots, (u_{50}, v_{50})$ be observed values of a BIEGIKw-Weibull random variable (U, V) with parameters $(\alpha, \beta, \gamma, \lambda, \delta, \phi, \delta_1, \delta_2, \delta_3)$, where U is the processor lifetime and V is the memory lifetime; see Table 3 and Figure 4. The maximum likelihood method is conducted to estimate the parameters with AIC and BIC; see Table 4. The joint density function, hazard function, and marginals for the estimated parameters are given in Figure 5.

8. Conclusions

Analysis of correlated data is one of the most important problems in statistics and data science. Here, we introduced a new bivariate distribution named BIEGIKw-Weibull. The proposed model is with nine parameters. It is a flexible one. Theoretical properties including density function, cumulative function, marginals, copula function, conditional distributions, and conditional moments have been derived explicitly. The new model exhibits very rich characteristics that differ according to the parameters. That supports the applicability of the model for a large set of correlated data with various properties. Simulation clearly verified the theoretical properties and the richness of its preferable characteristics. For different values of the parameters, the distribution has extremely different properties that are clear from Figures 1–3. For a set of values, we can observe symmetry and close surface for density function side by side with unimodality (Figure 1). All these properties deformed for

another set of parameters (Figure 3). The bivariate hazard function exhibited different shapes. Monte Carlo simulation and real data application prove the applicability of the new distribution and the availability of the maximum likelihood method.

Data Availability

The data used are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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